

ISOLATED BOUNDARY SINGULARITIES OF SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. Given a smooth domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \partial\Omega$ and given a nonnegative smooth function ζ on $\partial\Omega$, we study the behavior near 0 of positive solutions of $-\Delta u = u^q$ in Ω such that $u = \zeta$ on $\partial\Omega \setminus \{0\}$. We prove that if $\frac{N+1}{N-1} < q < \frac{N+2}{N-2}$, then $u(x) \leq C|x|^{-\frac{2}{q-1}}$ and we compute the limit of $|x|^{\frac{2}{q-1}} u(x)$ as $x \rightarrow 0$. We also investigate the case $q = \frac{N+1}{N-1}$. The proofs rely on the existence and uniqueness of solutions of related equations on spherical domains.

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1. INTRODUCTION AND MAIN RESULTS

Let Ω be a smooth open subset of \mathbb{R}^N , with $N \geq 2$, such that $0 \in \partial\Omega$. Given $q > 1$ and $\zeta \in C^\infty(\partial\Omega)$ with $\zeta \geq 0$ on $\partial\Omega$, consider the problem

$$(1.1) \quad \begin{cases} -\Delta u = u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = \zeta & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

By a solution of (1.1) we mean a function $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ which satisfies (1.1) in the classical sense. A solution may develop an isolated singularity at 0. Our main goal in this paper is to describe the behavior of u in a neighborhood of this point.

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In the study of boundary singularities of (1.1), one finds three critical exponents; namely,

$$q_1 = \frac{N+1}{N-1}, \quad q_2 = \frac{N+2}{N-2} \quad \text{and} \quad q_3 = \frac{N+1}{N-3},$$

with the usual convention if $N = 2$ or $N = 3$. When $1 < q < q_1$, it is proved by Bidaut-Véron-Vivier [8] that for every solution u of (1.1) there exists $\alpha \geq 0$ (depending on u) such that

$$u(x) = \alpha |x|^{-N} \text{dist}(x, \partial\Omega) (1 + o(1)) \quad \text{as } x \rightarrow 0.$$

In this paper we mainly investigate the case $q_1 \leq q < q_3$.

The counterpart of (1.1) for an interior singularity,

$$-\Delta u = u^q \quad \text{in } \Omega \setminus \{x_0\},$$

where $x_0 \in \Omega$, was studied by P.-L. Lions [18] in the subcritical case $1 < q < \frac{N}{N-2}$, by Aviles [2] when $q = \frac{N}{N-2}$ and by Gidas-Spruck [13] in the range $\frac{N}{N-2} < q < \frac{N+2}{N-2}$. We prove some counterparts of the works of Gidas-Spruck and Aviles in the framework of boundary singularities.

When (1.1) is replaced by an equation with an absorption term,

$$(1.2) \quad -\Delta u + u^q = 0 \quad \text{in } \Omega,$$

the problem has been first addressed by Gmira-Véron [14] (and later to nonsmooth domains in [12]). These results are important in the theory of boundary trace of positive solutions of (1.2) which was developed by Marcus-Véron [19, 20, 21] using analytic tools and by Le Gall [17] and Dynkin-Kuznetsov [10, 11] with a probabilistic approach. We refer the reader to Véron [25] for the case of interior singularities of (1.2).

Let us first consider the case where Ω is the upper-half space \mathbb{R}_+^N , and we look for solutions of (1.1) of the form

$$u(x) = |x|^{-\frac{2}{q-1}} \omega\left(\frac{x}{|x|}\right).$$

By an easy computation, ω must satisfy

$$(1.3) \quad \begin{cases} -\Delta' \omega = \ell_{N,q} \omega + \omega^q & \text{in } S_+^{N-1}, \\ \omega \geq 0 & \text{in } S_+^{N-1}, \\ \omega = 0 & \text{on } \partial S_+^{N-1}, \end{cases}$$

where Δ' denotes the Laplace-Beltrami operator in the unit sphere S^{N-1} ,

$$\ell_{N,q} = \frac{2(N-q(N-2))}{(q-1)^2} \quad \text{and} \quad S_+^{N-1} = S^{N-1} \cap \mathbb{R}_+^N.$$

Concerning equation (1.3), we prove

Theorem 1.1.

- (i) If $1 < q \leq q_1$, then (1.3) admits no positive solution.
- (ii) If $q_1 < q < q_3$, then (1.3) admits a unique positive solution.
- (iii) If $q \geq q_3$, then (1.3) admits no positive solution.

In Section 3 we study uniqueness of solutions of (1.3) with $\ell_{N,q}$ replaced by any $\ell \in \mathbb{R}$. The proofs are inspired from some interesting ideas taken from Kwong [15] and Kwong-Li [16]. The nonexistence of solutions of (1.3) when $q \geq q_3$ is based on a Pohožaev identity for spherical domains; see Theorem 2.1 below.

We now consider the case where $\Omega \subset \mathbb{R}^N$ is a smooth domain such that $0 \in \partial\Omega$. Without loss of generality, we may assume that $-\mathbf{e}_N$ is the outward unit normal vector of $\partial\Omega$ at 0. We prove the following classification of isolated singularities of solutions of (1.1):

Theorem 1.2. *Assume that $q_1 < q < q_2$. If u satisfies (1.1), then either u can be continuously extended at 0 or for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \Omega \setminus \{0\}$, $\frac{x}{|x|} \in S_+^{N-1}$ and $|x| < \delta$,*

$$(1.4) \quad \left| |x|^{\frac{2}{q-1}} u(x) - \omega\left(\frac{x}{|x|}\right) \right| < \varepsilon,$$

where ω is the unique positive solution of (1.3).

When $q_2 < q < q_3$, we have a similar conclusion provided u satisfies the estimate

$$u(x) \leq C|x|^{-\frac{2}{q-1}} \quad \forall x \in \Omega,$$

for some constant $C > 0$; see Proposition 8.1 below. In the critical case $q = q_1$ there is a superposition of the linear and nonlinear effects since their characteristic exponents $\frac{2}{q-1}$ and $N-1$ coincide. The counterpart of Theorem 1.2 in this case is the following:

Theorem 1.3. *Assume that $q = q_1$. If u satisfies (1.1), then either u can be continuously extended at 0 or for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \Omega \setminus \{0\}$ and $|x| < \delta$,*

$$(1.5) \quad \left| |x|^{N-1} \left(\log \frac{1}{|x|} \right)^{\frac{N-1}{2}} u(x) - \kappa \frac{x_N}{|x|} \right| < \varepsilon,$$

where κ is a positive constant depending only on the dimension N .

Our characterization of boundary isolated singularities is complemented by the existence of singular solutions which has been recently obtained by del Pino-Musso-Pacard [23]. We recall their result:

Theorem 1.4. *Assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. There exists $p \in (q_1, q_2)$ such that for every $q_1 \leq q < p$ and for every $\xi_1, \xi_2, \dots, \xi_k \in \partial\Omega$, there exists a positive function $u \in C(\bar{\Omega} \setminus \{\xi_j\}_{j=1}^k)$, vanishing on $\partial\Omega \setminus \{\xi_j\}_{j=1}^k$, solution of $-\Delta u = u^q$ in Ω , such that*

$$u(x) \rightarrow +\infty \quad \text{as } x \rightarrow \xi_j \text{ nontangentially for every } i = 1, 2, \dots, k.$$

In view of Theorems 1.2 and 1.3 any such solution must have the singular behavior we have obtained therein. In [23], the authors conjecture that such solutions exist for every $q_1 \leq q < q_2$.

Some of the main ingredients in the proofs of Theorems 1.2 and 1.3 are Theorem 1.1 above concerning existence and uniqueness of positive solutions of (1.3), a removable singularity result (see Theorems 7.1 and 7.2 below) and the following a priori bound of solutions of (1.1):

Theorem 1.5. *Assume that $1 < q < q_2$. Then, every solution of (1.1) satisfies*

$$(1.6) \quad u(x) \leq C|x|^{-\frac{2}{q-1}} \quad \forall x \in \Omega,$$

for some constant $C > 0$ independent of the solution.

We establish this estimate using a topological argument, called the *Doubling lemma* (see Lemma 5.1 below), introduced by Poláčik-Quittner-Souplet [24].

Theorems 1.2, 1.3 and 1.5 have been announced in [6].

2. POHOŽAEV IDENTITY IN SPHERICAL DOMAINS

We first prove the following Pohožaev identity in spherical domains.

Theorem 2.1. *Let $q > 1$, $\ell \in \mathbb{R}$ and S be a smooth domain in S_+^{N-1} . If $v \in C^2(S) \cap C(\bar{S})$ satisfies*

$$(2.1) \quad \begin{cases} -\Delta' v = \ell v + |v|^{q-1} v & \text{in } S, \\ v = 0 & \text{on } \partial S, \end{cases}$$

then

$$(2.2) \quad \left(\frac{N-3}{2} - \frac{N-1}{q+1}\right) \int_S |\nabla' v|^2 \phi \, d\sigma - \frac{N-1}{2} \left(\frac{\ell(q-1)+N-1}{q+1}\right) \int_S v^2 \phi \, d\sigma = \frac{1}{2} \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle \, d\tau,$$

where ν is the outward unit normal vector on ∂S , ∇' the tangential gradient to S^{N-1} , and ϕ is a first eigenfunction of the Laplace-Beltrami operator $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$.

We recall that the first eigenvalue of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ is $N-1$ and the eigenspace associated to this eigenvalue is spanned by the function $\phi(x) = \frac{x_N}{|x|}$.

Proof. Let

$$P = \langle \nabla' \phi, \nabla' v \rangle \nabla' v.$$

By the Divergence theorem,

$$(2.3) \quad \int_S \operatorname{div} P \, d\sigma = \int_{\partial S} \langle P, \nu \rangle \, d\tau.$$

Note that

$$\operatorname{div} P = \langle \nabla' v, \nabla' \phi \rangle \Delta' v + D^2 v(\nabla' v, \nabla' \phi) + D^2 \phi(\nabla' v, \nabla' v).$$

where $D^2 v$ is the Hessian operator. Now,

$$D^2 v(\nabla' v, \nabla' \phi) = \frac{1}{2} \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle.$$

Using the classical identity

$$D^2 \phi + \phi g = 0$$

where $g = (g_{i,j})$ is the metric tensor on S^{N-1} , we get

$$D^2 \phi(\nabla' v, \nabla' v) = -g(\nabla' v, \nabla' v) \phi = -|\nabla' v|^2 \phi.$$

We replace these identities in the expression of $\operatorname{div} P$,

$$\operatorname{div} P = -\langle \nabla' v, \nabla' \phi \rangle (\ell v + |v|^{q-1} v) + \frac{1}{2} \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle - |\nabla' v|^2 \phi.$$

Integrating over S , we obtain

$$\int_S \operatorname{div} P \, d\sigma = - \int_S \langle \nabla' v, \nabla' \phi \rangle (\ell v + |v|^{q-1} v) \, d\sigma + \frac{1}{2} \int_S \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle \, d\sigma - \int_S |\nabla' v|^2 \phi \, d\sigma.$$

Note that

$$\begin{aligned} \int_S \langle \nabla' v, \nabla' \phi \rangle (\ell v + |v|^{q-1} v) d\sigma &= \int_S \left\langle \nabla' \left(\frac{\ell}{2} v^2 + \frac{1}{q+1} |v|^{q+1} \right), \nabla' \phi \right\rangle d\sigma \\ &= - \int_S \left(\frac{\ell}{2} v^2 + \frac{1}{q+1} |v|^{q+1} \right) \Delta' \phi d\sigma \\ &= (N-1) \int_S \left(\frac{\ell}{2} v^2 + \frac{1}{q+1} |v|^{q+1} \right) \phi d\sigma, \end{aligned}$$

and

$$\begin{aligned} \int_S \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle d\sigma &= - \int_S |\nabla' v|^2 \Delta' \phi d\sigma + \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau \\ &= (N-1) \int_S |\nabla' v|^2 \phi d\sigma + \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

These identities imply

$$(2.4) \quad \begin{aligned} \int_S \operatorname{div} P d\sigma &= -\frac{\ell(N-1)}{2} \int_S v^2 \phi d\sigma - \frac{N-1}{q+1} \int_S |v|^{q+1} \phi d\sigma + \frac{N-3}{2} \int_S |\nabla' v|^2 \phi d\sigma + \\ &\quad + \frac{1}{2} \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

On the other hand, since v satisfies (2.1),

$$\begin{aligned} \int_S (\ell v^2 + |v|^{q+1}) \phi d\sigma &= - \int_S (\Delta' v) v \phi d\sigma \\ &= \int_S \langle \nabla' v, \nabla' (v\phi) \rangle d\sigma = \int_S |\nabla' v|^2 \phi d\sigma + \int_S \langle \nabla' v, \nabla' \phi \rangle v d\sigma. \end{aligned}$$

Since $v \nabla' v = \frac{1}{2} \nabla' (v^2)$ and $\Delta' \phi = -(N-1)\phi$,

$$\int_S \langle \nabla' v, \nabla' \phi \rangle v d\sigma = \frac{1}{2} \int_S \langle \nabla' (v^2), \nabla' \phi \rangle d\sigma = \frac{N-1}{2} \int_S v^2 \phi d\sigma.$$

Thus,

$$\int_S (\ell v^2 + |v|^{q+1}) \phi d\sigma = \int_S |\nabla' v|^2 \phi d\sigma + \frac{N-1}{2} \int_S v^2 \phi d\sigma.$$

This implies

$$\int_S |v|^{q+1} \phi d\sigma = \int_S |\nabla' v|^2 \phi d\sigma + \left(\frac{N-1}{2} - \ell \right) \int_S v^2 \phi d\sigma.$$

Inserting this identity in (2.4), we obtain

$$(2.5) \quad \begin{aligned} \int_S \operatorname{div} P d\sigma &= \left(\frac{N-3}{2} - \frac{N-1}{q+1} \right) \int_S |\nabla' v|^2 \phi d\sigma - \left(\frac{\ell(N-1)}{2} + \frac{N-1}{q+1} \left(\frac{N-1}{2} - \ell \right) \right) \int_S v^2 \phi d\sigma + \\ &\quad + \frac{1}{2} \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

Since v vanishes on ∂S , $\nabla'v = \langle \nabla'v, \nu \rangle \nu$ and, in particular, $|\nabla'v| = |\langle \nabla'v, \nu \rangle|$. Thus,

$$(2.6) \quad \begin{aligned} \int_{\partial S} \langle P, \nu \rangle d\tau &= \int_{\partial S} \langle \nabla' \phi, \nabla' v \rangle \langle \nabla' v, \nu \rangle d\tau = \int_{\partial S} (\langle \nabla' v, \nu \rangle)^2 \langle \nabla' \phi, \nu \rangle d\tau \\ &= \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

Combining (2.3), (2.5) and (2.6), we get the Pohožaev identity. \square

Using the Pohožaev identity on S_+^{N-1} we can prove that the Dirichlet problem (2.1) can only have trivial solutions for suitable values of q and ℓ .

Corollary 2.1. *Let $N \geq 4$. If $q \geq q_3$ and $\ell \leq -\frac{N-1}{q-1}$, then the function identically zero is the only solution in $C^2(S^{N-1}) \cap C^2(\overline{S^{N-1}})$ of the Dirichlet problem*

$$\begin{cases} -\Delta'v = \ell v + |v|^{q-1}v & \text{in } S_+^{N-1}, \\ v = 0 & \text{on } \partial S_+^{N-1}. \end{cases}$$

Proof. Let v be a solution of the Dirichlet problem. Applying the Pohožaev identity with $\phi(x) = \frac{x_N}{|x|}$, then the left-hand side of the Pohožaev identity is nonnegative, while its right-hand side is nonpositive. Thus, both sides are zero. If at least one of the inequalities $q \geq q_3$ or $\ell \leq -\frac{N-1}{q-1}$ is strict, then we immediately deduce that $v = 0$ in S_+^{N-1} .

If $q = q_3$ and $\ell = -\frac{N-1}{q-1}$, then

$$\int_{\partial S_+^{N-1}} |\nabla'v|^2 \langle \nabla' \phi, \nu \rangle d\tau = 0.$$

Since $\langle \nabla' \phi, \nu \rangle < 0$ on ∂S_+^{N-1} , we conclude that $\nabla'v = 0$ on ∂S_+^{N-1} . Define the function $\tilde{v} : S^{N-1} \rightarrow \mathbb{R}$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in S_+^{N-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, \tilde{v} satisfies (in the sense of distributions)

$$-\Delta' \tilde{v} = \ell \tilde{v} + |\tilde{v}|^{q-1} \tilde{v} \quad \text{in } S^{N-1}.$$

Since \tilde{v} vanishes in an open subset of S^{N-1} , by the unique continuation principle we have $\tilde{v} = 0$ in S^{N-1} and the conclusion follows. \square

Remark. When $S \subsetneq S_+^{N-1}$ and $q > q_3$ the previous non-existence result can be improved if we define

$$(2.7) \quad \lambda(S, \phi) = \sup \left\{ \mu \geq 0 : \int_S |\nabla' \zeta|^2 \phi d\sigma \geq \mu \int_S \zeta^2 \phi d\sigma, \quad \forall \zeta \in C_0^\infty(S) \right\}.$$

This constant $\lambda(S, \phi)$ is actually zero if $S = S_+^{N-1}$. With this inequality (2.2) turns into

$$(2.8) \quad \left[\left(\frac{N-3}{2} - \frac{N-1}{q+1} \right) \lambda(S, \phi) - \frac{N-1}{2} \left(\frac{\ell(q-1)+N-1}{q+1} \right) \right] \int_S v^2 \phi d\sigma \leq \frac{1}{2} \int_{\partial S} |\nabla'v|^2 \langle \nabla' \phi, \nu \rangle d\tau.$$

Therefore, the statement of Corollary 2.1 still holds if $q > q_3$ and

$$(2.9) \quad \ell(q-1) \geq 1 - N + \frac{q(N-3) - N - 1}{N-1} \lambda(S, \phi).$$

Note that $\lambda(S, \phi)$ tends to infinity if S shrinks to a point.

3. UNIQUENESS OF SOLUTIONS OF A PDE IN S_+^{N-1}

In this section we address the question of uniqueness of positive solutions of the Dirichlet problem

$$(3.1) \quad \begin{cases} -\Delta' v = \ell v + v^q & \text{in } S_+^{N-1}, \\ v \geq 0 & \text{in } S_+^{N-1}, \\ v = 0 & \text{on } \partial S_+^{N-1}, \end{cases}$$

where $\ell \in \mathbb{R}$. A solution of (3.1) is understood in the classical sense.

We shall prove the following results:

Theorem 3.1. *Assume that $N = 2$. If $q > 1$, then for every $\ell \in \mathbb{R}$ the Dirichlet problem (3.1) has at most one positive solution.*

Theorem 3.2. *Assume that $N \geq 4$. If $1 < q < q_3$, then for every $\ell \in \mathbb{R}$ the Dirichlet problem (3.1) has at most one positive solution.*

Theorem 3.3. *Assume that $N = 3$. Then, the Dirichlet problem (3.1) has at most one positive solution under one of the following assumptions:*

- for every $1 < q \leq 5$ and $\ell \in \mathbb{R}$,
- for every $q > 5$ and $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$.

Remark 3.1. In dimension $N = 3$ we do not know whether the Dirichlet problem (3.1) has a unique positive solution if $q > 5$ and $\ell > \frac{2(3-q)}{(q+3)(q-1)}$.

We first show that the graphs of two positive solutions of (3.1) must cross.

Lemma 3.1. *Assume that v_1 and v_2 are positive solutions of (3.1). If $v_1 \leq v_2$ in S_+^{N-1} , then $v_1 = v_2$.*

Proof. Multiplying by v_2 the equation satisfied by v_1 and integrating by parts, we get

$$\int_{S_+^{N-1}} \langle \nabla v_1, \nabla v_2 \rangle d\sigma = \int_{S_+^{N-1}} (\ell v_1 + (v_1)^q) v_2 d\sigma.$$

Reversing the roles of v_1 and v_2 , we also have

$$\int_{S_+^{N-1}} \langle \nabla v_2, \nabla v_1 \rangle d\sigma = \int_{S_+^{N-1}} (\ell v_2 + (v_2)^q) v_1 d\sigma.$$

Subtracting these identities, we have

$$\int_{S_+^{N-1}} (v_1^{q-1} - v_2^{q-1}) v_1 v_2 d\sigma = 0.$$

Since the integrand is nonnegative we must have $v_1^{q-1} - v_2^{q-1} = 0$ and the conclusion follows. \square

We consider first the case $N = 2$. The precise structure of the set of all signed solutions defined on \mathbb{R} is already established in [4, Lemma 1.1], see also Theorem 1.1 therein for the main result. In this paper the proof is based upon the fact that the equation is autonomous. Here we use another argument which is in the line of the one developed in the cases $N \geq 3$ studied below.

Proof of Theorem 3.1. Denoting by

$$\theta = \arccos \frac{x_2}{|x|},$$

then a solution of (3.1) satisfies

$$\begin{cases} v_{\theta\theta} + \ell v + v^q = 0 & \text{in } (0, \frac{\pi}{2}), \\ v_{\theta}(0) = 0, \quad v(\frac{\pi}{2}) = 0. \end{cases}$$

Moreover, for every $\theta \in (0, \frac{\pi}{2}]$, $v_{\theta}(\theta) < 0$; indeed $v_{\theta}(\frac{\pi}{2}) < 0$, and if $\rho = \inf\{\theta > 0 : v_{\theta}(\theta) < 0\} > 0$, then from uniqueness $\rho = \frac{\pi}{4}$, $v(\theta) = v(\frac{\pi}{2} - \theta)$, hence $v_{\theta}(0) > 0$, contradiction (notice that this argument is the 1-dim moving plane method). Thus, v is decreasing. Let $V : [0, v(0)] \rightarrow \mathbb{R}$ be the function defined by

$$(3.2) \quad V(\xi) = v_{\theta}(v^{-1}(\xi)).$$

Then, V is of class C^1 in $[0, v(0))$. Since for every $\xi \in [0, v(0))$,

$$(v^{-1})_{\xi}(\xi) = \frac{1}{v_{\theta}(v^{-1}(\xi))} = \frac{1}{V(\xi)},$$

we deduce that

$$(3.3) \quad (V^2)_{\xi} = 2VV_{\xi} = 2V(v_{\theta\theta} \circ v^{-1})(v^{-1})_{\xi} = 2(v_{\theta\theta} \circ v^{-1}) = -2(\ell\xi + \xi^q).$$

Assume by contradiction that (3.1) has two distinct positive solutions, say v_1 and v_2 . We may assume they are both defined in terms of the variable θ . Then, there exists $c_1 \in (0, \frac{\pi}{2})$ such that $v_1(c_1) = v_2(c_1)$. Let $c_2 \in (c_1, \frac{\pi}{2}]$ be the smallest number such that $v_1(c_2) = v_2(c_2)$ (this point c_2 exists since $v_{1\theta}(c_1) \neq v_{2\theta}(c_1)$). Without loss of generality, we may assume that, for every $\theta \in (c_1, c_2)$,

$$v_1(\xi) < v_2(\xi).$$

Let V_1 and V_2 be the functions given by (3.2) corresponding to v_1 and v_2 , respectively. For $i \in \{1, 2\}$, let

$$\alpha_i = v_1(c_i) = v_2(c_i).$$

By (3.3), for every $\xi \in (\alpha_2, \alpha_1)$,

$$(V_1^2)_{\xi}(\xi) = -2(\ell\xi + \xi^q) = (V_2^2)_{\xi}(\xi).$$

Hence, the function $V_1^2 - V_2^2$ is constant. On the other hand, since $v_1 < v_2$ and v_1, v_2 are both decreasing, by uniqueness of the Cauchy problem,

$$v_{1\theta}(c_1) < v_{2\theta}(c_1) < 0 \quad \text{and} \quad v_{2\theta}(c_2) < v_{1\theta}(c_2) < 0.$$

Thus,

$$V_1^2(\alpha_1) - V_2^2(\alpha_1) > 0 \quad \text{and} \quad V_1^2(\alpha_2) - V_2^2(\alpha_2) < 0.$$

This is a contradiction. We conclude that problem (3.1) cannot have more than one positive solution. \square

Remark. The proofs in [4] as well as the one here are valid for equation

$$(3.4) \quad v_{\theta\theta} + \ell v + g(v) = 0$$

where $g \in C^1(\mathbb{R})$, $g(0) = 0$ and $r \mapsto \frac{g(r)}{r}$ is increasing on $(0, \infty)$. In [5, Prop 4.4] a more general, result is obtained.

In order to study (3.1) in the case of higher dimensions, the first step is to rewrite the Dirichlet problem in terms of an ODE. By an adaptation of the moving planes method to S^{N-1} (see [22]), any positive solution v of (3.1) depends only on the geodesic distance to the North pole:

$$\theta = \arccos \frac{x_N}{|x|}$$

and v decreasing with respect to θ . Since in this case

$$\Delta' v = \frac{1}{(\sin \theta)^{N-2}} \frac{d}{d\theta} ((\sin \theta)^{N-2} v_\theta),$$

every solution of (3.1) satisfies the following ODE in terms of the variable θ :

$$(3.5) \quad \begin{cases} v_{\theta\theta} + (N-2) \cot \theta v_\theta + \ell v + v^q = 0 & \text{in } (0, \frac{\pi}{2}), \\ v_\theta(0) = 0, \quad v(\frac{\pi}{2}) = 0. \end{cases}$$

The heart of the matter is then to apply some ideas from Kwong [15] and Kwong-Li [16], originally dealing with positive solutions of

$$(3.6) \quad \begin{cases} u_{rr} + (N-2) \frac{1}{r} u_r + \ell u + u^q = 0 & \text{in } (0, a), \\ u_r(0) = 0, \quad u(a) = 0. \end{cases}$$

By Lemma 3.1 and the discussion above, the graphs of two positive solutions of (3.5) must intersect in $(0, \frac{\pi}{2})$. Of course, the number of intersection points could be arbitrarily large (but always finite in view of the uniqueness of the Cauchy problem). The next lemma allows us to reduce the problem to the case where there could be only one intersection point. The argument relies on the shooting method and continuous dependence arguments; we only give a sketch of the proof.

Lemma 3.2. *Assume that (3.5) has two distinct positive solutions. Then, there exists two positive solutions of (3.5) the graph of which intersect only once in the interval $(0, \frac{\pi}{2})$.*

Sketch of the proof. For each $\alpha > 0$ let v^α be the (unique) maximal solution of

$$\begin{cases} v_{\theta\theta} + (N-2) \cot \theta v_\theta + \ell v + |v|^{q-1} v = 0 & \text{in } I_\alpha = (0, m_\alpha) \subset (0, \pi), \\ v_\theta(0) = 0, \quad v(0) = \alpha. \end{cases}$$

Then $v = v^\alpha$ is obtained by the contraction mapping principle on some interval $[0, \tau_\alpha]$, by the formula

$$(3.7) \quad v(\theta) = \alpha - \int_0^\theta (\sin \sigma)^{2-N} \int_0^\sigma (\sin \tau)^{N-2} (\ell v + |v|^{q-1} v)(\tau) d\tau d\sigma.$$

It is extended to its maximal interval I_α , and by a standard concavity argument, $m_\alpha = \sup I_\alpha = \pi$. Notice that only a solution which vanishes at $\theta = \frac{\pi}{2}$ can be extended by continuity at $\theta = \pi$. By a standard argument $v^\alpha(\theta)$ depends

continuously on α , uniformly when $\theta \in [0, \pi - \epsilon_0]$ for any $\epsilon_0 > 0$ small enough, and $\epsilon_0 = \frac{\pi}{4}$ will be good enough. Since $v_\theta = v_\theta^\alpha$ satisfies

$$v_\theta(\theta) = -(\sin \theta)^{2-N} \int_0^\theta (\sin \tau)^{N-2} (\ell v + |v|^{q-1} v)(\tau) d\tau,$$

it follows that v^α depends continuously of α in the $C^1([0, \frac{3\pi}{4}])$ -topology. If v_1 and v_2 are two distinct solutions of (3.5) we can suppose that $v_2(0) > v_1(0)$. We assume now that their graph have more than one intersection and denote by σ_1 and σ_2 respectively their first and second intersections in $(0, \frac{\pi}{2})$. If $\alpha \in (0, v_2(0))$, we denote by $\sigma_j(\alpha)$, $j = 1, 2, \dots$, the finite and increasing sequence of intersections, if any, of the graphs of v_2 and v^α in $(0, \frac{\pi}{2})$. Then $\sigma_1(v_1(0)) = \sigma_1$ and $\sigma_2(v_1(0)) = \sigma_2$. Since the derivatives of v_2 and v^α at $\sigma_j(\alpha)$ differ, it follows from implicit function theorem that the mapping $\alpha \mapsto \sigma_j(\alpha)$ is continuous. Then, if $\sigma_2(\alpha) < \frac{\pi}{2}$ for any $\alpha \in (0, v_1(0))$, $\sigma_1(\alpha)$ satisfies the same upper bound,. Since $v^\alpha \rightarrow 0$ uniformly on $[0, \frac{\pi}{2}]$ and $v_{2\theta}(\frac{\pi}{2}) < 0$, this implies

$$\lim_{\alpha \rightarrow 0} \sigma_1(\alpha) = \sigma_2(\alpha) = \frac{\pi}{2}.$$

By the mean value theorem there exists $\tau(\alpha) \in (\sigma_2(\alpha), \sigma_1(\alpha))$ where $v_{2\theta}(\tau(\alpha)) = v_\theta^\alpha(\tau(\alpha))$. This is impossible as $\tau(\alpha) \rightarrow 0$ and

$$\lim_{\alpha \rightarrow 0} v_{2\theta}(\tau(\alpha)) = v_{2,\theta}(\frac{\pi}{2}) \neq \lim_{\alpha \rightarrow 0} v_\theta^\alpha(\tau(\alpha)) = 0.$$

Thus there exists $\tilde{\alpha} \in (0, v_1(0))$ such that $\sigma_2(\tilde{\alpha}) = \frac{\pi}{2}$. Moreover $\sigma_1(\tilde{\alpha}) < \frac{\pi}{2}$ otherwise we would have $v_{2\theta}(\frac{\pi}{2}) = v_\theta^{\tilde{\alpha}}(\frac{\pi}{2})$ as above, and $v_2 = v^{\tilde{\alpha}}$. Therefore $v^{\tilde{\alpha}}$ is a solution of ((3.5)) which intersects only once v_2 in $(0, \frac{\pi}{2})$. \square

The next result is standard but we present a proof for the convenience of the reader.

Lemma 3.3. *Assume that v_1 and v_2 are positive solutions of (3.5) whose graphs coincide at a single point of $(0, \frac{\pi}{2})$. If $v_1(0) > v_2(0)$, then the function*

$$\theta \in (0, \frac{\pi}{2}) \mapsto \frac{v_2(\theta)}{v_1(\theta)}$$

is increasing.

Proof. Let $J : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ be the function defined as $J = v_1 v_{2\theta} - v_2 v_{1\theta}$. To prove the lemma, it suffices to show that $J > 0$ in $(0, \frac{\pi}{2})$. Using the equations satisfied by v_1 and v_2 , one finds

$$J_\theta = -(N-2) \cot \theta J + (v_1^{q-1} - v_2^{q-1}) v_1 v_2.$$

Thus,

$$\frac{1}{(\sin \theta)^{N-2}} ((\sin \theta)^{N-2} J)_\theta = (v_1^{q-1} - v_2^{q-1}) v_1 v_2.$$

Let $\sigma \in (0, \frac{\pi}{2})$ be such that $v_1(\sigma) = v_2(\sigma)$. Since $v_{1\theta}(\sigma) \neq v_{2\theta}(\sigma)$, we have $v_1 > v_2$ in $(0, \sigma)$ and $v_1 < v_2$ in $(\sigma, \frac{\pi}{2})$, we conclude that the function

$$\theta \in [0, \frac{\pi}{2}] \mapsto (\sin \theta)^{N-2} J(\theta)$$

is increasing in $(0, \sigma)$ and decreasing in $(\sigma, \frac{\pi}{2})$. Since it vanishes at 0 and $\frac{\pi}{2}$, we have

$$(\sin \theta)^{N-2} J > 0 \quad \text{in } (0, \frac{\pi}{2}).$$

Thus $J > 0$ in $(0, \frac{\pi}{2})$ and the conclusion follows. \square

The following identity will be needed in the proofs of Theorems 3.2 and 3.3.

Lemma 3.4. *Let v be a solution of (3.5), $\alpha = \frac{2(N-2)}{q+3}$ and $\beta = \frac{2(N-2)(q-1)}{q+3}$. Set*

$$(3.8) \quad w(\theta) = (\sin \theta)^\alpha v(\theta)$$

Let $E : (0, \frac{\pi}{2}) \mapsto \mathbb{R}$ and $G : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$ be the functions defined by

$$(3.9) \quad E(\theta) = (\sin \theta)^\beta \frac{w_\theta^2}{2} + G(\theta) \frac{w^2}{2} + \frac{w^{q+1}}{q+1},$$

$$(3.10) \quad G(\theta) = \left((\alpha(N-2-\alpha) + \ell)(\sin \theta)^2 + \alpha(\alpha+3-N) \right) (\sin \theta)^{\beta-2}.$$

Then,

$$(3.11) \quad E_\theta = G_\theta \frac{w^2}{2}.$$

Proof. Let $w : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$ be the function defined by (3.8). Then,

$$w_{\theta\theta} + (N-2-2\alpha) \cot \theta w_\theta + \left(\alpha(N-2-\alpha) + \ell + \frac{\alpha(\alpha+3-N)}{(\sin \theta)^2} \right) w + \frac{w^q}{(\sin \theta)^{\alpha(q-1)}} = 0.$$

Multiplying this identity by $(\sin \theta)^\beta$, we get

$$(\sin \theta)^\beta w_{\theta\theta} + (N-2-2\alpha)(\sin \theta)^{\beta-1} \cos \theta w_\theta + G(\theta) w + (\sin \theta)^{\beta-\alpha(q-1)} w^q = 0$$

where G is defined by (3.10). We now observe that α and β satisfy

$$N-2-2\alpha = \frac{\beta}{2} \quad \text{and} \quad \beta - \alpha(q-1) = 0.$$

The identity satisfied by w becomes

$$(\sin \theta)^\beta w_{\theta\theta} + \frac{\beta}{2} (\sin \theta)^{\beta-1} \cos \theta w_\theta + G(\theta) w + w^q = 0.$$

Since

$$\frac{d}{d\theta} \left((\sin \theta)^\beta \frac{(w_\theta)^2}{2} \right) = \left((\sin \theta)^\beta w_{\theta\theta} + \frac{\beta}{2} (\sin \theta)^{\beta-1} \cos \theta w_\theta \right) w_\theta$$

and

$$\frac{d}{d\theta} \left(G(\theta) \frac{w^2}{2} \right) = G(\theta) w w_\theta + G_\theta(\theta) \frac{w^2}{2}$$

identity (3.11) follows. \square

The following proof is inspired from Kwong-Li [16].

Proof of Theorem 3.2. We use the notation of Lemma 3.4. We observe that E can be continuously extended at 0 and $\frac{\pi}{2}$. This is clear at $\frac{\pi}{2}$, where we take

$$(3.12) \quad E\left(\frac{\pi}{2}\right) = \frac{(w_\theta(\frac{\pi}{2}))^2}{2} = \frac{(v_\theta(\frac{\pi}{2}))^2}{2}.$$

To reach the conclusion at 0, it suffices to observe that for every $\theta \in (0, \frac{\pi}{2})$,

$$\begin{aligned} (\sin \theta)^\beta (w_\theta(\theta))^2 &= (\sin \theta)^\beta \left(\alpha (\sin \theta)^{\alpha-1} \cos \theta v(\theta) + (\sin \theta)^\alpha v_\theta(\theta) \right)^2 \\ &= (\sin \theta)^{2\alpha+\beta-2} \left(\alpha \cos \theta v(\theta) + \sin \theta v_\theta(\theta) \right)^2. \end{aligned}$$

Since $N \geq 4$,

$$2\alpha + \beta - 2 = \frac{2(N-3)}{q+3} \left(q + \frac{N-5}{N-3} \right) > 0,$$

the right-hand side of the previous expression converges to 0 as $\theta \rightarrow 0$. We can then set $E(0) = 0$. Notice that

$$G_\theta(\theta) = \left[(\alpha(N-2-\alpha) + \ell)\beta(\sin \theta)^2 + \alpha(\alpha+3-N)(\beta-2) \right] (\sin \theta)^{\beta-3} \cos \theta.$$

By the choices of α and β ,

$$\alpha(\alpha+3-N)(\beta-2) = \frac{4(N-2)(N-3)^2}{(q+3)^3} \left(q + \frac{N-5}{N-3} \right) \left(\frac{N+1}{N-3} - q \right).$$

Since $N \geq 4$ and $1 < q < \frac{N+1}{N-3}$, this quantity is positive. Hence, there exists $\varepsilon > 0$ such that

$$G_\theta(\theta) > 0 \quad \forall \theta \in (0, \varepsilon).$$

In view of the expression of G_θ , we have the following possibilities: either

$$(i) \quad G_\theta > 0 \text{ in } (0, \frac{\pi}{2}),$$

or

$$(ii) \quad \text{there exists } c \in (0, \frac{\pi}{2}) \text{ such that } G_\theta > 0 \text{ in } (0, c) \text{ and } G_\theta < 0 \text{ in } (c, \frac{\pi}{2}).$$

Assume by contradiction that (3.1) has more than one solution, hence by Lemma 3.2 problem (3.5) has two positive solutions v_1 and v_2 whose graphs intersect exactly once in the interval $(0, \frac{\pi}{2})$. Without loss of generality, we may assume that $v_1(0) > v_2(0)$. For $i \in \{1, 2\}$, define w_i and E_i accordingly.

First, assume that G satisfies property (i) above. Let

$$\gamma = \frac{v_{2\theta}(\frac{\pi}{2})}{v_{1\theta}(\frac{\pi}{2})}.$$

We have from (3.12)

$$(3.13) \quad (E_2 - \gamma^2 E_1)(0) = 0 = (E_2 - \gamma^2 E_1)\left(\frac{\pi}{2}\right).$$

On the other hand, by Lemma 3.3 the function

$$\theta \in (0, \frac{\pi}{2}) \mapsto \frac{v_2(\theta)}{v_1(\theta)}$$

is increasing. In particular, for every $\theta \in [0, \frac{\pi}{2})$,

$$\frac{v_2(\theta)}{v_1(\theta)} < \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{v_2(\theta)}{v_1(\theta)} = \frac{v_{2\theta}(\frac{\pi}{2})}{v_{1\theta}(\frac{\pi}{2})} = \gamma.$$

Hence,

$$(w_2)^2 - \gamma^2 (w_1)^2 = (\sin \theta)^{2\alpha} \left((v_2)^2 - \gamma^2 (v_1)^2 \right) < 0 \quad \text{in } (0, \frac{\pi}{2}).$$

Thus, by Lemma 3.4 and by assumption (i), we have for every $\theta \in (0, \frac{\pi}{2})$,

$$(E_2 - \gamma^2 E_1)_\theta(\theta) = G_\theta(\theta) \left((w_2)^2 - \gamma^2 (w_1)^2 \right) < 0.$$

This contradicts (3.13). Therefore, problem (3.1) cannot have two distinct positive solutions if G satisfies (i).

Next, we assume that G satisfies property (ii) for some point c . Let

$$\tilde{\gamma} = \frac{v_2(c)}{v_1(c)}.$$

As in the previous case, $(E_2 - \tilde{\gamma}^2 E_1)(0) = 0$. By Lemma 3.3, we have

$$\frac{v_2}{v_1} < \tilde{\gamma} \quad \text{in } (0, c) \quad \text{and} \quad \frac{v_2}{v_1} > \tilde{\gamma} \quad \text{in } (c, \frac{\pi}{2}).$$

Hence $(E_2 - \tilde{\gamma}^2 E_1)(\frac{\pi}{2}) > 0$. By Lemma 3.4 and by assumption (ii), we have for every $\theta \in (0, \frac{\pi}{2})$,

$$(E_2 - \tilde{\gamma}^2 E_1)_\theta(\theta) = G_\theta(\theta)((w_2)^2 - \tilde{\gamma}^2(w_1)^2) \leq 0.$$

This is still a contradiction. Therefore, if G satisfies (ii), then problem (3.1) has a unique positive solution. The proof of Theorem 3.2 is complete. \square

When $N = 3$, the proof of uniqueness of positive solutions of (3.1) is inspired from Kwong-Li [16] (Case 1 below) and Kwong [15] (Case 2 below).

Proof of Theorem 3.3. We split the proof in two cases:

Case 1. $q > 1$ and $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$.

Let $G : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$ be the function defined by (3.10). Since $N = 3$, we have $\alpha = \frac{2}{q+3}$ and $\beta = \frac{2(q-1)}{q+3}$. Thus,

$$\alpha(\alpha + 3 - N)(\beta - 2) = \alpha^2(\beta - 2) = -\frac{32}{(q+3)^3} < 0.$$

Moreover, since by assumption $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$, we have

$$(\alpha(N - 2 - \alpha) + \ell)\beta + \alpha(\alpha + 3 - N)(\beta - 2) = \frac{2(q-1)}{q+3} \left[\frac{2(q-3)}{(q+3)(q-1)} + \ell \right] \leq 0.$$

Therefore, G satisfies

(iii) $G_\theta < 0$ in $(0, \frac{\pi}{2})$.

We still consider the function E defined by (3.9), and astisfying (3.11). We observe that E can still be continuously extended at $\frac{\pi}{2}$ by (3.12), but not at 0 since $E(\theta)$ diverges to $+\infty$ as $\theta \rightarrow 0$.

Assume by contradiction that (3.1) has more than one solution, hence as above problem (3.5) has two positive solutions v_1 and v_2 whose graphs intersect exactly once in the interval $(0, \frac{\pi}{2})$, and $v_1(0) > v_2(0)$. For $i \in \{1, 2\}$, define w_i and E_i accordingly.

Let

$$\hat{\gamma} = \frac{v_2(0)}{v_1(0)}.$$

By Lemma 3.3 we find

$$(w_2)^2 - \hat{\gamma}^2(w_1)^2 = (\sin \theta)^{2\alpha}((v_2)^2 - \hat{\gamma}^2(v_1)^2) > 0 \quad \text{in } (0, \frac{\pi}{2}).$$

By Lemma 3.4 and by assumption (iii), we have for every $\theta \in (0, \frac{\pi}{2})$,

$$(3.14) \quad (E_2 - \hat{\gamma}^2 E_1)_\theta(\theta) = G_\theta(\theta)((w_2)^2 - \hat{\gamma}^2(w_1)^2) < 0.$$

By Lemma 3.3,

$$(E_2 - \hat{\gamma}^2 E_1)\left(\frac{\pi}{2}\right) = \frac{(v_{2\theta}(\frac{\pi}{2}))^2 - \hat{\gamma}^2 (v_{1\theta}(\frac{\pi}{2}))^2}{2} > 0.$$

Although E_1 and E_2 cannot be continuously extended at 0, one checks that

$$\lim_{\theta \rightarrow 0} (E_2(\theta) - \hat{\gamma}^2 E_1(\theta)) = 0,$$

by expanding the v_i up to the order 2 at $\theta = 0$. This contradicts (3.14). Therefore, equation (3.1) has at most one positive solution.

Case 2. $1 < q \leq 5$ and $\ell > \frac{2(3-q)}{(q+3)(q-1)}$.

Since $1 < q \leq 5$, we have $\ell > -\frac{1}{8}$, in particular $\ell \geq -\frac{1}{4}$. The remaining of the argument only requires $1 < q \leq 5$ and $\ell \geq -\frac{1}{4}$.

Let $z : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$ be the function defined as

$$z(\theta) = (\sin \theta)^{\frac{1}{2}} v(\theta).$$

Then, z satisfies

$$(3.15) \quad z_{\theta\theta} + \left(\ell + \frac{1}{4} + \frac{1}{4(\sin \theta)^2} \right) z + \frac{z^q}{(\sin \theta)^{\frac{q-1}{2}}} = 0.$$

Assume by contradiction that equation (3.5) has two positive distinct solutions v_1 and v_2 intersecting at some point $\sigma_0 \in (0, \frac{\pi}{2})$, with $v_1(0) > v_2(0)$. Define z_1 and z_2 accordingly. Then $z_1 > z_2$ on $(0, \sigma_0)$, $z_1 < z_2$ on $(\sigma_0, \frac{\pi}{2})$ and $z_1(0) = z_2(0) = z_1(\frac{\pi}{2}) = z_2(\frac{\pi}{2}) = 0$. Let $\xi_0 = z_1(\sigma_0) = z_2(\sigma_0)$.

As a first claim, we show that z_1 and z_2 cannot be both decreasing in $[\sigma_0, \frac{\pi}{2}]$. Indeed, if it holds, we may consider their inverses $z_i^{-1} : [0, \xi_0] \rightarrow [\sigma_0, \frac{\pi}{2}]$. For $i \in \{1, 2\}$, let $Z_i : [0, \xi_0] \rightarrow \mathbb{R}$ be the function given by

$$Z_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))$$

(Z_i is well-defined since $\sigma_1 > 0$). Since

$$z_{1\theta}(\sigma_0) < z_{2\theta}(\sigma_0) < 0 \quad \text{and} \quad z_{2\theta}(\frac{\pi}{2}) < z_{1\theta}(\frac{\pi}{2}) < 0,$$

we have

$$(Z_1(\xi_0))^2 > (Z_2(\xi_0))^2 \quad \text{and} \quad (Z_1(0))^2 < (Z_2(0))^2.$$

From the Mean value theorem, there exists $\eta \in (0, \xi_0)$ such that

$$(3.16) \quad (Z_1^2)_\xi(\eta) > (Z_2^2)_\xi(\eta).$$

On the other hand, for $i \in \{1, 2\}$ and for every $\xi \in (0, \xi_0)$,

$$(3.17) \quad Z_i Z_{i\xi} = z_{i\theta\theta}(z_i^{-1}(\xi)) = - \left(\ell + \frac{1}{4} + \frac{1}{4(\sin z_i^{-1}(\xi))^2} \right) \xi - \frac{\xi^q}{(\sin z_i^{-1}(\xi))^{\frac{q-1}{2}}}.$$

Since $z_1^{-1}(\xi) < z_2^{-1}(\xi)$ in $(0, \xi_0)$, we deduce that

$$(Z_1^2)_\xi = 2Z_1 Z_{1\xi} < 2Z_2 Z_{2\xi} = (Z_2^2)_\xi.$$

This contradicts (3.16) and prove the claim.

As a second claim, we now show that z_1 and z_2 cannot be both increasing in $(0, \sigma_0)$. Assuming that it holds, we may consider their inverses $z_i^{-1} : [0, \xi_0] \rightarrow [0, \sigma_0]$. For $i \in \{1, 2\}$, let $Y_i : [0, \xi_0] \rightarrow \mathbb{R}$ be the function defined as

$$Y_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi)).$$

Observe that Y_i can be continuously extended to 0 by taking $Y_i(0) = 0$. Since $z_2 < z_1$ in $(0, \sigma_0)$, we have

$$(3.18) \quad (Y_1(0))^2 = (Y_2(0))^2 = 0 \quad \text{and} \quad (Y_2(\xi_0))^2 > (Y_1(\xi_0))^2.$$

On the other hand, for $i \in \{1, 2\}$,

$$\begin{aligned} Y_{i\xi} &= \left(z_{i\theta\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi)) + z_{i\theta}(z_i^{-1}(\xi))(\cos z_i^{-1}(\xi)) \right) \frac{1}{z_{i\theta}(z_i^{-1}(\xi))} \\ &= \left(z_{i\theta\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi))^2 \right) \frac{1}{Y_i} + \cos z_i^{-1}(\xi). \end{aligned}$$

Thus,

$$Y_i Y_{i\xi} - Y_i \cos z_i^{-1}(\xi) = - \left((\ell + \frac{1}{4})(\sin z_i^{-1}(\xi))^2 + \frac{1}{4} \right) \xi - (\sin z_i^{-1}(\xi))^{\frac{5-q}{2}} \xi^q.$$

Since $z_2^{-1}(\xi) > z_1^{-1}(\xi)$ in $(0, \xi_0)$ and $\ell \geq -\frac{1}{4}$,

$$(\ell + \frac{1}{4})(\sin z_2^{-1}(\xi))^2 \geq (\ell + \frac{1}{4})(\sin z_1^{-1}(\xi))^2.$$

Since $q \leq 5$,

$$(\sin z_2^{-1}(\xi))^{\frac{5-q}{2}} \geq (\sin z_1^{-1}(\xi))^{\frac{5-q}{2}}.$$

We deduce that

$$Y_2 Y_{2\xi} - Y_2 \cos z_2^{-1}(\xi) \leq Y_1 Y_{1\xi} - Y_1 \cos z_1^{-1}(\xi).$$

Hence,

$$\begin{aligned} ((Y_2)^2 - (Y_1)^2)_\xi &\leq 2(Y_2 \cos z_2^{-1}(\xi) - Y_1 \cos z_1^{-1}(\xi)) \\ &\leq 2 \cos z_1^{-1}(\xi) (Y_1 - Y_2) \\ &\leq \frac{2 \cos z_1^{-1}(\xi)}{Y_1 + Y_2} ((Y_1)^2 - (Y_2)^2). \end{aligned}$$

Let $f : (0, \xi_0) \rightarrow \mathbb{R}$ be the function defined by

$$f(\xi) = \frac{2 \cos z_1^{-1}(\xi)}{Y_1(\xi) + Y_2(\xi)}.$$

Using this notation,

$$((Y_2)^2 - (Y_1)^2)_\xi \leq f(\xi) ((Y_2)^2 - (Y_1)^2).$$

Thus, for every $\xi \in [0, \xi_0]$,

$$((Y_2)^2 - (Y_1)^2)(\xi) \geq ((Y_2)^2 - (Y_1)^2)(\xi_0) e^{\int_\xi^{\xi_0} f(\tau) d\tau}.$$

This clearly contradicts (3.18) and the second claim is proved.

We can now conclude the proof. It follows from equation (3.15) that both z_1 and z_2 are concave. Since z_1 and z_2 cannot be simultaneously increasing on $(0, \sigma_0)$ or decreasing on $(\sigma_0, \frac{\pi}{2})$, at their intersection point there holds

$$z_{1\theta}(\sigma_0) < 0 < z_{2\theta}(\sigma_0).$$

Therefore, the maximum of z_1 is achieved in $(0, \sigma_0)$ while the maximum of z_2 is achieved in $(\sigma_0, \frac{\pi}{2})$.

Denote the maximum of z_i by m_i . We first show that $m_2 > m_1$. Indeed, assume by contradiction that $m_2 \leq m_1$. Let $\tilde{\sigma}_2 \in (\sigma_0, \frac{\pi}{2})$ be such that

$$z_2(\tilde{\sigma}_2) = m_2.$$

Let $\tilde{\sigma}_1$ be the largest number in $(0, \frac{\pi}{2})$ such that

$$z_1(\tilde{\sigma}_1) = m_2.$$

The restrictions $z_i : [\tilde{\sigma}_i, \frac{\pi}{2}] \rightarrow [0, m_2]$ are both decreasing. Let $\tilde{Z}_i : [0, m_2] \rightarrow \mathbb{R}$ be the function defined as

$$\tilde{Z}_i(\xi) = z_{i\theta}(z_i^{-1}(\xi)).$$

In the interval $[0, m_2]$ we have $z_1^{-1}(\xi) < z_2^{-1}(\xi)$ and (3.17), thus, as in the first claim,

$$(\tilde{Z}_1^2)_\xi < (\tilde{Z}_2^2)_\xi.$$

Since

$$(\tilde{Z}_1(0))^2 < (\tilde{Z}_2(0))^2 \quad \text{and} \quad (\tilde{Z}_1(m_2))^2 \geq 0 = (\tilde{Z}_2(m_2))^2,$$

we have a contradiction.

We now show that $m_1 \geq m_2$. Assume by contradiction that $m_1 < m_2$. Let $\hat{\sigma}_1 \in (0, \sigma)$ be such that

$$z_1(\hat{\sigma}_1) = m_1.$$

Let $\hat{\sigma}_2$ be the smallest number in $(0, \frac{\pi}{2})$ such that

$$z_2(\hat{\sigma}_1) = m_1.$$

The restrictions $z_i : [0, \hat{\sigma}_i] \rightarrow [0, m_1]$ are both increasing. Let $\hat{Y}_i : [0, m_1] \rightarrow \mathbb{R}$ be the function defined as

$$\hat{Y}_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi))$$

if $\xi \neq 0$ and $\hat{Y}_i(0) = 0$. Then, \hat{Y}_i is continuous. In the interval $[0, \hat{\sigma}_i]$ we have $z_1^{-1}(\xi) < z_2^{-1}(\xi)$, thus, as in the second claim,

$$((\hat{Y}_2)^2 - (\hat{Y}_1)^2)_\xi \leq \frac{2 \cos z_1^{-1}(\xi)}{\hat{Y}_2 + \hat{Y}_1} ((\hat{Y}_2)^2 - (\hat{Y}_1)^2).$$

This contradicts

$$((\hat{Y}_2)^2 - (\hat{Y}_1)^2)(0) = 0 \quad \text{and} \quad ((\hat{Y}_2)^2 - (\hat{Y}_1)^2)(m_1) > 0.$$

Finally $m_2 > m_1 \geq m_1 > 0$, which is a contradiction. Therefore, problem (3.1) can have at most one positive solution. \square

4. PROOF OF THEOREM 1.1

Proof of (i). Assume that $1 < q \leq q_1$. Let ϕ be a positive eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ associated to the first eigenvalue $N - 1$, and let ω be a solution of (1.3). Using ϕ as test function, we get

$$\int_{S_+^{N-1}} \langle \nabla' \omega, \nabla' \phi \rangle d\sigma = \int_{S_+^{N-1}} (\ell_{N,q} \omega + \omega^q) \phi d\sigma.$$

On the other hand, since ϕ is an eigenfunction of $-\Delta'$,

$$\int_{S_+^{N-1}} \langle \nabla' \omega, \nabla' \phi \rangle d\sigma = (N - 1) \int_{S_+^{N-1}} \omega \phi d\sigma.$$

Thus,

$$(4.1) \quad (N-1-\ell_{N,q}) \int_{S_+^{N-1}} \omega \phi \, d\sigma = \int_{S_+^{N-1}} \omega^q \phi \, d\sigma.$$

Since $q \leq q_1$, we have

$$N-1-\ell_{N,q} = \frac{(N-1)(q+1)}{(q-1)^2} \left(q - \frac{N+1}{N-1} \right) \leq 0.$$

Hence, the left-hand side of (4.1) is nonpositive while the right-hand side is nonnegative. Thus,

$$\int_{S_+^{N-1}} \omega^q \phi \, d\sigma = 0$$

We conclude that $\omega = 0$ in S_+^{N-1} . Hence, problem (1.3) has no positive solution. \square

Proof of (ii). Since $q > q_1$,

$$N-1-\ell_{N,q} = \frac{(N-1)(q+1)}{(q-1)^2} \left(q - \frac{N+1}{N-1} \right) > 0.$$

Thus, the functional $\mathcal{J} : W_0^{1,2}(S_+^{N-1}) \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}(w) = \int_{S_+^{N-1}} (|\nabla' w|^2 - \ell_{N,q} w^2) \, d\sigma$$

is bounded from below by 0. On the other hand, since $q < q_3$ we can minimize \mathcal{J} over the set

$$\left\{ w \in W_0^{1,2}(S_+^{N-1}) ; \int_{S_+^{N-1}} (w^+)^{q+1} \, d\sigma = 1 \right\}.$$

Let w be a minimizer. Then, w^+ is also a minimizer, whence $w = w^+$ and this function satisfies

$$-\Delta' w - \ell_{N,q} w = \lambda w^q \quad \text{in } S_+^{N-1}$$

for some $\lambda > 0$. By standard elliptic regularity theory, w is smooth and vanishes on ∂S_+^{N-1} in the classical sense. The function $\lambda^{\frac{1}{q-1}} w$ is therefore a solution of (1.3). For uniqueness, one applies Theorem 3.1 and Theorem 3.2 in the case $N \neq 3$. If $N = 3$ we can apply Theorem 3.3 since $\ell_{q,3} = \frac{2(3-q)}{(q-1)^2}$ always satisfies the assumption therein. \square

Proof of (iii). We may assume that $N \geq 4$, for otherwise there is nothing to prove. Note that if $q \geq q_3$,

$$\frac{N-1}{q-1} - \ell_{N,q} = -\frac{N-3}{(q-1)^2} \left(q - \frac{N+1}{N-3} \right) \leq 0.$$

Applying Corollary 2.1, we deduce that (1.3) has no positive solution. \square

5. THE A PRIORI ESTIMATE

In this section we establish Theorem 1.5 whose proof is based on the following result.

Proposition 5.1. *Assume that $1 < q < q_2$. Let $0 < r < \frac{1}{2} \text{diam } \Omega$ and $\zeta \in C^\infty(\partial\Omega)$ with $\zeta \geq 0$ on $\partial\Omega$. Then, every solution of*

$$(5.1) \quad \begin{cases} -\Delta u = u^q & \text{in } \Omega \cap (B_{2r} \setminus \overline{B}_r), \\ u \geq 0 & \text{in } \Omega \cap (B_{2r} \setminus \overline{B}_r), \\ u = \zeta & \text{on } \partial\Omega \cap (B_{2r} \setminus \overline{B}_r), \end{cases}$$

satisfies

$$(5.2) \quad u(x) \leq C [\text{dist}(x, \Gamma_r)]^{-\frac{2}{q-1}} \quad \forall x \in \Omega \cap (B_{2r} \setminus \overline{B}_r),$$

where $\Gamma_r = \overline{\Omega} \cap (\partial B_{2r} \cup \partial B_r)$ and $C > 0$ is a constant independent of u .

We denote by B_r the ball of radius r centered at 0. The proof of this estimate is based on two results: a Liouville theorem for the equation $-\Delta u = u^q$ in \mathbb{R}^N or in \mathbb{R}_+^N (see [9]) and the Doubling lemma of Poláčik-Quittner-Souplet [24] which we recall:

Lemma 5.1. *Let (X, d) be a complete metric space, $\Gamma \subsetneq X$ and $\gamma : X \setminus \Gamma \rightarrow (0, +\infty)$. Assume that γ is bounded on all compact subsets of $X \setminus \Gamma$. Given $k > 0$, let $y \in X \setminus \Gamma$ be such that*

$$\gamma(y) \text{dist}(y, \Gamma) > 2k.$$

Then, there exists $x \in X \setminus \Gamma$ such that

- $\gamma(x) \text{dist}(x, \Gamma) > 2k$;
- $\gamma(x) \geq \gamma(y)$;
- $2\gamma(x) \geq \gamma(z)$, $\forall z \in B_{k/\gamma(x)}(x)$.

Proof of Proposition 5.1. To simplify the notation we may assume that $\zeta \equiv 0$. Assume by contradiction that (5.2) is false. Then, for every integer $k \geq 1$ there exist $0 < r_k < \frac{1}{2} \text{diam } \Omega$, a solution u_k of (5.1) with $r = r_k$, and $y_k \in \Omega \cap (B_{2r_k} \setminus \overline{B}_{r_k})$ such that

$$u_k(y_k) > (2k)^{\frac{2}{q-1}} [\text{dist}(y_k, \Gamma_{r_k})]^{-\frac{2}{q-1}}.$$

Applying the previous lemma with

$$X = \overline{\Omega} \cap (\overline{B}_{2r_k} \setminus B_{r_k}) \quad \text{and} \quad \gamma = u_k^{\frac{q-1}{2}},$$

one finds $x_k \in X \setminus \Gamma_{r_k}$ such that

- (i) $u_k(x_k) > (2k)^{\frac{2}{q-1}} [\text{dist}(x_k, \Gamma_{r_k})]^{-\frac{2}{q-1}}$;
- (ii) $u_k(x_k) \geq u_k(y_k)$;
- (iii) $2^{\frac{2}{q-1}} u_k(x_k) \geq u_k(z)$, $\forall z \in B_{R_k}(x_k) \cap \Omega$, with $R_k = k[u_k(x_k)]^{-\frac{q-1}{2}}$.

By (i) we have $R_k < \frac{1}{2} \text{dist}(x_k, \Gamma_{r_k})$ and thus

$$B_{R_k}(x_k) \cap \Gamma_{r_k} = \emptyset.$$

Since $\text{dist}(x_k, \Gamma_{r_k}) \leq \frac{1}{2} r_k < \frac{1}{4} \text{diam } \Omega$, we also deduce from (i) that

$$u_k(x_k) \geq \left(\frac{8k}{\text{diam } \Omega} \right)^{\frac{2}{q-1}}.$$

In particular,

$$u_k(x_k) \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

For every $k \geq 1$, let

$$t_k = [u_k(x_k)]^{-\frac{q-1}{2}},$$

$$D_k = \left\{ \xi \in \mathbb{R}^N; |\xi| \leq k \text{ and } x_k + t_k \xi \in \Omega \right\}$$

and

$$v_k(\xi) = \frac{1}{u_k(x_k)} u_k(x_k + t_k \xi) \quad \forall \xi \in D_k.$$

Then, v_k satisfies

$$-\Delta v_k = v_k^q, \quad 0 \leq v_k \leq 2^{\frac{2}{q-1}} \quad \text{and} \quad v_k(0) = 1.$$

Passing to a subsequence if necessary, we may assume that either

(A) for every $a > 0$ there exists $k_0 \geq 1$ such that if $k \geq k_0$, then $B_{at_k}(x_k) \cap \partial\Omega = \emptyset$,

or

(B) there exists $a_0 > 0$ such that for every $k \geq 1$, $B_{a_0 t_k}(x_k) \cap \partial\Omega \neq \emptyset$.

Since the sequence (v_k) is uniformly bounded, it follows that (Δv_k) is also uniformly bounded. In both cases, by elliptic (interior and boundary) estimates, we have for every $1 < p < +\infty$ and every $s > 0$,

$$\|v_k\|_{W^{2,p}(D_k \cap B_s)} \leq C_{s,p}.$$

If (A) holds, then up to a subsequence (v_k) converges locally uniformly in \mathbb{R}^N to some smooth function v such that

$$-\Delta v = v^q, \quad 0 \leq v \leq 2^{\frac{2}{q-1}} \quad \text{and} \quad v(0) = 1.$$

On the other hand, if (B) holds, then up to a subsequence and a rotation of the domain there exists some smooth function v defined in \mathbb{R}_+^N such that (v_k) converges locally uniformly to v . Since the sequence (v_k) is equicontinuous and for every $k \geq 1$, $v_k(0) = 0$, we have $v(0) = 0$.

In both cases, we deduce that v is a nontrivial bounded solution of

$$-\Delta v = v^q$$

in \mathbb{R}^N or in \mathbb{R}_+^N , which is impossible (see [9]). Therefore, estimate (5.2) must hold. \square

Proof of Theorem 1.5. It suffices to establish (1.6) if $x \in \Omega$ and $|x| < \frac{3}{4} \text{diam}\Omega$. For this purpose, we apply Proposition 5.1 with $r = \frac{2}{3}|x|$. Since $\text{dist}(x, \Gamma_r) = \frac{1}{3}r$, we deduce that

$$u(x) \leq C [\text{dist}(x, \Gamma_r)]^{-\frac{2}{q-1}} = C \left(\frac{r}{3}\right)^{-\frac{2}{q-1}} = \tilde{C} |x|^{-\frac{2}{q-1}}.$$

This establishes the result. \square

6. THE GEOMETRIC AND ANALYTIC FRAMEWORK

We recall some of the preliminaries and the geometric framework in [14] which will be used in the remaining of the paper.

We denote by (x_1, \dots, x_N) the coordinates of $x \in \mathbb{R}^N$ and by $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ the canonical orthonormal basis in \mathbb{R}^N . Since we are assuming that the outward unit normal vector is $-\mathbf{e}_N$, $\partial\Omega$ is the graph of a smooth function in a neighborhood of 0. In other words, there exist a neighborhood G of 0 and a smooth function $\phi : G \cap T_0\Omega \rightarrow \mathbb{R}$ such that

$$G \cap \partial\Omega = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} ; x' \in G \cap T_0\Omega \text{ and } x_N = \phi(x') \right\}.$$

Furthermore,

$$\phi(0) = 0 \quad \text{and} \quad \nabla\phi(0) = 0.$$

Setting $\Phi(x) = y$, with $y_i = x_i$ if $i = 1, \dots, N-1$ and $y_N = x_N - \phi(x')$, we can assume that Φ is a C^∞ diffeomorphism from G to $\tilde{G} = \Phi(G)$, and $\Phi(\Omega \cap G) = \tilde{G} \cap \mathbb{R}_+^N$. To avoid introducing some additional notation, we will assume that

$$\tilde{G} = B_1.$$

Given $\zeta \in C^\infty(\partial\Omega)$, let z be the harmonic extension of ζ in Ω . For every solution u of (1.1), we denote

$$u(x) - z(x) = \tilde{u}(y), \quad z(x) = \tilde{z}(y) \quad \text{and} \quad \zeta(x) = \tilde{\zeta}(y),$$

for every $x = \Phi^{-1}(y)$ with $y \in \tilde{G} \cap \mathbb{R}_+^N$. Since u is superharmonic and $u = z$ on $\partial\Omega$, we have $\tilde{u} \geq 0$. On the other hand, a straightforward computation yields

$$\Delta u = \Delta \tilde{u} + |\nabla\phi|^2 \tilde{u}_{y_N, y_N} - 2\langle \nabla\phi, \nabla \tilde{u}_{y_N} \rangle - \tilde{u}_{y_N} \Delta\phi$$

Thus, \tilde{u} satisfies the equation

$$-\Delta \tilde{u} - |\nabla\phi|^2 \tilde{u}_{y_N, y_N} + 2\langle \nabla\phi, \nabla \tilde{u}_{y_N} \rangle + \tilde{u}_{y_N} \Delta\phi = (\tilde{u} + \tilde{z})^q.$$

Rewriting this equation in terms of spherical coordinates, one obtains

$$\begin{aligned} (1 + \eta_1) \tilde{u}_{rr} + \frac{1}{r^2} \Delta' \tilde{u} + \frac{N-1 + \eta_2}{r} \tilde{u}_r + (\tilde{u} + \tilde{z})^q &= \\ &= \frac{1}{r^2} \langle \nabla' \tilde{u}, \vec{\eta}_3 \rangle + \frac{1}{r} \langle \nabla' \tilde{u}_r, \vec{\eta}_4 \rangle + \frac{1}{r^2} \langle \nabla' \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle, \vec{\eta}_5 \rangle. \end{aligned}$$

where

$$\begin{aligned} \eta_1 &= -2\phi_r \langle \mathbf{n}, \mathbf{e}_N \rangle + |\nabla\phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle^2, \\ \eta_2 &= -r \langle \mathbf{n}, \mathbf{e}_N \rangle \Delta\phi - 2\langle \nabla' \langle \mathbf{n}, \mathbf{e}_N \rangle, \nabla' \phi \rangle + r |\nabla\phi|^2 \langle \nabla' \langle \mathbf{n}, \mathbf{e}_N \rangle, \mathbf{e}_N \rangle, \\ \vec{\eta}_3 &= - \left(2\phi_r - |\nabla\phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle - r \Delta\phi \right) \mathbf{e}_N, \\ \vec{\eta}_4 &= - \left(|\nabla\phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle - 2\phi_r \right) \mathbf{e}_N + \frac{2}{r} \langle \mathbf{n}, \mathbf{e}_N \rangle \nabla' \phi, \\ \vec{\eta}_5 &= - |\nabla\phi|^2 \mathbf{e}_N + \frac{2}{r} \nabla' \phi. \end{aligned}$$

Taking into account the fact that $\phi(0) = 0$ and $\nabla\phi(0) = 0$,

$$|\phi(x)| \leq Cr^2, \quad |D\phi(x)| \leq Cr \quad \text{and} \quad |D^2\phi| \leq C.$$

Thus, for every $j = 1, \dots, 5$,

$$\|\eta_j(r, \cdot)\|_{L^\infty} \leq Cr \quad \forall r \in (0, 1).$$

Lemma 6.1. *Let*

$$(6.1) \quad t = \log \frac{1}{r}, \quad v(t, \sigma) = r^{\frac{2}{q-1}} \tilde{u}(r, \sigma) \quad \text{and} \quad \alpha(t, \sigma) = r^{\frac{2}{q-1}} \tilde{z}(r, \sigma).$$

Then, v satisfies

$$(6.2) \quad (1 + \epsilon_1) v_{tt} + \Delta' v - \left(N - \frac{2(q+1)}{q-1} + \epsilon_2 \right) v_t + (\ell_{N,q} + \epsilon_3) v + (v + \alpha)^q = \\ = \langle \nabla' v, \vec{\epsilon}_4 \rangle + \langle \nabla' v_t, \vec{\epsilon}_5 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6 \rangle,$$

where ϵ_j are functions defined in $(0, +\infty) \times S_+^{N-1}$ satisfying the estimates

$$(6.3) \quad \|\epsilon_j(t, \cdot)\|_{L^\infty} \leq C e^{-t} \quad \forall t \geq 0,$$

for every $j = 1, \dots, 6$.

We refer the reader to [14] for the proof of Lemma 6.1 and for the explicit expressions of the functions ϵ_j .

For every $T \geq 0$ and $\delta > 0$, let

$$Q_T = (T, +\infty) \times S_+^{N-1} \quad \text{and} \quad Q_{T,\delta} = (T - \delta, T + \delta) \times S_+^{N-1}.$$

We have the following $W^{2,p}$ -estimates satisfied by v :

Proposition 6.1. *Let v be defined as in Lemma 6.1. If v is uniformly bounded in Q_0 , then for every $1 < p < +\infty$,*

$$(6.4) \quad \|v\|_{W^{2,p}(Q_{T,1})} \leq C \left(\|v\|_{L^2(Q_{T,2})} + e^{-\frac{2T}{q-1}} \right) \quad \forall T \geq 2,$$

for some positive constant depending on $\|v\|_{L^\infty}$ and on p .

Proof. Since Δ' is uniformly elliptic and Φ is a diffeomorphism, the operator L given by

$$L(v) = (1 + \epsilon_1) v_{tt} + \Delta' v - \left(N - \frac{2(q+1)}{q-1} + \epsilon_2 \right) v_t + \\ - \langle \nabla' v, \vec{\epsilon}_4 \rangle - \langle \nabla' v_t, \vec{\epsilon}_5 \rangle - \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6 \rangle$$

is uniformly elliptic. Let $\delta > 0$. By the Agmon-Douglis-Nirenberg estimates (see [1]) applied to the restriction of v on the set $Q_{T,1+\delta}$,

$$\|v\|_{W^{2,p}(Q_{T,1+\delta})} \leq C \left(\|v\|_{L^p(Q_{T,1+2\delta})} + \|(\alpha + v)^q\|_{L^p(Q_{T,1+2\delta})} \right).$$

Since α and v are uniformly bounded in Q_0 , for every $s \in (1, 2)$ we have

$$\|(\alpha + v)^q\|_{L^p(Q_{T,s})} \leq \|\alpha + v\|_{L^\infty(Q_{T,s})}^{q-1} \|\alpha + v\|_{L^p(Q_{T,s})} \\ \leq C \left(\|\alpha\|_{L^p(Q_{T,s})} + \|v\|_{L^p(Q_{T,s})} \right).$$

Since \tilde{z} is uniformly bounded in Ω ,

$$\|\alpha\|_{L^p(Q_{T,s})} \leq C e^{-\frac{2T}{q-1}} \|\tilde{z}\|_{L^\infty(\Omega)} \leq C e^{-\frac{2T}{q-1}}.$$

Thus,

$$(6.5) \quad \|v\|_{W^{2,p}(Q_{T,1+\delta})} \leq C \left(\|v\|_{L^p(Q_{T,1+2\delta})} + e^{-\frac{2T}{q-1}} \right).$$

In particular,

$$\|v\|_{W^{2,p}(Q_{T,1})} \leq C \left(\|v\|_{L^p(Q_{T,\frac{3}{2}})} + e^{-\frac{2T}{q-1}} \right).$$

By a bootstrap argument based on the estimate (6.5) above and the Sobolev imbedding, we also have

$$\|v\|_{L^p(Q_{T,\frac{3}{2}})} \leq C \left(\|v\|_{L^2(Q_{T,2})} + e^{-\frac{2T}{q-1}} \right).$$

Combining these inequalities, the estimate follows. \square

7. REMOVABLE SINGULARITIES AT 0

The goal of this section is to show that solutions of (1.1) which are not too large in a neighborhood of 0 must be continuous at 0.

Theorem 7.1. *Let $q > q_1$ and let u be a solution of (1.1). If*

$$(7.1) \quad \lim_{x \rightarrow 0} |x|^{\frac{2}{q-1}} u(x) = 0,$$

then u can be continuously extended at 0.

Proof. Let v be the function given by (6.1). By assumption (7.1), we have

$$(7.2) \quad \lim_{t \rightarrow +\infty} v(t, \cdot) = 0 \quad \text{uniformly in } S_+^{N-1}.$$

We now rewrite (6.2) under the form

$$(7.3) \quad v_{tt} - \left(N - \frac{2(q+1)}{q-1} \right) v_t + \ell_{N,q} v + \Delta' v + (v + \alpha)^q = H,$$

where H is given by

$$(7.4) \quad H = -\epsilon_1 v_{tt} + \epsilon_2 v_t - \epsilon_3 v + \langle \nabla' v, \vec{\epsilon}_4 \rangle + \langle \nabla' v_t, \vec{\epsilon}_5 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6 \rangle.$$

Thus,

$$(7.5) \quad \int_{S_+^{N-1}} v v_{tt} d\sigma - \left(N - \frac{2(q+1)}{q-1} \right) \int_{S_+^{N-1}} v v_t d\sigma + \ell_{N,q} \int_{S_+^{N-1}} v^2 d\sigma + \int_{S_+^{N-1}} v \Delta' v d\sigma + \int_{S_+^{N-1}} v (v + \alpha)^q d\sigma = \int_{S_+^{N-1}} v H d\sigma.$$

Let

$$X(t) = \|v(t, \cdot)\|_{L^2(S_+^{N-1})} \quad \forall t \geq 0.$$

Note that for every $t > 0$,

$$(7.6) \quad X X_t = \int_{S_+^{N-1}} v v_t d\sigma.$$

Using Hölder's inequality we have

$$|X X_t| \leq \|v(t, \cdot)\|_{L^2(S_+^{N-1})} \|v_t(t, \cdot)\|_{L^2(S_+^{N-1})}.$$

Thus,

$$(7.7) \quad |X_t| \leq \|v_t(t, \cdot)\|_{L^2(S_+^{N-1})}.$$

Computing the derivative with respect to t on both sides of identity (7.6), we get

$$(X_t)^2 + XX_{tt} = \int_{S_+^{N-1}} (v_t)^2 d\sigma + \int_{S_+^{N-1}} vv_{tt} d\sigma = \|v_t(t, \cdot)\|_{L^2(S_+^{N-1})}^2 + \int_{S_+^{N-1}} vv_{tt} d\sigma.$$

From this identity and estimate (7.7), we deduce that

$$(7.8) \quad XX_{tt} \geq \int_{S_+^{N-1}} vv_{tt} d\sigma.$$

On the other hand, since the first eigenvalue of the Laplace-Beltrami operator $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ is $N-1$,

$$(N-1)X^2 \leq \int_{S_+^{N-1}} |\nabla' v|^2 d\sigma = - \int_{S_+^{N-1}} v \Delta' v d\sigma.$$

By Hölder's inequality,

$$\int_{S_+^{N-1}} vH d\sigma \leq X \|H(t, \cdot)\|_{L^2(S_+^{N-1})}.$$

From the elementary inequality

$$(v + \alpha)^q \leq 2^q(v^q + \alpha^q),$$

we get

$$\int_{S_+^{N-1}} v(v + \alpha)^q d\sigma \leq 2^q \int_{S_+^{N-1}} (v^{q+1} + v\alpha^q) d\sigma$$

It follows from Hölder's inequality that

$$(7.9) \quad \int_{S_+^{N-1}} v(v + \alpha)^q d\sigma \leq 2^q \left(X^2 \|v(t, \cdot)\|_{L^\infty(S_+^{N-1})}^{q-1} + X \|\alpha(t, \cdot)\|_{L^{2q}(S_+^{N-1})}^q \right).$$

We may assume that u is a nontrivial solution of (1.1). By the strong maximum principle, we have $u > 0$ in Ω , thus $X > 0$. Combining (7.5), (7.6) and (7.8)–(7.9), one gets

$$\begin{aligned} X_{tt} - \left(N - \frac{2(q+1)}{q-1} \right) X_t + \left(\ell_{N,q} - N + 1 + 2^q \|v(t, \cdot)\|_{L^\infty}^{q-1} \right) X &\geq \\ &\geq - \left(\|H(t, \cdot)\|_{L^2} + 2^q \|\alpha(t, \cdot)\|_{L^{2q}}^q \right) \end{aligned}$$

(to simplify the notation we drop the explicit dependence of the set S_+^{N-1}). From the definition of the function α , there exists $C > 0$ such that

$$2^q \|\alpha(t, \cdot)\|_{L^{2q}}^q \leq C e^{-\frac{2qt}{q-1}}.$$

In view of (7.2), given $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$2^q \|v(t, \cdot)\|_{L^\infty}^{q-1} \leq \varepsilon \quad \text{on } [t_0, \infty).$$

We deduce that for every $t \geq t_0$ we have

$$X_{tt} - \left(N - \frac{2(q+1)}{q-1} \right) X_t + (\ell_{N,q} - N + 1 + \varepsilon) X \geq -\|H(t, \cdot)\|_{L^2} - C e^{-\frac{2qt}{q-1}}.$$

We shall show that

$$X(t) \leq Ce^{-\frac{2t}{q-1}} \quad \forall t \geq 0,$$

and the conclusion will now follow from a bootstrap argument. Note that the linear equation

$$Z_{tt} - \left(N - \frac{2(q+1)}{q-1}\right) Z_t + (\ell_{N,q} - N + 1) Z = 0$$

has two linearly independent solutions:

$$Z_1(t) = e^{-\frac{q+1}{q-1}t} \quad \text{and} \quad Z_2(t) = e^{(N - \frac{q+1}{q-1})t}.$$

We can then take $\varepsilon > 0$ small enough so that the linear equation

$$Z_{tt} - \left(N - \frac{2(q+1)}{q-1}\right) Z_t + (\ell_{N,q} - N + 1 + \varepsilon) Z = 0$$

has two linearly independent solutions:

$$Z_{1,\varepsilon}(t) = e^{r_{1,\varepsilon}t} \quad \text{and} \quad Z_{2,\varepsilon}(t) = e^{r_{2,\varepsilon}t}$$

such that

$$r_{1,\varepsilon} < -\frac{2}{q-1} \quad \text{and} \quad r_{2,\varepsilon} > 0.$$

In particular,

$$Z_{2,\varepsilon}(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

From assumption (7.1), v is bounded. In view of (6.3) and Proposition 6.1 with $p = 2$, there exists $C_1 > 0$ such that

$$\|H(t, \cdot)\|_{L^2} \leq C_1 e^{-t} \quad \forall t \geq 0.$$

Thus,

$$X_{tt} - \left(N - \frac{2(q+1)}{q-1}\right) X_t + (\ell_{N,q} - N + 1) X \geq -\hat{C}_1 e^{-t}.$$

Since

$$X(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

from the maximum principle there exists a constant $\tilde{C}_1 > 0$ such that

$$X(t) \leq \tilde{C}_1 (Z_{1,\varepsilon}(t) + e^{-t}).$$

If $r_{1,\varepsilon} \geq -1$, then

$$X(t) \leq 2\tilde{C}_1 Z_{1,\varepsilon}(t).$$

Since $r_{1,\varepsilon} < -\frac{2}{q-1}$, the estimate above implies that u is bounded and thus by standard elliptic estimates u is continuous. Otherwise $r_{1,\varepsilon} < -1$, in which case,

$$X(t) \leq 2\tilde{C}_1 e^{-t}.$$

Thus, by Proposition 7.1 for every $T \geq 2$,

$$\|v\|_{W^{2,2}(Q_{T,2})} \leq \tilde{C}_1 e^{-T}.$$

In view of (6.3), there exists $C_2 > 0$ such that

$$\|H(t, \cdot)\|_{L^2} \leq C_2 e^{-2t} \quad \forall t \geq 0.$$

Thus,

$$X_{tt} - \left(N - \frac{2(q+1)}{q-1}\right) X_t + (\ell_{N,q} - N + 1) X \geq -\hat{C}_2 e^{-2t}.$$

This implies as before that

$$X(t) \leq \tilde{C}_2 (Z_{1,\varepsilon}(t) + e^{-2t}).$$

If $r_{1,\varepsilon} \geq -2$, then

$$X(t) \leq 2\tilde{C}_2 Z_{1,\varepsilon}(t)$$

and u is bounded. Otherwise $r_{1,\varepsilon} < -2$, in which case,

$$X(t) \leq 2\tilde{C}_2 e^{-2t}.$$

We can continue this argument and deduce in finitely many steps that

$$X(t) \leq 2\tilde{C}_k Z_{1,\varepsilon}(t).$$

Applying Proposition 6.1 with $p > \frac{N}{2}$, we deduce that for every $T \geq 2$,

$$\|v\|_{W^{2,p}(Q_{T,1})} \leq C \left(Z_{1,\varepsilon}(T) + e^{-\frac{2T}{q-1}} \right) \leq C e^{-\frac{2T}{q-1}}.$$

Thus, by Morrey's embedding,

$$\|v\|_{L^\infty(Q_{T,1})} \leq C e^{-\frac{2T}{q-1}}.$$

This implies that u is bounded and hence continuous in $\bar{\Omega}$. \square

The conclusion of Theorem 7.1 is false with the critical exponent $q = q_1$. In fact, combining Theorem 1.3 and the result of del Pino-Musso-Pacard mentioned in the Introduction (Theorem 1.4), when $q = q_1$ there exist solutions of (1.1) such that

$$u(x) \sim x_N |x|^{-N} \left(\log \frac{1}{|x|} \right)^{-\frac{N-1}{2}}$$

in a neighborhood of 0. These solutions are necessarily discontinuous at 0 but, since $\frac{2}{q_1-1} = N-1$,

$$\lim_{x \rightarrow 0} |x|^{\frac{2}{q_1-1}} u(x) = 0.$$

The right statement in this case is the following:

Theorem 7.2. *Let $q = q_1$ and let u be a nonnegative solution of (1.1). If*

$$\lim_{x \rightarrow 0} |x|^{N-1} \left(\log \frac{1}{|x|} \right)^{\frac{N-1}{2}} u(x) = 0,$$

then u can be continuously extended at 0.

Proof. Let

$$W(t) = t^{\frac{N-1}{2}} \|v(t, \cdot)\|_{L^2(S_+^{N-1})} \quad \forall t \geq 0,$$

where v is the function given by (6.1). By assumption, $W(t) \rightarrow 0$ as $t \rightarrow +\infty$. As in the proof of Theorem 7.1, for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for every $t \geq t_0$,

$$\begin{aligned} W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(-\frac{N(N-1)}{2} + \varepsilon + \frac{N^2-1}{4t}\right) W &\geq \\ &\geq -t^{\frac{N-1}{2}} \|H(t, \cdot)\|_{L^2} - C t^{\frac{N-1}{2}} e^{-(N+1)t}. \end{aligned}$$

The linear equation

$$W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(-\frac{N(N-1)}{2} + \frac{N^2-1}{4t}\right) W = 0$$

has two linearly independent solutions W_1 and W_2 such that for t sufficiently large (see Lemma A.2 below)

$$W_1(t) = t^{\frac{N-1}{2}} e^{-Nt} (1 + o(1)) \quad \text{and} \quad W_2(t) = t^{\frac{N-1}{2}} (1 + o(1)).$$

We can then take $\varepsilon > 0$ small enough so that the linear equation

$$W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(\frac{-N(N-1)}{2} + \varepsilon + \frac{N^2-1}{4t}\right) W = 0$$

has two linearly independent solutions $W_{1,\varepsilon}$ and $W_{2,\varepsilon}$ such that

$$W_{1,\varepsilon}(t) \leq Ct^{\frac{N-1}{2}} e^{-(N-1)t}$$

and

$$W_{2,\varepsilon}(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

In view of (6.3) and Proposition 6.1 with $p = 2$, there exists $C_1 > 0$ such that

$$\|H(t, \cdot)\|_{L^2} \leq C_1 t^{-\frac{N-1}{2}} e^{-t} \quad \forall t > 0.$$

Thus,

$$W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(\frac{-N(N-1)}{2} + \varepsilon + \frac{N^2-1}{4t}\right) W \geq -Ce^{-t}.$$

Since

$$W(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

from the maximum principle there exists a constant $\tilde{C}_1 > 0$ such that

$$W(t) \leq \tilde{C}_1 (W_{1,\varepsilon}(t) + e^{-t}).$$

Thus,

$$W(t) \leq \hat{C}_1 e^{-t}.$$

Thus, by Proposition 6.1 with $p = 2$, for every $T \geq 2$,

$$\|v\|_{W^{2,2}(Q_{T,2})} \leq \hat{C}_1 t^{-\frac{N-1}{2}} e^{-T}.$$

In view of (6.3), there exists $\hat{C}_2 > 0$ such that

$$\|H(t, \cdot)\|_{L^2} \leq \hat{C}_2 t^{-\frac{N-1}{2}} e^{-2t} \quad \forall t \geq 0.$$

We can continue this argument as in the previous theorem and deduce after finitely many steps that

$$W(t) \leq \hat{C}_k W_{1,\varepsilon}(t) \leq \tilde{C} t^{\frac{N-1}{2}} e^{-(N-1)t},$$

which implies that u is bounded and hence continuous in $\bar{\Omega}$. \square

8. PROOF OF THEOREM 1.2

We first establish the following

Proposition 8.1. *Let $q_1 \leq q < q_3$, with $q \neq q_2$. If u is a solution of (1.1) such that*

$$|x|^{\frac{2}{q-1}} u(x) \text{ is bounded in } \Omega,$$

then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \Omega \setminus \{0\}$, $\frac{x}{|x|} \in S_+^{N-1}$ and $|x| < \delta$, then

$$(8.1) \quad \left| |x|^{\frac{2}{q-1}} u(x) - w\left(\frac{x}{|x|}\right) \right| < \varepsilon$$

where w is a solution of (1.3).

Proof. Let v be the function given by (6.1). We first rewrite equation (6.2) under the form

$$(8.2) \quad v_{tt} + \ell_{N,q}v + \Delta'v + (v + \alpha)^q - \left(N - \frac{2(q+1)}{q-1}\right)v_t = H,$$

where H is given by (7.4). Multiplying (8.2) by v_t and integrating over S_+^{N-1} yields

$$\begin{aligned} \int_{S_+^{N-1}} v_t v_{tt} d\sigma + \ell_{N,q} \int_{S_+^{N-1}} v_t v d\sigma + \int_{S_+^{N-1}} v_t \Delta'v d\sigma + \int_{S_+^{N-1}} v_t (v + \alpha)^q d\sigma + \\ - \left(N - \frac{2(q+1)}{q-1}\right) \int_{S_+^{N-1}} (v_t)^2 d\sigma = \int_{S_+^{N-1}} v_t H d\sigma. \end{aligned}$$

Thus,

$$(8.3) \quad \begin{aligned} \frac{d}{dt} \int_{S_+^{N-1}} \left[\frac{(v_t)^2}{2} + \frac{\ell_{N,q}v^2}{2} - \frac{|\nabla'v|^2}{2} + \frac{(v + \alpha)^{q+1}}{q+1} \right] d\sigma - \left(N - \frac{2(q+1)}{q-1}\right) \int_{S_+^{N-1}} (v_t)^2 d\sigma = \\ = \int_{S_+^{N-1}} [v_t H + \alpha_t (v + \alpha)^q] d\sigma. \end{aligned}$$

From our assumption on u , v is bounded. It follows from (6.4) and the Sobolev imbedding that v , v_t and $\nabla'v$ are uniformly bounded in $S_+^{N-1} \times \mathbb{R}_+$. Integrating (8.3) from 0 to T , for any $T > 0$, one deduces that

$$\int_{S_+^{N-1}} \left| \frac{(v_t)^2}{2} + \frac{\ell_{N,q}v^2}{2} - \frac{|\nabla'v|^2}{2} + \frac{(v + \alpha)^{q+1}}{q+1} \right| d\sigma \leq C \quad \text{in } \mathbb{R}_+$$

for some constant $C > 0$. On the other hand,

$$\int_{S_+^{N-1}} |v_t H| d\sigma \leq C e^{-t}.$$

Moreover, since v is bounded and α satisfies (6.5), we have

$$\int_{S_+^{N-1}} |\alpha_t| (v + \alpha)^q d\sigma \leq C e^{-\frac{2t}{q-1}}.$$

Thus, integrating (8.3) on $(0, +\infty)$, we obtain

$$\left| N - \frac{2(q+1)}{q-1} \right| \int_0^{+\infty} \int_{S_+^{N-1}} v_t^2 d\sigma < +\infty.$$

Since $q \neq q_2$, $N - \frac{2(q+1)}{q-1} \neq 0$. Hence,

$$\int_0^{+\infty} \int_{S_+^{N-1}} v_t^2 d\sigma < +\infty.$$

By (6.4) and Morrey's estimates, v_t is uniformly continuous on Q_0 . We deduce that

$$v_t(t, \cdot) \rightarrow 0 \quad \text{uniformly in } S_+^{N-1} \text{ as } t \rightarrow +\infty.$$

We now prove that

$$v(t, \cdot) \rightarrow w \quad \text{uniformly in } S_+^{N-1} \text{ as } t \rightarrow +\infty,$$

where w is a nonnegative solution of (1.3). For this purpose, we study the limit set of the trajectories of v , namely the set

$$\Gamma = \bigcap_{\tau > 0} \overline{\bigcup_{t \geq \tau} \{v(t, \cdot)\}},$$

where the closure is computed with respect to the usual norm in $C^0(S_+^{N-1})$. Since Γ is the intersection of a decreasing family of closed connected subsets of $C^0(S_+^{N-1})$, Γ is closed and connected. In addition, since v is uniformly continuous in Q_0 , it follows from the Arzelà-Ascoli theorem that Γ is also compact and nonempty.

We claim that every $w \in \Gamma$ satisfies problem (1.3). Indeed, let (t_k) be a sequence of nonnegative real numbers such that $t_k \rightarrow +\infty$ and

$$v(t_k, \cdot) \rightarrow w \quad \text{uniformly in } S_+^{N-1}.$$

Clearly, w is nonnegative and $w = 0$ on ∂S_+^{N-1} . For each $k \geq 1$, let

$$V_k : (s, \sigma) \in [0, 1] \times S_+^{N-1} \mapsto v(t_k + s, \sigma).$$

For every $\varphi \in C_0^\infty(S_+^{N-1})$ and for every $\varepsilon \in (0, 1)$, from the equation satisfied by v we have

$$\begin{aligned} \int_0^\varepsilon \int_{S_+^{N-1}} \left[(V_k)_{tt} \varphi + \ell_{N,q} V_k \varphi + V_k \Delta' \varphi + (V_k + \alpha)^q \varphi - \left(N - \frac{2(q+1)}{q-1} \right) (V_k)_t \varphi \right] d\sigma dt &= \\ &= \int_{t_k}^{t_k + \varepsilon} \int_{S_+^{N-1}} H \varphi d\sigma dt. \end{aligned}$$

As $k \rightarrow +\infty$,

$$\int_{t_k}^{t_k + \varepsilon} \int_{S_+^{N-1}} H \varphi d\sigma dt \rightarrow 0.$$

Since $v_t \rightarrow 0$ uniformly as $t \rightarrow +\infty$, we also have

$$\int_0^\varepsilon \int_{S_+^{N-1}} (V_k)_t \varphi d\sigma dt \rightarrow 0.$$

Note that

$$\int_0^\varepsilon \int_{S_+^{N-1}} (V_k)_{tt} \varphi d\sigma d\tau = \int_{S_+^{N-1}} [v_t(t_k + \varepsilon, \sigma) - v_t(t_k, \sigma)] \varphi d\sigma \rightarrow 0.$$

Since the sequence (V_k) is bounded in C^1 , passing to a subsequence if necessary, we may assume that for some continuous function W ,

$$V_k \rightarrow W \quad \text{uniformly in } [0, 1] \times S_+^{N-1}.$$

We conclude that for every $\varepsilon \in (0, 1)$,

$$\int_0^\varepsilon \int_{S_+^{N-1}} \left[\ell_{N,q} W \varphi - W \Delta' \varphi + W^q \varphi \right] d\sigma dt = 0.$$

Dividing both sides by ε and letting $\varepsilon \rightarrow 0$, we get

$$\int_{S_+^{N-1}} \left[\ell_{N,q} W(0, \sigma) \varphi - W(0, \sigma) \Delta' \varphi + (W(0, \sigma))^q \varphi \right] d\sigma = 0.$$

Since $w = W(0, \cdot)$, we conclude that w satisfies (1.3). Hence, every element of Γ is a nonnegative solution of (1.3). Since these solutions form a discrete subset of $C^0(S_+^{N-1})$ and Γ is connected (in our case, the set of nonnegative solutions is $\{0, \omega\}$, where ω is the unique positive solution of (1.3)), Γ contains a single element. In particular,

$$v(t, \cdot) \rightarrow w \quad \text{uniformly in } S_+^{N-1} \text{ as } t \rightarrow +\infty.$$

The proposition follows from this convergence. \square

Proof of Theorem 1.2. Let u be a solution of (1.1). Since $q < q_2$, by Theorem 1.5 there exists $C > 0$ such that for every $x \in \Omega$,

$$0 \leq |x|^{\frac{2}{q-1}} u(x) \leq C.$$

Thus, by Proposition 8.1, there exists a solution w of (1.3) such that (8.1) holds. Either w is the unique positive solution of (1.3) (see Theorem 1.1) or $w = 0$. If $w = 0$, then

$$\lim_{x \rightarrow 0} |x|^{\frac{2}{q-1}} u(x) = 0.$$

Hence, by Theorem 7.1 u can be continuously extended at 0. \square

9. PROOF OF THEOREM 1.3

We first prove an estimate which improves Theorem 1.5 when $q = q_1$, except that we do not know whether the constant C below can be chosen independently of the solution.

Theorem 9.1. *Assume that $q = q_1$. Then, every solution of (1.1) satisfies*

$$u(x) \leq C |x|^{-(N-1)} \left(\log \frac{1}{|x|} \right)^{-\frac{N-1}{2}} \quad \forall x \in \Omega,$$

for some constant $C > 0$ possibly depending on the solution.

In the proof of this result we need the following lemma:

Lemma 9.1. *Let $a = q_1$ and $E = \ker [\Delta' + (N-1)I]$. Given a solution of (1.1), denote by v the function given by (6.1). If*

$$v = v_1 + v_2$$

is the decomposition of v as the orthogonal projections in $L^2(S_+^{N-1})$ onto E and E^\perp , respectively, then

$$(9.1) \quad \|v_1(t, \cdot)\|_{L^2(S_+^{N-1})} \leq C t^{-\frac{N-1}{2}} \quad \text{and} \quad \|v_2(t, \cdot)\|_{L^2(S_+^{N-1})} \leq C e^{-\frac{t}{2}} \quad \forall t > 0.$$

Proof. Denoting by ϕ_1 the first eigenfunction of Δ' with $\|\phi_1\|_{L^1} = 1$, we have

$$v_1(t, \sigma) = y(t)\phi_1(\sigma) \quad \text{where} \quad y(t) = \int_{S_+^{N-1}} v(t, \sigma)\phi_1(\sigma) d\sigma.$$

Since $q = q_1$, equation (7.3) becomes

$$(9.2) \quad v_{tt} + Nv_t + (N-1)v + \Delta'v + (v + \alpha)^{q_1} = H,$$

with H defined in (7.4). Since $\alpha \geq 0$, we have $(v + \alpha)^{q_1} \geq v^{q_1}$. Thus,

$$v_{tt} + Nv_t + (N-1)v + \Delta'v + v^{q_1} \leq H.$$

By Jensen's inequality,

$$y^{q_1} \leq \int_{S_+^{N-1}} v^{q_1} \phi_1 d\sigma.$$

Multiplying (9.2) by ϕ_1 and integrating over S_+^{N-1} , we get

$$y'' + Ny' + y^{q_1} \leq \int_{S_+^{N-1}} H\phi_1 d\sigma.$$

By Theorem 1.5, v is uniformly bounded in $\mathbb{R}_+ \times S_+^{N-1}$. In particular, by (6.3) and Proposition 6.1 with $p = 2$, we have for every $t \geq 0$,

$$\int_{S_+^{N-1}} H\phi_1 d\sigma \leq C e^{-t}.$$

Thus,

$$y'' + Ny' + y^{q_1} \leq C e^{-t}.$$

Applying Lemma A.1 we deduce that

$$y(t) \leq C t^{-\frac{N-1}{2}} \quad \forall t > 0.$$

This concludes the proof of the first estimate in (9.1).

In order to prove the estimate for v_2 , let

$$Y(t) = \|v_2(t, \cdot)\|_{L^2(S_+^{N-1})} \quad \forall t \geq 0.$$

Since $v(t, \sigma) = y(t)\phi_1(\sigma) + v_2(t, \sigma)$, we have

$$v_t = y_t\phi_1 + (v_2)_t \quad \text{and} \quad v_{tt} = y_{tt}\phi_1 + (v_2)_{tt}.$$

Using the orthogonality between ϕ_1 and v_2 ,

$$YY_t = \int_{S_+^{N-1}} v_2(v_2)_t d\sigma = \int_{S_+^{N-1}} v_2[y_t\phi_1 + (v_2)_t] d\sigma = \int_{S_+^{N-1}} v_2v_t d\sigma.$$

From the first equality, we have

$$|Y_t| \leq \|v_2(t, \cdot)\|_{L^2}.$$

One also shows that

$$YY_{tt} \geq \int_{S_+^{N-1}} v_2v_{tt} d\sigma.$$

On the other hand, since the second eigenvalue of the Laplace-Beltrami operator $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ is $2N$,

$$2NY^2 \leq \int_{S_+^{N-1}} |\nabla' v_2|^2 d\sigma = - \int_{S_+^{N-1}} v_2 \Delta' v_2 d\sigma = - \int_{S_+^{N-1}} v_2 \Delta' v d\sigma.$$

Multiply (9.2) by v_2 and integrate over S_+^{N-1} . As in the proof of Theorem 7.1, for every $\varepsilon > 0$ there exists $t_1 > 0$ such that for every $t \geq t_1$,

$$Y_{tt} + NY_t - (N + 1 - \varepsilon)Y \geq -C e^{-t}.$$

Note that for $\varepsilon > 0$ small the linear equation

$$Z_{tt} + NZ_t - (N + 1 - \varepsilon)Z = 0$$

has two linearly independent solutions $Z_{1,\varepsilon}$ and $Z_{2,\varepsilon}$ such that

$$Z_{1,\varepsilon}(t) = e^{r_{1,\varepsilon}t} \quad \text{and} \quad Z_{2,\varepsilon}(t) = e^{r_{2,\varepsilon}t}$$

with

$$r_{1,\varepsilon} \leq -\frac{1}{2} \quad \text{and} \quad r_{2,\varepsilon} > 0.$$

Since $Y(t) \rightarrow 0$ as $t \rightarrow +\infty$, applying the maximum principle one deduces that

$$Y(t) \leq C(Z_{1,\varepsilon}(t) + e^{-t}).$$

In particular,

$$Y(t) \leq C e^{-\frac{t}{2}}.$$

This gives the estimate for v_2 . □

Proof of Theorem 9.1. By Lemma 9.1 above, we have

$$\|v(t, \cdot)\|_{L^2} \leq C t^{-\frac{N-1}{2}} \quad \forall t > 0.$$

Inserting this estimate into estimate (6.4) for some $p > \frac{N}{2}$ the result follows. □

Proof of Theorem 1.3. By Theorem 9.1, the function $w : [0, +\infty) \rightarrow \mathbb{R}$ given by

$$w(t, \sigma) = t^{\frac{N-1}{2}} v(t, \sigma)$$

is bounded. By a straightforward computation, w satisfies

$$(9.3) \quad w_{tt} + \left(N - \frac{N-1}{t}\right) w_t + \left(N - 1 + \frac{N^2-1}{4t^2}\right) w + \Delta' w + \frac{1}{t} \left(w^{q_1} - \frac{N(N-1)}{2} w\right) = t^{\frac{N-1}{2}} H,$$

where H is given by (7.4). Let $\phi : S_+^{N-1} \rightarrow \mathbb{R}$ be the function defined by $\phi(\sigma) = \frac{\sigma_N}{|\sigma|}$; we recall that ϕ is an eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ associated to the first eigenvalue $N - 1$. Let

$$z(t) = \int_{S_+^{N-1}} w(t, \sigma) \phi(\sigma) d\sigma \quad \forall t \geq 0.$$

Multiplying (9.3) by ϕ and integrating over S_+^{N-1} , we obtain the following equation satisfied by z :

$$z_{tt} + \left(N - \frac{N-1}{t}\right) z_t + \frac{N^2-1}{4t^2} z + \frac{1}{t} \int_{S_+^{N-1}} w^{q_1} \phi \, d\sigma - \frac{N(N-1)}{2t} z = t^{\frac{N-1}{2}} \int_{S_+^{N-1}} H \phi \, d\sigma.$$

Thus,

$$z_{tt} + \left(N - \frac{N-1}{t}\right) z_t + \frac{1}{t} \left(\theta z^{q_1} - \frac{N(N-1)}{2} z\right) = \Psi,$$

where

$$\theta = \int_{S_+^{N-1}} \phi^{q_1+1} \, d\sigma$$

and

$$\Psi = t^{\frac{N-1}{2}} \int_{S_+^{N-1}} H \phi \, d\sigma - \frac{N^2-1}{4t^2} z + \frac{1}{t} \int_{S_+^{N-1}} [(z\phi)^{q_1} - w^{q_1}] \phi \, d\sigma.$$

By Lemma 9.1, we have

$$(9.4) \quad \|z(t)\phi - w(t, \cdot)\|_{L^2} \leq C t^{\frac{N-1}{2}} e^{-\frac{t}{2}}.$$

Since

$$|(z\phi)^{q_1} - w^{q_1}| \leq q_1 |z\phi - w| [(z\phi)^{q_1-1} + w^{q_1-1}],$$

z is bounded in \mathbb{R}_+ and w is bounded in $\mathbb{R}_+ \times S_+^{N-1}$,

$$\begin{aligned} \int_{S_+^{N-1}} |(z\phi)^{q_1} - w^{q_1}| \phi \, d\sigma &\leq \|z\phi - w\|_{L^2} \left[z^{q_1-1} \|\phi^{q_1-1}\|_{L^2} + \|w^{q_1-1}\|_{L^2} \right] \\ &\leq C t^{\frac{N-1}{2}} e^{-\frac{t}{2}}. \end{aligned}$$

By Proposition 9.1, (6.3) and Proposition 6.1 with $p = 2$,

$$\|H(t, \cdot)\|_{L^2} \leq C t^{-\frac{N-1}{2}} e^{-t}.$$

Thus,

$$\|\Psi(t, \cdot)\|_{L^\infty} \leq C \left(e^{-t} + t^{-2} + t^{\frac{N-3}{2}} e^{-\frac{t}{2}} \right) \leq \tilde{C} t^{-2}.$$

By a straightforward modification of the end of the proof of [7, Corollary 4.2], z admits a limit $\kappa \geq 0$ when $t \rightarrow +\infty$, where κ satisfies

$$\theta \kappa^{q_1} - \frac{N(N-1)}{2} \kappa = 0.$$

Therefore, either $\kappa = 0$ or $\kappa = \left(\frac{N(N-1)}{2\theta}\right)^{\frac{N-1}{2}}$.

By (9.4) we deduce that, as $t \rightarrow +\infty$,

$$t^{\frac{N-1}{2}} v(t, \cdot) \rightarrow \kappa \phi \quad \text{in } L^2(S_+^{N-1}).$$

By Proposition 6.1 with $p > \frac{N}{2}$ and Morrey's estimates, we conclude that

$$t^{\frac{N-1}{2}} v(t, \cdot) \rightarrow \kappa \phi \quad \text{uniformly in } S_+^{N-1}.$$

Rewriting the convergence in terms of u , we conclude that either (1.5) holds or

$$(9.5) \quad |x|^{N-1} \left(\log \frac{1}{|x|}\right)^{\frac{N-1}{2}} u(x) \rightarrow 0 \text{ as } x \rightarrow 0.$$

If (9.5) holds, then u must be continuous in view of Theorem 7.2. \square

APPENDIX A. SOME ODE LEMMAS

We gather in this section a couple of ODE results which are used in this paper. These results are presumably well-known to specialists:

Lemma A.1. *Given $T > 0$, let $y \in C^2([T, +\infty))$ be a nonnegative function such that*

$$\begin{cases} y_{tt} + ay_t + by^q \leq ce^{-t} & \text{in } (T, +\infty), \\ \lim_{t \rightarrow +\infty} y(t) = 0, \end{cases}$$

where $q, a > 1$ and $b, c > 0$. Then, there exists $C > 0$ such that

$$(A.1) \quad 0 \leq y(t) \leq Ct^{-\frac{1}{q-1}} \quad \forall t \geq T.$$

Proof. Given $A > 0$, let

$$z(t) = y(t) + Ae^{-t} \quad \forall t \geq T.$$

Then, z satisfies

$$z_{tt} + az_t + bz^q \leq [c - (a-1)A]e^{-t} + b(z^q - y^q).$$

By convexity of the function $t \in \mathbb{R}_+ \mapsto t^q$,

$$y^q \geq z^q - qz^{q-1}Ae^{-t}.$$

Thus,

$$(A.2) \quad z_{tt} + az_t + bz^q \leq [c - (a-1 + bqz^{q-1})A]e^{-t}.$$

Since $a > 1$ and $z(t) \rightarrow 0$ as $t \rightarrow \infty$, we can choose $T_1 > T$ and $A > 0$ sufficiently large so that the right-hand side of (A.2) is negative on $[T_1, \infty)$. Thus,

$$(A.3) \quad z_{tt} + az_t + bz^q \leq 0 \quad \text{in } [T_1, \infty).$$

Let $w = z^{1-q}$. By a straightforward computation, we have

$$(A.4) \quad w_{tt} + aw_t \geq -(q-1) \frac{z_{tt} + az_t}{z^q}.$$

Combining (A.3)–(A.4), we deduce that

$$w_{tt} + aw_t \geq b(q-1) \quad \text{in } [T_1, \infty).$$

The function $x = w_t$ satisfies

$$x_t + ax \geq b(q-1) \quad \text{in } [T_1, \infty).$$

Thus, taking $T_2 > T_1$ sufficiently large,

$$x(t) \geq \frac{b(q-1)}{a} + c_1 e^{-at} \geq \frac{b(q-1)}{2a} \quad \forall t \geq T_2.$$

Since $w_t = x$, choosing $T_3 > T_2$ large enough, we then get

$$w(t) \geq \frac{b(q-1)}{4a} t \quad \forall t \geq T_3.$$

Therefore,

$$z(t) \leq \left(\frac{4a}{b(q-1)} t^{-1} \right)^{\frac{1}{q-1}} \quad \forall t \geq T_3.$$

We can now enlarge the constant in the right-hand side so that this estimate holds for every $t \geq T$. This immediately implies (A.1). \square

Lemma A.2. *Let $a, a_1, b, b_1 \in \mathbb{R}$ with $a \neq 0$. Then, the equation*

$$y_{tt} + \left(a - \frac{a_1}{t}\right)y_t + \frac{1}{t}\left(b - \frac{b_1}{t}\right)y = 0 \quad \text{in } (0, +\infty),$$

has two linearly independent solutions y_1 and y_2 such that

$$y_1(t) = t^{a_1 + \frac{b}{a}} e^{-at} (1 + o(1)) \quad \text{and} \quad y_2(t) = t^{-\frac{b}{a}} (1 + o(1))$$

for t sufficiently large.

Proof. Let

$$z(t) = e^{\frac{at}{2}} t^{-\frac{a_1}{2}} y(t).$$

Then, z satisfies the equation

$$z_{tt} - \left(\frac{a^2}{4} - \frac{A_1}{t} + \frac{A_2}{t^2}\right)z = 0,$$

where $A_1 = b + \frac{aa_1}{2}$ and $A_2 = b_1 + \frac{a_1}{2} + \frac{a_1^2}{4}$. By [3, pp. 126–127], the equation satisfied by z has two linearly independent solutions with the following asymptotic behaviors as $t \rightarrow +\infty$:

$$z_1(t) = e^{-\frac{at}{2}} t^{\frac{A_1}{a}} (1 + o(1)) \quad \text{and} \quad z_2(t) = e^{\frac{at}{2}} t^{-\frac{A_1}{a}} (1 + o(1)).$$

Rewriting these formulas in terms of the function y , the result follows. \square

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