

# BOUNDING THE RATIONAL SUMS OF SQUARES OVER TOTALLY REAL FIELDS

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ABSTRACT. Let  $K$  be a totally real Galois number field. C. J. Hillar proved that if  $f \in \mathbb{Q}[x_1, \dots, x_n]$  is a sum of  $m$  squares in  $K[x_1, \dots, x_n]$ , then  $f$  is a sum of  $N(m)$  squares in  $\mathbb{Q}[x_1, \dots, x_n]$ , where  $N(m) \leq 2^{[K:\mathbb{Q}]+1} \cdot \binom{[K:\mathbb{Q}]+1}{2} \cdot 4m$ , the proof being constructive.

We show in fact that  $N(m) \leq (4[K:\mathbb{Q}]-3) \cdot m$ , the proof being constructive as well.

## 1. INTRODUCTION

In the theory of semidefinite linear programming, there is a question by Sturmfels

**Question 1.1** (Sturmfels). If  $f \in \mathbb{Q}[x_1, \dots, x_n]$  is a sum of squares in  $\mathbb{R}[x_1, \dots, x_n]$ , then is  $f$  also a sum of squares in  $\mathbb{Q}[x_1, \dots, x_n]$  ?

Hillar ([3]) answers the question in the case where the sum of squares has coefficients in a totally real Galois number field :

**Theorem 1.2** (Hillar). *Let  $f \in \mathbb{Q}[x_1, \dots, x_n]$  be a sum of  $m$  squares in  $K[x_1, \dots, x_n]$  where  $K$  is a totally real Galois extension of  $\mathbb{Q}$ . Then,  $f$  is a sum of*

$$2^{[K:\mathbb{Q}]+1} \cdot \binom{[K:\mathbb{Q}]+1}{2} \cdot 4m$$

*squares in  $\mathbb{Q}[x_1, \dots, x_n]$ .*

The aim of this note is to show, modifying a little bit Hillar's proof, that only  $(4[K:\mathbb{Q}]-3) \cdot m$  squares are needed (that is Theorem 3.1). Moreover, as in [3], the argument is constructive.

## 2. HILLAR'S METHOD

Having in mind the Lagrange's four squares Theorem, we focus ourselves on *rational sum of squares* i.e. linear combination of squares with positive rational coefficients.

Let  $K$  be a totally real Galois extension of  $\mathbb{Q}$  which we write  $K = \mathbb{Q}(\theta)$  with  $\theta$  a real algebraic number, all of whose conjugates are also real. We set  $r = [K:\mathbb{Q}]$  and  $G = \text{Gal}(K/\mathbb{Q})$ .

Let  $f \in \mathbb{Q}[x_1, \dots, x_n]$  be a sum of  $m$  squares in  $K[x_1, \dots, x_n]$ , namely  $f = \sum_{k=1}^m f_k^2$ , with  $f_k \in K[x_1, \dots, x_n]$ . Summing over all actions of  $G$  (i.e. "averaging"), we get

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*Date:* July 14, 2009.

*2000 Mathematics Subject Classification.* 12Y05, 12F10, 11E25, 13B24.

*Key words and phrases.* Rational sum of squares, semidefinite programming, totally real number field.

$$(1) \quad f = \frac{1}{|G|} \sum_{k=1}^m \sum_{\sigma \in G} (\sigma f_k)^2$$

Next, we write each  $f_k$  in the form

$$f_k = \sum_{i=0}^{r-1} q_i \theta^i$$

where  $q_i \in \mathbb{Q}[x_1, \dots, x_n]$ . Then,

$$(2) \quad \sum_{\sigma \in G} (\sigma f_k)^2 = \sum_{j=1}^r \left( \sum_{i=0}^{r-1} q_i (\sigma_j \theta)^i \right)^2$$

We may write this sum of squares as the following product of matrices

$$\begin{pmatrix} q_0 \\ \vdots \\ q_{r-1} \end{pmatrix}^T \begin{pmatrix} 1 & \sigma_1 \theta & \dots & (\sigma_1 \theta)^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_r \theta & \dots & (\sigma_r \theta)^{r-1} \end{pmatrix}^T \begin{pmatrix} 1 & \sigma_1 \theta & \dots & (\sigma_1 \theta)^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_r \theta & \dots & (\sigma_r \theta)^{r-1} \end{pmatrix} \begin{pmatrix} q_0 \\ \vdots \\ q_{r-1} \end{pmatrix}$$

We obtain what is called a Gram matrix (cf [1]) associated to the sum of squares in (2). Let

$$G = \begin{pmatrix} 1 & \sigma_1 \theta & \dots & (\sigma_1 \theta)^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_r \theta & \dots & (\sigma_r \theta)^{r-1} \end{pmatrix}^T \begin{pmatrix} 1 & \sigma_1 \theta & \dots & (\sigma_1 \theta)^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_r \theta & \dots & (\sigma_r \theta)^{r-1} \end{pmatrix}$$

Note that the entries of  $G$  are in  $\mathbb{Q}$  since they are invariant under the  $\sigma_j$ 's.

Now, we come to the slight modification of the proof of Hillar that will improve the bound.

### 3. LU-DECOMPOSITION OF THE GRAM MATRIX

If  $u(x)$  denotes the minimal polynomial of the Galois extension  $K$  over  $\mathbb{Q}$ , then the  $(i, j)$ -th entry of the matrix  $G$  is the  $i + j - 2$ -th Newton sum of  $(\sigma_1, \dots, \sigma_r)$  the roots of  $u(x)$ . It is well known that the rank of  $G$  is equal to  $r$  and its signature (the difference between the positive eigenvalues and the negative ones) is equal to the number of real roots of  $u(x)$  (see for instance [2, Theorem 4.57]). In our case, we readily deduce that  $G$  is a positive definite matrix since  $K$  is totally real. Thus, all its principal minors are different from zero (they are strictly positive !) and  $G$  admits a LU-decomposition which we may put in a symmetric form

$$G = U^T D U$$

where  $D$  is diagonal and  $U$  is upper triangular with diagonal identity, and  $U, D$  have rational entries.

We may view this decomposition as a matricial realization of the Gauss algorithm which reduce the quadratic form given by  $G$ .

Now, if we denote by  $f_1, \dots, f_r$  the polynomials in  $\mathbb{Q}[x_1, \dots, x_r]$  appearing as the rows of the matrix  $U \times \begin{pmatrix} q_0 \\ \vdots \\ q_{r-1} \end{pmatrix}$  and by  $d_1, \dots, d_r$  the rational entries onto the diagonal of  $D$ , then we get from (2) the identity :

$$(3) \quad \frac{1}{|G|} \sum_{\sigma \in G} (\sigma f_k)^2 = \frac{d_1}{|G|} g_1^2 + \dots + \frac{d_r}{|G|} g_r^2$$

This construction leads to

**Theorem 3.1.** *Let  $f \in \mathbb{Q}[x_1, \dots, x_n]$  be a sum of  $m$  squares in  $K[x_1, \dots, x_n]$ , where  $K$  is a totally real Galois extension of  $\mathbb{Q}$ . Then,  $f$  is a sum of  $(4[K : \mathbb{Q}] - 3) \cdot m$  squares in  $\mathbb{Q}[x_1, \dots, x_n]$ .*

*Proof.* By (1) and (3), it suffices to apply Lagrange's four squares Theorem to get that  $f$  is a sum of  $4[K : \mathbb{Q}] \cdot m$  squares in  $\mathbb{Q}[x_1, \dots, x_n]$ .

But let us note that the first diagonal entry of  $D$  is always  $d_1 = r = [K : \mathbb{Q}]$ . Then, by the averaging process the first coefficient appearing in the rational sum of squares in (3) is  $\frac{d_1}{|G|} = 1$  : already a square in  $\mathbb{Q}$  ! Whereas the others coefficients  $\frac{d_i}{|G|}$  in the rational sum of squares could be any positive rational which we rewrite as a sum of 4 squares. This concludes the proof.  $\square$

*Remark 3.2.* Beware that if we perform the Cholesky algorithm to the matrix  $G$  instead of the LU-decomposition, it yields a factorisation  $G = U^T U$  where  $U$  is lower triangular but with entries in  $\mathbb{Q}[\sqrt{d_1}, \dots, \sqrt{d_r}]$  for some integers  $d_1, \dots, d_r$ . Then, an averaging argument would produce identities over  $\mathbb{Q}$  but will raise the number of squares by an unexpected multiplicative factor  $2^{[K:\mathbb{Q}]}$ .

Let us consider as an example, the simple case of quadratic extensions :

**Example 3.3.** Let  $K = \mathbb{Q}(\sqrt{d})$  where  $d \in \mathbb{Q}$  is not a square. The extension  $K$  is always Galois, and it is totally real if  $d \geq 0$ .

Let  $f \in \mathbb{Q}(x_1, \dots, x_n)$  be such that  $f = \sum_{k=1}^m (a_k + b_k \sqrt{d})^2$  with  $a_k, b_k \in \mathbb{Q}(x_1, \dots, x_n)$ . Since  $f$  has rational coefficients, by averaging we get

$$f = \frac{1}{2} \sum_{k=1}^m (a_k + b_k \sqrt{d})^2 + (a_k - b_k \sqrt{d})^2 = \sum_{k=1}^m (a_k^2 + db_k^2)$$

It remains to write  $d$  as a sum of  $l \leq 4$  squares of rationals, and we get that  $f$  is a sum of at most  $(1+l) \cdot m$  squares in  $\mathbb{Q}(x_1, \dots, x_n)$ .

As another illustration, we apply our method to [3, Example 1.7] :

**Example 3.4.** Consider the polynomial

$$f = 3 - 12y - 6x^3 + 18y^2 + 3x^6 + 12x^3y - 6xy^3 + 6x^2y^4$$

which is the following sum of squares

$$f = (x^3 + \alpha^2 y + \beta xy^2 - 1)^2 + (x^3 + \beta^2 y + \gamma xy^2 - 1)^2 + (x^3 + \gamma^2 y + \alpha xy^2 - 1)^2$$

in  $\mathbb{Q}(\alpha)[x, y]$  where  $\alpha, \beta, \gamma$  are the real roots of the polynomial  $u(x) = x^3 - 3x + 1$ .

We do not need to average and directly compute the matrix  $G$  and its symmetric LU-decomposition

$$\begin{pmatrix} 3 & 0 & 6 \\ 0 & 6 & -3 \\ 6 & -3 & 18 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Because of the relations  $\beta = 2 - \alpha - \alpha^2$  and  $\gamma = \alpha^2 - 2$ , the vector of polynomials  $q = (q_0, q_1, q_2)^T$  is  $q = (x^3 + 2xy^2 - 1, -xy^2, y - xy^2)^T$  and hence

$$f = 3 \left( (x^3 + 2xy^2 - 1) + 2(y - xy^2) \right)^2 + 6 \left( -xy^2 - \frac{1}{2}(y - xy^2) \right)^2 + \frac{9}{2} (y - xy^2)^2$$

a rational sum of 3 squares, to compare with the rational sum of 6 squares obtained in [3].

#### REFERENCES

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