FANO MANIFOLDS OF DEGREE TEN AND EPW SEXTICS

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Abstract. O'Grady showed that certain special sextics in $\mathbb{P}^5$ called EPW sextics admit smooth double covers with a holomorphic symplectic structure. We propose another perspective on these symplectic manifolds, by showing that they can be constructed from the Hilbert schemes of conics on Fano fourfolds of degree ten. As applications, we construct families of Lagrangian surfaces in these symplectic fourfolds, and related integrable systems whose fibers are intermediate Jacobians.

1. Introduction

EPW sextics (named after their discoverers, Eisenbud, Popescu and Walter) are some special hypersurfaces of degree six in $\mathbb{P}^5$, first introduced in [EPW] as examples of Lagrangian degeneracy loci. These hypersurfaces are singular in codimension two, but O'Grady realized in [OG1, OG2, OG3] that they admit smooth double covers which are irreducible holomorphic symplectic fourfolds. In fact, the first examples of such double covers were discovered by Mukai in [Mu2], who constructed them as moduli spaces of stable rank two vector bundles on a polarized K3 surface of genus six. From this point of view, the symplectic structure is induced from the K3 surface. It carries over to double covers of EPW sextics by a deformation argument.

The main goal of this paper is to provide another point of view on this symplectic structure. Our starting point will be smooth Fano fourfolds Z of index two, obtained by cutting the six dimensional Grassmannian $G(2,5)$, considered in its Plücker embedding, by a hyperplane and a quadric. Our main observation is that the Hodge number $h^{3,1}(Z)$ equals one (Lemma 4.1). By the results of e.g. [KM], a generator of $H^{3,1}(Z)$ induces a closed holomorphic two-form on the smooth part of any Hilbert scheme of curves on Z. We focus on the case of conics. The most technical part of the paper consists in proving that for Z general, the Hilbert scheme $F_g(Z)$ of conics in Z is smooth (Theorem 3.2). It is thus endowed with a canonical (up to scalar) global holomorphic two-form.

Since $F_g(Z)$ has dimension five, it can certainly not be a symplectic variety. However, it admits a natural map to a sextic hypersurface $Y_Z^\vee$ in $\mathbb{P}^5$. We consider the Stein factorization

$$F_g(Z) \to \tilde{Y}_Z^\vee \to Y_Z^\vee.$$

It turns out that $\tilde{Y}_Z^\vee$ is a smooth fourfold, over which $F_g(Z)$ is essentially a smooth fibration in projective lines. Thus the two-form on $F_g(Z)$ descends to $\tilde{Y}_Z^\vee$. We show that this makes of $\tilde{Y}_Z^\vee$ a holomorphic symplectic fourfold (Theorem 4.13). Moreover the map $\tilde{Y}_Z^\vee \to Y_Z^\vee$ is a double cover, such that the associated involution of $\tilde{Y}_Z^\vee$ is anti-symplectic. This implies that $Y_Z^\vee$ is an EPW sextic (Proposition 4.17), and that $\tilde{Y}_Z^\vee$ does indeed coincide with the double cover constructed by O'Grady (Proposition 4.18).

Apart from making O'Grady's construction more transparent, at least from our point of view, our approach has several interesting consequences.

First, it shows that double covers of EPW sextics are very close to another classical example of symplectic fourfolds, namely the Fano varieties of lines on cubic fourfolds. Indeed, a smooth cubic fourfold Z also has $h^{3,1}(Z) = 1$, and the symplectic form on its Fano scheme of lines $F(Z)$ can be seen as induced from a generator of $H^{3,1}(Z)$, exactly as above. Note that a similar line of ideas has been used to explain the existence of a non degenerate two-form on the symplectic fourfolds in $G(6, 10)$ recently discovered in [DV].

Second, it sheds some light on the intriguing interplay between the varieties of type $Z = G(2, 5) \cap Q \cap L$ of different dimensions N, where L denotes a linear space of dimension $N + 4$. If $N = 2$, one gets the genus six K3 surfaces which were, thanks to Mukai's observations, at the beginning of that story, but whose associated sextics form only a codimension one family in the moduli space of all EPW sextics (see [OG4]). If $N = 5$, it is very easy to see that there is an EPW sextic attached to Z; we explain this in Proposition 2.1, as a way to introduce these special sextics. The case $N = 3$ was the main theme of investigations of [Lo] and [DIM]; in these studies the surface of conics on Z played a crucial role; it is very closely related to the singular locus of the EPW sextic attached to Z. Finally, for $N = 4$ we have seen how to construct an EPW sextic from the family of conics on Z. In particular, we conclude that for any $N = 3, 4$ or 5, a general EPW sextic is attached to a general Z, in fact a certain family of such sextics. For sure there is more to understand about this, see section 4.5 for a tentative discussion.

Third, from the fourfold Z we obtain a rather concrete description of the symplectic form on $\tilde{Y}_Z^\vee$ (while in [OG3] its existence was only guaranteed by a deformation argument). This allows us to exhibit certain Lagrangian surfaces in $\tilde{Y}_Z^\vee$, that we construct either from threefolds that are hyperplane sections of Z (Proposition 5.2), or fivefolds that contain Z as a hyperplane section (Proposition 5.6). More, we are able to construct, over the moduli stacks parametrizing these families of threefolds (respectively, fivefolds), two integrable systems whose Liouville tori are the corresponding intermediate Jacobians (Theorems 5.3 and 5.7). Again, this is strikingly similar to the constructions of [IM2], of two integrable systems over the moduli stacks parametrizing cubic threefolds (respectively, fivefolds) contained in (respectively, containing) a given cubic fourfold.

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Notation.

V_5 is a five-dimensional complex vector space. The Grassmannian $G = G(2, 5) = G(2, V_5)$ parametrizes two-dimensional vector spaces in V_5.

$Z = G \cap Q \cap H$ is the intersection of G, considered in its Plücker embedding, with a quadric Q and a hyperplane $H = \mathbb{P}V_9$, where $V_9 \subset \wedge^2 V_5$.

$I_2(Z)$ denotes the linear system of quadrics containing Z. The hyperplane of quadrics containing G, called Pfaffian quadrics, is $I_2(G) \simeq V_5$. The hyperplane of Pfaffian quadrics in the projectivization $I = \mathbb{P}(I_2(G)) \simeq \mathbb{P}^5$ is denoted H_P. In the dual projective space $I^\vee$, it defines a point h_P called the Plücker point.

$Y_Z \subset I$ denotes the closure of the locus of singular non Pfaffian quadrics. The projectively dual hypersurface is $Y_Z^\vee \subset I^\vee$. The variety $\hat{Y}_Z^\vee$ parametrizes pairs $(h, V_4) \in I^\vee \times \mathbb{P}V_5^\vee$ such that a quadric in h cuts $\mathbb{P}(\wedge^2 V_4) \cap H$ along a singular quadric.

$F_g(G)$ is the Hilbert scheme of conics in G, $F(G)$ is the nested Hilbert scheme of pairs $(c, V_4) \in F_g(G) \times \mathbb{P}V_5^\vee$ such that $c \subset G(2, V_4)$.

$F_g(Z)$ is the Hilbert scheme of conics in Z, $F(Z)$ its preimage in $F(G)$.

For c a generic conic in Z, there is a unique V_4 such that $(c, V_4) \in F(Z)$, and we denote $G_c = G(2, V_4)$, $P_c = G(2, V_4) \cap H$ and $S_c = P_c \cap Q$. In the pencil of quadrics containing S_c, the unique quadric containing the plane $\langle c \rangle$ spanned by c is denoted Q_{c,V_4}.

This defines maps $F(Z) \to \hat{Y}_Z^\vee$ and $F_g(Z) \to Y_Z^\vee$. The varieties $\bar{Y}_Z^\vee$ and $\tilde{Y}_Z^\vee$ are defined by the Stein factorizations $F(Z) \to \bar{Y}_Z^\vee \to \hat{Y}_Z^\vee$ and $F_g(Z) \to \tilde{Y}_Z^\vee \to Y_Z^\vee$.

2. EPW sextics in duality

2.1. **Quadratic sections of $G(2,5)$.** Let V_5 be a five dimensional complex vector space. Denote by $G = G(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)$ the Grassmannian of planes in V_5, considered in the Plücker embedding. Let $X = G \cap Q$ be a general quadric section: this is a Fano fivefold of index three and degree ten. In the sequel, when we will talk about a Fano manifold of degree ten, this will always mean a variety of this type, or possibly a linear section.

Let $I = |I_X(2)|$ denote the linear system of quadrics containing X. Then $I \simeq \mathbb{P}^5$ is generated by Q and the hyperplane $H_P = |I_G(2)|$ of Pfaffian quadrics. Note that once we have chosen an isomorphism $\wedge^5 V_5 \simeq \mathbb{C}$, there is a natural isomorphism

$$V_5 \simeq I_G(2), \qquad v \mapsto P_v(x) = v \wedge x \wedge x.$$

To be more precise, $H_P \simeq \wedge^4 V_5^\vee \simeq V_5 \otimes \det(V_5^\vee)$.

The Pfaffian quadrics P_v are all of rank six. Therefore, the divisor D_X of degree ten parametrizing singular quadrics in I decomposes as

$$D_X = 4H_P + Y_X,$$

for some sextic hypersurface $Y_X \subset I$.

On the other hand, consider a hyperplane V_4 of V_5. Then the Plücker quadrics cut $\mathbb{P}(\wedge^2 V_4) \subset \mathbb{P}(\wedge^2 V_5)$ along the same quadric, namely the Grassmannian $G(2, V_4)$. Therefore the quadrics in $|I_X(2)|$ cut out a pencil of quadrics in $\mathbb{P}(\wedge^2 V_4)$. If V_4 is general, the generic quadric in this pencil is smooth, and there is a finite number of hyperplanes in $|I_X(2)|$ restricting to singular quadrics in $\mathbb{P}(\wedge^2 V_4)$. This condition defines a hypersurface $Y_X^\vee \subset I^\vee$. The following statement is essentially contained in [OG3] (see in particular Propositions 7.1 and 3.1).

Proposition 2.1. *The two hypersurfaces $Y_X \subset I$ and $Y_X^\vee \subset I^\vee$ are projectively dual EPW sextics.*

First we need to recall briefly the definition of an EPW sextic (for more details see [EPW, OG3]; the version we give here follows [OG4], section 3.2). One starts with a six-dimensional vector space U_6. Then $\wedge^3 U_6$ is twenty-dimensional and admits a natural non degenerate skew-symmetric form (once we have fixed a generator of $\wedge^3 U_6 \simeq \mathbb{C}$). Let then $A \subset \wedge^3 U_6$ be a ten-dimensional Lagrangian subspace. The associated EPW sextic $Y_A^\vee \subset \mathbb{P}(U_6^\vee)$ is defined as

$$Y_A^\vee = \{H \subset U_6, \quad \wedge^3 H \cap A \neq 0\}.$$

If A is general enough, then $Y_A^\vee$ is singular exactly along

$$S_A = \{H \subset U_6, \quad \dim(\wedge^3 H \cap A) \geq 2\},$$

which is a smooth surface.

Proof. The quadric Q in $G(2, V_5)$ is defined by a tensor in $S^2(\wedge^2 V_5)^\vee$ modded out by the space of Pfaffian quadrics. We choose a representative Q_0 in $S^2(\wedge^2 V_5)^\vee$. In particular, the choice of Q_0 induces a decomposition $I = H_P \oplus \mathbb{C}Q_0$, hence a decomposition

$$\wedge^3 I \simeq \wedge^3 H_P \oplus \wedge^2 H_P \otimes Q_0.$$

Observe that if we let $D = \det V_5^\vee$, then $H_P \simeq V_5 \otimes D$, hence $\wedge^2 H_P \simeq \wedge^2 V_5 \otimes D^2$ and $\wedge^3 H_P \simeq \wedge^3 V_5 \otimes D^3 \simeq \wedge^2 V_5^\vee \otimes D^2$. We can therefore attach to Q_0 the subspace $A(Q_0)$ of $\wedge^3 I$ defined as

$$A(Q_0) := \left\{(Q_0(x, \bullet) \otimes d^2, x \otimes d^2 \otimes Q_0), \quad x \in \wedge^2 V_5\right\},$$

where d is some generator of D. Then $A(Q_0)$ is a Lagrangian subspace of $\wedge^3 I$, canonically attached to the point defined by Q_0 in $I - H_P \simeq \mathbb{C}^5$. Consider the EPW sextic $Y_{A(Q_0)}^\vee \subset I^\vee$.

Lemma 2.2. $Y_{A(Q_0)}^\vee \simeq Y_X^\vee$.

Proof. We prove that $Y_{A(Q_0)} \supset Y_X^\vee$. Since they are both sextic hypersurfaces, this will imply the claim.

A point of $Y_X^\vee$ is defined by a hyperplane $H \subset I$ parametrizing quadrics that are all singular when restricted to $\mathbb{P}(\wedge^2 V_4)$, for some hyperplane $V_4 \subset V_5$. If H is not the Pfaffian hyperplane H_P, we can define it as the space of quadrics of the form $Q_v := P_v - \lambda(v)Q_0$, for some linear form λ on V_5. By the hypothesis, there exists some non zero $p \in \wedge^2 V_4$ such that $Q_v(p, q) = 0$ for any $q \in \wedge^2 V_4$. Generically, this p will not be contained in the cone over $G(2, V_4)$. Otherwise said, p has rank four, $p \wedge p \neq 0$, and V_4 is defined uniquely by p.

Observe that the kernel of λ must be V_4. Indeed, if $\lambda(v) = 0$, we get that $P_v(p, p) = v \wedge p \wedge p = 0$. But this implies that v belongs to V_4.

The subspace $\wedge^3 H$ of $\wedge^3 I$ is generated by the tensors $P_u \wedge P_v \wedge (P_w - \lambda(w)Q_0)$, for $u, v \in V_4$ and $w \in V_5$. We can see it as the graph Γ of the map $\wedge^3 H_P \to \wedge^2 H_P \otimes Q_0$ induced by the map $H_P \to \mathbb{C}Q_0$ sending P_v to $\lambda(v)Q_0$. The Lemma is a consequence of the following assertion.

Claim. The point $(Q_0(p,\bullet), p\otimes Q_0)$ belongs to $\Gamma\cap A(Q_0)$.

This is clearly a point of $A(Q_0)$, so we just need to check that it belongs to Γ. Observe that Γ contains the points $(p\wedge w, \lambda(w)p\otimes Q_0)$, for all $w\in V_5$, so we just need to prove that there exists some non zero w such that

$$R_w(\bullet) := p\wedge w\wedge \bullet - \lambda(w)Q_0(p,\bullet) = 0.$$

Here R_w is to be considered as a linear form on $\wedge^2 V_5$, and we know that it vanishes on $\wedge^2 V_4$. But the orthogonal to $\wedge^2 V_4$ in $\wedge^2 V_5$ is isomorphic to $V_4^\vee$ (once we have chosen a generator of V_5/V_4), which means that we can identify R_w with a linear form $r(w)$ on V_4, depending linearly on w. But then the linear map $r: V_5\to V_4^\vee$ must have a non trivial kernel, and we are done. □

Lemma 2.3. *$Y_X^\vee$ is dual to Y_X.*

Proof. Consider a general point of Y_X, defined by a non Pfaffian singular quadric Q, with singular point p. Suppose that the infinitesimally near quadric $Q+\delta Q$ remains singular at the point $p+\delta p$. We may suppose that $\delta Q = P_{\delta v}$ and we get the order one condition that

$$(Q+P_{\delta v})(p+\delta p,\bullet) = Q(\delta p,\bullet) + P_{\delta v}(p,\bullet) = 0.$$

Generically, $p\in\wedge^2 V_5$ has rank four, hence belongs to $\wedge^2 V_4$ for a unique hyperplane V_4 of V_5. We claim that the quadrics $Q+\delta Q$ are all singular at p, after restriction to $\wedge^2 V_4$. That is, we claim that

$$(Q+\delta Q)(p,q) = 0 \qquad \forall q\in\wedge^2 V_4.$$

Indeed, $\wedge^4 V_4$ is one dimensional and generated by $p\wedge p$, hence $p\wedge q = \alpha(q)p\wedge p$ for some linear form α on $\wedge^2 V_4$. Then, by the identity above, and the fact that $Q(p,\bullet)=0$ since Q is singular at p, we get

$$\begin{aligned}(Q+\delta Q)(p,q) &= \delta Q(p,q)\\ &= P_{\delta v}(p,q)\\ &= \delta v\wedge p\wedge q\\ &= \alpha(q)\delta v\wedge p\wedge p\\ &= \alpha(q)P_{\delta v}(p,p)\\ &= -\alpha(q)Q(\delta p,p)\\ &= 0.\end{aligned}$$

This means that the generic tangent hyperplane to Y_X defines a point of $Y_X^\vee$, hence that $Y_X^\vee$ is projectively dual to Y_X.

This concludes the proof of the Lemma, and of the Proposition as well. □

2.2. **Variants.** Consider now a smooth degree ten variety X of dimension $5-k$, defined as the intersection of $G(2,V_5)$ with a quadric and a codimension k linear subspace $\mathbb{P}V_{10-k}$ of $\mathbb{P}(\wedge^2 V_5)$. As before we denote by $I\simeq\mathbb{P}^5$ the linear system of quadrics in $\mathbb{P}V_{10-k}$ containing X, and by H_P the Pfaffian hyperplane. Generically the Pfaffian quadrics have rank 6, hence corank $4-k$. Hence the hypersurface D_X of degree $10-k$, parametrizing singular quadrics in I, decomposes as

$$D_X = (4-k)H_P + Y_X,$$

where Y_X is again a sextic hypersurface.

As before, we can also define a hypersurface $Y_X^\vee \subset I^\vee$, parametrizing the hyperplanes in I made of quadrics whose restrictions to some $\mathbb{P}(\wedge^2 V_4 \cap V_{10-k})$ are all singular, V_4 being a hyperplane in V_5. The same proof as for the $k=0$ case yields the following result.

Proposition 2.4. *The two hypersurfaces $Y_X \subset I$ and $Y_X^\vee \subset I^\vee$ are projectively dual sextics.*

Proof. The only thing we have to prove is that $Y_X^\vee$ has degree six. For this we describe, following [Lo], this hypersurface as the image of a degeneracy locus, defined as follows. Consider over $\mathbb{P} = \mathbb{P}V_5^\vee$ the rank two vector bundle $F = \mathcal{O}_\mathbb{P} \oplus \mathcal{O}_\mathbb{P}(1)$, and the rank $6-k$ vector bundle M whose fiber over V_4 is $\wedge^2 V_4 \cap V_{10-k}$. For a generic V_{10-k} this is indeed a vector bundle, at least for $0 \leq k \leq 2$. Let $\mathcal{O}_F(-1)$ be the tautological line bundle over $\mathbb{P}(F)$. There is a morphism of vector bundles

$$\eta : \mathcal{O}_F(-1) \to S^2 M^\vee,$$

defined by mapping a pair $(z,v) \in \mathbb{C} \oplus V_5$ to the restriction of the quadric $zQ + P_v$ to M_{V_4}. Since this restriction does only depend on the class of v modulo V_4, this mapping factors through η.

The first degeneracy locus $\hat{Y}_X^\vee$ of η, defined by the condition that the resulting quadric be singular, is a divisor linearly equivalent to

$$[\hat{Y}_X^\vee] = 2c_1(M^\vee) - (6-k)c_1(\mathcal{O}_F(-1)).$$

One easily computes that $c_1(M^\vee) = 3h$, where h denotes the hyperplane class of $\mathbb{P}(V_5)$.

Observe that there is a natural map from $\mathbb{P}(F)$ to $I^\vee$. Indeed, a point of $\mathbb{P}(F)$ over some V_4 is of the form $[\lambda, \phi]$, for $\lambda \in \mathbb{C}$ and ϕ a linear form on V_5 vanishing on V_4. It defines the hyperplane in I consisting of quadrics of the form $zQ + P_v$ for $\lambda z + \phi(v) = 0$.

In fact this map $\mathbb{P}(F) \to I^\vee$ is just the blow-up of the point h_P in $I^\vee$ defined by the Plücker hyperplane. This yields a basis of the Picard group of $\mathbb{P}(F)$ consisting of the exceptional divisor E, and the pull-back H of the hyperplane class of $I^\vee$. A standard computation yields $c_1(\mathcal{O}_F(-1)) = -E$ and $h = H - E$. Hence

$$[\hat{Y}_X^\vee] = 6h + (6-k)E = 6H - kE.$$

But the hypersurface $Y_X^\vee$ is just the image of $\hat{Y}_X^\vee$ in $I^\vee$. Therefore, this formula reads as follows: $Y_X^\vee$ is a degree six hypersurface having multiplicity k at h_P. □

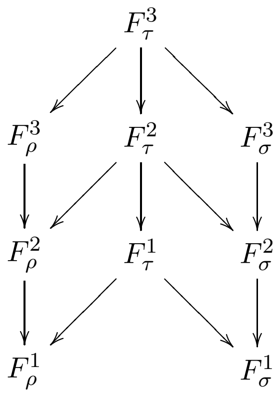
3. Conics on Fano fourfolds of degree ten

3.1. **Conics on $G(2,5)$.** Consider the Hilbert scheme $F_g(G)$ parametrizing conics in $G = G(2,V_5)$. In order to study this scheme, we first recall that conics in G can be partitioned into three different classes, according to the type of their supporting plane:

(1) τ-conics are conics spanning a plane which is not contained in $G(2,V_5)$; any smooth τ-conic can be parametrized by $(s,t) \mapsto (sv_1 + tv_2) \wedge (sv_3 + tv_4)$ for some linearly independent vectors v_1, v_2, v_3, v_4 in V_5;

(2) σ-conics are conics parametrizing lines passing through a common point; any smooth σ-conic can be parametrized by $(s,t) \mapsto v_1 \wedge (s^2 v_2 + stv_3 + t^2 v_4)$ for some linearly independent vectors v_1, v_2, v_3, v_4 in V_5;
(3) ρ-conics are conics parametrizing lines contained in a common plane; any smooth ρ-conic can be parametrized by $(s,t) \mapsto (sv_1 + tv_2) \wedge (sv_2 + tv_3)$ for some linearly independent vectors v_1, v_2, v_3 in V_5.

Each of these three classes of conics is partitioned into three orbits of $PGL(V_5)$, consisting of smooth conics, singular but reduced conics, and double lines. In particular $F_g(G)$ has exactly nine $PGL(V_5)$-orbits. Exactly two are closed: the orbits F^1_ρ and F^1_σ parametrizing double-lines of type ρ and of type σ. They are isomorphic, respectively, with the partial flag varieties $F(2,3,V_5)$ and $F(1,3,4,V_5)$; their dimensions are 8 and 9. The incidence diagram is the following one:



Theorem 3.1. *The Hilbert scheme $F_g(G)$ of conics in $G = G(2,V_5)$ is irreducible and smooth, of dimension* 13*.*

Proof. The dimension count is straightforward. The singular locus being closed, it is enough to check the smoothness at one point of each of the two closed orbits F^1_ρ and F^1_σ. Such a point represents a double-line ℓ. Recall (e.g. from [Se]) that the Zariski tangent space to $F_g(G)$ at the point represented by ℓ is given by

$$T_{[\ell]}F_g(G) = Hom_{\mathcal{O}_G}(\mathcal{I}_\ell, \mathcal{O}_\ell).$$

What we need to check is that this vector space has dimension 13. Since $F_g(G)$ is certainly connected, its smoothness will imply its irreducibility.

Double-line of type σ. We choose for the support of ℓ the σ-plane generated by $v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4$, for some basis $v_1, \ldots, v_5$ of V_5, and we choose in this plane the double line ℓ of equations

$$p_{14}^2 = 0, \quad p_{15} = p_{23} = p_{24} = p_{25} = p_{34} = p_{35} = p_{45} = 0,$$

where the p_{ij}'s denote the Plücker coordinates on $G(2,V_5)$ associated to our choice of basis.

We first compute $\mathcal{H}om_{\mathcal{O}_G}(\mathcal{I}_\ell, \mathcal{O}_\ell)$ in the affine neighborhood of $v_1 \wedge v_2$ parametrizing planes which are transverse to $\langle v_3, v_4, v_5 \rangle$. Such a plane has a unique basis of the form

$$\begin{aligned} u_1 &= v_1 + x_3v_3 + x_4v_4 + x_5v_5, \\ u_2 &= v_2 + y_3v_3 + y_4v_4 + y_5v_5. \end{aligned}$$

In these coordinates we have $\mathcal{I}_\ell = \langle y_4^2, y_5, x_3, x_4, x_5\rangle$. An element ϕ of $\mathcal{H}om_{\mathcal{O}_G}(\mathcal{I}_\ell, \mathcal{O}_\ell)$ associates to each of these generators a section of $\mathcal{O}_\ell$, which can be represented as $p(y_3) + y_4p'(y_3)$ for some polynomials p and p'.

We can make the same analysis in the affine neighborhood of $v_1 \wedge v_3$ parametrizing planes which are transverse to $\langle v_2, v_4, v_5\rangle$. Such a plane has a unique basis of the form

$$\begin{aligned} w_1 &= v_1 + z_2v_2 + z_4v_4 + z_5v_5, \\ w_3 &= v_3 + t_2v_2 + t_4v_4 + t_5v_5. \end{aligned}$$

In these coordinates we have $\mathcal{I}_\ell = \langle t_4^2, t_5, z_2, z_4, z_5\rangle$. An element ψ of $\mathcal{H}om_{\mathcal{O}_G}(\mathcal{I}_\ell, \mathcal{O}_\ell)$ associates to each of these generators a section of $\mathcal{O}_\ell$, which can be represented as $q(t_3) + t_4q'(t_3)$ for some polynomials q and q'.

Now, we want such a ψ do be defined globally along ℓ, which means that it must extend to a regular morphism over the previous neighborhood. Over $t_2 \neq 0$, the formulas for the change of coordinates are the following:

$$\begin{aligned} x_3 = -\frac{z_2}{t_2}, \quad x_4 = z_4 - \tfrac{z_2}{t_2}t_4, \quad x_5 = z_5 - \frac{z_2}{t_2}t_5, \\ y_3 = \frac{1}{t_2}, \qquad y_4 = \tfrac{t_4}{t_2}, \qquad y_5 = \frac{t_5}{t_2}. \end{aligned}$$

Suppose that ψ maps t_4^2 to $q(t_2) + t_4q'(t_2)$. Then it maps $y_4^2 = t_4^2/t_2^2$ to

$$t_2^{-2}q(t_2) + t_4t_2^{-2}q'(t_2) = y_3^2q(y_3^{-1}) + y_4y_3q'(y_3^{-1}).$$

Therefore $y_3^2q(y_3^{-1})$ and $y_3q'(y_3^{-1})$ must be regular, which means that q is at most quadratic and q' is affine. Treating the other conditions similarly we check that ψ must be of the following form:

$$\begin{aligned} t_4^2 &\mapsto \psi_1 + \psi_2t_2 + \psi_3t_2^2 + (\psi_4 + \psi_5t_2)t_4, \\ t_5 &\mapsto \psi_6 + \psi_7t_2 + \psi_8t_4, \\ z_2 &\mapsto \psi_9 + \psi_{10}t_2 + \psi_{11}t_4, \\ z_4 &\mapsto \psi_{12} + \psi_{10}t_4, \\ z_5 &\mapsto \psi_{13} + \psi_{10}t_4. \end{aligned}$$

So there are exactly 13 free parameters $\psi_1, \dots, \psi_{13}$ for ψ, as required.

Double-line of type ρ. We choose for the support of ℓ the ρ-plane generated by $v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3$, for some basis $v_1, \dots, v_5$ of V_5, and we choose in this plane the double line ℓ of equations

$$p_{23}^2 = 0, \quad p_{14} = p_{24} = p_{15} = p_{25} = p_{34} = p_{35} = p_{45} = 0.$$

We compute in the same affine neighborhoods of $v_1 \wedge v_2$ and $v_1 \wedge v_3$. In the latter, we have $\mathcal{I}_\ell = \langle z_2^2, z_4, z_5, t_4, t_5\rangle$. An element ψ of $\mathcal{H}om_{\mathcal{O}_G}(\mathcal{I}_\ell, \mathcal{O}_\ell)$ associates to each of these generators a section of $\mathcal{O}_\ell$, which can be represented as $q(t_2) + z_2q'(t_2)$ for some polynomials q and q'.

A similar analysis as before shows that to be defined globally, such a morphism ψ must be of the following type:

$$\begin{array}{rcl} z_2^2 & \mapsto & \psi_1 + \psi_2 t_2 + \psi_3 t_2^2 + (\psi_4 + \psi_5 t_2) z_2, \\ t_4 & \mapsto & \psi_6 + \psi_7 t_2 + \psi_8 t_4, \\ t_5 & \mapsto & \psi_9 + \psi_{10} t_2 + \psi_{11} t_4, \\ z_4 & \mapsto & \psi_{12} + \psi_7 t_4, \\ z_5 & \mapsto & \psi_{13} + \psi_{10} t_4. \end{array}$$

Again there are exactly 13 free parameters $\psi_1, \ldots, \psi_{13}$ for ψ, as required. This concludes the proof. □

Remark. One can check that $F_g(G)$ is a spherical variety, which means that a Borel subgroup of $PGL(V_5)$ acts transitively on some open subset. Moreover the Picard number of $F_g(G)$ is three. Indeed, we can consider the nested Hilbert scheme $F(G)$ parametrizing pairs (c, V_4), for c a conic in G and $V_4 \subset V_5$ a hyperplane such that c is contained in the quadric $G(2, V_4)$. One can check that the forgetful map $F(G) \to F_g(G)$ is the blow-up of the codimension two smooth variety $F_\rho(G)$ parametrizing ρ-conics. In particular $F(G)$ contains two disjoint divisors $E_\rho(G)$ and $E_\sigma(G)$, the preimages of the subvarieties $F_\rho(G)$ and $F_\sigma(G)$ of $F_g(G)$ parametrizing ρ and σ conics, respectively. These two divisors can themselves be contracted to the variety $S(G)$ parametrizing pairs (P, V_4), where P is a projective plane inside $\mathbb{P}(\wedge^2 V_4)$. Of course $S(G)$ is a Grassmann bundle over $\mathbb{P}V_5^\vee$. The condition that P be contained inside $G(2, V_4)$ defines two subvarieties $S_\rho(G)$ and $S_\sigma(G)$ (according to the type of P), both of codimension six. Blowing-up $S(G)$ over their union gives $F(G)$. This is summarized by the following diagram:

$$\begin{array}{ccccc} & & F(G) & & \\ & \swarrow & & \searrow & \\ S(G) & & & & F_g(G) \\ \downarrow & & & & \\ \mathbb{P}V_5^\vee & & & & \end{array}$$

3.2. **Conics on the general Fano fourfold of degree ten.** Now let $Z = G(2,5) \cap H \cap Q$ be a general Fano fourfold of degree ten. We denote by $F_g(Z)$ the Hilbert scheme of conics in Z. In this section our main goal is to prove the following statement.

Theorem 3.2. *For a general Z, the Hilbert scheme $F_g(Z)$ of conics on Z is a smooth fivefold.*

The proof of this result will occupy the rest of the section. We will need several auxiliary results, with different techniques to handle the three types of conics and their possible singularities.

Reduced conics.

We begin with smooth conics. According to its type, the restriction to a smooth conic c on G, of the dual tautological bundle $T^\vee$, and of the quotient bundle Q, split as follows (we denote by $\mathcal{O}_c(1)$ the ample generator of the Picard group of c, so that $\mathcal{O}_Z(1)_{|c} = \mathcal{O}_c(2)$):

type	$T_c^\vee$	Q_c
τ	$\mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$	$\mathcal{O}_c(1) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c$
σ	$\mathcal{O}_c(2) \oplus \mathcal{O}_c$	$\mathcal{O}_c(1) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c$
ρ	$\mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$	$\mathcal{O}_c(2) \oplus \mathcal{O}_c \oplus \mathcal{O}_c$

This follows at once from the fact that $T_c^\vee$ and Q_c are globally generated and of degree two, and the definitions of the three types of conics. This gives the splitting of the tangent bundle $TG = T^\vee \otimes Q$ restricted to c. We can deduce the splitting type of the normal bundle:

type	$N_{c/G}$
τ	$\mathcal{O}_c(2) \oplus \mathcal{O}_c(2) \oplus \mathcal{O}_c(2) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$
σ	$\mathcal{O}_c(4) \oplus \mathcal{O}_c(2) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c$
ρ	$\mathcal{O}_c(4) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$

Then we consider the normal exact sequence for the triple $c \subset Z \subset G$,

$$0 \to N_{c/Z} \to N_{c/G} \stackrel{\theta}{\to} N_{Z/G|c} = \mathcal{O}_c(2) \oplus \mathcal{O}_c(4) \to 0.$$

Our aim is to deduce the possible splitting types of $N_{c/Z}$, and conclude that $H^1(N_{c/Z}) = 0$. This will ensure the smoothness of $F_g(Z)$ at $[c]$.

Lemma 3.3. *Let $c \subset Z$ be a smooth τ-conic. Then*

$$N_{c/Z} \simeq \mathcal{O}_c \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1).$$

In particular $F_g(Z)$ is smooth at $[c]$.

Proof. There exists a unique hyperplane $V_4 \subset V_5$ such that $c \subset G_c := G(2, V_4)$. Moreover c is a linear section of G_c, so that $N_{c/G_c} = \mathcal{O}_c(2) \oplus \mathcal{O}_c(2) \oplus \mathcal{O}_c(2)$, while $N_{G_c/G} = T^\vee_{|G_c}$ (since G_c is the zero locus of a section of $T^\vee$ on G) and $N_{G_c/G|c} = \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$. The normal exact sequence of the triple $c \subset G_c \subset G$ is split.

Now, c being contained in $Z = G \cap Q \cap H$, it must be contained in the quartic surface $S_c = G_c \cap Q \cap H$. We get the exact sequence

$$0 \to N_{c/S_c} \to N_{c/Z} \to N_{S_c/Z|c} = N_{G_c/G|c} = \mathcal{O}_c(1) \oplus \mathcal{O}_c(1) \to 0.$$

But $\omega_{S_c} = \mathcal{O}_{S_c}(-1)$, hence $\omega_{S_c|c} \simeq \omega_c$. Therefore $N_{c/S_c} \simeq \mathcal{O}_c$ and the exact sequence above must be split. This implies the lemma. ∎

Now suppose that c is a ρ-conic. Consider in the normal exact sequence for the triple $c \subset Z \subset G$, the component $\theta_{44} : \mathcal{O}_c(4) \to \mathcal{O}_c(4)$ of the morphism θ. We say that c is special if $\theta_{44} = 0$.

Lemma 3.4. *Let c be a non special smooth ρ-conic in Z. Then*

$$N_{c/Z} \simeq \mathcal{O}_c \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1).$$

In particular $F(Z)$ is smooth at $[c]$.

Proof. Since $\theta_{44} \neq 0$, it is an isomorphism and we get an exact sequence

$$0 \to N_{c/Z} \to \mathcal{O}_c(1)^{\oplus 4} \to \mathcal{O}_c(2) \to 0.$$

In particular $N^\vee_{c/Z}(1)$ is generated by global sections. This implies that $N_{c/Z} = \mathcal{O}_c(n_1) \oplus \mathcal{O}_c(n_2) \oplus \mathcal{O}_c(n_3)$ with $n_1, n_2, n_3 \leq 1$ and $n_1 + n_2 + n_3 = 2$. The only possibility is that, up to permutation, $n_1 = n_2 = 1$ and $n_3 = 0$. □

Lemma 3.5. *A general Z contains no special ρ-conic.*

Proof. One readily checks, with the previous notations, that $\theta_{44} = 0$ if and only if the quadric Q contains the plane spanned by the ρ-conic c. So, that plane must be contained in Z, which is not possible for a general Z, because of the next easy lemma. □

Lemma 3.6. *If Z is general, it does not contain any plane.*

Proof. There are two families of planes on $G(2, V_5)$: ρ-planes of the form $\mathbb{P}(\wedge^2 V_3)$, for $V_3 \subset V_5$, and σ-planes of the form $\mathbb{P}(V_1 \wedge V_4)$, for $V_1 \subset V_4 \subset V_5$. The family of ρ-planes is parametrized by $G(3, V_5)$, hence six-dimensional. The family of σ-planes is parametrized by the partial flag variety $F(1, 4, V_5)$, hence seven-dimensional.

Containing a projective plane imposes three conditions on hyperplanes, and six conditions on quadrics, hence nine conditions on Z. Since nine is bigger that seven, we are done. □

We can analyze the case of σ-conics in a similar way: we can define a σ-conic c in Z to be special if $\theta_{44} = 0$. As for ρ-conic, this implies that the plane spanned by c is contained in Z, which is not possible for a general Z.

For a non-special σ-conic c, we get an exact sequence

$$0 \to N_{c/Z} \to \mathcal{O}_c \oplus \mathcal{O}_c(1)^{\oplus 2} \oplus \mathcal{O}_c(2) \xrightarrow{\tau} \mathcal{O}_c(2) \to 0.$$

Again we have two cases, according to the vanishing of the component $\tau_{22} : \mathcal{O}_c(2) \to \mathcal{O}_c(2)$. We say that c is of the first kind if $\tau_{22} \neq 0$, and of the second kind otherwise. In the latter case, $N_{c/Z} = \mathcal{O}_c(2) \oplus N$, where N fits into an exact sequence

$$0 \to N \to \mathcal{O}_c \oplus \mathcal{O}_c(1)^{\oplus 2} \to \mathcal{O}_c(2) \to 0.$$

This rank two bundle N has degree zero and $N^\vee(1)$ is generated by global sections, which leaves only two possibilities: $N = \mathcal{O}_c \oplus \mathcal{O}_c$ or $N = \mathcal{O}_c(-1) \oplus \mathcal{O}_c(1)$. We have proved:

Lemma 3.7. *Let c be a non special smooth σ-conic in Z.*

If c is of the first kind,

$$N_{c/Z} \simeq \mathcal{O}_c \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1).$$

If c is of the second kind,

$$\begin{aligned} & N_{c/Z} \simeq \mathcal{O}_c(2) \oplus \mathcal{O}_c \oplus \mathcal{O}_c \\ \textit{or} \quad & N_{c/Z} \simeq \mathcal{O}_c(2) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(-1). \end{aligned}$$

In any case $F(Z)$ is smooth at $[c]$.

This analysis can be extended, with the same conclusions regarding the smoothness of $F(Z)$, to reduced singular conics. This was done in [IM1] in a similar case. We prefer to present a detailed treatment of the case of double lines, which requires a different type of arguments.

Double lines.

Lemma 3.8. *A general Z contains a two-dimensional family of double lines. This family contains a one dimensional sub-family of double-lines of type σ, and a finite number of double-lines of type ρ.*

Proof. This is just a dimension count. There is an eight-dimensional family of lines of G, parametrized by the partial flag variety $F(1,3,V_5)$. For each line ℓ, the set of double lines supported by ℓ is parametrized by a projective plane, with a line parametrizing double lines of type σ, and a unique point corresponding to a double line of type ρ.

More explicitly, if the line ℓ is generated by $e_1 \wedge e_2$ and $e_1 \wedge e_3$, a double line supported by ℓ spans a plane

$$P = \langle e_1 \wedge e_2, e_1 \wedge e_3, ze_2 \wedge e_3 + e_1 \wedge f\rangle,$$

where f is defined up to $\langle e_1, e_2, e_3\rangle$. We can thus parametrize P by the point $[z, \bar{f}] \in \mathbb{P}^2$, where $\bar{f}$ denotes the projection of f to $V_5/\langle e_1, e_2, e_3\rangle$. The corresponding double line has type σ for $z = 0$, and type ρ for $\bar{f} = 0$.

In particular we get a ten dimensional family of double lines on the Grassmannian G. Since containing any of these imposes eight conditions on Z, we are done. □

Now suppose that ℓ be a double line in Z. Denote by $\mathcal{I}_{\ell,Z} \subset \mathcal{O}_Z$ its ideal sheaf in Z, and by $\mathcal{I}_{\ell,G} \subset \mathcal{O}_G$ its ideal sheaf in G. The restriction map $\mathcal{I}_{\ell,G} \to \mathcal{I}_{\ell,Z}$ induces an exact sequence

$$0 \to T_{[\ell]}F_g(Z) = Hom(\mathcal{I}_{\ell,Z}, \mathcal{O}_\ell) \to$$
$$\to T_{[\ell]}F_g(G) = Hom(\mathcal{I}_{\ell,G}, \mathcal{O}_\ell) \xrightarrow{\phi} Hom(\mathcal{I}_{Z,G}, \mathcal{O}_\ell),$$

where $\mathcal{I}_{Z,G}$ denotes the ideal sheaf of Z in G. Since ℓ is a smooth point of $F_g(G)$, the Hilbert scheme $F_g(Z)$ is smooth at $[\ell]$ if and only if ϕ has rank eight.

Lemma 3.9. *Let ℓ be a double line of type τ in G. In the variety parametrizing the Fano fourfolds Z containing ℓ, the subvariety parametrizing those Z for which ℓ is a singular point of $F_g(Z)$, has codimension at least three.*

Proof. The proof is rather computational, see the Appendix. □

Double lines of type σ or ρ can be treated similarly. In fact they are easier to handle, since it is enough to show that for such double-lines, defining a singular point of $F_g(Z)$ impose at least two, resp. one, conditions on Z.

This concludes the proof of Theorem 4.3. □

Remarks.

1. The variety $F_g^\rho(Z)$ of ρ-conics in a general $Z = G \cap Q \cap H$ can be analyzed as follows. Since Z contains no plane, a ρ-conic in Z must be the trace of Q over a ρ-plane of G contained in H. Recall that $G \cap H$ can be interpreted as an isotropic Grassmannian $IG(2, V_5)$, with respect to

a maximal rank two-form ω on V_5. All ρ-planes in $IG(2, V_5)$ are of the form $\mathbb{P}(V_3)$ for $V_3 \subset V_5$ containing the kernel W_1 of ω. Taking the quotient by this kernel, this identifies the variety of ρ-planes in $IG(2, V_5)$, with a Lagrangian Grassmannian $LG(2, V_4)$, which is nothing else than a smooth three-dimensional quadric $\mathbb{Q}^3$. Hence

$$F_g^\rho(Z) \simeq \mathbb{Q}^3.$$

2. Similarly, a σ-conic in Z must be the trace of Q over a σ-plane of G contained in H. Such a σ-plane is defined by a flag $V_1 \subset V_4$, and it is contained in H if and only if $V_4 \subset V_1^\perp$, where the orthogonality is taken with respect to the two-form ω. There are two cases. If V_1 does not coincide with W_1, the kernel of ω, then it determines V_4 uniquely. If $V_1 = W_1$, then V_4 can be any hyperplane containing it. One easily concludes that

$$F_g^\sigma(Z) \simeq Bl_0\mathbb{P}^4,$$

the blow-up of $\mathbb{P}^4$ at one point.

4. A two-form on the Hilbert scheme of conics

Let $Z = G(2, V_5) \cap Q \cap H$ be a general smooth Fano fourfold of degree ten and index two.

4.1. **The Hodge numbers of Z.**

Lemma 4.1. *The Hodge diamond of Z is the following:*

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 22 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Proof. We write $Z = X \cap Q$ and $X = G(2, V_5) \cap H$, with $H = \mathbb{P}V_9$. In order to compute $h^{3,1}(Z) = h^1(Z, TZ(-2))$, we use the normal sequence

$$0 \to TZ \to TX_{|Z} \to \mathcal{O}_Z(2) \to 0.$$

The claim that $h^{3,1}(Z) = 1$ follows from the fact that $TX(-2)_{|Z}$ has no cohomology in degree zero and one, which itself follows from the fact that $TX(-2)$ and $TX(-4)$ have no cohomology in degree zero and one, and one and two, respectively. But X is a linear section of G, and by Bott's theorem $TG(-k)$ is acyclic for $1 \le k \le 4$. This implies that $TG(-k)_{|X}$ is acyclic for $1 \le k \le 3$, and then that $TX(-k)$ is acyclic for $k = 2, 3$. Finally, $TG(-5) \simeq \Omega^5_G$ has non-zero cohomology in degree five only, so $TG(-4)_{|X}$, and $TX(-4)$ a fortiori, have no cohomology in degree less than four.

Now we compute $h^{2,2}(Z) = h^2(Z, \Omega_Z^2) = \chi(Z, \Omega_Z^2)$. Observe that the conormal exact sequence of the inclusion $Z \subset G$ induces a filtration of $\Omega^2_{G|Z}$ with successive quotients Ω_Z^2, $\Omega_Z^1(-1) \oplus \Omega_Z^1(-2)$ and $\mathcal{O}_Z(-3)$, so

$$\chi(\Omega_Z^2) = \chi(\Omega^2_{G|Z}) - \chi(\Omega_Z^1(-1)) - \chi(\Omega_Z^1(-2)) - \chi(\mathcal{O}_Z(-3)).$$

Using the Koszul exact sequence we get that

$$\chi(\Omega^2_{G|Z}) = \chi(\Omega_G^2) - \chi(\Omega_G^2(-1)) - \chi(\Omega_G^2(-2)) + \chi(\Omega_G^2(-3)).$$

Bott's theorem yields $\chi(\Omega_G^2) = 2$, $\chi(\Omega_G^2(-1)) = \chi(\Omega_G^2(-2)) = 0$, and $\chi(\Omega_G^2(-3)) = -5$, hence $\chi(\Omega^2_{G|Z}) = -3$. Computing the other terms similarly we get $\chi(\Omega_Z^2) = 22$. □

4.2. **The induced form on $F_g(Z)$.** Since $h^{1,3}(Z) = 1$, there is a canonical (up to constant) holomorphic two-form induced on $F_g(Z)$. At a point defined by a smooth conic $c \subset Z$, this two-form can be defined on $T_{[c]}F_g(Z) = H^0(N_{c/Z})$ as follows (see [KM]). Choose a generator σ of $H^1(Z, \Omega_Z^3) = H^1(Z, TZ(-2))$. Then consider the composition

$$\begin{aligned}\phi_\sigma :\ \wedge^2 H^0(N_{c/Z}) &\to H^0(\wedge^2 N_{c/Z}) = H^0(N^\vee_{c/Z}(2)) \\ &\xrightarrow{\sigma} H^1(TZ \otimes N^\vee_{c/Z}(-2)) \to H^1(\omega_c) = \mathbb{C}.\end{aligned}$$

For the last arrow we used the natural quotient map $TZ_{|c} \to N_{c/Z}$. Note that rather than using this map, we can proceed as follows. If $X = G \cap H$, recall form the proof of Lemma 4.1 that a generator of $H^1(Z, \Omega_Z^3) = H^1(Z, TZ(-2))$ is given by the extension class of the normal exact sequence

$$0 \to TZ \to TX_{|Z} \to \mathcal{O}_X(2) \to 0.$$

On the conic c, after dualizing, twisting by $\mathcal{O}_X(1)_{|c} = \mathcal{O}_c(2)$ and passing to the normal bundles of c in Z and X, this induces an extension

$$(1) \qquad 0 \to \omega_c \to N^\vee_{c/X}(2) \to N^\vee_{c/Z}(2) \to 0.$$

We can use directly this extension to produce the map

$$H^0(N^\vee_{c/Z}(2)) \to H^1(\omega_c) = \mathbb{C}$$

which defines the two-form at $[c]$, at least up to constant.

Recall that the τ-conic is contained in a unique sub-Grassmannian $G(2, V_4)$ of $G(2, V_5)$, and that we denoted by S_c the quartic surface $G(2, V_4) \cap H \cap Q$.

Proposition 4.2. *Let c be a smooth τ-conic in Z. Then the line*

$$H^0(N_{c/S_c}) \subset H^0(N_{c/Z}) = T_{[c]}F_g(Z)$$

is contained in the kernel of ϕ_σ.

Proof. This means that the composition of maps above vanishes when restricted to $H^0(N_{c/S_c}) \wedge H^0(N_{c/Z}) \subset \wedge^2 H^0(N_{c/Z})$. Consider the commutative diagram

$$\begin{array}{ccccc} \wedge^2 H^0(N_{c/Z}) & \to & H^0(\wedge^2 N_{c/Z}) & = & H^0(N^\vee_{c/Z}(2)) \\ \uparrow & & \uparrow & & \uparrow \\ H^0(N_{c/S_c}) \wedge H^0(N_{c/Z}) & \to & H^0(N_{c/S_c} \wedge N_{c/Z}) & = & H^0(N^\vee_{S_c/Z|c}(2)). \end{array}$$

Since $N^\vee_{S_c/Z|c}(2)$ is the restriction to c of the vector bundle $N^\vee_{S_c/Z}(1)$ on S_c, and we can compute the remaining arrows on S_c before restricting to c. In other words, we can factor through the maps

$$H^0(N^\vee_{S_c/Z}(1)) \stackrel{\cup\sigma}{\to} H^1(TZ \otimes N^\vee_{S_c/Z}(-1)) \to H^1(\mathcal{O}_{S_c}(-1)) \to H^1(\mathcal{O}_c(-2)) = \mathbb{C}.$$

And the result is clearly zero, since $H^1(\mathcal{O}_{S_c}(-1)) = 0$. Indeed, this follows from the Kodaira vanishing theorem when the quartic surface S_c is smooth. By continuity, the same conclusion continues to hold when S_c is singular. □

Proposition 4.3. *Let c be a generic conic in Z. Then ϕ_σ has rank four at the corresponding point $[c]$ of $F_g(Z)$.*

Proof. We denote by P_c the three-dimensional quadric $G(2, V_4) \cap H$. We have $S_c = P_c \cap Q$, and an induced diagram

$$\begin{array}{ccccccc} 0 \to & \omega_c & \to & N^\vee_{c/X}(2) & \to & N^\vee_{c/Z}(2) & \to 0 \\ & || & & \downarrow & & \downarrow & \\ 0 \to & \omega_c & \to & N^\vee_{c/P_c}(2) & \to & N^\vee_{c/S_c}(2) & \to 0. \end{array}$$

Recall that $N_{c/S_c} \simeq \mathcal{O}_c$, so that the previous exact sequence induces a coboundary map

$$\kappa_c : H^0(\mathcal{O}_c(2)) \to H^1(\omega_c) = \mathbb{C},$$

which we can consider as a quadratic form on $H^0(\mathcal{O}_c(1))$.

Lemma 4.4. *The skew-symmetric form ϕ_σ has rank four at $[c]$ if the quadratic form κ_c is non degenerate.*

Proof. First recall $N_{S_c/Z} = T^\vee_{S_c}$. By the previous proposition, ϕ_σ factors as

$$\wedge^2 H^0(N_{c/Z}) \to H^0(\wedge^2 N_{c/Z}) \to H^0(\wedge^2 N_{S_c/Z|c}) = H^0(\mathcal{O}_c(2)) \stackrel{\kappa_c}{\to} H^1(\omega_c) = \mathbb{C}.$$

This should be interpreted as follows. We may suppose that c is a smooth τ-conic, in which case we know that $N_{c/Z} = \mathcal{O}_c \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$ by Lemma 3.3. Hence $H^0(N_{c/Z}) = \mathbb{C} \oplus A \oplus A$ if $A = H^0(\mathcal{O}_c(1))$. In this decomposition, the fact that ϕ_σ factors as we have seen means that its matrix is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \kappa_c \\ 0 & -\kappa_c & 0 \end{pmatrix}$$

It is thus clear that ϕ_σ has rank four if (and only if) κ_c has rank two. □

What remains to be proved is that, generically κ_c is non degenerate. For this we can focus on the following situation: we have a quartic surface S in $\mathbb{P}^4$ which is a general intersection of two quadrics Q, Q' and c is the general conic in S. We must prove that the exact sequence

$$0 \to \omega_c \to N^\vee_{c/Q}(2) \to N^\vee_{c/S}(2) = \mathcal{O}_c(2) \to 0$$

induces a non degenerate quadratic form κ_c on $H^0(\mathcal{O}_c(1))$.

This can be seen as follows. We may suppose that Q' contains the plane $\langle c \rangle$ spanned by c, whose linear equations are, say, $x_3 = x_4 = 0$. This means that Q' has an equation of the form $x_3 m_3 + x_4 m_4 = 0$, for some linear forms

m_3, m_4. Restricted to $\langle c\rangle$, these linear forms define two global sections q_3, q_4 of $\mathcal{O}_c(2)$, and the linear form κ_c is just the projection map

$$\kappa_c : H^0(\mathcal{O}_c(2)) \to H^0(\mathcal{O}_c(2))/\langle q_3, q_4\rangle \simeq \mathbb{C}.$$

In other words, κ_c is polar to the pencil $\langle q_3, q_4\rangle$. It is thus non degenerate as soon as this pencil has no base point, which is the general situation. □

4.3. **The dual sextic.** Recall that we denoted by I the linear system of quadrics containing Z. We have defined in section 2.2 the hypersurface $Y_Z^\vee$ in $I^\vee$ as follows. Let $\hat{Y}_Z^\vee \subset I^\vee \times \mathbb{P}(V_5^\vee)$ be the variety parametrizing pairs (h, V_4) such that quadrics in $h \subset I^\vee$ cut $\mathbb{P}(\wedge^2 V_4)\cap H$ along singular quadrics. Then $Y_Z^\vee$ is just the image of $\hat{Y}_Z^\vee$ by the first projection.

Note that generically, for (h, V_4) in $\hat{Y}_Z^\vee$, quadrics of the hyperplane h will restrict to a corank one quadric in $\mathbb{P}(\wedge^2 V_4) \cap H$. Let S_Z denote the locus where the corank of the restricted quadric is bigger than one.

Proposition 4.5. *For Z general, the variety $\hat{Y}_Z^\vee$ is an irreducible fourfold whose singular locus is exactly S_Z. Moreover S_Z is a smooth surface, and $\hat{Y}_Z^\vee$ has multiplicity two at any point of S_Z.*

Proof. As in [Lo], and as we have seen in the proof of Proposition 2.4, the variety $\hat{Y}_Z^\vee$ is a degeneracy locus for a section of a bundle of quadrics. The conclusion will thus follow from a transversality argument: if the section is general enough, such a degeneracy locus $\hat{Y}_Z^\vee$ is singular exactly along the next degeneracy locus (see [ACGH], Chapter 2), which is precisely S_Z. Since the next one has, generically, codimension three in S_Z, it is in fact empty, and S_Z must be smooth. Unfortunately, in our section we do not deal with a general section, and we will need to check the transversality condition explicitly.

We recall the setting: over $\mathbb{P} = \mathbb{P}V_5^\vee$, first consider the rank two vector bundle $F = \mathcal{O}_\mathbb{P} \oplus \mathcal{O}_\mathbb{P}(1)$, then the rank five vector bundle M whose fiber over V_4 is $\wedge^2 V_4 \cap V_9$. Let $\mathcal{O}_F(-1)$ be the tautological line bundle over $\mathbb{P}(F)$. Then we denoted by

$$\eta : \mathcal{O}_F(-1) \to S^2 M^\vee$$

the morphism of vector bundles defined by mapping a pair $(z, v) \in \mathbb{C} \oplus V_5$ to the restriction of the quadric $zQ + P_v$ to $\wedge^2 V_4 \cap V_9$.

We will only need to consider quadrics of corank one or two, since:

Lemma 4.6. *For a general Z, the image of η does not contain any quadric of corank three or more.*

Proof. A straightforward dimension count. □

Corank one.

Consider a point of $\hat{Y}_Z^\vee$ defined by a corank one quadric. We will prove it is a smooth point of $\hat{Y}_Z^\vee$. If this quadric is not the Plücker one, we may suppose, up to a change of notation, that Q itself cuts $\wedge^2 V_4 \cap V_9$ along a corank one quadric, singular at ω_0. We choose local coordinates on $\mathbb{P}(F)$ at the corresponding point as follows. First, we choose a supplement V_1 of V_4, so that any hyperplane in V_5 transverse to V_1 can be represented as the graph $V_4(\phi)$ of a morphism $\phi \in Hom(V_4, V_1)$. Then, we represent the trace of a hyperplane in I on $\wedge^2 V_4(\phi)\cap V_9$ by the restriction of the quadric $Q+P_v$, for some $v \in V_1$. Our local coordinates will be the pair (ϕ, v).

We want to represent the latter quadric $Q+P_v$ on $\wedge^2 V_4(\phi)\cap V_9$ by an isomorphic quadric $Q_{\phi,v}$ on $\wedge^2 V_4 \cap V_9$. To do this, we first observe that ϕ induces an isomorphism from $\wedge^2 V_4$ to $\wedge^2 V_4(\phi)$ sending ω to $\phi(\omega) := \omega + \phi\rfloor\omega$ (where the contraction map $\phi\rfloor$ maps $v\wedge v'$ to $\phi(v)\wedge v' + v\wedge\phi(v')$). Let h be an equation of H, and Ω in $\wedge^2 V_4$ be such that $h(\Omega)=1$. If ω belongs to $\wedge^2 V_4\cap V_9$, and $\omega' = \omega + t\Omega$, then $\phi(\omega')$ belongs to V_9 when $t = -h(\phi\rfloor\omega)$, up to terms of higher order in (ϕ, v). Up to such terms, we thus let

$$\begin{aligned} Q_{\phi,v}(\omega) &= (Q+P_v)(\phi(\omega')) \\ &= Q(\omega) + 2Q(\omega, \phi\rfloor\omega - h(\phi\rfloor\omega)\Omega) + v\wedge\omega\wedge\omega. \end{aligned}$$

Since Q is supposed to have corank one and kernel $\langle\omega_0\rangle$, the function $\det(Q_{\phi,v})$ is equal, up to a constant and higher order terms, to $Q_{\phi,v}(\omega_0)$, which is therefore a local equation of $\hat{Y}_Z^\vee$. For this equation to vanish identically at first order, we would need that

$$v\wedge\omega_0\wedge\omega_0 = 0 \quad \forall v\in V_1, \tag{2}$$

$$Q(\omega_0, \phi\rfloor\omega_0 - h(\phi\rfloor\omega_0)\Omega) = 0 \quad \forall\phi\in Hom(V_4, V_1). \tag{3}$$

Since V_1 is transverse to V_4, the first equation implies that $\omega_0\wedge\omega_0 = 0$, hence ω_0 has rank two and defines a point of $G(2,V_4)\cap H$. We will denote the corresponding plane by $V_2\subset V_4$. Observe that, when ϕ varies, $\phi\rfloor\omega_0$ describes $V_1\wedge V_2\subset\wedge^2 V_5$. Since, by the singularity condition, $Q(\omega_0,\omega)=0$ for any $\omega\in\wedge^2 V_4\cap V_9$, the second condition means that the linear form $Q(\omega_0,\omega)$ is proportional to h on $V_2\wedge V_1\oplus\wedge^2 V_4$. We claim that this implies that $[\omega_0]$ is a singular point of Z, thus leading to a contradiction. Indeed, the affine tangent space to G at ω_0 is $V_2\wedge V_5$, which is contained in $V_2\wedge V_1\oplus\wedge^2 V_4$. So the traces on this tangent space, of H and of the tangent space to Q, would coincide, and the intersection of G, H and Q would not be transverse at ω_0.

Suppose now that the Plücker quadric itself, $G(2,V_4)\cap H$, is singular at ω_0. Choose a generator v_1 of V_1. Local coordinates on $\mathbb{P}(F)$ are given by (t,ϕ), where $\phi\in Hom(V_4,V_1)$ as above and the quadric to consider on $\wedge^2 V_4(\phi)\cap V_9$ is $P_{v_1}+tQ$. As in the previous case we identify $\wedge^2 V_4(\phi)\cap V_9$ with $\wedge^2 V_4\cap V_9$, and the previous quadric on $\wedge^2 V_4(\phi)\cap V_9$ with the isomorphic one on $\wedge^2 V_4\cap V_9$ given by

$$Q_{t,\phi}(\omega) = v_1\wedge\omega\wedge\omega + tQ(\omega) - 2h(\phi\rfloor\omega_0)v_1\wedge\Omega\wedge\omega,$$

up to order one. As above, we want to exclude the possibility that

$$Q_{t,\phi}(\omega_0) = tQ(\omega_0) - 2h(\phi\rfloor\omega_0)v_1\wedge\Omega\wedge\omega_0 = 0$$

for all t and ϕ. Since $\Omega\wedge\omega_0\neq 0$ (otherwise $G(2,V_4)$ would be singular at $[\omega_0]$!), this would mean that $Q(\omega_0)=0$ and $h(V_2\wedge V_1)=0$ where, as above, V_2 is the plane defined by ω_0. The former condition means that ω_0 defines a point of Z. The latter one implies that H is tangent to G at $[\omega_0]$. As in the previous case we would therefore conclude that Z is singular at $[\omega_0]$, a contradiction.

Corank two.

Since $G(2,V_4)$ is smooth, a hyperplane section cannot have corank two. We may therefore suppose that Q itself cuts $\wedge^2 V_4 \cap V_9$ along a corank two quadric, singular along the line $\langle \omega_0, \omega_1 \rangle$. Considering the intersection of this line with the quadric $G(2,V_4) \cap H$, we may suppose that ω_0 and ω_1 have rank two. We choose local coordinates (v, ϕ) on $\mathbb{P}(F)$ as above and we consider the same quadric $Q_{v,\phi}$. The required transversality condition can be expressed by the condition that the map

$$(v,\phi) \mapsto \begin{pmatrix} Q_{v,\phi}(\omega_0,\omega_0) & Q_{v,\phi}(\omega_0,\omega_1) \\ Q_{v,\phi}(\omega_0,\omega_1) & Q_{v,\phi}(\omega_1,\omega_1) \end{pmatrix}$$

have rank three. If $\omega_0 \wedge \omega_1 \neq 0$, the off diagonal term $Q_{v,\phi}(\omega_0,\omega_1)$ will contribute by one to the rank, through its terms involving v. So what we need to avoid is that $Q(\omega_0, \phi\rfloor\omega_0 - h(\phi\rfloor\omega_0)\Omega)$ and $Q(\omega_1, \phi\rfloor\omega_1 - h(\phi\rfloor\omega_1)\Omega)$ impose linearly dependent conditions. Since we have supposed that ω_0 and ω_1 represent transverse planes in V_4, this must be the case, except if one of these linear forms is zero. This can be excluded, for a general Z, by a simple dimension count: for a given H, we have four parameters for V_4, then three for each of $[\omega_0]$ and $[\omega_1]$, which belong to the three-dimensional quadric $G(2,V_4) \cap H$. But we get eleven linear conditions on Q.

Of course we also need to consider the degenerate cases for which the line $\langle \omega_0, \omega_1 \rangle$ is tangent to $G(2,V_4) \cap H$, or even contained in $G(2,V_4) \cap H$ (this would mean that $\omega_0 \wedge \omega_1 = 0$). A similar dimension count leads in both cases to the same conclusion. □

Proposition 4.7. *The projection map $\hat{Y}_Z^\vee \to Y_Z^\vee$ is the blow-up of the point of $Y_Z^\vee$ defined by the Plücker hyperplane.*

Proof. This follows from the proof of Proposition 2.4. Indeed, we have seen in this proof that $\hat{Y}_Z^\vee \to Y_Z^\vee$ is the restriction of the blow-up of the point h_P in $I^\vee$ defined by the Plücker hyperplane. Moreover, we are in the case $k = 1$ of the Proposition, and the proof shows that $Y_Z^\vee$ has multiplicity one at this point. Otherwise said, this is a smooth point of $Y_Z^\vee$. This is enough to ensure that the projection map $\hat{Y}_Z^\vee \to Y_Z^\vee$ is just the blowing-up of h_P.

Note that the preimage of h_P in $\hat{Y}_Z^\vee$ is easily determined. It is simply the hyperplane in $\mathbb{P}V_5^\vee$, defined as the set of those hyperplanes that contain the kernel of the (degenerate) two-form defining H. □

4.4. **The symplectic structure.** Consider the nested Hilbert scheme $F(Z)$ parametrizing pairs (c, V_4) such that c be a conic in Z and V_4 a hyperplane in V_5 such that the quadric $G(2,V_4)$ contains c. Recall that V_4 is uniquely determined by c except if c is a ρ-conic, in which case there is a projective line of possible V_4's. Otherwise said, the forgetful map

$$\pi : F(Z) \to F_g(Z)$$

is an isomorphism outside $F_g^\rho(Z)$, and contracts a divisor onto $F_g^\rho(Z)$.

Lemma 4.8. *For Z general, $F(Z)$ is smooth.*

Proof. The nested Hilbert scheme $F(Z)$ is a subscheme of $F_g(Z) \times \mathbb{P}V_5^\vee$. Its Zariski tangent space at a point (c, V_4) can be described by the following exact sequence, where we let $G_c = G(2, V_4)$:

$$0 \to T_{(c,V_4)}F(Z) \to H^0(N_{c/Z}) \oplus H^0(N_{G_c/G}) \to H^0(N_{G_c/G|c}).$$

Since we already know that $F_g(Z)$ is smooth at c, the smoothness of $F(Z)$ at (c, V_4) is equivalent to the surjectivity of the rightmost arrow.

First observe that $N_{G_c/G} = T^\vee_{|G_c}$, so that $H^0(N_{G_c/G})$ is simply $V_4^\vee$, and the restriction map $H^0(N_{G_c/G}) \to H^0(N_{G_c/G|c})$ is already surjective if c is not a ρ-conic.

So suppose that c be a ρ-conic, spanning a plane $G(2, V_3)$ with $V_4 \supset V_3$. In this case the restriction map $H^0(N_{G_c/G}) \to H^0(N_{G_c/G|c})$ has for image a hyperplane H_c, and we need to check that the image of the other map $H^0(N_{c/Z}) \to H^0(N_{G_c/G|c})$ is not contained in H_c.

This condition amounts to the fat that a matrix of size 8×4 be of maximal rank. If this matrix is sufficiently general, this fails to happen in codimension $8 - 4 + 1 = 5 > 4$. Our claim follows. □

Remark. One can show that the map $\pi : F(Z) \to F_g(Z)$ is simply the blow-up of the codimension two subvariety $F_\rho(Z)$ parametrizing ρ-conics in Z. This subvariety is smooth for Z general enough.

Our next goal is to construct a morphism

$$\alpha : F(Z) \longrightarrow \hat{Y}_Z^\vee.$$

For a point (c, V_4) in $F(Z)$, we have two quadratic hypersurfaces inside $\mathbb{P}(\wedge^2 V_4) \cap H \simeq \mathbb{P}^4$: the intersection P_{V_4} of the Plücker quadric $G(2, V_4)$ with H, and the trace Q_{V_4} of the quadric Q defining Z. The pencil $\langle P_{V_4}, Q_{V_4}\rangle$ is uniquely defined by Z (and V_4). Any quadric in this pencil contains the conic c, but the generic one does not contain the plane $\langle c\rangle$ spanned by c, since that plane cannot be contained in Z by Lemma 3.6. This implies that there is a unique quadric $Q_{c,V_4} \in \langle P_{V_4}, Q_{V_4}\rangle$ containing $\langle c\rangle$. Moreover this quadric must be singular, since a smooth three dimensional quadric does not contain any plane. Therefore Q_{c,V_4} defines a point in $\hat{Y}_Z^\vee$: this is $\alpha(c, V_4)$.

Proposition 4.9. *Let y be a point of $\hat{Y}_Z^\vee$.*

(1) *If $y \in S_Z$, the set-theoretical fiber $\alpha^{-1}(y)$ is a projective line.*
(2) *If $y \notin S_Z$, the fiber $\alpha^{-1}(y)$ is a disjoint union of two projective lines.*

Proof. Suppose that $y = \alpha(c, V_4)$ does not belong to S_Z. This means that the quadric Q_{c,V_4} has rank four; otherwise said, it is a cone over a smooth quadratic surface. The projective planes L in Q_{c,V_4} are then cones over the lines in this surface, and are parametrized by two projective lines. Any such plane L, cut out with any other quadric in the pencil $\langle P_{V_4}, Q_{V_4}\rangle$, gives a conic $c(L)$ such that $Q_{c(L),V_4} = Q_{c,V_4}$, hence $\alpha(c(L), V_4) = \alpha(c, V_4) = y$. This implies that $\alpha^{-1}(y)$ is the disjoint union of these two projective lines.

If $y = \alpha(c, V_4)$ does belong to S_Z, the quadric Q_{c,V_4} has rank three; it is a double cone over a smooth conic. The projective planes L in Q_{c,V_4} are then parametrized by that single conic. We can make with these planes the same construction as above, but we end up with a single projective line parametrizing $\alpha^{-1}(y)$. □

Remark. Observe what happens over conics not of type τ. First recall that $F^\sigma(Z) \simeq F^\sigma_g(Z) \simeq Bl_0\mathbb{P}^4$, the blow-up of $\mathbb{P}^4$ at one point. This blow-up is a $\mathbb{P}^1$-fibration of $\mathbb{P}^3$, and the restriction of α to $F^\sigma(Z)$ coincides with this fibration. Second, recall that $F^\sigma_g(Z) \simeq \mathbb{Q}^3$, coincides with the isotropic Grassmannian $IG(3, V_5)$. Its preimage $F^\sigma(Z)$ in $F(Z)$ is the variety $IF(3, 4, V_5)$ of flags $V_3 \subset V_4$ with V_3 isotropic (recall that this implies that V_3 contains W_1, the kernel of the two-form ω on V_5 defining H). The restriction of the map α to $F^\sigma(Z)$ simply forgets V_3. In particular its image is the space of hyperplanes in V_5 containing W_1, hence a copy of $\mathbb{P}^3$.

Proposition 4.10. *Any fiber of α is a smooth curve in $F(Z)$.*

Proof. Let (c, V_4) be a point of $F(Z)$. We want to prove that the corresponding fiber F_c of α is smooth at that point. Set-theoretically, we have seen that the plane $\langle c\rangle$ is contained in a unique (singular) quadric Q_{c,V_4} of the pencil of quadrics obtained by restricting I to $\mathbb{P}(\wedge^2 V_4) \cap H$. This plane $\langle c\rangle$ varies in a family of planes in Q_{c,V_4} parametrized by a projective line, and we get a map $\mathbb{P}^1 \to F_c$, which we shall prove to be a local isomorphism.

Observe that the tangent space to $F_c \subset F(Z)$ is $H^0(N_{c/S}) \subset H^0(N_{c/Z})$, where S is the quartic surface cut out by Z on $\mathbb{P}(\wedge^2 V_4) \cap H$. This surface is the intersection $S = Q_0 \cap Q_{c,V_4}$ of two quadrics. We can choose linear coordinates $x_0, \ldots, x_4$ such that $\langle c\rangle$ be defined by $x_2 = x_3 = 0$, and write

$$\begin{aligned} Q_0 &= x_3\ell_3 + x_4\ell_4 + q(x_0, x_1, x_2), \\ Q_{c,V_4} &= x_3m_3 + x_4m_4, \end{aligned}$$

for some linear forms ℓ_3, ℓ_4, m_3, m_4. Note that $q(x_0, x_1, x_2)$ (which is non zero since Z contains no plane) is an equation of c in $\langle c\rangle$.

Now we can see very explicitly that the map $T_{\langle c\rangle}\mathbb{P}^1 \to H^0(N_{c/S})$ is non zero, which will prove our claim. Indeed, an infinitesimal deformation of $\langle c\rangle$ in Q_{c,V_4} is simply obtained by $\epsilon \mapsto \langle c\rangle(\epsilon)$, the plane defined by the two equations $x_3 + \epsilon m_4 = x_4 - \epsilon m_3 = 0$. It is mapped to a global element θ of $H^0(N_{c/S}) = Hom_{\mathcal{O}_S}(\mathcal{I}_c, \mathcal{O}_c)$ defined by

$$\begin{aligned} x_3 &\mapsto m_4, \\ x_4 &\mapsto -m_3 \\ q &\mapsto m_3\ell_4 - m_4\ell_3. \end{aligned}$$

Indeed, $\mathcal{I}_c$ is generated by x_3, x_4 and q at any point (strictly speaking, to make sense of this we need to divide them by some linear, respectively quadratic form not vanishing at the point considered), and although ℓ_3, ℓ_4, m_3, m_4 are not uniquely defined, m_4, m_3 and $m_3\ell_4 - m_4\ell_3$ are uniquely defined when restricted to c.

There just remains to check that θ cannot be zero. This would mean that m_3 and m_4 vanish identically on c, hence that they are linear combinations of x_3 and x_4. But then Q_{c,V_4} would have rank at most two, and by Lemma 4.6, this is not possible for a general Z. □

Consider the Stein factorization of α:

$$F(Z) \xrightarrow{\beta} \bar{Y}^\vee_Z \xrightarrow{\gamma} \hat{Y}^\vee_Z.$$

By the previous proposition, γ has degree two, and ramifies precisely over S_Z. By the previous Proposition the reduced fibers of β are smooth projective lines, and in fact β is a $\mathbb{P}^1$-bundle, since an application of [AW, Theorem 4.1] yields:

Proposition 4.11. *The variety $\bar{Y}_Z^\vee$ is smooth.*

Note that conics in Z which are not τ conics are sent to the Plücker hyperplane in $Y_Z^\vee$. In particular the map $F(Z) \to Y_Z^\vee$ factorizes through $F_g(Z)$. Taking the Stein factorization of the induced map $F_g(Z) \to Y_Z^\vee$, we get a commutative diagram

$$\begin{array}{ccccc} F(Z) & \to & \bar{Y}_Z^\vee & \to & \hat{Y}_Z^\vee \\ \downarrow & & \downarrow & & \downarrow \\ F_g(Z) & \to & \tilde{Y}_Z^\vee & \to & Y_Z^\vee \end{array}$$

A consequence of the previous lemma is that:

Lemma 4.12. *The projection map $\bar{Y}_Z^\vee \to \tilde{Y}_Z^\vee$ is the blow-up of the two points of $\tilde{Y}_Z^\vee$ in the preimage of the Plücker hyperplane. In particular $\tilde{Y}_Z^\vee$ is smooth.*

Now we can prove the main result of this section:

Theorem 4.13. *The variety $\tilde{Y}_Z^\vee$ is a smooth symplectic fourfold.*

Proof. We can use our two-form ϕ_σ on $F_g(Z)$ and lift it to $F(Z)$. Since the fibers of β are projective lines, the induced two-form on $F(Z)$ descends to a globally defined two-form Φ_σ on $\bar{Y}_Z^\vee$, which remains a closed form. The generic rank of Φ_σ is four since the generic rank of ϕ_σ is four by Proposition 4.3. Since the projection to $\tilde{Y}_Z^\vee$ is birational, we also get a closed two-form $\tilde{\Phi}_\sigma$ on $\tilde{Y}_Z^\vee$, generically non-degenerate.

But the canonical class of $\tilde{Y}_Z^\vee$ is trivial, implying that $\tilde{\Phi}_\sigma$ is in fact everywhere non-degenerate. Indeed, the sextic $Y_Z^\vee$ is smooth in codimension one, hence normal. Its canonical class is trivial. The map $\tilde{Y}_Z^\vee \to Y_Z^\vee$ is finite of degree two, ramified on the surface S_Z only, so the canonical class of $\tilde{Y}_Z^\vee$ is simply the pull-back of that of $Y_Z^\vee$. Hence the claim and the theorem. □

4.5. **EPW sextics attached to Fano manifolds of different dimensions.** Let us elaborate on what we have proved at this point. Consider a general variety $X = G \cap Q \cap \mathbb{P}V_{10-k}$, of dimension $N = 5 - k$, and the associated sextic hypersurface Y_X. We have recalled in Proposition 2.1 that for $k = 0$, Y_X is an EPW sextic. We have proved it is also the case for $k = 1$. This is also true for $k = 3$, in which case X is a generic polarized K3 surface of degree ten [Mu1]. Mukai showed ([Mu2], Ex. 5.17) that the natural double cover $\tilde{Y}_X$ of the sextic Y_X can be identified with the moduli space of stable rank two vector bundles E on X with Chern classes $c_1(E) = \mathcal{O}_X(1)$ and $c_2(E) = 5$. This explains the existence of a symplectic structure on $\tilde{Y}_X$, directly inherited from that of X.

Remark. O'Grady proved that in the (irreducible) family of EPW sextics, those coming from polarized K3 surfaces of degree ten form a codimension one family ([OG4], Proposition 3.3). Note that the dual $\tilde{Y}_X^\vee$ has a point of multiplicity three, a special property already observed in [OG3], Proposition 6.1, and that we have met in the proof of Proposition 2.4.

What about the missing case $k = 2$? And can we understand the relations between the families of EPW sextics obtained from different values of k?

To answer the latter question, we can use the construction of Gushel threefolds as degenerations of non Gushel Fano threefolds of degree ten [Gu]. To be more specific, consider the projective cone over G, that we denote by $CG \subset \mathbb{P}(\mathbb{C} \oplus \wedge^2 V_5)$. Let p_0 denote the vertex of this cone. Now we cut CG by a general quadric Q, and a general linear space $\mathbb{P}V_{10-k}$, of codimension $k+1$. We get a variety Z of dimension $5-k$. There are two cases:

- $p_0 \notin \mathbb{P}V_{10-k}$: then Z is isomorphic with the intersection X of G with the projection of $\mathbb{P}V_{10-k}$ to $\mathbb{P}(\wedge^2 V_5)$, and a quadric Q';
- $p_0 \in \mathbb{P}V_{10-k}$, that is, $\mathbb{P}V_{10-k}$ is a cone over some $\mathbb{P}V_{9-k} \subset \mathbb{P}(\wedge^2 V_5)$: then Z is a double cover of $G \cap \mathbb{P}V_{9-k}$, branched over its intersection X with a quadric Q'.

Obviously the second case is a degeneration of the first one. We call the corresponding Z *Gushel varieties.*

Lemma 4.14. *If Z is Gushel, the sextics Y_Z and Y_X are equal.*

Proof. Take coordinates (t, ω) on $\mathbb{C} \oplus \wedge^2 V_5$ and write the equation of the quadric Q as $Q(t, \omega) = t^2 + 2\ell(\omega)t + q(\omega)$. Note that we can choose for the quadric Q' defining X, the discriminant $Q'(\omega) = q(\omega) - \ell(\omega)^2$.

Suppose that $Q + P_v$ defines a singular quadric in $\mathbb{P}(\mathbb{C} \oplus \wedge^2 V_5)$. This means that we can find a point (t_0, ω_0) such that $Q((t_0, \omega_0), (t, \omega)) + P_v(\omega_0, \omega) = 0$ for any (t, ω). That is, we must have $t_0 + \ell(\omega_0) = 0$ and

$$t_0 \ell(\omega) + q(\omega_0, \omega) + P_v(\omega_0, \omega) = 0$$

for any ω. But this implies that ω_0 defines a singular point of Q'. Hence $Y_Z \subset Y_X$, and since they are both sextics hypersurfaces, they are equal. □

Now consider the Gushel manifold Z as a degeneration of a family of non Gushel manifolds. Suppose that Z be defined by a quadric $Q(t, \omega)$ as above, and a linear space $\mathbb{P}V_{10-k}$ through p_0, defined by the $k+1$ equations $h_0(\omega) = \cdots = h_k(\omega) = 0$. Then we define $Z(\epsilon)$ by the same quadric, and the linear space $\mathbb{P}V_{10-k}(\epsilon)$ with equations

$$h_0(\omega) = \epsilon t, \qquad h_1(\omega) = \cdots = h_k(\omega) = 0.$$

For $\epsilon \neq 0$, $\mathbb{P}V_{10-k}(\epsilon)$ does not contain p_0. Hence $Z(\epsilon)$ is isomorphic with the intersection $Z^*(\epsilon)$ of G with the linear space $\mathbb{P}V^*_{10-k}(\epsilon)$ of equations $h_1(\omega) = \cdots = h_k(\omega) = 0$, and the quadric $Q(\epsilon^{-1}h_0(\omega), \omega) = 0$.

Lemma 4.15. *The sextic Y_Z is a degeneration of the sextics $Y_{Z^*(\epsilon)}$.*

Proof. This is rather clear. The sextic $Y_{Z^*(\epsilon)}$ is defined by the condition that the quadric $zQ(\epsilon^{-1}h_0(\omega), \omega) + P_v(\omega)$ be singular on $\mathbb{P}V^*_{10-k}(\epsilon)$, or equivalently, that the quadric $zQ(t, \omega) + P_v(\omega)$ be singular on $\mathbb{P}V_{10-k}(\epsilon)$. Letting ϵ tend to zero, we get the sextic Y_Z as a degeneration of the sextics $Y_{Z^*(\epsilon)}$. □

We can conclude inductively that for any $k \geq 0$, and any Fano manifold X of degree ten and dimension $5-k$, the associated sextic Y_X is a possibly degenerate EPW sextic.

Remark. For *k odd*, the general quadric in Y_X, having corank one, is a cone over a smooth quadric of even dimension. Such a quadric has two rulings by maximal linear subspaces, and this induces the double cover $\tilde{Y}_X \to Y_X$, branched over the locus parametrizing quadrics of corank at least two. By the preceding construction, this remark can be extended to the case where k is even. The double covering $\tilde{Y}_X$ is endowed, by a deformation argument, with a symplectic structure, for X general of any dimension.

Proposition 4.16. *For X a general Fano threefold of degree ten, the associated sextic Y_X is a general EPW sextic.*

Proof. This follows from a dimension count. Remember that EPW sextics have 20 moduli. On the other hand, if Y is an EPW sextic, and X is a general Fano threefold such that $Y_X \simeq Y$, then we know from [Lo] that the singular locus of Y is a smooth surface isomorphic with the Fano surface of conics in X (more precisely, with the quotient of the minimal model of that surface, by a base point free involution). Moreover, Logachev's reconstruction theorem (see the Appendix of [DIM]) implies that there is only a two dimensional family of Fano threefolds X with the same Fano surface, and a fortiori with the same associated sextic Y. Since Fano threefolds of degree ten have 22 moduli, this implies that Y lives in a 20-dimensional family, hence must be a generic EPW sextic. □

Corollary 4.17. *For any $N = 3, 4, 5$, and X a general Fano manifold of degree ten and dimension N, the associated sextic Y_X is a general EPW sextic.*

Proof. This is a direct consequence of the previous degeneration argument to Gushel type manifolds. □

The next obvious question to ask is: which are the Fano manifolds X of degree ten and dimension N, whose associated sextic Y_X is a given general EPW sextic Y? Denote by m_N the dimension of the moduli space[1] of Fano manifolds of degree ten and dimension N. An easy computation shows that

$$m_2 = 19, \quad m_3 = 22, \quad m_4 = 24, \quad m_5 = 25.$$

The relative dimension r_N of the map to the moduli space of EPW sextics is therefore given by

$$r_2 = -1, \quad r_3 = 2, \quad r_4 = 4, \quad r_5 = 5.$$

For $N = 3$, we have seen that the family $EPW_N^{-1}(Y)$ of Fano threefolds X whose associated EPW sextic is isomorphic with Y, is essentially the surface $S(Y) = Sing(Y^\vee)$. It is tempting to imagine that a similar phenomenon should hold for $N = 4$ or 5.

Question. If Y is a generic EPW sextic, is it true that

$$EPW_4^{-1}(Y) \simeq Y^\vee - S(Y) \qquad \text{and} \qquad EPW_5^{-1}(Y) \simeq \mathbb{P}^5 - Y^\vee?$$

Indeed, for $N = 4$, once we have a representation of Y as Y_Z, or equivalently, of $Y^\vee$ as $Y_Z^\vee$, the Plücker point, which belongs to $Y^\vee - S(Y)$, is given

[1] We suppose implicitly that this moduli space does exist. We hope to come back to this question in a future paper.

a special role. We believe that specifying that point in $Y^\vee - S(Y)$ should be equivalent to specifying Z. The same phenomenon should hold for $N = 5$, except that in that case the Plücker point does not belong to $Y^\vee$.

4.6. **O'Grady's double overs.** We can now prove that for Z a general Fano fourfold of degree ten, our double cover $\tilde{Y}_Z^\vee$ of the general EPW sextic $Y_Z^\vee$ coincides with the double cover constructed by O'Grady (see [OG3], §4). We denote the latter by $\tilde{Y}_{Z,O}^\vee$.

Proposition 4.18. *The symplectic manifolds $\tilde{Y}_Z^\vee$ and $\tilde{Y}_{Z,O}^\vee$ are isomorphic. In particular $\tilde{Y}_Z^\vee$ is an irreducible symplectic manifold.*

Proof. Recall that we denoted by S_Z the singular locus of $Y_Z^\vee$. For a general Z this is a smooth surface. Since $\tilde{Y}_{Z,O}^\vee$ is simply-connected, the étale double cover $\tilde{Y}_{Z,O}^\vee - \tilde{S}_{Z,O} \to Y_Z^\vee - S_Z$ (where $\tilde{S}_{Z,O}$ denotes the preimage of S_Z) is the universal covering, and in particular $\pi_1(Y_Z^\vee - S_Z) = \mathbb{Z}_2$ (see [OG4], §3.2). Then since $\tilde{Y}_Z^\vee - \tilde{S}_Z \to Y_Z^\vee - S_Z$ is also a non trivial étale double cover, it lifts to an isomorphism between $\tilde{Y}_{Z,O}^\vee - \tilde{S}_{Z,O}$ and $\tilde{Y}_Z^\vee - \tilde{S}_Z$. So $\tilde{Y}_{Z,O}^\vee$ and $\tilde{Y}_Z^\vee$ are birational, and in particular $h^{2,0}(\tilde{Y}_Z^\vee) = h^{2,0}(\tilde{Y}_{Z,O}^\vee) = 1$.

Moreover, being birational, $\tilde{Y}_{Z,O}^\vee$ and $\tilde{Y}_Z^\vee$ are also deformation equivalent [Hu, Theorem 4.6]. Therefore $\tilde{Y}_Z^\vee$ is, as $\tilde{Y}_{Z,O}^\vee$, a numerical $(K3)^{[2]}$, according to O'Grady's terminology. And we can conclude the proof by applying Theorem 1.1 in [OG3], once we know that:

Lemma 4.19. *The defining involution ι of the double covering $\tilde{Y}_Z^\vee \to Y_Z^\vee$ is anti-symplectic.*

Proof. Let c be a general conic in Z, and V_4 the corresponding hyperplane in V_5. Let y be the image of $[c]$ in $\tilde{Y}_Z^\vee$. Recall that the quartic surface $S_c = G(2, V_4) \cap Q \cap H$ has normal bundle $N_{S_c/Z} \simeq T_{S_c}^\vee$. Moreover the normal sequence to the triple $(c \subset S_c \subset Z)$, gives rise to the exact sequence

$$0 \to H^0(N_{c/S_c}) \to H^0(N_{c/Z}) \to H^0(N_{S_c/Z|c}) \to 0,$$

which must be interpreted as the tangent sequence of the map $F_g(Z) \to \tilde{Y}_Z^\vee$. In particular, this yields a natural identification

$$T_y\tilde{Y}_Z^\vee \simeq H^0(N_{S_c/Z|c}) \simeq H^0(T_c^\vee) \simeq V_4^\vee,$$

and the symplectic form $\tilde{\Phi}_\sigma$ can be defined at y as the composition

$$\wedge^2 T_y\tilde{Y}_Z^\vee \simeq \wedge^2 H^0(T_c^\vee) \to H^0(\wedge^2 T_c^\vee) = H^0(\mathcal{O}_c(2)) \stackrel{\kappa_c}{\to} \mathbb{C}.$$

Here, recall that the quadratic form κ_c is induced by the twisted conormal sequence

$$0 \to \omega_c \to N_{c/Q_{c,V_4}}^\vee(2) \to \mathcal{O}_c(2) = \mathcal{O}_Z(1)_{|c} \to 0,$$

where $Q_{c,V_4} = G(2, V_4) \cap H$.

Take another conic c' in Z such that the image of $[c']$ in $\tilde{Y}_Z^\vee$ be the point $y' = \iota(y)$. Recall that this means that the planes $\langle c\rangle$ and $\langle c'\rangle$ spanned by the two conics are contained in the same (singular) quadric $Q_{c,V_4} = Q_{c',V_4}$ of the pencil of quadrics we have in $\mathbb{P}(\wedge^2 V_4) \cap H$, but do not belong to the same ruling. In particular the two planes $\langle c\rangle$ and $\langle c'\rangle$ meet along a line, and the three-plane $\langle c, c'\rangle$ cuts Z along the degenerate elliptic curve

$$e = c \cup c' = G(2, V_4) \cap Q \cap \langle c, c'\rangle.$$

Now, if we take an element of $\wedge^2 V_4^\vee$ and apply $\tilde{\Phi}_{\sigma,y} + \tilde{\Phi}_{\sigma,y'}$, we first get an element of $H^0(\mathcal{O}_c(2)) \oplus H^0(\mathcal{O}_{c'}(2))$ which defines a global section of $H^0(\mathcal{O}_Z(1)_{|e})$. Then we apply the coboundary maps defined by the twisted conormal sequences of c and c'. But the analogous conormal sequence for e,

$$0 \to \omega_e \to N^\vee_{e/Q_e}(2) \to \mathcal{O}_Z(1)_{|e} \to 0,$$

splits since e is a complete intersection curve in $Q_e = Q_{c,V_4} = Q_{c',V_4}$. This implies that the associated coboundary operator is zero, which means that $\tilde{\Phi}_{\sigma,y} + \tilde{\Phi}_{\sigma,y'}$ is zero. Otherwise stated,

$$\tilde{\Phi}_\sigma + \iota^* \tilde{\Phi}_\sigma = 0,$$

or else, ι is anti-symplectic. □

5. Two integrable systems

5.1. **Fano threefolds contained in Z.** Let $W = G \cap Q \cap H' \cap H''$ be a general Fano threefold of degree ten contained in the general Fano fourfold $Z = G \cap Q \cap H$ as a hyperplane section. The linear systems $I_2(Z)$ and $I_2(W)$ of quadrics containing them are naturally identified. As before, we associate to W the hypersurface $\hat{Y}_W^\vee \subset I^\vee$ and its singular locus, the surface S_W.

Lemma 5.1. *The surface S_W is contained in $Y_Z^\vee$.*

Proof. By definition, a point h in S_W is a hyperplane in I such that for some hyperplane V_4 of V_5, quadrics in h restrict to a corank two quadric in $\mathbb{P}(\wedge^2 V_4) \cap H' \cap H''$. This is a hyperplane in $\mathbb{P}(\wedge^2 V_4) \cap H$, and it follows that quadrics in h cut $\mathbb{P}(\wedge^2 V_4) \cap H$ along a quadric with some corank two hyperplane section. But this is possible only if that quadric is itself singular. This precisely means that h defines a point of $Y_Z^\vee$. □

As explained in [Lo], and as we already mentioned, the surface S_W is closely related to conics in W. Indeed, the Hilbert scheme $F_g(W)$ parametrizing conics in W is a smooth surface, containing a unique ρ-conic, and a line of σ-conics. This line is an exceptional curve which may be contracted. The resulting surface $F_m(W)$ is then endowed with a fixed-point free involution whose quotient is precisely $S_W = F_\iota(W)$. It follows that $F_m(W)$ can be seen as the pull-back $\tilde{S}_W$ of S_W inside $\tilde{Y}_Z^\vee$, and that we have a commutative diagram

$$\begin{array}{ccccc} F_g(W) & \to & F_m(W) = \tilde{S}_W & \to & F_\iota(W) = S_W \\ \downarrow & & \downarrow & & \downarrow \\ F_g(Z) & \to & \tilde{Y}_Z^\vee & \to & Y_Z^\vee, \end{array}$$

where the vertical maps are injections.

Proposition 5.2. *The surface $\tilde{S}_W$ is a Lagrangian subvariety of $\tilde{Y}_Z^\vee$.*

Proof. We just need to prove that $F_g(W)$ is isotropic with respect to the two-form ϕ_σ on $F_g(Z)$. Otherwise said, for a general conic c in W, we must check that ϕ_σ vanishes on the subspace $T_{[c]}F_g(W) = H^0(N_{c/W})$ of $T_{[c]}F_g(Z) = H^0(N_{c/Z})$. Since W is a hyperplane section of Z, the conormal sequence of the triple (c, W, Z) is just

$$0 \to N_{c/W} \to N_{c/Z} \to \mathcal{O}_c(2) \to 0.$$

In particular $N_{c/W}$ has degree zero (in fact $N_{c/W}$ is trivial for a general conic, but we will not need that – see [DIM] for more details). We have a commutative diagram

$$\begin{array}{ccccc}
\wedge^2 H^0(N_{c/W}) & \hookrightarrow & \wedge^2 H^0(N_{c/Z}) & & \\
\downarrow & & \downarrow & & \\
H^0(\wedge^2 N_{c/W}) & \to & H^0(\wedge^2 N_{c/Z}) & & \\
|| & & || & & \\
H^0(\mathcal{O}_c) & \hookrightarrow & H^0(N^\vee_{c/Z}(2)) & \to & H^1(\omega_c) = \mathbb{C},
\end{array}$$

and we need to prove that the composition $\wedge^2 H^0(N_{c/W}) \to H^1(\omega_c)$ is zero. But recall that the map $H^0(N^\vee_{c/Z}(2)) \to H^1(\omega_c)$ was induced by the twisted conormal sequence of the triple $(c, Z, X = G \cap H)$. This sequence fits with the conormal sequence of the triple $(c, W, Y = G \cap H' \cap H'')$ into the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_c & = & \mathcal{O}_c & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \omega_c & \to & N^\vee_{c/X}(2) & \to & N^\vee_{c/Z}(2) & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & \omega_c & \to & N^\vee_{c/Y}(2) & \to & N^\vee_{c/W}(2) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The map $H^0(N^\vee_{c/Z}(2)) \to H^1(\omega_c)$ is the coboundary map of the middle exact sequence. But this diagram shows that the sequence is split over the factor $\mathcal{O}_c = \wedge^2 N_{c/W}$ of $N^\vee_{c/Z}(2)$. Therefore the coboundary map vanishes on $H^0(\mathcal{O}_c) \subset H^0(N^\vee_{c/Z}(2))$, and our claim follows. □

We are thus in the situation where we can use the results of Donagi and Markman about deformations of a Lagrangian subvariety S of a symplectic variety Y [DM]: over the Hilbert scheme $\mathcal{B}$ parametrizing smooth deformations of S in Y (which are non-obstructed), there exists an integrable system, otherwise said a Lagrangian fibration, whose Liouville tori are the Albanese varieties $Alb(S)$.

In our setting, note that the Abel-Jacobi mapping $AJ : F_g(W) \to J(W)$ factorizes through $F_m(W) = \tilde{S}_W$ and induces an isomorphism (see [Lo, DIM])

$$alb(AJ) : \;\; Alb(\tilde{S}_W) \simeq J(W).$$

Denote by $U_Z \subset \mathbb{P}V_9^\vee$ the open subset parametrizing smooth hyperplane section W of Z. We deduce the following statement:

Theorem 5.3. *For a general Fano fourfold $Z = G \cap H \cap Q$ of degree ten, the set U_Z parametrizing smooth Fano threefolds $W = G \cap H' \cap H'' \cap Q$ in Z, is contained in the base $\mathcal{B}$ of an integrable system, in such a way that over U_Z the Liouville tori are the intermediate Jacobians $J(W)$.*

An interesting point here is that U_Z has dimension eight, while $\mathcal{B}$ is ten dimensional. In particular, the deformations of $S = \tilde{S}_W$ in $Y = \tilde{Y}_Z^\vee$ are not all obtained by deforming W in Z. This is certainly related to the fact that the representation $Y = \tilde{Y}_Z^\vee$ does not defined Z uniquely, as we have stressed in section 4.5. Deforming Z without changing Y, and taking hyperplane sections, we should get more deformations of S.

To be more specific, we can observe that the representation $Y = \tilde{Y}_Z^\vee$ gives a special role to the two Plücker points (the two preimages of the Plücker point in $Y_Z^\vee$), and that the surfaces $\tilde{S}_W$ always contain these points (since W always contain conics of type ρ or σ). Therefore, deforming W in Z should be equivalent to deforming $S = \tilde{S}_W$ in $Y = \tilde{Y}_Z^\vee$ with the Plücker points fixed.

5.2. **Fano fivefolds containing** Z**.** Now we consider the moduli stack $\mathcal{B}$ parametrizing smooth fivefolds $X = G \cap Q$ containing a fixed fourfold $Z = G \cap Q \cap H$ as a hyperplane section. By [B], the tangent space to $\mathcal{B}$ at the point defined by the fivefold Z can be identified with $H^1(X, TX(-1))$.

Lemma 5.4. *The Zariski tangent space $H^1(X, TX(-1))$ to $\mathcal{B}$ at $[X]$ is naturally isomorphic with $H^2(X, \Omega_X^3)$. Its dimension is ten.*

Proof. Since $\omega_X = \mathcal{O}_X(-3)$, we have $H^2(X, \Omega_X^3) = H^2(X, \wedge^2 TX(-3))$. The normal exact sequence of the inclusion $X \subset G$ induces the exact sequence

$$0 \to \wedge^2 TX \to \wedge^2 TG_{|X} \to TX(2) \to 0.$$

By Bott's theorem $\wedge^2 TG(-3)$ is acyclic, and $\wedge^2 TG(-5) = \Omega_G^4$ has non zero cohomology only in degree four. Therefore $\wedge^2 TG(-3)_{|X}$ has non zero cohomology only in degree three. This implies that $H^1(X, TX(-1)) \simeq H^2(X, \wedge^2 TX(-3)$, as claimed. □

Now consider the EPW sextic Y_X and its singular locus S_X, which is a smooth surface.

Proposition 5.5. *The surface S_X is contained in $\hat{Y}_Z^\vee$.*

Proof. Recall that S_X parametrizes pairs (h, V_4) made of hyperplanes h in $I_2(X)$, and hyperplanes $V_4 \subset V_5$, such that the pencil of quadrics on $\mathbb{P}(\wedge^2 V_4)$ obtained by restricting $I_2(X)$, contains a quadric of rank four whose preimage in $I_2(X)$ is precisely h. Cutting with the hyperplane H defining Z, we remain with a quadric of rank at most four, which implies that the point (h, V_4) belongs to $\hat{Y}_Z^\vee$. □

Now we lift this surface to $\tilde{Y}_Z^\vee$. We get the diagram:

$$\begin{array}{ccc} \tilde{S}_X & \hookrightarrow & \tilde{Y}_Z^\vee \\ \downarrow & & \downarrow \\ S_X & \hookrightarrow & \hat{Y}_Z^\vee. \end{array}$$

Proposition 5.6. *The surface $\tilde{S}_X$ is a Lagrangian subvariety of $\tilde{Y}_Z^\vee$.*

Proof. A point $(h, V_4) \in S_X$ defines a corank two quadric in a $\mathbb{P}(\wedge^2 V_4)$, and such a quadric contains two pencils of three-planes. Each of these three-planes will cut X along quadratic surface.

A general quadratic surface Σ in G is given by two transverse planes V_2 and V_2', as the image of the obvious map $\mathbb{P}(V_2)\times\mathbb{P}(V_2') \hookrightarrow G(2, V_2\oplus V_2') \subset G$. The corresponding parameter space is and open subset of $Sym^2 G$ and has dimension 12. The normal bundle of Σ in G is easily seen to decompose as

$$N_{\Sigma/G} = \mathcal{O}_\Sigma(1,1)^{\oplus 2} \oplus \mathcal{O}_\Sigma(1,0) \oplus \mathcal{O}_\Sigma(0,1).$$

If $\Sigma \subset X$, its normal bundle is the kernel of the induced exact sequence

$$0 \to N_{\Sigma/X} \to N_{\Sigma/G} \to \mathcal{O}_\Sigma(2,2) \to 0.$$

Generically $h^1(N_{\Sigma/X}) = 0$ and $h^0(N_{\Sigma/X}) = 12 - 3\times 3 = 3$, so that there is a smooth three-dimensional family of quadratic surfaces in X. There is a natural map from this family to $F_g(Z)$, defined by cutting a quadratic surface Σ with the hyperplane H spanned by Z, to get a conic $c = \Sigma \cap H$. We are then reduced to showing that the image of the restriction map

$$H^0(N_{\Sigma/X}) \to H^0(N_{\Sigma/X|c}) = H^0(N_{c/Z})$$

is isotropic with respect to the two-form ϕ_σ.

But this is easy: recall that if $Y = G \cap H$, so that $Z = Y \cap Q$, the two-form ϕ_σ was defined with the help of the normal exact sequence of the triple (c, Z, Y). But this is the restriction to c of the normal exact sequence of the triple (Σ, X, G), which reads, after dualizing and twisting,

$$0 \to \mathcal{O}_\Sigma(-1,-1) \to N_{\Sigma/G}^\vee(1) \to N_{\Sigma/X}^\vee(1) \to 0.$$

Otherwise said, there is a commutative diagram

$$\begin{array}{ccccc} \wedge^2 H^0(N_{c/Z}) & \to & H^0(\wedge^2 N_{c/Z}) = H^0(N_{c/Z}^\vee(1)) & \to & H^1(\omega_c) = \mathbb{C} \\ \uparrow & & \uparrow & & \uparrow \\ \wedge^2 H^0(N_{\Sigma/X}) & \to & H^0(\wedge^2 N_{\Sigma/X}) = H^0(N_{\Sigma/X}^\vee(1)) & \to & H^1(\mathcal{O}_\Sigma(-1,-1)). \end{array}$$

The first line defines ϕ_σ, and the last line, its restriction to $H^0(N_{\Sigma/X})$. Since $H^1(\mathcal{O}_\Sigma(-1,-1)) = 0$, this restriction vanishes, and we are done. □

Theorem 5.7. *For a general Fano fourfold $Z = G\cap H\cap Q$ of degree ten, the moduli stack $\mathcal{B}$ parametrizing smooth Fano fivefolds $X = G\cap Q$ containing Z is the base of an integrable system whose Liouville tori are the intermediate Jacobians $J(X)$.*

Proof. This can be proved as in [Ma1] for K3-Fano flags, or as in [Ma2] for cubic fivefolds containing a given cubic fourfold. Let us briefly recall the argument, which goes back to [DM], with the necessary (minor) modifications.

A first observation is that the normal exact sequence of the pair (Z, X) induces isomorphisms

$$H^1(\Omega^4_X(Z)) \simeq H^1(\Omega^4_X(Z)_{|Z}) \simeq H^1(\Omega^3_Z) \simeq \mathbb{C}.$$

(For the first two isomorphisms, there are some easy vanishing to verify. For the last one see Lemma 4.1.) One then checks that tensoring with a generator ω_Z of $H^1(\Omega^4_X(Z))$ defines an isomorphism

$$H^1(TX(-Z)) \simeq H^2(\Omega^3_X).$$

The left hand side is to be interpreted as the tangent space to $\mathcal{B}$ at the point defined by Z. The right hand side is the fiber of the Hodge bundle $\mathcal{H}^{3,2}(\mathcal{X}/\mathcal{B})$. The dual vector bundle $\mathcal{E}$ on $\mathcal{B}$ is thus endowed with a natural symplectic form, and one must check that this form descends to the intermediate Jacobian bundle, its quotient by the locally constant bundle of integral forms. For this, one has to normalize the isomorphism $\mathcal{H}^{3,2}(\mathcal{X}/\mathcal{B}) \simeq \Omega^1_{\mathcal{B}}$ by requiring that over Z, it is defined by a generator ω_Z of $H^1(\Omega^4_X(Z))$ restricting to a fixed generator of $H^1(\Omega^3_Z)$. Then the proof [Ma2], Theorem 2.3, applies verbatim. □

It is probably possible to deduce Theorem 5.7 directly from Proposition 5.6, as we deduced Theorem 5.3 from Proposition 5.2. Indeed the general results of Donagi-Markman imply that one can define over $\mathcal{B}$ an integrable system whose fiber over X is the Albanese variety $Alb(\tilde{S}_X)$.

On the other hand, consider the Hilbert scheme $F_{qs}(X)$ parametrizing quadratic surfaces in X. Once a point is chosen in this scheme, the Abel-Jacobi mapping gives a morphism

$$AJ : F_{qs}(X) \to J(X).$$

By the previous arguments $F_{qs}(X)$ is a $\mathbb{P}^1$-bundle over $\tilde{S}_X$, and since every map from $\mathbb{P}^1$ to a complex torus is constant, we get an induced morphism $AJ : \tilde{S}_X \to J(X)$. Hence, for the Albanese variety, a morphism

$$alb(AJ) : \quad Alb(\tilde{S}_X) \to J(X).$$

Very probably, this morphism should be an isomorphism. But this seems technically much more difficult to check than to prove Theorem 5.7 as we did above.

Appendix : Proof of Lemma 3.9

We can choose a basis of V_5 such that ℓ be the double line defined as the intersection of $G(2, V_5)$ with the plane $P = \langle v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3 + v_1 \wedge v_4 \rangle$. Around $v_1 \wedge v_3$ we have affine coordinates on $G(2, V_5)$ such that a plane transverse to $\langle v_2, v_4, v_5 \rangle$ has a basis of the form

$$\begin{aligned} w_1 &= v_1 + z_2 v_2 + z_4 v_4 + z_5 v_5, \\ w_3 &= v_3 + t_2 v_2 + t_4 v_4 + t_5 v_5. \end{aligned}$$

In these coordinates, $\mathcal{I}_{\ell,G}$ is generated by $z_4, z_5, t_4 - z_2, t_5$ and t_4^2. We check that an element $\psi \in Hom(\mathcal{I}_{\ell,G}, \mathcal{O}_\ell)$ is of the following form:

$$\begin{aligned} t_4^2 &\mapsto \psi_1 + \psi_2 t_2 + \psi_3 t_2^2 + \psi_4 t_4 + \psi_5 t_2 t_4, \\ t_5 &\mapsto \psi_6 + \psi_7 t_2 + \psi_8 t_4, \\ t_4 - z_2 &\mapsto \psi_9 + \psi_{10} t_2 + \psi_{11} t_4, \\ z_5 &\mapsto \psi_{12} + \psi_7 t_4, \\ z_4 &\mapsto \psi_{13} + \psi_3 t_2 + (\psi_5 - \psi_{10}) t_4. \end{aligned}$$

Now, locally around $e_1 \wedge e_3$ the ideal sheaf $\mathcal{I}_{Z,G}$ is generated by h/p_{13} and q/p_{13}^2, where h and q are the equations of the hyperplane H and of the quadric Q, expressed in terms of Plücker coordinates. Write

$$h = \sum_{i<j} h_{ij} p_{ij}, \qquad q = \sum_{i<j, k<\ell} q_{ij,kl} p_{ij} p_{kl}.$$

For Z to contain ℓ we first need that H contains P, that is,

$$h_{12} = h_{13} = h_{14} + h_{23} = 0,$$

and that the equation of Q restricted to the plane P reduces to that of the double line ℓ, which gives

$$\begin{gathered} q_{12,12} = q_{12,13} = q_{13,13} = 0, \\ q_{12,14} + q_{12,23} = q_{13,14} + q_{13,23} = 0. \end{gathered}$$

Using these relations, we must express h/p_{13} and q/p_{13}^2 in terms of our preferred local generators of $\mathcal{I}_{\ell,G}$ and deduce their images by the morphism ψ. We find that

$$\begin{aligned} h/p_{13} &\mapsto A + Bt_2 + Ct_4, \\ q/p_{13}^2 &\mapsto D + Et_2 + Ft_2^2 + Gt_4 + Ht_2 t_4 \end{aligned}$$

where the quantities A, B, C, D, E, F, G, H are given by the following formulas:

$$\begin{aligned} A &= h_{15}\psi_6 + h_{14}\psi_9 + h_{24}\psi_1 - h_{34}\psi_{13} - h_{35}\psi_{12}, \\ B &= h_{15}\psi_7 + h_{14}\psi_{10} + h_{24}(\psi_2 - \psi_{13}) - h_{34}\psi_3 - h_{25}\psi_{12}, \\ C &= h_{15}\psi_3 + h_{14}\psi_{11} + h_{24}(\psi_4 - \psi_9) - h_{34}(\psi_5 - \psi_{10}) \\ &\qquad -h_{35}\psi_7 + h_{25}\psi_6 - h_{45}\psi_{12}, \\ D &= b_{15}\psi_6 + (b_{24} + g)\psi_1 - b_{34}\psi_{13} - b_{35}\psi_{12} + e\psi_9, \\ E &= a_{15}\psi_6 + b_{15}\psi_7 + a_{24}\psi_1 + b_{24}(\psi_2 - \psi_{13}) - (a_{35} + b_{25})\psi_{12} \\ &\qquad -a_{34}\psi_{13} - b_{34}\psi_3 + d\psi_9 + e\psi_{10} + g\psi_2, \\ F &= a_{15}\psi_7 + a_{24}(\psi_2 - \psi_{13}) - a_{25}\psi_{12} + (g - a_{34})\psi_3 + d\psi_{10}, \\ G &= b_{15}\psi_8 + (b_{25} + c_{15})\psi_6 + c_{24}\psi_1 + b_{24}(\psi_4 - \psi_9) - b_{34}(\psi_5 - \psi_{10}) \\ &\qquad -c_{34}\psi_{13} - b_{35}\psi_7 - c_{35}\psi_{12} + f\psi_9 + e\psi_{11} + g\psi_4, \\ H &= a_{15}\psi_8 + c_{15}\psi_7 + a_{24}(\psi_4 - \psi_9) + c_{24}(\psi_2 - \psi_{13}) + a_{25}\psi_6 - c_{25}\psi_{12} \\ &\qquad -a_{34}(\psi_5 - \psi_{10}) - c_{34}\psi_3 - a_{35}\psi_7 + g\psi_5 + f\psi_{10} + d\psi_{11}. \end{aligned}$$

In these formulas we have set $a_{ij}=q_{12,ij}$, $b_{ij}=q_{13,ij}$, $c_{ij}=2q_{14,ij}=2q_{23,ij}$, $d=q_{12,14}=-q_{12,23}$, $e=q_{13,14}=-q_{13,23}$, $f=-2q_{23,23}$ and $g=q_{14,14}+q_{23,23}$.

So the rank of ϕ is equal to the rank of the 13×8 matrix

$$\begin{pmatrix}
b_{24}+g & a_{24} & 0 & c_{24} & 0 & h_{24} & 0 & 0\\
0 & b_{24}+g & a_{24} & 0 & c_{24} & 0 & h_{24} & 0\\
0 & -b_{34} & g-a_{34} & 0 & -c_{34} & 0 & -h_{34} & 0\\
0 & 0 & 0 & b_{24}+g & a_{24} & 0 & 0 & h_{24}\\
0 & 0 & 0 & -b_{34} & g-a_{34} & 0 & 0 & -h_{34}\\
b_{15} & a_{15} & 0 & c_{15}+b_{15} & -a_{25} & h_{15} & 0 & h_{25}\\
0 & b_{15} & a_{15} & -b_{35} & c_{15}-a_{35} & 0 & h_{15} & -h_{35}\\
0 & 0 & 0 & b_{15} & a_{15} & 0 & 0 & h_{15}\\
e & d & 0 & f-b_{24} & -a_{24} & h_{14} & 0 & -h_{24}\\
0 & e & d & b_{34} & f+a_{34} & 0 & h_{14} & h_{34}\\
0 & 0 & 0 & e & d & 0 & 0 & h_{14}\\
-b_{35} & -b_{25}-a_{35} & -a_{25} & -c_{35} & -c_{25} & -h_{35} & -h_{25} & -h_{45}\\
-b_{34} & -b_{24}-a_{34} & -a_{24} & -c_{34} & -c_{24} & -h_{34} & -h_{24} & 0
\end{pmatrix}$$

We need to show that this matrix has full rank outside a locus of codimension at least three. For this we let $d_{24}=b_{24}+g$, $d_{34}=a_{34}+f$, $k=-a_{34}-b_{24}$. After permuting lines and columns we get the following matrix M:

$$\begin{pmatrix}
d_{24} & a_{24} & 0 & h_{24} & 0 & c_{24} & 0 & 0\\
0 & d_{24} & a_{24} & 0 & h_{24} & 0 & c_{24} & 0\\
0 & 0 & 0 & 0 & 0 & d_{24} & a_{24} & h_{24}\\
b_{15} & a_{15} & 0 & h_{15} & 0 & c_{15}+b_{15} & -a_{25} & h_{25}\\
0 & b_{15} & a_{15} & 0 & h_{15} & -b_{35} & c_{15}-a_{35} & -h_{35}\\
0 & 0 & 0 & 0 & 0 & b_{15} & a_{15} & h_{15}\\
e & d & 0 & h_{14} & 0 & d_{24}+k & -a_{24} & -h_{24}\\
0 & e & d & 0 & h_{14} & b_{34} & d_{34} & h_{34}\\
0 & 0 & 0 & 0 & 0 & e & d & h_{14}\\
0 & 0 & 0 & 0 & 0 & -b_{34} & d_{24}+k & -h_{34}\\
-b_{35} & -b_{25}-a_{35} & -a_{25} & -h_{35} & -h_{25} & -c_{35} & -c_{25} & -h_{45}\\
-b_{34} & k & -a_{24} & -h_{34} & -h_{24} & -c_{34} & -c_{24} & 0\\
0 & -b_{34} & d_{24}+k & 0 & -h_{34} & 0 & -c_{34} & 0
\end{pmatrix}$$

Observe the role of the matrix

$$m=\begin{pmatrix} d_{24} & a_{24} & h_{24}\\ b_{15} & a_{15} & h_{15}\\ e & d & h_{14}\end{pmatrix}$$

Its rank is at least two in codimension three. If it is equal to three, then clearly ϕ has full rank. So we may suppose that the rank of m is equal to two, which occurs in codimension one. Let A, B, C denote the three-dimensional spaces corresponding to columns 124, 235 and 678 respectively. Observing the three first lines of the matrix, we see that they can be written as $p(v_1)+v_1', q(v_1)+v_1'', v_1$, where v_1, v_1', v_1'' belong to C and $p: C\to A$, $q: C\to B$ are isomorphisms. Moreover the same is true for the two next groups of three lines, for some vectors $v_2, v_3, ..$ in C. Our hypothesis on m is that the span of v_1, v_2, v_3 is two-dimensional. So there is a relation $\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3=0$, and combining our lines accordingly we get the vectors $\alpha_1 v_1'+\alpha_2 v_2'+\alpha_3 v_3'$ and $\alpha_1 v_1''+\alpha_2 v_2''+\alpha_3 v_3''$, which belong to C.

Since the tenth line of the matrix M is also a vector of C, it is easy to conclude that in codimension at least three, C is contained is the span of the lines of M.

Then we can focus on the first five columns and forget the other ones. We know that the first nine lines of M span a space of the form $p(L)+q(L)$ for some plane L in C. If $p(L)$ and $q(L)$ meet, this can be only inside $A\cap B$, which is one dimensional. This is easy to exclude in codimension two. Then $p(L)+q(L)$ has codimension one, and to ensure that the matrix has full rank, it is enough to check that the the last three lines contribute, that is, they are not contained in $p(L)+q(L)$. Since the entries of these lines do not appear in the remaining of the matrix, except for a_{24} and d_{24}, this is also easy. □

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