hal-00403670, version 1 - 16 Jul 2009

GENERALIZED BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES WITH NON-LIPSCHITZ COEFFICIENTS [*]

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Abstract

We study the homogenization problem of semilinear reflected partial We prove an existence and uniqueness result for generalized backward doubly stochastic differential equations driven by Lévy processes with non-Lipschitz assumptions.

AMS Subject Classification: Primary: 60F05, 60H15; Secondary: 60J30

Keywords: Backward doubly stochastic differential equations; Lévy processes; non-Lipschitz coefficients, Teugel martingale

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) have been introduced by Pardoux and Peng [10]. The original motivation for the study of this kind of equations was to provide probabilistic interpretation for solutions of both parabolic and elliptic semi linear partial differential equations (see Pardoux and Peng [11], Peng [14]). Thanks to its link with the finance [3], the stochastic control and stochastic game theory (see [6] and references therein), the theory of BSDEs quickly tooks a real enthusiasm since 1990.

Moreover, in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs in short), Pardoux and Peng [12] considered a new kind of BSDEs, called backward doubly stochastic differential equations (BDSDEs in short). There exist two different direction of stochastic integral driven respectively by two independent Brownian motion. The first integral is the well-know backward Itô integral and the second, the forward one. Following it, Bally and Matoussi [1] gave the probabilistic representation of the weak solutions to parabolic semilinear SPDEs in Sobolev spaces by means of BDSDEs. Furthermore, Boufoussi et al. [2] recommended a class of generalized

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[*] This works is partially support by Fellowship grant of AMMSI

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BDSDEs (GBDSDEs in short) which involved an other integral with respect to an adapted continuous increasing process and gave the probabilistic representation for stochastic viscosity solutions of semi-linear SPDEs with a Neumann boundary condition.
Recently, Ren et al. [5] showed the existence and uniqueness of solutions to GBDSDEs driven by Teugels martingales associated with Lévy process and gave probabilistic interpretation for solutions to a class of stochastic partial differential integral equations (SPDIEs in short) with a nonlinear Neumann boundary condition. These results are obtained with strong conditions on the coefficients, those are Lipschitz conditions and monotony ones. Recently, N'zi and Owo [9] proved an existence and uniqueness result of solutions for BDSDEs with non-Lipschitz conditions.

Inspired by this work, the aim of this paper is to extend the study of GBDSDEs driven by Lévy processes introduced by in Ren et al. [5]. We prove an existence and uniqueness result in the non-Lipschitz case.

The rest of the paper is organized as follows. In section 2, we introduce some preliminaries and notations. Section 3 is devoted to the proof of the existence and uniqueness of the solutions to GBDSDEs driven by Lévy processes with non-Lipschitz coefficients.

2 Preliminaries and notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which are defined all the processes considered and T be a fixed final time.
Let $\{B_t; 0 \leq t \leq T\}$ be a standard Brownian motion, with values in $\mathbb{R}$ and $\{L_t; 0 \leq t \leq T\}$ be a $\mathbb{R}$-valued Lévy process independent of $\{B_t; 0 \leq t \leq T\}$ corresponding to a standard Lévy measure ν such that $\int_{\mathbb{R}} (1 \wedge y)\nu(dy) < \infty$.
Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \overset{\Delta}{=} \mathcal{F}_t^{L} \vee \mathcal{F}_{t,T}^{B},$$

where for any process $\{\eta_t\}$; $\mathcal{F}_{s,t}^{\eta} = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^{\eta} = \mathcal{F}_{0,t}^{\eta}$.
Note that $\{\mathcal{F}_t^{L}, t \in [0, T]\}$ is an increasing filtration and $\{\mathcal{F}_{t,T}^{B}, t \in [0, T]\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing so that it does not constitute a filtration.
Let ℓ^2 denote the set of real valued sequences $x = (x^{(i)})_{i \geq 1}$ such that $\|x\|^2 = \sum_{i=1}^{\infty} |x^{(i)}|^2 < \infty$.
We will denote by $\mathcal{M}^2(0, T, \ell^2)$ the set of (class of $dP \otimes dt$ a.e. equal) ℓ^2-valued process which satisfy

(i) $\|\varphi\|^2_{\mathcal{M}^2(\ell^2)} = \mathbb{E}\left(\int_0^T \|\varphi_t\|^2 dt\right) < \infty$.

(ii) φ_t is $\mathcal{F}_t$-measurable, for a.e. $t \in [0, T]$.

Similarly, $\mathcal{S}^2(0, T)$ stands for the set of real valued random processes which satisfy:

(i) $\|\varphi\|^2_{\mathcal{S}^2} = \mathbb{E}\left(\sup_{0 \leq t \leq T} |\varphi_t|^2\right) < \infty$

(ii) φ_t is $\mathcal{F}_t$-measurable, for any $t \in [0, T]$.

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In the sequel, let $\{A_t; 0 \leq t \leq T\}$ be a continuous and increasing real valued process such that A_t is $\mathcal{F}_t$-measurable, for any $t \in [0,T]$ and $A_0 = 0$.
Let $\mathcal{A}^2(0,T)$ denote the set of (class of $dP \otimes dA_t$ a.e. equal) real valued measurable random processes $\{\varphi_t; 0 \leq t \leq T\}$ such that $\mathbb{E}\left(\int_0^T |\varphi_t|^2 dA_t\right) < \infty$.
We will denote by $\mathcal{E}(0,T) = \left(\mathcal{S}^2(0,T) \cap \mathcal{A}^2(0,T)\right) \times \mathcal{M}^2(0,T,\ell^2)$ the set of $\mathbb{R} \times \ell^2$-valued processes (Y,Z) defined on $\Omega \times [0,T]$ which satisfy condition (ii) as above and such that

$$\|(Y,Z)\|^2_{\mathcal{E}} = \mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Y_s|^2 dA_s + \int_0^T \|Z_s\|^2 ds\right) < \infty.$$

$\mathcal{E}(0,T)$ endowed with the norm $\|.\|_{\mathcal{E}}$ is a Banach space.
Let denote by $(H^{(i)})_{i\geq 1}$ the Teugels Martingale associated with the Lévy process $\{L_t; 0 \leq t \leq T\}$. More precisely

$$H^{(i)}_t = c_{i,i} T^{(i)}_t + c_{i,i-1} T^{(i-1)}_t + \ldots + c_{i,1} T^{(1)}_t,$$

where $T^{(i)}_t = L^{(i)}_t - \mathbb{E}(L^{(i)}_t) = L^{(i)}_t - t\mathbb{E}(L^{(i)}_1)$ for all $i \geq 1$ and $L^{(i)}_t$ are power jump processes so that $L^{(1)}_t = L_t$ and $L^{(i)}_t = \sum_{0<s\leq t} (\Delta L_s)^i$ for $i \geq 2$, with $L_{t^-} = \lim_{s \nearrow t} L_s$ and $\Delta L_s = L_s - L_{s^-}$.
Nualart and Schoutens have proved in [8] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1,\ x,\ x^2, \cdots$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$:

$$q_i(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

The martingale $(H^{(i)})_{i\geq 1}$ can be chosen to be pairwise strongly orthonormal martingale. That is for all $i, j,\ \langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij} t$.

Remark 2.1. If μ only has mass at 1, we are in the Poisson case N_t with parameter $\lambda > 0$; here $H^{(1)}_t = \frac{N_t - \lambda t}{\lambda}$ and $H^{(i)} = 0,\ \ i = 2,\ 3, \cdots$. This case is degenerate in this Lévy framework.

Definition 2.2. A pair $(Y,Z) : \Omega \times [0,T] \to \mathbb{R} \times \ell^2$ of processes is called solution of GBDSDE(ξ, f, g, h, A) driven by Lévy processes if $(Y,Z) \in \mathcal{E}(0,T)$ such that

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_{s^-}, Z_s)ds + \int_t^T h(s, Y_{s^-})dA_s + \int_t^T g(s, Y_{s^-}, Z_s)\overleftarrow{dB_s} \\ &\quad - \sum_{i=1}^{\infty} \int_t^T Z^{(i)}_s dH^{(i)}_s,\ \ t \in [0,T]. \end{aligned} \tag{2.1}$$

Here the integral with respect to $\{B_t\}$ is the classical backward Itô integral (see Kunita [7]) and the integral with respect to $\{(H^{(i)}_t)_{i\geq 1}\}$ is a standard forward Itô-type semi martingale integral (see Gong [4]).

First, let us recall the extension of the well-known Itô formula on which depend strongly our results. Its proof follows the same program as Lemma 2.5 in [2] or Lemma 1.3 in [12].

Lemma 2.3. *Let* α, β *and* γ *in* $\mathcal{S}^2(0,T)$, $\eta \in \mathcal{A}^2(0,T)$ *and* $\zeta \in \mathcal{M}^2(0,T,\ell^2)$ *such that*

$$\alpha_t = \alpha_0 + \int_t^T \beta_s ds + \int_t^T \eta_s dA_s + \int_t^T \gamma_s dB_s - \sum_{i=1}^{\infty} \int_t^T \zeta^{(i)}_s dH^{(i)}_s,\ \ t \in [0,T].$$

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Then

$$\begin{aligned} |\alpha_t|^2 &= |\xi|^2 + 2\int_t^T \alpha_s\beta_s ds + 2\int_t^T \alpha_s\eta_s dA_s + 2\int_t^T \alpha_s\gamma_s dB_s \\ &\quad -2\sum_{i=1}^{\infty}\int_t^T \alpha_s\zeta_s^i dH_s^{(i)} + \int_t^T |\gamma_s|^2 ds - \sum_{i,j=1}^{\infty}\int_t^T \zeta_s^i\zeta_s^j d[H_s^{(i)}, H_s^{(j)}]. \end{aligned}$$

Note that $\left(\int_t^T \alpha_s\gamma_s dB_s\right)_{0\le t\le T}$, $\left(\int_0^t \alpha_s\zeta_s^{(i)} dH_s^{(i)}\right)_{0\le t\le T}$ *for all* $i\ge 1$ *and* $\left(\int_0^t \zeta_s^{(i)}\zeta_s^{(j)} d[H_s^{(i)}, H_s^{(j)}]\right)_{0\le t\le T}$ *for* $i\ne j$ *are uniformly integrable martingale and* $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t$, *we have*

$$\begin{aligned} \mathbb{E}|\alpha_t|^2 &= \mathbb{E}|\alpha_0|^2 + 2\mathbb{E}\int_t^T \alpha_s\beta_s ds + 2\mathbb{E}\int_t^T \alpha_s\eta_s dA_s + \mathbb{E}\int_t^T |\gamma_s|^2 ds \\ &\quad -\mathbb{E}\left(\int_t^T \sum_{i=1}^{\infty} |\zeta_s^{(i)}|^2 ds\right), \ \ t\in[0,T]. \end{aligned}$$

Next, let us recall the existence and uniqueness result on GBDSDE(ξ, f, g, h, A) in the Lipschitz and monotony context. This work is due to Ren et al. [5]. They use the following assumptions:

(A1) The terminal value $\xi \in \mathrm{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$ such that for all $\lambda > 0$

$$\mathbb{E}(e^{\lambda A_T}|\xi|^2) < \infty.$$

(A2) The coefficients $f, g : \Omega\times[0,T]\times\mathbb{R}\times\ell^2 \to \mathbb{R}$ and $h : \Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy, for some $\beta_1\in\mathbb{R}$, $K>0$, $0<\alpha<1$ and $\beta_2<0$, three $\mathcal{F}_t$-adapted processes $\{f_t, g_t, h_t : 0\le t\le T\}$ with values in $[1,\infty[$ and for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\ell^2$, $\lambda>0$

(i) $f(.,y,z), g(.,y,z)$ and $h(.,y)$ are progressively measurable,

(ii) $$\begin{cases} |f(t,y,z)| \le f_t + K(|y| + \|z\|) \\ |g(t,y,z)| \le g_t + K(|y| + \|z\|) \\ |h(t,y)| \le h_t + K|y| \end{cases}$$

(iii) $\mathbb{E}(\int_0^T e^{\lambda A_t} f_t^2 dt + \int_0^T e^{\lambda A_t} g_t^2 dt + \int_0^T e^{\lambda A_t} h_t^2 dA_t) < \infty$

(iv) $\langle y-y', f(t,y,z) - f(t,y',z)\rangle \le \beta_1 \mid y-y' \mid^2$

(vi) $\mid f(t,y,z) - f(t,y,z') \mid^2 \le K\|z-z'\|^2$

(vii) $\langle y-y', h(t,y) - h(t,y')\rangle \le \beta_2 \mid y-y' \mid^2$

(vii) $\mid g(t,y,z) - g(t,y',z') \mid^2 \le K \mid y-y' \mid^2 + \alpha\|z-z'\|^2$

(viii) $y \mapsto (f(t,y,z), g(t,y,z), h(t,y))$ is continuous for all z, (ω, t).

(A3) $\mid f(t,y,z) - f(t,y',z) \mid^2 + \mid h(t,y) - h(t,y') \mid^2 \le K \mid y-y' \mid^2$.

Lemma 2.4 (Ren et al. [5])**.** *Under the assumptions* ***(A1)***, ***(A2)*** *and* ***(A3)***, *the GBDSDE*(ξ, f, g, h, A) *has a unique solution*

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3 Existence and uniqueness result in non-Lipschtz case

In order to attain the solution of GBDSDE (ξ, f, g, h, A), we stand the following assumptions. The coefficients $f, g : \Omega \times [0,T] \times \mathbb{R} \times \ell^2 \to \mathbb{R}$, $h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ and the terminal value ξ satisfy:

(H1) $f(.,y,z), g(.,y,z)$ and $h(.,y)$ are progressively measurable such that

$$0 < \mathbb{E}\left(\int_0^T |f(s,0,0)|^2 ds + \int_0^T |h(s,0)|^2 dA_s + \int_0^T |g(s,0,0)|^2 ds\right) < \infty.$$

(H2) For some $K > 0$ and three $\mathcal{F}_t$-adapted processes $\{f_t, g_t, h_t : 0 \le t \le T\}$ with values in $[1,\infty[$ and for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \ell^2$, $\lambda > 0$

$$\begin{cases} |f(t,y,z)| \le f_t + K(|y| + \|z\|) \\ |g(t,y,z)| \le g_t + K(|y| + \|z\|) \\ |h(t,y)| \le h_t + K|y| \\ \mathbb{E}\left(\int_0^T e^{\lambda A_t} f_t^2 dt + \int_0^T e^{\lambda A_t} g_t^2 dt + \int_0^T e^{\lambda A_t} h_t^2 dA_t\right) < \infty \end{cases}$$

(H3) For some $\beta < 0$ and for all $y_1,\ y_2 \in \mathbb{R}$ and $t \in [0,T]$,

$$\langle y_1 - y_2, h(t,y_1) - h(t,y_2)\rangle \le \beta \mid y_1 - y_2 \mid^2$$

(H4) For all $(y_1,z_1),\ (y_2,z_2) \in \mathbb{R} \times \ell^2$ and $t \in [0,T]$,

$$\begin{cases} |h(t,y_1) - h(t,y_2)| \le K \mid y_1 - y_2 \mid \\ \mid f(t,y_1,z_1) - f(t,y_2,z_2) \mid^2 \le \rho(t, \mid y_1 - y_2 \mid^2) + C\|z_1 - z_2\|^2 \\ \mid g(t,y_1,z_1) - g(t,y_2,z_2) \mid^2 \le \rho(t, \mid y_1 - y_2 \mid^2) + \alpha\|z_1 - z_2\|^2 \end{cases},$$

where $C > 0$ and $0 < \alpha < 1$ are two constants and $\rho : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

(i) for fixed $t \in [0,T]$, $\rho(t,.)$ is a concave and non-decreasing function such that $\rho(t,0) = 0$.

(ii) for fixed u, $\int_0^T \rho(t,u)dt < +\infty$.

(iii) for any $M > 0$, the following ODE

$$\begin{cases} u' = -M\rho(t,u) \\ u(T) = 0 \end{cases}$$

has a unique solution $u(t) \equiv 0,\ \ t \in [0,T]$.

(H5) $\xi \in \mathrm{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$ such that for all $\lambda > 0$

$$\mathbb{E}(e^{\lambda A_T} |\xi|^2) < \infty.$$

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Under above assumptions, we now construct an approximate sequence using a Picard-type iteration with the help of Lemma 2.4. Let $Y_t^0 = 0$ and $(Y^n, Z^n)_{n\geq 1}$ be a sequence in $\mathcal{E}^2(0,T)$ defined recursively by

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_s^{n-1}, Z_s^n)ds + \int_t^T h(s, Y_s^n)dA_s + \int_t^T g(s, Y_s^{n-1}, Z_s^n)\overleftarrow{dB_s} \\ &\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{n(i)} dH_s^{(i)}. \qquad (3.1) \end{aligned}$$

Indeed, for each $n \geq 1$ and fixed Y^{n-1} in $\mathcal{S}^2(0,T)$, BDSDE (3.1) satisfies assumptions **(A1)**, **(A2)** and **(A3)**. So, by Lemma 2.4, the BDSDE (3.1) has a unique solution $(Y^n, Z^n) \in \mathcal{E}^2(0,T)$.
Our purpose is to prove that the sequence $(Y^n, Z^n)_{n\geq 0}$ converges in $\mathcal{E}^2(0,T)$ to the unique solution of BDSDEs (2.1). We begin with some preliminary results.

Lemma 3.1. *Let **(H1)**, **(H3)** and **(H4)** be satisfied. Then for all $0 \leq t \leq T$, $n, m \geq 1$, we have*

$$\mathbb{E}\left|Y_t^{n+m} - Y_t^n\right|^2 \leq e^{\frac{CT}{1-\alpha}}\left(\frac{1-\alpha}{C} + 1\right)\int_t^T \rho(s, \mathbb{E}\left|Y_s^{n+m-1} - Y_s^{n-1}\right|^2)ds.$$

Proof. In view of Itô's formula, we have

$$\begin{aligned} &\mathbb{E}\left|Y_t^{n+m} - Y_t^n\right|^2 + \mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds \\ = \;& 2\mathbb{E}\int_t^T \left\langle Y_s^{n+m} - Y_s^n, f(s, Y_s^{n+m-1}, Z_s^{n+m}) - f(s, Y_s^{n-1}, Z_s^n)\right\rangle ds \\ &+2\mathbb{E}\int_t^T \left\langle Y_s^{n+m} - Y_s^n, h(s, Y_s^{n+m}) - h(s, Y_s^n)\right\rangle dA_s \\ &+\mathbb{E}\int_t^T \left|g(s, Y_s^{n+m-1}, Z_s^{n+m}) - g(s, Y_s^{n-1}, Z_s^n)\right|^2 ds. \end{aligned}$$

In view of **(H3)**, and Young's inequality $2ab \leq \frac{1}{\theta}a^2 + \theta b^2$, for any $\theta > 0$, we have

$$\begin{aligned} &\mathbb{E}\left|Y_t^{n+m} - Y_t^n\right|^2 + \mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds + 2|\beta|\mathbb{E}\int_t^T \left|Y_s^{n+m} - Y_s^n\right|^2 dA_s \\ \leq \;& \frac{1}{\theta}\mathbb{E}\int_t^T \left|Y_s^{n+m} - Y_s^n\right|^2 ds + (\theta+1)\mathbb{E}\int_t^T \rho(s, \left|Y_s^{n+m-1} - Y_s^{n-1}\right|^2)ds \\ &+(\theta C + \alpha)\mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds. \end{aligned}$$

Choosing $\theta = \frac{1-\alpha}{C} > 0$, it follows from Gronwall's inequality and Jensen's inequality that

$$\mathbb{E}\left|Y_t^{n+m} - Y_t^n\right|^2 \leq e^{\frac{CT}{1-\alpha}}\left(\frac{1-\alpha}{C} + 1\right)\int_t^T \rho(s, \mathbb{E}\left|Y_s^{n+m-1} - Y_s^{n-1}\right|^2)ds.$$

□

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Lemma 3.2. *Let* **(H1)**, **(H3)** *and* **(H4)** *be satisfied. Then, there exists* $T_1 \in [0,T[$ *and a constant* $M_1 \geq 0$ *such that for all* $t \in [T_1,T]$, *each* $n \geq 1$, $\mathbb{E}\,|Y_t^n|^2 \leq M_1$.

Proof. In view of Itô's formula, we have

$$\begin{aligned}
&\mathbb{E}\,|Y_t^n|^2 + \mathbb{E}\int_t^T \|Z_s^n\|^2 ds \\
= \;& \mathbb{E}\,|\xi|^2 + 2\mathbb{E}\int_t^T \left\langle Y_s^n, f(s,Y_s^{n-1},Z_s^n)\right\rangle ds + 2\mathbb{E}\int_t^T \langle Y_s^n, h(s,Y_s^n)\rangle\, dA_s \\
&+\mathbb{E}\int_t^T \left|g(s,Y_s^{n-1},Z_s^n)\right|^2 ds.
\end{aligned}$$

By virtue of **(H3)**, **(H4)** and Young's inequality $2ab \leq \frac{1}{\theta}a^2 + \theta b^2$, for any $\theta > 0$, we have

$$\begin{aligned}
2\left\langle Y_s^n, f(s,Y_s^{n-1},Z_s^n)\right\rangle &\leq \frac{1}{\theta}\,|Y_s^n|^2 + \theta\left|f(s,Y_s^{n-1},Z_s^n)\right|^2 \\
&\leq \frac{1}{\theta}\,|Y_s^n|^2 + 2\theta\rho(s,\left|Y_s^{n-1}\right|^2) + 2\theta C\|Z_s^n\|^2 + 2\theta\,|f(s,0,0)|^2,
\end{aligned}$$

$$\begin{aligned}
2\,\langle Y_s^n, h(s,Y_s^n)\rangle &\leq 2\beta\,|Y_s^n|^2 + 2\,\langle Y_s^n, h(s,0)\rangle \\
&\leq -|\beta|\,|Y_s^n|^2 + \frac{1}{|\beta|}\,|h(s,0)|^2,
\end{aligned}$$

$$\left|g(s,Y_s^{n-1},Z_s^n)\right|^2 \leq (1+\theta)\rho(s,\left|Y_s^{n-1}\right|^2) + (1+\theta)\alpha\|Z_s^n\|^2 + (1+\frac{1}{\theta})\,|g(s,0,0)|^2.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\,|Y_t^n|^2 + [1-2\theta C-(1+\theta)\alpha]\mathbb{E}\int_t^T \|Z_s^n\|^2 ds + |\beta|\mathbb{E}\int_t^T |Y_s^n|^2\, dA_s \\
\leq\;& \mathbb{E}\,|\xi|^2 + \frac{1}{\theta}\mathbb{E}\int_t^T |Y_s^n|^2\, ds + (3\theta+1)\int_t^T \rho(s,\mathbb{E}\left|Y_s^{n-1}\right|^2) ds \\
&+\mathbb{E}\int_t^T [2\theta\,|f(s,0,0)|^2 + (1+\frac{1}{\theta})\,|g(s,0,0)|^2] ds + \frac{1}{|\beta|}\mathbb{E}\int_t^T |h(s,0)|^2\, dA_s.
\end{aligned}$$

We choose $\theta = \frac{1-\alpha}{2C+\alpha} > 0$, then

$$\begin{aligned}
\mathbb{E}\,|Y_t^n|^2 \;\leq\;& \mathbb{E}\,|\xi|^2 + \frac{2C+\alpha}{1-\alpha}\mathbb{E}\int_t^T |Y_s^n|^2\, ds + \left(3\frac{1-\alpha}{2C+\alpha}+1\right)\int_t^T \rho(s,\mathbb{E}\left|Y_s^{n-1}\right|^2) ds \\
&+\mathbb{E}\int_t^T \left[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + (\frac{1+2C}{1-\alpha})\,|g(s,0,0)|^2\right] ds \\
&+\frac{1}{|\beta|}\mathbb{E}\int_t^T |h(s,0)|^2\, dA_s.
\end{aligned}$$

Now, in view of Gronwall's inequality, we derive

$$\mathbb{E}\,|Y_t^n|^2 \;\leq\; \mu_t^1 + \left(3\frac{1-\alpha}{2C+\alpha}+1\right) e^{\frac{(2C+\alpha)T}{1-\alpha}} \int_t^T \rho(s,\mathbb{E}\left|Y_s^{n-1}\right|^2) ds \qquad (3.2)$$

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where

$$\mu_t^1 = e^{\frac{(2C+\alpha)T}{1-\alpha}} \left(\mathbb{E}\,|\xi|^2 + \mathbb{E}\int_t^T \left[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + (\frac{1+2C}{1-\alpha})\,|g(s,0,0)|^2 \right] ds \right.$$
$$\left. + \frac{1}{|\beta|}\mathbb{E}\int_t^T |h(s,0)|^2\, dA_s \right),$$

Let

$$M = \max\left\{ \left(3\frac{1-\alpha}{2C+\alpha}+1\right) e^{\frac{(2C+\alpha)T}{1-\alpha}}, \left(\frac{1-\alpha}{C}+1\right) e^{\frac{CT}{1-\alpha}} \right\} > 0. \tag{3.3}$$

and

$$M_1 = 2\mu_0^1 = 2e^{\frac{(2C+\alpha)T}{1-\alpha}} \left(\mathbb{E}\,|\xi|^2 + \mathbb{E}\int_0^T \left[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 \right.\right.$$
$$\left.\left. + \ (\frac{1+2C}{1-\alpha})\,|g(s,0,0)|^2 \right] ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T |h(s,0)|^2\, dA_s \right) \geq 0.$$

By virtue of **(H4)**, $\int_0^T \rho(s,M_1)ds < +\infty$, so we can find T_1 such that

$$\int_{T_1}^T \rho(s,M_1)ds \leq \frac{\mu_0^1}{M}.$$

Now, we complete the proof as in N'zi and Owo [9].

□

With the help of the above Lemmas, we can now prove existence and uniqueness which is our main result.

Theorem 3.3. *Let **(H1)-(H5)** be satisfied. Then the equation* (2.1) *has an unique solution* $(Y,Z) \in \mathcal{E}^2(0,T)$.

Proof. Existence. For all $n \geq 1$, and $t \in [0,T]$, we let

$$\phi_0(t) = M\int_t^T \rho(s,M_1)ds \ \text{ and } \ \phi_{n+1}(t) = M\int_t^T \rho(s,\phi_n(s))ds.$$

N'zi and Owo proved in [9] that $(\phi_n(t))_{n\geq 0}$ is non-increasing and converges uniformly to 0 for all $t \in [T_1,T]$.
In view of Lemmas 3.1 and 3.2, we conclude as in [9] that for all $t \in [T_1,T]$, $n,\ m \geq 1$,

$$\mathbb{E}\left|Y_t^{n+m} - Y_t^n\right|^2 \leq \phi_{n-1}(t) \leq M_1. \tag{3.4}$$

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Using Itô's formula, we deduce from assumptions **(H3)**, **(H4)** and Young's inequality $2ab \leq \frac{1}{\theta}a^2+\theta b^2,\ \theta>0$, that for all $t\in[T_1,T]$

$$\begin{aligned} &\left|Y_t^{n+m}-Y_t^n\right|^2-(\theta C+\alpha)\int_t^T \|Z_s^{n+m}-Z_s^n\|^2 ds+2|\beta|\int_t^T \left|Y_s^{n+m}-Y_s^n\right|^2 dA_s\\ \leq\ & \frac{1}{\theta}\int_t^T \left|Y_s^{n+m}-Y_s^n\right|^2 ds+(\theta+1)\int_t^T \rho(s,\left|Y_s^{n+m-1}-Y_s^{n-1}\right|^2)ds\\ &+2\int_t^T \left\langle Y_s^{n+m}-Y_s^n,(g(s,Y_s^{n+m-1},Z_s^{n+m})-g(s,Y_s^{n-1},Z_s^n))\overleftarrow{dB_s}\right\rangle\\ &-2\sum_{i,j=1}^{\infty}\int_t^T \left\langle Y_s^{n+m}-Y_s^n, Z_s^{n+m(i)}-Z_s^{n(i)}\right\rangle dH_s^{(i)}-\sum_{i,j=1}^{\infty}\int_t^T Z_s^iZ_s^j d[H_s^i,H_s^j]. \end{aligned}$$

Note that $\left(\int_t^T \left\langle Y_s^{n+m}-Y_s^n,(g(s,Y_s^{n+m-1},Z_s^{n+m})-g(s,Y_s^{n-1},Z_s^n))\overleftarrow{dB_s}\right\rangle\right)_{0\leq t\leq T}$, $\left(\int_t^T \left\langle Y_s^{n+m}-Y_s^n, Z_s^{n+m(i)}-Z_s^{n(i)}\right\rangle dH_s^{(i)}\right)_{0\leq t\leq T}$ for all $i\geq 1$ and $\left(\int_t^T Z_s^iZ_s^j d[H_s^i,H_s^j]\right)_{0\leq t\leq T}$ for $i\neq j$ are uniformly integrable martingale.

Therefore, taking expectation and Jensen inequality, we obtain from inequality (3.4),

$$\begin{aligned} &\mathbb{E}\left|Y_t^{n+m}-Y_t^n\right|^2+(1-\theta C-\alpha)\mathbb{E}\int_t^T \|Z_s^{n+m}-Z_s^n\|^2 ds+2|\beta|\mathbb{E}\int_t^T \left|Y_s^{n+m}-Y_s^n\right|^2 dA_s\\ \leq\ & \frac{1}{\theta}\mathbb{E}\int_t^T \left|Y_s^{n+m}-Y_s^n\right|^2 ds+(\theta+1)\int_t^T \rho(s,\mathbb{E}\left|Y_s^{n+m-1}-Y_s^{n-1}\right|^2)ds\\ \leq\ & \frac{1}{\theta}\int_t^T \phi_{n-1}(s)ds+\frac{\theta+1}{M}\phi_{n-1}(t). \end{aligned}$$

Thus, choosing $\theta=\frac{1-\alpha}{2C}$, we get

$$\begin{aligned} &\sup_{T_1\leq t\leq T}\left(\mathbb{E}\left|Y_t^{n+m}-Y_t^n\right|^2\right)+\frac{1-\alpha}{2}\mathbb{E}\int_{T_1}^T \|Z_s^{n+m}-Z_s^n\|^2 ds+2|\beta|\mathbb{E}\int_{T_1}^T \left|Y_s^{n+m}-Y_s^n\right|^2 dA_s\\ \leq\ & \left(\frac{T-T_1}{\theta}+\frac{\theta+1}{M}\right)\phi_{n-1}(T_1). \end{aligned}$$

Now, in view of this inequality, we deduce by Burkhölder-Davis-Gundy's inequality that

$$\mathbb{E}\left(\sup_{T_1\leq t\leq T}\left|Y_t^{n+m}-Y_t^n\right|^2\right)+\mathbb{E}\int_{T_1}^T \|Z_s^{n+m}-Z_s^n\|^2 ds+\mathbb{E}\int_{T_1}^T \left|Y_s^{n+m}-Y_s^n\right|^2 dA_s\leq K\phi_{n-1}(T_1),$$

where K is positive constant depending on C, T_1, T, α, $|\beta|$ and M.

Since $\phi_n(t)\longrightarrow 0$, for all $t\in[T_1,T]$, as $n\longrightarrow\infty$, it follows that (Y^n,Z^n) is a Cauchy sequence in $\mathcal{E}^2(T_1,T)$. Now, set

$$Y=\lim_{n\to+\infty}Y^n,\quad Z=\lim_{n\to+\infty}Z^n.$$

Then, as $\mathcal{E}^2(T_1,T)$ is a Banach space, $(Y,Z)\in\mathcal{E}^2(T_1,T)$.

Passing to the limit in (3.1), we prove that (Y,Z) satisfies the BDSDE (2.1) on $[T_1,T]$.

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If $T_1 = 0$, then we have proved the existence result. If $T_1 \neq 0$, we consider the following equation

$$\begin{aligned} Y_t &= Y_{T_1} + \int_t^{T_1} f(s, Y_{s^-}, Z_s)ds + \int_t^{T_1} h(s, Y_{s^-})dA_s + \int_t^{T_1} g(s, Y_{s^-}, Z_s)\overleftarrow{dB_s} \\ &\quad - \sum_{i=1}^{\infty} \int_t^{T_1} Z_s^{(i)} dH_s^{(i)}, \ \ t \in [0, T_1]. \end{aligned} \tag{3.5}$$

We construct the Picard approximate sequence of equation (3.5), as in (3.1). Using the same procedure as in the proof of Lemmas 3.1 and Lemma 3.2, for all $t \in [T_1, T]$, $n, \ m \geq 1$, we establish that

$$\mathbb{E}\left|Y_t^{n+m} - Y_t^n\right|^2 \leq e^{\frac{CT}{1-\alpha}} \left(\frac{1-\alpha}{C} + 1\right) \int_t^{T_1} \rho(s, \mathbb{E}\left|Y_s^{n+m-1} - Y_s^{n-1}\right|^2)ds,$$

and

$$\mathbb{E}\left|Y_t^n\right|^2 \quad \leq \quad \mu_t^2 + M\int_t^{T_1} \rho(s, \mathbb{E}\left|Y_s^{n-1}\right|^2)ds$$

where

$$\begin{aligned} \mu_t^2 &= e^{\frac{(2C+\alpha)T}{1-\alpha}} \Bigg(\mathbb{E}\left|Y_{T_1}\right|^2 + \mathbb{E}\int_t^T \left[\frac{2(1-\alpha)}{2C+\alpha}\left|f(s,0,0)\right|^2\right. \\ &\qquad \left. + \left(\frac{1+2C}{1-\alpha}\right)\left|g(s,0,0)\right|^2\right] ds + \frac{1}{|\beta|}\mathbb{E}\int_t^T \left|h(s,0)\right|^2 dA_s\Bigg), \end{aligned}$$

Let

$$\begin{aligned} M_2 = 2\mu_0^2 &= 2e^{\frac{(2C+\alpha)T}{1-\alpha}} \Bigg(\mathbb{E}\left|Y_{T_1}\right|^2 + \mathbb{E}\int_0^T \left[\frac{2(1-\alpha)}{2C+\alpha}\left|f(s,0,0)\right|^2\right. \\ &\qquad \left. + \left(\frac{1+2C}{1-\alpha}\right)\left|g(s,0,0)\right|^2\right] ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T \left|h(s,0)\right|^2 dA_s\Bigg). \end{aligned}$$

We can also find $T_2 \in [0, T_1[$ such that

$$\mathbb{E}\left|Y_t^n\right|^2 \quad \leq \quad M_2, \ \ n \geq 1, \ t \in [T_2, T_1].$$

Here $T_2 = 0$ or $T_2 \in]0, T_1[$ such that $\displaystyle\int_{T_2}^{T_1} \rho(s, M_2)ds = \frac{\mu_0^2}{M}$. As above, we prove the existence of the solution to BDSDE (3.5) on $[T_2, T_1]$. If $T_2 = 0$, the proof of the existence is complete. Overwise, we repeat the above procedures. Thus, we obtain a sequence $\{T_p, \ \mu_t^p, \ M_p, \ \ p \geq$

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$1\}$ defined by

$$
\begin{aligned}
&0 \leq T_p < T_{p-1} < \ldots < T_1 < T_0 = T,\\
&\mu_t^p = e^{\frac{(2C+\alpha)T}{1-\alpha}} \left[\mathbb{E}\left|Y_{T_{p-1}}\right|^2 + \mathbb{E}\int_t^T \left(\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 \right.\right.\\
&\qquad\qquad \left.\left. + \left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2 \right) ds + \frac{1}{|\beta|}\mathbb{E}\int_t^T |h(s,0)|^2\,dA_s \right],\\
&M_p = 2\mu_0^p = 2e^{\frac{(2C+\alpha)T}{1-\alpha}} \left[\mathbb{E}\left|Y_{T_{p-1}}\right|^2 + \mathbb{E}\int_0^T \left(\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 \right.\right.\\
&\qquad\qquad \left.\left. + \left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2 \right) ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T |h(s,0)|^2\,dA_s \right],\\
&\text{ and } \quad \int_{T_p}^{T_{p-1}} \rho(s,M_p)\,ds = \frac{\mu_0^p}{M}.
\end{aligned}
$$

Therefore, by iteration, we deduce the existence of a solution to BDSDE (2.1) on $[T_p,T]$.
Finally, setting

$$
A = 2e^{\frac{(2C+\alpha)T}{1-\alpha}} \left[\mathbb{E}\int_0^T \left(\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + \left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2 \right) ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T |h(s,0)|^2\,dA_s \right]
$$

and using the same argument as in [9], we prove the existence of a finite $p \geq 1$ such that $T_p = 0$. Thus, we obtain the existence of the solution on $[0,T]$.

Uniqueness. Let $(Y,Z)\ ,(Y',Z') \in \mathcal{S}^2([0,T];\mathbb{R}^k)\times\mathcal{M}^2(0,T;\mathbb{R}^{k\times d})$ be two solutions of BDSDE (2.1).
Let $\theta > 0$. By virtue of Itô's formula, we have

$$
\begin{aligned}
&\mathbb{E}|Y_t - Y_t'|^2 e^{\theta t} + \theta\mathbb{E}\int_t^T |Y_s - Y_s'|^2 e^{\theta s} ds + \mathbb{E}\int_t^T \|Z_s - Z_s'\|^2 e^{\theta s} ds\\
&= 2\mathbb{E}\int_t^T \left\langle Y_s - Y_s',\, f(s,Y_s,Z_s) - f(s,Y_s',Z_s') \right\rangle e^{\theta s} ds + 2\mathbb{E}\int_t^T \left\langle Y_s - Y_s', h(s,Y_s) - h(s,Y_s') \right\rangle e^{\theta s} dA_s\\
&+\mathbb{E}\int_t^T |g(s,Y_s,Z_s) - g(s,Y_s',Z_s')|^2 e^{\theta s} ds.
\end{aligned}
$$

By virtue of the assumption **(H3)**, **(H4)** and Young's inequality $2ab \leq \frac{1}{\theta}a^2 + \theta b^2$, we derive

$$
\begin{aligned}
&\mathbb{E}|Y_t - Y_t'|^2 e^{\theta t} + (1-\alpha-\frac{1}{\theta}C)\mathbb{E}\int_t^T \|Z_s - Z_s'\|^2 e^{\theta s} ds + 2|\beta|\mathbb{E}\int_t^T |Y_s - Y_s'|^2 e^{\theta s} dA_s\\
&\leq \left(\frac{1}{\theta}+1\right)\mathbb{E}\int_t^T \rho(s,|Y_s - Y_s'|^2) e^{\theta s} ds.
\end{aligned}
$$

Choosing $\theta > \frac{C}{1-\alpha}$ and noting that $1 \leq e^{\theta t} \leq e^{\theta T},\ \forall\, t \in [0,T]$, we get

$$
\begin{aligned}
&\mathbb{E}|Y_t - Y_t'|^2 + (1-\alpha-\frac{1}{\theta}C)\mathbb{E}\int_t^T \|Z_s - Z_s'\|^2 ds + 2|\beta|\mathbb{E}\int_t^T |Y_s - Y_s'|^2 dA_s \qquad (3.6)\\
&\leq \left(\frac{1}{\theta}+1\right) e^{\theta T}\mathbb{E}\int_t^T \rho(s,|Y_s - Y_s'|^2) ds.
\end{aligned}
$$

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Therefore

$$\mathbb{E}|Y_t - Y_t'|^2 \leq \left(\frac{1}{\theta}+1\right) e^{\theta T} \int_t^T \rho(s, \mathbb{E}|Y_s - Y_s'|^2) ds.$$

In view of the comparison Theorem for ODE, we have

$$\mathbb{E}|Y_t - Y_t'|^2 \leq r(t), \ \ \forall\, t \in [0,T],$$

where $r(t)$ is the maximum left shift solution of the following equation:

$$\left\{ \begin{array}{rcl} u' & = & -(\frac{1}{\theta}+1)e^{\theta T}\rho(t,u); \\ u(T) & = & 0. \end{array} \right.$$

By virtue of the assumption **(H3)**, $r(t) = 0, t \in [0,T]$. Thus $\mathbb{E}|Y_t - Y_t'|^2 = 0, \ t \in [0,T]$, this means $Y_t = Y_t'$, a.s.. It then follows from (3.6) that $Z_t = Z_t'$, a.s., for any $t \in [0,T]$. □

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