Reflected generalized backward doubly SDEs driven by Lévy processes and Applications

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Abstract

In this paper, a class of reflected generalized backward doubly stochastic differential equations (reflected GBDSDEs in short) driven by Teugels martingales associated with Lévy process and the integral with respect to an adapted continuous increasing process is investigated. We obtain the existence and uniqueness of solutions to these equations. A probabilistic interpretation for solutions to a class of reflected stochastic partial differential integral equations (PDIEs in short) with a nonlinear Neumann boundary condition is given.

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1 Introduction

Backward stochastic differential equations (BSDEs, in short) have been first introduced by Pardoux and Peng [13] in order to give a probabilistic interpretation (Feynman-Kac formula) for the solutions of semilinear parabolic PDEs, one can see Peng [17], Pardoux and Peng [14]. Recently, a new class of BSDEs, named backward doubly stochastic differential equations (BDSDEs in short) has been introduced by Pardoux and Peng [15] in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs in short). Following it, Bally and Matoussi [1] gave the probabilistic representation of the weak solutions to parabolic semilinear SPDEs in Sobolev spaces by means of BDSDEs. Furthermore, Pardoux and Zhang [16] gave a probabilistic formula for the viscosity solution of a system of PDEs with a nonlinear Neumann boundary condition by introducing a generalized BSDEs (GBSDEs, in short) which involved an integral with respect to an adapted continuous increasing process. Its extension to an obstacle problem for PDEs with a nonlinear Neumann boundary condition was given in Ren and Xia [20] by reflected GBSDEs. Motivated by the above works, especially by [15] and [16], Boufoussi

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et al. [3] recommended a class of generalized BDSDEs (GBDSDEs in short) and gave the probabilistic representation for stochastic viscosity solutions of semi-linear SPDEs with a Neumann boundary condition. The main tool in the theory of BSDEs is the martingale representation theorem, which is well known for martingale which adapted to the filtration of the Brownian motion or that of Poisson point process (Pardoux and Peng [13], Tang and Li [21]) or that of a Poisson random measure (see Ouknine [12]). Recently, Nualart and Schoutens [10] gave a martingale representation theorem associated to Lévy process. Furthermore, they showed the existence and uniqueness of solutions to BSDEs driven by Teugels martingales associated with Lévy process with moments of all orders in [11]. The results were important from a pure mathematical point of view as well as in the world of finance. It could be used for the purpose of option pricing in a Lévy market and related PDEs which provided an analogue of the famous Black-Scholes formula. Further, Hu and Yong considered respectively BDSDEs and generalized BDSDE driven by Lévy processes and its applications in [8] and [19].

Motivated by the above works, especially by [19] the purpose of the present paper is to consider reflected GBDSDEs driven by Lévy processes of the kind considered in Nualart and Schoutens [10]. Our aim is to give a probabilistic interpretation for the solutions to a class of reflected stochastic PDIEs with a nonlinear Neumann boundary condition.

The paper is organized as follows. In Section 2, we introduce some preliminaries and notations. Section 3 is devoted to GBDSDEs driven by Lévy processes and the comparison theorem related to it. In Section 4, we give existence and uniqueness result for the reflected GBDSDE. Finally Section 5 point out a probabilistic interpretation of solutions to a class of reflected stochastic PDIEs with a nonlinear Neumann boundary condition.

2 Preliminaries and Notations

The scalar product of the space $\mathbb{R}^d (d \geq 2)$ will be denoted by $< . >$ and the associated Euclidian norm by $\|.\|$.

In what follows let us fix a positive real number $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, L_t : t \in [0,T])$ be a complete Wiener-Lévy space in $\mathbb{R} \times \mathbb{R} \backslash \{0\}$, with Levy measure ν, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\{\mathcal{F}_t : t \in [0,T]\}$ is a right-continuous increasing family of complete sub σ-algebras of $\mathcal{F}$, $\{B_t : t \in [0,T]\}$ is a standard Wiener process in $\mathbb{R}$ with respect to $\{\mathcal{F}_t : t \in [0,T]\}$ and $\{L_t : t \in [0,T]\}$ is a $\mathbb{R}$-valued Lévy process independent of $\{B_t : t \in [0,T]\}$, which has only m jumps size and no continuous part and corresponding to a standard Lévy measure ν satisfying the following conditions: $\int_{\mathbb{R}} (1 \wedge y) \nu(dy) < \infty$,

Let $\mathcal{N}$ denote the totality of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define that

$$\mathcal{F}_t = \mathcal{F}_t^L \vee \mathcal{F}_{t,T}^B$$

where for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^{\eta} = \sigma(\eta_r - \eta_s, s \leq r \leq t) \vee \mathcal{N}$, $\mathcal{F}_t^{\eta} = \mathcal{F}_{0,t}^{\eta}$.

Let us remark that the collection $\mathbf{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$ is neither increasing nor decreasing and it does not constitute a filtration.

We denote by $(H^{(i)})_{i \geq 1}$ the Teugels Martingale associated with the Lévy process $\{L_t : t \in [0,T]\}$. More precisely

$$H^{(i)} = c_{i,i} Y^{(i)} + c_{i,i-1} Y^{(i-1)} + \cdots + c_{i,1} Y^{(1)}$$

where $Y_t^{(i)} = L_t^i - \mathbb{E}(L_t^i) = L_t^i - t\mathbb{E}(L_t^1)$ for all $i \geq 1$ and L_t^i are power-jump processes. That is $L_t^1 = L_t$ and $L_t^i = \sum_{0<s<t}(\Delta L_s)^i$ for all $i \geq 2$, where $X_{t^-} = \lim_{s \nearrow t} X_s$ and $\Delta X_t = X_t - X_{t^-}$. It was shown in Nualart and Schoutens [10] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = x^2 d\nu(x) + \sigma^2 \delta_0(dx)$:

$$q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x^1.$$

The martingale $(H^{(i)})_{i \geq 1}$ can be chosen to be pairwise strongly orthonormal martingale.

Remark 2.1. 1. If μ only has mass at 1, we are in the Poisson case; here $H_t^{(i)} = 0, i = 2, \cdots$. This case is degenerate in this Lévy framework

2. Generally, if the Lévy process L has only m different jump sizes, then

$(i)\, H^{(k)} = 0, \forall\, k \geq m+1$, if L has no continuous part;

$(ii)\, H^{(k)} = 0, \forall\, k \geq m+2$, if L has continuous part.

In the sequel, let $\{A_t,\ 0 \leq t \leq T\}$ be a continuous, increasing and $\mathbf{F}$-adapted real valued with bounded variation on $[0,T]$ such that $A_0 = 0$.

For any $d \geq 1$, we consider the following spaces of processes:

1. $\mathcal{M}^2(\mathbb{R}^d)$ denote the space of real valued, square integrable and $\mathcal{F}_t$-predictable processes $\varphi = \{\varphi_t; t \in [0,T]\}$ such that

   $\|\varphi\|^2_{\mathcal{M}^2} = \mathbb{E}\int_0^T \|\varphi_t\|^2 dt < \infty.$

2. $\mathcal{S}^2(\mathbb{R})$ is the subspace of $\mathcal{M}^2(\mathbb{R})$ formed by the $\mathcal{F}_t$-adapted processes $\varphi = \{\varphi_t; t \in [0,T]\}$ right continuous with left limit (rcll) such that

$$\|\varphi\|^2_{\mathcal{S}^2} = \mathbb{E}\left(\sup_{0\leq t\leq T}|\varphi_t|^2 + \int_0^T |\varphi_t|^2 dA_t\right) < \infty.$$

3. $\mathcal{A}^2(\mathbb{R})$ is the set of $\mathcal{F}_t$-measurable, continuous, real-valued, increasing process $\varphi = \{\varphi_t; t \in [0,T]\}$ such that $K_0 = 0$, $\mathbb{E}|K_T|^2 < \infty$

Finally we denote by $\mathcal{E}^{2,m} = \mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m) \times \mathcal{A}^2(\mathbb{R})$ endowed with the norm

$$\|(Y,Z,K)\|^2_{\mathcal{E}} = \mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Y_t|^2 dA_t + \int_0^T \|Z_t\|^2 dt + |K_T|^2\right).$$

Then, the couple $(\mathcal{E}^{2,m}, \|.\|_{\mathcal{E}^{2,m}})$ is a Banach space.

To end this section, let us give following needed assumptions

(**H1**) ξ is a square integrable random variable which is $\mathcal{F}_T$-measurable such that for all $\mu > 0$

$$\mathbb{E}\left(e^{\mu A_T}|\xi|^2\right) < \infty.$$

(**H2**) $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$, $g:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$, and $\phi:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$, are three functions such that:

(a) There exist $\mathcal{F}_t$-adapted processes $\{f_t,\phi_t,g_t : 0\le t\le T\}$ with values in $[1,+\infty)$ and with the property that for any $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, and $\mu>0$, the following hypotheses are satisfied for some strictly positive finite constant K:

$$\begin{cases} f(t,y,z),\ \phi(t,y),\ \text{and}\, g(t,y,z)\ \text{are}\ \mathcal{F}_t\text{-measurable processes}, \\ |f(t,y,z)|\le f_t+K(|y|+\|z\|), \\ |\phi(t,y)|\le\phi_t+K|y|, \\ |g(t,y)|\le g_t+K|y|, \\ \mathbb{E}\left(\int_0^T e^{\mu A_t}f_t^2dt+\int_0^T e^{\mu A_t}g_t^2dt+\int_0^T e^{\mu A_t}\phi_t^2dA_t\right)<\infty. \end{cases}$$

(b) There exist constants $c>0,\beta<0$ and $0<\alpha<1$ such that for any $(y_1,z_1),\,(y_2,z_2)\in\mathbb{R}\times\mathbb{R}^m$,

$$\begin{cases} (i)\,|f(t,y_1,z_1)-f(t,y_2,z_2)|^2\le c(|y_1-y_2|^2+\|z_1-z_2\|^2), \\ (ii)\,|g(t,y_1)-g(t,y_2)|^2\le c|y_1-y_2|^2, \\ (iii)\ \langle y_1-y_2,\phi(t,y_1)-\phi(t,y_2)\rangle\le\beta|y_1-y_2|^2. \end{cases}$$

(H3) The obstacle $\{S_t, 0\le t\le T\}$, is a $\mathcal{F}_t$-progressively measurable real-valued process satisfying

$$E\left(\sup_{0\le t\le T}|S_t^+|^2\right)<\infty.$$

We shall always assume that $S_T\le\xi$ $a.s.$

3 Generalized backward doubly stochastic differential equations driven by Lévy processes

In this section, we present existence and uniqueness results for GBDSDEs driven by Lévy processes and we prove a comparison theorem which is an important tool in the proofs for results of Sections 4. The existence and uniqueness result is a direct consequence of Theorem 3.2 in [8].

Proposition 3.1. *Given standard parameter* (ξ,f,ϕ,g), *there exists* $(Y,Z)\in\mathcal{S}^2(\mathbb{R})\times\mathcal{M}^2(\mathbb{R}^m)$ *to the following GBDSDEs driven by the Lévy processes*

$$\begin{aligned} Y_t &= \xi+\int_t^T f(s,Y_{s^-},Z_s)ds+\int_t^T\phi(s,Y_{s^-})dA_s+\int_t^T g(s,Y_{s^-})\,dB_s \\ &\quad -\sum_{i=1}^m\int_t^T Z_s^{(i)}dH_s^{(i)},\ 0\le t\le T. \end{aligned} \tag{3.1}$$

Here the integral with respect to $\{B_t\}$ is the classical backward Itô integral (see Kunita [7]) and the integral with respect to $\{H_t^{(i)}\}$ is a standard forward Itô-type semimartingale integral.

The comparison theorem is one of the principal tools in the theories of the BSDEs. But it does not hold in general for solutions of BSDEs with jumps (see the counter-example in Barles et al. [2]). In the following we prove, with the additional property of the jumps size, the comparison theorem for solution of GBDSDEs driven by Lévy processes. Let note that in the standard BSDE case i.e $g=\phi=0$, comparison theorem has already been established by Qing Zhou [18] with this property of jumps size.

Theorem 3.2. *Assume that L has only n different jump sizes and has no continuous part. Let (ξ^1, f^1, ϕ, g) and (ξ^2, f^2, ϕ, g) be two standard parameters of BSDE (4.2) and let (Y^1, Z^1) and (Y^2, Z^2) be the associated square-integrable solutions. Suppose that*

1. $\xi^1 \geq \xi^2$, $\mathbb{P}$ *a.s.,*
2. $f^1(t,y,z) \geq f^2(t,y,z)$, $\mathbb{P}$ *a.s. for all* $y \in \mathbb{R}$, $z \in \mathbb{R}^m$*,*
3. $\beta_t^i = \dfrac{f(t, Y_{t^-}^2, \tilde{Z}_t^{(i-1)}) - f^2(t, Y_{t^-}^2, \tilde{Z}_t^{(i)})}{Z_t^{1(i)} - Z_t^{2(i)}} \mathbf{1}_{\{Z_t^{1(i)} - Z_t^{2(i)} \neq 0\}}$

where

$$\begin{aligned}
\tilde{Z}^i &= \left(Z^{2(1)}, Z^{2(2)}, \cdots, Z^{2(i)}, Z^{1(i+1)}, \cdots, Z^{1(n)}\right) \\
\tilde{Z}^{i-1} &= \left(Z^{2(1)}, Z^{2(2)}, \cdots, Z^{2(i-1)}, Z^{1(i)}, Z^{1(i+1)}, \cdots, Z^{1(n)}\right),
\end{aligned}$$

satisfying that $\sum_{i=1}^{m} \beta_t^i \Delta H_t^i > -1$, $dt \otimes d\mathbb{P}$ a.s. Then we have that almost surely for any time t, $Y_t^1 \geq Y_t^2$ and that if $\mathbb{P}(\xi^1 > \xi^2) > 0$ then $\mathbb{P}(Y_t^1 > Y_t^2) > 0$.

Proof. Denote

$$\begin{aligned}
\hat{\xi} &= \xi^1 - \xi^2, \quad \hat{Y}_t = Y_t^1 - Y_t^2, \quad \hat{V}_t = V_t^1 - V_t^2, \quad \hat{Z}_t = Z_t^1 - Z_t^2 \\
\hat{f}_t &= f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2),
\end{aligned}$$

and

$$\begin{aligned}
a_t &= [f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^1)]/(Y_t^1 - Y_t^2)\mathbf{1}_{\{Y_t^1 \neq Y_t^2\}}, \\
b_t &= [\phi(t, Y_t^1) - \phi(t, Y_t^2,)]/(Y_t^1 - Y_t^2)\mathbf{1}_{\{Y_t^1 \neq Y_t^2\}}, \\
c_t &= [g(t, Y_t^1) - g(t, Y_t^2)]/(Y_t^1 - Y_t^2)\mathbf{1}_{\{Y_t^1 \neq Y_t^2\}}.
\end{aligned}$$

Then

$$\hat{Y}_t = \hat{\xi} + \int_t^T [a_s \hat{Y}_{s^-} + \sum_{i=1}^m \beta_s^i \hat{Z}_s^{(i)} + \hat{f}_s] ds + \int_t^T b_s \hat{Y}_{s^-} dA_s + \int_t^T c_s \hat{Y}_{s^-} dB_s - \sum_{i=1}^m \int_t^T \hat{Z}_s^{(i)} dH_s^{(i)}$$

is a linear GBDSDE driven the Lévy processes.

Let $\Gamma_t = 1 + \int_0^t \Gamma_{s^-} dX_s$, where

$$X_t = \int_0^t a_s ds + \int_0^t b_s dA_s + \int_0^t c_s dB_s - \int_0^t |c_s|^2 ds + \sum_{i=1}^m \int_0^t \beta_s^i dH_s^{(i)}.$$

Then we have $\Delta X_t = \sum_{i=1}^m \beta_t^i \Delta H_t^{(i)} > -1$. Note that $|a_t| \le C$, $|b_t| \le C$, $|c_t| \le C$, $|\beta_t^i| \le C$, for all $0 \le t \le T$, a.s., $i = 1, \cdots, m$. Then by the Doléans-Dade exponential formula and the Gronwall inequality, we conclude that $\Gamma_t > 0$ and $\sup_{0\le t\le T} \mathbb{E}[\Gamma_t^2] \le C_1$. Thus, $\mathbb{E}[\int_0^T \Gamma_{s^-}^2 ds] \le C_1$, where C_1 is a positive constant. Then applying Itô's formula to $\Gamma_s \hat{Y}_s$ from $s = t$ to $s = T$, it follows that

$$\begin{aligned}
\Gamma_T \hat{\xi} - \Gamma_t \hat{Y}_t &= \int_t^T \Gamma_{s^-} \hat{Y}_{s^-} d\hat{Y}_s + \int_t^T \hat{Y}_{s^-} \Gamma_{s^-} d\Gamma_s + \int_t^T d[\Gamma, \hat{Y}]_s \\
&= -\int_t^T \Gamma_{s^-}[a_s \hat{Y}_{s^-} + \sum_{i=1}^m \beta_s^i \hat{Z}_s^{(i)} + \hat{f}_s] ds - \int_t^T \Gamma_{s^-} b_s \hat{Y}_{s^-} dA_s - \int_t^T \Gamma_{s^-} \hat{Y}_{s^-} c_s dB_s \\
&\quad + \sum_{i=1}^m \int_t^T \Gamma_{s^-} \hat{Z}_s^{(i)} dH_s^{(i)} + \int_t^T a_s \hat{Y}_{s^-} \Gamma_{s^-} ds + \int_t^T \hat{Y}_{s^-} \Gamma_{s^-} b_s dA_s + \int_t^T \hat{Y}_{s^-} \Gamma_{s^-} c_s dB_s \\
&\quad + \sum_{i=1}^m \int_t^T \hat{Y}_{s^-} \Gamma_{s^-} \beta_s^i dH_s^{(i)} + \sum_{i=1}^m \sum_{j=1}^m \int_t^T \Gamma_{s^-} \beta_s^i \hat{Z}_s^{(j)} d[H^{(i)}, H^{(j)}]_s \\
&= -\int_t^T \Gamma_{s^-}[\sum_{i=1}^m \beta_s^i \hat{Z}_s^{(i)} + \hat{f}_s] ds + \sum_{i=1}^m \int_t^T \Gamma_{s^-} \hat{Z}_s^{(i)} dH_s^{(i)} + \sum_{i=1}^m \int_t^T \hat{Y}_{s^-} \Gamma_{s^-} \beta_s^i dH_s^{(i)} \\
&\quad + \sum_{i=1}^m \sum_{j=1}^m \int_t^T \Gamma_{s^-} \beta_s^i \hat{Z}_s^{(j)} d[H^{(i)}, H^{(j)}]_s. \qquad (3.2)
\end{aligned}$$

By Davis's inequality, we know that $\sum_{i=1}^m \int_t^T \Gamma_{s^-} \hat{Z}_s^{(i)} dH_s^{(i)}$ and $\sum_{i=1}^m \int_t^T \hat{Y}_{s^-} \Gamma_{s^-} \beta_s^i dH_s^{(i)}$ are martingales. Since $\sum_{i=1}^m \int_t^T \hat{Z}_s^{(i)} dH_s^{(i)}$ and $\sum_{i=1}^m \int_t^T \Gamma_{s^-} \beta_s^i dH_s^{(i)}$ are square integrable martingales, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^m \int_t^T \Gamma_{s^-} \beta_s^i \hat{Z}_s^{(j)} d[H^{(i)}, H^{(j)}]_s | \mathcal{F}_t\right] &= \mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^m \int_t^T \Gamma_{s^-} \beta_s^i \hat{Z}_s^{(j)} d\langle H^{(i)}, H^{(j)}\rangle_s | \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\sum_{i=1}^m \int_t^T \Gamma_{s^-} \beta_s^i \hat{Z}_s^{(i)} ds | \mathcal{F}_t\right].
\end{aligned}$$

Thus, from (3.2), and taking conditional expectation w.r.t. $\mathcal{F}_t$, we conclude that

$$\Gamma_t \hat{Y}_t = \mathbb{E}\left[\Gamma_T \hat{\xi} + \int_t^T \Gamma_{s^-} \hat{f}_s ds | \mathcal{F}_t\right] \ge 0.$$

It is clear that $\hat{Y}_t \ge 0$ and that if $\mathbb{P}(\hat{\xi} > 0) > 0$ then $\mathbb{P}(\hat{Y}_t > 0) > 0$. The proof of the theorem is complete. □

4 Reflected generalized backward doubly stochastic differential equation driven by Lévy processes

This section is devoted to the study of reflected GBDSDEs driven by the Lévy processes (4.1), one of our main goal in this paper. First of all let us give a definition to the solution of this reflected GBDSDEs driven by Lévy processes.

Definition 4.1. By a solution of the reflected GBDSDE (ξ, f, ϕ, g, S) driven by Lévy processes we mean a triplet of processes $(Y,Z,K) \in \mathcal{E}$, which satisfied

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_{s^-}, Z_s) ds + \int_t^T \phi(s, Y_{s^-}) dA_s + \int_t^T g(s, Y_{s^-}) \, dB_s \\ &\quad - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)} + K_T - K_t, \; 0 \le t \le T. \end{aligned} \tag{4.1}$$

such that the following holds $\mathbb{P}$-a.s

(*i*) $Y_t \ge S_t, \;\; 0 \le t \le T,$

(*ii*) $\displaystyle\int_0^T (Y_{t^-} - S_t) \, dK_t = 0.$

In the sequel, C denotes a finite constant which may take different values from line to line and usually is strictly positive.

Theorem 4.2. *Under the hypotheses* (**H1**), (**H2**) *and* (**H3**), *there exists a unique solution for the reflected generalized BDSDE* (ξ, f, ϕ, g, S) *driven by Lévy processes.*

Our proof is based on a penalization method from El Karoui et al [6].

For each $n \in \mathbb{N}^*$ we set

$$f_n(s,y,z) = f(s,y,z) + n(y - S_s)^- \tag{4.2}$$

and let (Y^n, Z^n) be the $\mathcal{F}_t$-progressively measurable process with values in $\mathbb{R} \times \mathbb{R}^m$ unique solution of the GBDSDE with (ξ, f_n, g) driven by the Lévy processes. It exists according to Proposition 3.1. So

$$\mathbb{E}\left(\sup_{0 \le t \le T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 \, ds \right) < \infty,$$

and

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_{s^-}^n, Z_s^n) ds + \int_t^T (Y_{s^-}^n - S_s)^- ds + \int_t^T \phi(s, Y_{s^-}^n) dA_s \\ &\quad + \int_t^T g(s, Y_{s^-}^n) \, dB_s - \sum_{i=1}^{m} \int_t^T (Z_s^n)^{(i)} dH_s^{(i)}. \end{aligned} \tag{4.3}$$

Set

$$K_t^n = n \int_0^t (Y_{s^-}^n - S_s)^- ds, \; 0 \le t \le T. \tag{4.4}$$

In order to prove Theorem 4.2, we state the following lemma that will be useful.

Lemma 4.3. *Let us consider* $(Y^n, Z^n) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m)$ *solution of GBDSDE* (4.3)*. Then there exists* $C > 0$ *such that,*

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\sup_{0\le t\le T}|Y_t^n|^2 + \int_t^T |Y_s^n|^2\,dA_s + \int_t^T \|Z_s^n\|^2\,ds + |K_T^n|^2\right) < C$$

Proof. From Itô's formula, we have

$$\begin{aligned} |Y_t^n|^2 &= |\xi|^2 + 2\int_t^T Y_{s^-}^n f(s,Y_{s^-}^n,Z_s^n)ds + 2\int_t^T Y_{s^-}^n \phi(s,Y_{s^-}^n)dA_s \\ &+ \int_t^T |g(s,Y_{s^-}^n)|^2 ds + 2\int_t^T Y_{s^-}^n dK_s^n + 2\int_t^T Y_{s^-}^n g(s,Y_{s^-}^n)dB_s \\ &- 2\sum_{i=1}^m \int_t^T Y_{s^-}^n (Z_s^n)^{(i)} dH_s^{(i)} - \sum_{i,j=1}^m \int_t^T (Z_s^n)^{(i)}(Z_s^n)^{(j)} d[H_s^{(i)},H_s^{(j)}]. \end{aligned} \tag{4.5}$$

Note that $\int_t^T Y_{s^-}^n g(s,Y_{s^-}^n)dB_s$, $\int_t^T Y_{s^-}^n (Z_s^n)^{(i)} dH_s^{(i)}$, for $i \ge 1$ and $\int_t^T (Z_s^n)^{(i)}(Z_s^n)^{(j)} d[H_s^{(i)},H_s^{(j)}]$ for $i \neq j$ are uniformly integrable martingales. Taking the expectation, we get

$$\begin{aligned} &\mathbb{E}\,|Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds \\ &\le |\xi|^2 + 2\mathbb{E}\int_t^T Y_{s^-}^n f(s,Y_{s^-}^n,Z_s^n)ds + 2\mathbb{E}\int_t^T Y_{s^-}^n \phi(s,Y_{s^-}^n)dA_s \\ &+ \mathbb{E}\int_t^T |g(s,Y_{s^-}^n)|^2 ds + 2\mathbb{E}\int_t^T Y_{s^-}^n dK_s^n, \end{aligned}$$

where we have used $\int_t^T (Y_{s^-}^n - S_s)dK_s^n \le 0$ and the fact that

$$\int_t^T Y_s^n dK_s^n \;=\; \int_t^T (Y_{s^-}^n - S_s)dK_s^n + \int_t^T S_s dK_s^n \le \int_t^T S_s dK_s^n.$$

Using (**H2**) and the elementary inequality $2ab \le \gamma a^2 + \frac{1}{\gamma}b^2,\ \forall\gamma > 0$,

$$\begin{aligned} 2Y_s^n f(s,Y_s^n,Z_s^n) &\le (c\gamma_1 + \frac{1}{\gamma_1})|Y_s^n|^2 + 2c\gamma_1\|Z_s^n\|^2 + 2\gamma_1 f_s^2, \\ 2Y_s^n \phi(s,Y_s^n) &\le (\gamma_2 - 2|\beta|)|Y_s^n|^2 + \frac{1}{\gamma_2}\phi_s^2, \\ |g(s,Y_s^n)|^2 &\le 2c|Y_s^n|^2 + 2g_s^2. \end{aligned}$$

Taking expectation in both sides of the inequality (4.5) and choosing $\gamma_1 = \dfrac{1}{4c}$, $\gamma_2 = |\beta|$, we obtain for all $\varepsilon > 0$

$$\begin{aligned} &\mathbb{E}\,|Y_t^n|^2 + |\beta|\mathbb{E}\int_t^T |Y_s^n|^2\,dA_s + \frac{1}{2}\mathbb{E}\int_t^T \|Z_s^n\|^2\,ds \\ &\le C\mathbb{E}\left\{|\xi|^2 + \int_0^t |Y_s^n|^2 ds + \int_0^t f_s^2 ds + \int_0^t \phi_s^2 dA_s + \int_0^t g_s^2 ds\right\} \\ &+ \frac{1}{\varepsilon}\mathbb{E}\left(\sup_{0\le s\le t}(S_s^+)^2\right) + \varepsilon\mathbb{E}\,(K_T^n - K_t^n)^2. \end{aligned} \tag{4.6}$$

On the other hand, we get from (4.3) that for all $0 \leq t \leq T$,

$$K_t^n = Y_t^n - \xi - \int_0^t f(s, Y_{s^-}^n, Z_s^n)ds - \int_0^t \phi(s, Y_{s^-}^n)dA_s - \int_0^t g(s, Y_{s^-}^n)dB_s + \sum_{i=1}^m \int_0^t (Z_s^n)^{(i)} dH_s^{(i)}. \tag{4.7}$$

So by used standard computations, we get

$$\begin{aligned}\mathbb{E}(K_T^n - K_t^n)^2 \leq C\mathbb{E}\Bigg\{ &|\xi|^2 + \int_0^t f_s^2 ds + \int_0^t \phi_s^2 dA_s + \int_0^t g_s^2 ds + \int_0^t |Y_s^n|^2 ds \\ &+ \mathbb{E}\left(\sup_{0\leq s\leq t} (S_s^+)^2 \right) + \int_0^t |Y_s^n|^2 dA_s + \int_0^t \|Z_s^n\|^2 ds \Bigg\}. \end{aligned} \tag{4.8}$$

Substituting Equation (4.8) to Equation (4.6) and choosing ε small enough such that $\varepsilon C < \min(1/2, |\beta|)$, yields

$$\begin{aligned}&\mathbb{E}\left\{ |Y_t^n|^2 + \int_t^T |Y_s^n|^2 dA_s + \int_t^T \|Z_s^n\|^2 ds + |K_T^n|^2 \right\} \\ &\leq C\mathbb{E}\left\{ |\xi|^2 + \int_0^T f_s^2 ds + \int_0^T \phi_s^2 dA_s + \int_0^T g_s^2 ds + \sup_{0\leq t\leq T} (S_t^+)^2 \right\}.\end{aligned}$$

From this, Gronwall's inequality and the Burkholder-Davis-Gundy inequality [4], we get

$$\begin{aligned}\mathbb{E}\left\{ \sup_{0\leq t\leq T} |Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds + |K_T^n|^2 \right\} \quad \leq \quad & C\mathbb{E}\Bigg\{ |\xi|^2 + \int_0^T f_s^2 ds + \int_0^T \phi_s^2 dA_s \\ & + \int_0^T g_s^2 ds + \sup_{0\leq t\leq T} (S_t^+)^2 \Bigg\},\end{aligned}$$

which end the proof of this Lemma. □

Proof of Theorem 4.2. **Existence** The proof of existence will be divided in two steps.
Step 1. g does not dependent on (Y, Z). More precisely, we consider the following equation

$$\begin{aligned}Y_t \quad = \quad & \xi + \int_t^T f(s, Y_{s^-}, Z_s)ds + \int_t^T \phi(s, Y_{s^-})dA_s + \int_t^T g(s)\, dB_s \\ & - \sum_{i=1}^m \int_t^T Z_s^{(i)} dH_s^{(i)} + K_T - K_t,\ 0 \leq t \leq T.\end{aligned} \tag{4.9}$$

The penalized equation is given by

$$\begin{aligned}Y_t^n \quad = \quad & \xi + \int_t^T f(s, Y_{s^-}^n, Z_s^n)ds + n\int_t^T (Y_{s^-}^n - S_s)^- ds + \int_t^T \phi(s, Y_{s^-}^n)dA_s \\ & + \int_t^T g(s)\, dB_s - \sum_{i=1}^m \int_t^T (Z_s^n)_s^{(i)} dH_s^{(i)},\ 0 \leq t \leq T.\end{aligned} \tag{4.10}$$

Since the sequence of functions $(y \mapsto n(y - S_t)^-)_{n\geq 1}$ is nondecreasing, then thanks to the comparison theorem 3.2, the sequence $(Y^n)_{n>0}$ is non-decreasing. Hence, Lemma 4.3 implies that there exists a $\mathcal{F}_t$- progressively measurable process Y such that $Y_t^n \nearrow Y_t$ *a.s.* Recall that $Y_t^n \nearrow Y_t$ *a.s.* Then, Fatou's lemma and Lemma 4.3 ensure

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_t|^2\right) < +\infty,$$

It then follows from Lemma 4.3 and Lebegue's dominated convergence theorem that

$$\mathbb{E}\left(\int_0^T |Y_s^n - Y_s|^2\,ds\right) \longrightarrow 0, \text{ as } n \to \infty. \tag{4.11}$$

Next, for $n \ge p \ge 1$, by Itô's formula and together with assumptions (**H2**), yields

$$\begin{aligned}
&\mathbb{E}\left\{\left|Y_t^n - Y_t^p\right|^2 + \int_t^T |Y_s^n - Y_s^p|^2\,dA_s + \int_t^T \|Z_s^n - Z_s^p\|^2\,ds\right\}\\
\le\ & C\mathbb{E}\left\{\int_t^T |Y_s^n - Y_s^p|^2 ds + \sup_{0\le s\le T}(Y_s^n - S_s)^- K_T^p + \sup_{0\le s\le T}(Y_s^p - S_s)^- K_T^n\right\},
\end{aligned}$$

which, by Gronwall lemma, Hölder inequality and Lemma 4.3 respectively, implies

$$\begin{aligned}
\mathbb{E}\left\{\left|Y_t^n - Y_t^p\right|^2 + \int_t^T \|Z_s^n - Z_s^p\|^2\,ds\right\} \ \le\ & C\left\{\mathbb{E}\left(\sup_{0\le s\le T}|(Y_s^n - S_s)^-|^2\right)\right\}^{1/2}\\
&+C\left\{\mathbb{E}\left(\sup_{0\le s\le T}|(Y_s^p - S_s)^-|^2\right)\right\}^{1/2}
\end{aligned} \tag{4.12}$$

Let us admit for the moment the following result.

Lemma 4.4. *If g does not dependent on (Y,Z), then for each $n \in \mathbb{N}^*$,*

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|(Y_t^n - S_t)^-\right|^2\right) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

We can now conclude. Indeed, it follows from Lemma 4.4 that,

$$\mathbb{E}\left\{|Y_s^n - Y_s^p|^2 + \int_t^T \|Z_s^n - Z_s^p\|^2\,ds\right\} \longrightarrow 0, \text{ as } n,p \longrightarrow \infty.$$

Finally, from Burkhölder-Davis-Gundy's inequality, we obtain

$$\mathbb{E}\left(\sup_{0\le s\le T}|Y_s^n - Y_s^p|^2 + \int_t^T \|Z_s^n - Z_s^p\|^2\,ds\right) \longrightarrow 0, \text{ as } n,p \longrightarrow \infty,$$

and from (4.7) we can deduce

$$\mathbb{E}\left\{\sup_{0\le s\le T}|K_s^n - K_s^p|^2\right) \longrightarrow 0, \text{ as } n,p \to \infty,$$

which provides that the sequence of processes (Y^n, Z^n, K^n) is Cauchy in the Banach space $\mathcal{E}^{2,m}$. Consequently, there exists a triplet $(Y,Z,K) \in \mathcal{E}^{2,m}$ such that

$$\mathbb{E}\left\{\sup_{0\le s\le T}|Y_s^n - Y_s|^2 + \int_t^T \|Z_s^n - Z_s\|^2\,ds + \sup_{0\le s\le T}|K_s^n - K_s|^2\right) \to 0, \text{ as } n \to \infty.$$

It remains to show that (Y,Z,K) solves the reflected GBDSDE driven by Lévy processes (ξ,f,ϕ,g,S). In this fact, since $(Y_t^n,K_t^n)_{0\le t\le T}$ tends to $(Y_t,K_t)_{0\le t\le T}$ uniformly in t in probability, the measure dK^n converges to dK weakly in probability, so that $\int_t^T(Y_{s^-}^n-S_s)dK_s^n\to\int_t^T(Y_{s^-}-S_s)dK_s$ in probability as $n\to\infty$. Obviously, $\int_t^T(Y_{s^-}-S_s)dK_s\ge 0$, while, on the other hand, for all $n\ge 0$, $\int_t^T(Y_{s^-}^n-S_s)dK_s^n\le 0$.

Hence

$$\int_t^T(Y_{s^-}-S_s)dK_s=0,\ \ a.s$$

Finally, passing to the limit in (4.10) we proved that (Y,Z,K) verifies (4.9) and is the solution of the reflected GBDSDE (ξ,f,g,S) driven by the Lévy processes. We finally return to the proof of Lemma 2.2.

Proof of Lemma 4.4. Since $Y_t^n\ge Y_t^0$, we can w.l.o.g. replace S_t by $S_t\vee Y_t^0$, i.e. we may assume that $\mathbb{E}(\sup_{0\le t\le T}S_t^2)<\infty$. We want to compare a.s. Y_t and S_t for all $t\in[0,T]$. In this, let us introduce the following processes

$$\begin{cases}\overline{\xi}:=\xi+\int_t^T g(s)\,dB_s\\ \overline{S}_t:=S_t+\int_t^T g(s)\,dB_s\\ \overline{Y}_t^n:=Y_t^n+\int_t^T g(s)\,dB_s.\end{cases}$$

Hence,

$$\overline{Y}_t^n=\overline{\xi}+\int_t^T f(s,Y_{s^-}^n,Z_s^n)\,ds+n\int_t^T\left(\overline{Y}_{s^-}^n-\overline{S_s}\right)^- ds+\int_t^T\phi(s,Y_{s^-}^n)\,dA_s-\sum_{i=1}^m\int_t^T(Z_s^n)^{(i)}dH_s^{(i)}. \tag{4.13}$$

and we define $\quad \overline{Y}_t:=\sup_n\overline{Y}_t^n$.

From Theorem 3.2, we have that a.s., $\overline{Y}_t^n\ge\widetilde{Y}_t^n$, $0\le t\le T$, $n\in\mathbb{N}^*$, where $\left\{(\widetilde{Y}_t^{\,n},\widetilde{Z}_t^n),\ 0\le t\le T\right\}$ is the unique solution of the GBDSDE driven by the Lévy processes

$$\widetilde{Y}_t^n\ =\ \overline{S}_T+\int_t^T f(s,Y_{s^-}^n,Z_s^n)\,ds+n\int_t^T(\overline{S}_s-\widetilde{Y}_{s^-}^n)ds+\int_t^T\phi(s,Y_{s^-}^n)\,dA_s-\sum_{i=1}^m\int_t^T(\widetilde{Z}_s^n)^{(i)}dH_s^{(i)}.$$

Now let $\mathbf{G}=(\mathcal{G}_t)_{0\le t\le T}$ be a filtration defined by $\mathcal{G}_t=\mathcal{F}_t^L\vee\mathcal{F}_T^B$ and ν a $\mathbf{G}$-stopping time such that $0\le\nu\le T$. Then, applying It formula to $\widetilde{Y}_t^n e^{-n(t-\nu)}$, we have

$$\begin{aligned}\widetilde{Y}_\nu^n\ =\ \mathbb{E}\Big\{&e^{-n(T-\nu)}\overline{S}_T+\int_\nu^T e^{-n(\nu-s)}f(s,Y_{s^-}^n,Z_s^n)ds+n\int_\nu^T e^{-n(\nu-s)}\overline{S}_s ds\\ &+\int_\nu^T e^{-n(\nu-s)}\phi(s,Y_{s^-}^n)dA_s\mid\mathcal{G}_\nu\Big\}.\end{aligned}$$

It is easily seen that

$$e^{-n(T-\nu)}\overline{S}_T+n\int_\nu^T e^{-n(s-\nu)}\overline{S}_s ds\to\overline{S}_\nu\mathbf{1}_{\{\nu<T\}}+\overline{S}_T\mathbf{1}_{\{\nu=T\}}\ a.s.,\text{ and in }L^2(\Omega)\text{ as }n\to\infty,$$

and the conditional expectation converges also in $L^2(\Omega)$. Moreover, we get

$$\mathbb{E}\left(\int_\nu^T e^{-n(s-\nu)} f(s,Y^n_{s^-},Z^n_s)ds\right)^2 \quad\leq\quad \frac{C}{2n}\mathbb{E}\left(\int_0^T (f_s^2+|Y^n_s|^2+\|Z^n_s\|^2)ds\right),$$

and

$$\mathbb{E}\left(\int_\nu^T e^{-n(s-\nu)}\phi(s,Y^n_s)dA_s\right)^2 \quad\leq\quad \mathbb{E}\left[|A_T|\left(\int_0^T(\phi_s^2+K^2|Y^n_s|^2)dA_s\right)\right]<C,$$

which provide

$$\mathbb{E}\left(\int_\nu^T e^{-n(\nu-s)} f(s,Y^n_{s^-},Z^n_s)ds+\int_\nu^T e^{-n(s-\nu)}\phi(s,Y^n_s)dA_s|\mathcal{G}_\nu\right)\longrightarrow 0$$

in $L^2(\Omega)$ as $n\to\infty$.

Consequently,

$$\widetilde{Y}^n_\nu \longrightarrow \overline{S}_\nu \mathbf{1}_{\{\nu<T\}}+\overline{S}_T\mathbf{1}_{\{\nu=T\}} \ \text{ in} L^2(\Omega), \text{ as } n\to\infty.$$

Therefore $Y_\nu\geq S_\nu$ a.s. From this and the section theorem [4], we deduce that $Y_t\geq S_t$ for all $t\in[0,T]$ and then

$$(Y^n_t-S_t)^-\searrow 0,\ \ 0\leq t\leq T,\ \ a.s.$$

Since $(Y^n_t-S_t)^-\leq(S_t-Y^0_t)^+\leq|S_t|+\left|Y^0_t\right|$and the result follows from the dominated convergence theorem. □

Step 2. The general case. In light of the above step, and for any $(\overline{Y},\overline{Z})\in\mathcal{S}^2(\mathbb{R})\times\mathcal{M}^2(\mathbb{R}^m)$, the reflected GBDSDE driven by Lévy processes

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T\phi(s,Y_s)dA_s+\int_t^T g(s,\overline{Y}_s)\,dB_s-\sum_{i=1}^m\int_t^T Z^{(i)}_s dH^{(i)}_s+K_T-K_t$$

has a unique solution (Y,Z,K). So, we can define the mapping

$$\begin{array}{rrcl}\Psi: & \mathcal{S}^2(\mathbb{R})\times\mathcal{M}^2(\mathbb{R}^m) & \longrightarrow & \mathcal{S}^2(\mathbb{R})\times\mathcal{M}^2(\mathbb{R}^m)\\ & (\overline{Y},\overline{Z}) & \longmapsto & (Y,Z)=\Psi(\overline{Y},\overline{Z}).\end{array}$$

Now, let (Y,Z), (Y',Z') in $\mathcal{S}^2(\mathbb{R})\times\mathcal{M}^2(\mathbb{R}^m)$ and $(\overline{Y},\overline{Z}),(\overline{Y'},\overline{Z'})$ in $\mathcal{S}^2(\mathbb{R})\times\mathcal{M}^2(\mathbb{R}^m)$ such that $(Y,Z)=\Psi(\overline{Y},\overline{Z})$ and $(Y',Z')=\Psi(\overline{Y'},\overline{Z'})$. Putting $\Delta\eta=\eta-\eta'$ for any process η, and by virtue of Itô's formula, we have

$$\begin{aligned}&\mathbb{E}e^{-\mu t}|\Delta Y_t|^2+\mathbb{E}\int_t^T e^{-\mu s}\|\Delta Z_s\|^2ds\\ &=2\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s\left\{f(s,Y_{s^-},Z_s)-f(s,Y'_{s^-},Z'_s)\right\}ds+2\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s\left\{\phi(s,Y_{s^-})-\phi(s,Y'_{s^-})\right\}dA_s\\ &+2\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s d(\Delta K_s)+\int_t^T e^{-\mu s}\left|g(s,\overline{Y}_{s^-})-g(s,\overline{Y'}_{s^-})\right|^2ds-\mu\mathbb{E}\int_t^T e^{-\mu s}|\Delta Y_s|^2ds.\end{aligned}$$

But since $\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s d(K_s-K'_s)\leq 0$, then from (**H2**) there exists constant γ such that,

$$\begin{aligned}
&(\mu-\gamma)\mathbb{E}\int_t^T e^{-\mu s}|\Delta Y_s|^2\,ds+\frac{1}{2}\mathbb{E}\int_t^T e^{-\mu s}\|\Delta Z_s\|^2 ds\\
&\leq c\mathbb{E}\left(\int_t^T e^{-\mu s}|\Delta\bar{Y}_s|^2\,ds\right)
\end{aligned}$$

Now choose $\mu=\gamma+2c$ and define $\bar{c}=2c$, we obtain

$$\begin{aligned}
&\bar{c}\mathbb{E}\int_0^t e^{-\mu s}|\Delta Y_s|^2\,ds+\frac{1}{2}\mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2 ds\\
&\leq \frac{1}{2}\left(\bar{c}\mathbb{E}\int_0^t e^{-\mu s}|\Delta\bar{Y}_s|^2\,ds+\frac{1}{2}\mathbb{E}\int_0^t e^{-\mu s}|\Delta\bar{Z}_s|^2\,ds\right).
\end{aligned}$$

Consequently, Ψ is a strict contraction on $\mathcal{S}^2(\mathbb{R})\times\mathcal{M}^2(\mathbb{R}^m)$ equipped with the norm

$$\|Y,Z)\|^2=\bar{c}\mathbb{E}\int_0^t e^{-\mu s}|Y_s|^2\,ds+\frac{1}{2}\mathbb{E}\int_0^t e^{-\mu s}\|Z_s\|^2 ds$$

and it has a unique fixed point, which is the unique solution our BDSDE.

Uniqueness Assume $(Y_t,Z_t,K_t)_{0\leq t\leq T}$ and $(Y'_t,Z'_t,K'_t)_{0\leq t\leq T}$ are two solutions of the reflected GBDSDE (ξ,f,g,ϕ,S) driven by Lévy processes. Set $\Delta Y_t=Y_t-Y'_t$, $\Delta Z_t=Z_t-Z'_t$ and $\Delta K_t=K_t-K'_t$. Applying It's formula to $(\Delta Y)^2$ on the interval $[t,T]$ and taking expectation on both sides, it follows that

$$\begin{aligned}
&\mathbb{E}\,|\Delta Y_t|^2+\mathbb{E}\int_t^T\|\Delta Z_s\|^2 ds\\
=\;&2\mathbb{E}\int_t^T\Delta Y_s(f(s,Y_{s^-},Z_s)-f(s,Y'_{s^-},Z'_s))ds+2\mathbb{E}\int_t^T|g(s,Y_{s^-})-g(s,Y'_{s^-})|^2 ds\\
&+2\mathbb{E}\int_t^T\Delta Y_s(\phi(s,Y_{s^-})-\phi(s,Y'_{s^-}))dA_s+2\mathbb{E}\int_t^T\Delta Y_s d(\Delta K_s)\\
\leq\;&4c^2\mathbb{E}\int_t^T|\Delta Y_s|^2 ds+\frac{1}{4c^2}\mathbb{E}\int_t^T|f(s,Y_{s^-},Z_s)-f(s,Y'_{s^-},Z'_s)|^2 ds\\
&+\beta\mathbb{E}\int_t^T|\Delta Y_s|^2 ds+c\mathbb{E}\int_t^T|\Delta Y_s|^2 ds\\
\leq\;&4c^2\mathbb{E}\int_t^T|\Delta Y_s|^2 ds+\frac{2c^2}{4c^2}\mathbb{E}\int_t^T|\Delta Y_s|^2 ds+\frac{2c^2}{4c^2}\mathbb{E}\int_t^T\|\Delta Z_s\|^2 ds\\
&+\beta\mathbb{E}\int_t^T|\Delta Y_s|^2 ds+c\mathbb{E}\int_t^T|\Delta Y_s|^2 ds\\
\leq\;&(4c^2+c+\frac{1}{2})\mathbb{E}\int_t^T|\Delta Y_s|^2 ds+\frac{1}{2}\mathbb{E}\int_t^T\|\Delta Z_s\|^2 ds,
\end{aligned}$$

here we have used the assumption (**H2**), the inequality $2ab\leq\frac{a^2}{\gamma}+\gamma b^2$ $(\forall\,\gamma>0)$ and the fact that

$$\int_0^T\Delta Y_s d(\Delta K_s)\leq 0.$$

So, we have

$$\mathbb{E}|\Delta Y_t|^2 \quad \leq \quad (4c^2+c+\frac{1}{2})\mathbb{E}\int_t^T |\Delta Y_s|^2 ds.$$

Henceforth, from Gronwall's inequality, it follows that $\mathbb{E}|\Delta Y_t|^2 = \mathbb{E}|Y_t - Y_t'|^2 = 0, 0 \leq t \leq T$, that is, $Y_t = Y_t'$ a.s. Then, we also have $\mathbb{E}\int_t^T \|\Delta Z_s\|^2 ds = \mathbb{E}\int_t^T \|Z_s - Z'\|^2 ds = 0$ and $Z_t = Z_t'$, $K_t = K_t'$ follows. The proof is complete now. □

5 Connection to reflected stochastic PDIEs with nonlinear Neumann boundary condition

In this section, we study the link between reflected GBDSDEs driven by Lévy processes and the solution of a class of reflected stochastic PDIEs with a nonlinear Neumann boundary condition. Suppose that our Lévy processes L has bounded jump and has the following Lévy decomposition:

$$L_t = bt + \int_{|z|\leq 1} z(N_t(.,dz) - t\nu(dz))$$

where $N_t(\omega, dz)$ denotes the random measure such that $\int_\Lambda N_t(.,dz)$ is a Poisson process with parameter $\nu(\Lambda)$ for all set Λ $(0 \notin \Lambda)$.

Let $\Theta = (-\theta, \theta)$ and $e : [-\theta, \theta] \to \mathbb{R}$ such that $e(-\theta) = 1$ and $e(\theta) = -1$. Consider the following reflected SDE:

$$X_t = x + \int_t^T \sigma(X_{s^-}) dL_s + \eta_t, \tag{5.1}$$

and

$$\eta_t = \int_0^t e(X_s) d|\eta|_s, \text{ with } |\eta|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial\Theta\}} d|\eta|_s. \tag{5.2}$$

Under adequate conditions (see [5] or [9]), there exists a unique pair of progressively measurable processes (X, η) that satisfies (5.1) and (5.2), and for any progressively measurable process V which is right continuous having left-hand limits and take its values in $\overline{\Theta}$, we have

$$\int_0^T (X_s - V_s) d|\eta|_s \geq 0.$$

In order to attain our main result in this section, we give a Lemma appeared in [11].

Lemma 5.1. *let $c : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$|c(s,y)| \leq a_s(y^2 \wedge |y|) \;\; a.s.,$$

where $\{a_s, s \in [0,T]\}$ is a non-negative predictable process such that $E\int_0^T a_s^2 ds < \infty$. Then, for each $0 \leq t \leq T$, we have

$$\sum_{t\leq s\leq T} c(s, \Delta L_s) = \sum_{i=1}^m \int_t^T \langle c(s,.), p_i \rangle_{L^2(\nu)} dH_s^{(i)} + \int_t^T \int_{\mathbb{R}} c(s,y) d\nu(y) ds$$

Let $l : \mathbb{R} \to \mathbb{R},\ h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that

(i) $\mathbb{E}\left(|l(X_T)|^2 + \sup_{0\le t\le T} |h(t,X_t)|^2\right) < \infty,$

(ii) $l(x) \ge h(T,x)$, for all $x \in \mathbb{R}$.

Next, consider the following reflected GBDSDE:

$$\begin{aligned} Y_t &= l(X_T) + \int_t^T f(s,X_{s^-}Y_{s^-},Z_s)ds + \int_t^T \phi(s,X_{s^-},Y_{s^-})d|\eta|_s + \int_t^T g(s,X_{s^-},Y_{s^-})\,dB_s \\ &\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)} + K_T - K_t,\ 0 \le t \le T, \end{aligned} \tag{5.3}$$

such that the following holds $\mathbb{P}$-a.s

(i) $Y_t \ge h(t,X_t),\ \ 0 \le t \le T,$

(ii) $\int_0^T (Y_{t^-} - h(t,X_t))\,dK_t = 0.$

Define

$$u^1(t,x,y) = u(t,x+y) - u(t,x) - \tfrac{\partial u}{\partial x}(t,x)y,$$

where u is the solution of the following reflected stochastic PDIE with a nonlinear Neumann boundary condition:

$$\begin{cases} \min\Big\{u(t,x) - h(t,x),\ \dfrac{\partial u}{\partial t}(t,x) + a'\sigma(x)\dfrac{\partial u}{\partial x}(t,x) + f(t,x,u(t,x),(u^i(t,x))_i^m) \\ \qquad + \int_{\mathbb{R}} u^1(t,x,y)d\nu(y) + g(t,x,u(t,x))dB_t\Big\} = 0,\ \ (t,x) \in [0,T]\times\Theta \\ e(x)\dfrac{\partial u}{\partial x}(t,x) + \phi(t,x,u(t,x)) = 0,\ \ (t,x) \in [0,T]\times\{-\theta,\theta\}, \\ u(T,x) = l(x), \quad x \in \Theta, \end{cases} \tag{5.4}$$

where $a' = a + \int_{\{|y|\ge 1\}} y\nu(dy),\ dB_t = \dot{B}_t$ denotes a white noise and

$u^{(1)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(t,x)(\int_{\mathbb{R}} y^2\nu(dy))^{1/2}$

and for $2 \le i \le m$, $u^{(i)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_i(y)\nu(dy).$

Suppose that u is $\mathcal{C}^{1,2}$ function such that $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ is bounded by polynomial function of x, uniformly in t. Then we have the following

Theorem 5.2. *The unique adapted solution of* (5.9) *is given by*

$$\begin{aligned} Y_t &= u(t,X_t), \\ Z_t^{(1)} &= \int_{\mathbb{R}} u^1(t,X_{t^-},y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}\sigma(X_{t^-})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2} \\ Z_t^{(i)} &= \int_{\mathbb{R}} u^1(t,X_{t^-},y)p_i(y)\nu(dy),\ 2 \le i \le m, \end{aligned}$$

Proof. For each $n \geq 1$, let $\{^nY_s, ^nZ_s,\ 0 \leq s \leq T\}$ denote the solution of the GBDSDE

$$\begin{aligned} ^nY_s &= l(X_T) + \int_s^T f(r, X_{r^-}, ^nY_{r^-}, ^nZ_r)dr + n\int_s^T (^nY_{r^-} - h(r, X_r))^- dr \\ &\quad + \int_s^T \phi(r, X_{r^-}, ^nY_{r^-})d|\eta|_r \int_s^T g(r, X_{r^-}, ^nY_{r^-})dB_r - \sum_{i=1}^m \int_s^T {}^nZ_r^{(i)} dH_r^{(i)}. \end{aligned}$$

It is know from Hu and Yong [8] that

$$\begin{aligned} ^nY_t &= u_n(t, X_t), \\ ^nZ_t^{(1)} &= \int_{\mathbb{R}} u_n^1(t, X_{t^-}, y)p_1(y)\nu(dy) + \frac{\partial u_n}{\partial x}\sigma(X_{t^-})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2} \\ ^nZ_t^{(i)} &= \int_{\mathbb{R}} u_n^1(t, X_{t^-}, y)p_i(y)\nu(dy),\ 2 \leq i \leq m, \end{aligned}$$

where u_n is the classical solution of stochastic PDIE:

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(t,x) + a'\sigma(x)\dfrac{\partial u_n}{\partial x}(t,x) + f_n(t,x,u_n(t,x),(u_n^i(t,x))_i^m) \\ \qquad + \int_{\mathbb{R}} u_n^1(t,x,y)d\nu(y) + g(t,x,u_n(t,x))dB_t = 0,\ \ (t,x) \in [0,T]\times\Theta \\ e(x)\dfrac{\partial u_n}{\partial x}(t,x) + \phi(t,x,u_n(t,x)) = 0,\ \ (t,x) \in [0,T]\times\{-\theta,\theta\}, \\ u_n(T,x) = l(x), \quad x \in \Theta, \end{cases} \tag{5.5}$$

where $f_n(t,x,y,z) = f(t,x,y,z) + n(y - h(t,x))^-$.

Applying Itô's formula to $u_n(s, X_s)$, we obtain

$$\begin{aligned} u_n(T,X_T) - u_n(t,X_t) &= \int_t^T \frac{\partial u_n}{\partial s}(s,X_{s^-})ds + \int_t^T e(X_s)\frac{\partial u_n}{\partial x}(s,X_s)d|\eta|_s \\ &\quad + \int_t^T \sigma(X_{s^-})\frac{\partial u_n}{\partial x}(s,X_{s^-})dL_s \\ &\quad + \sum_{t\leq s\leq T}[u_n(s,X_s) - u_n(s,X_{s^-}) - \frac{\partial u_n}{\partial x}(s,X_{s^-})\Delta X_s]. \end{aligned} \tag{5.6}$$

Lemma 4.1 applied to $u_n(s,X_{s^-}+y) - u_n(s,X_{s^-}) - \frac{\partial u_n}{\partial x}(s,X_{s^-})y$ shows

$$\begin{aligned} \sum_{t\leq s\leq T}[u_n(s,X_s) - u_n(s,X_{s^-}) - \frac{\partial u_n}{\partial x}(s,X_{s^-})\Delta X_s] &= \sum_{i=1}^m \int_t^T \left(\int_{\mathbb{R}} u_n^1(s,X_{s^-},y)p_i(y)\nu(dy)\right) dH^{(i)} \\ &\quad + \int_t^T \left(\int_{\mathbb{R}} u_n^1(s,X_{s^-},y)\nu(dy)\right) ds. \end{aligned} \tag{5.7}$$

Note that

$$L_t = Y_t^{(1)} + t\mathbb{E}L_1 = \left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2} H^{(1)} + t\mathbb{E}L_1, \tag{5.8}$$

where $\mathbb{E}L_1 = a + \int_{\{|y|\geq 1\}} y\nu(dy)$. Hence, substituting (5.2), (5.7) and (5.8) into (5.6) together with (5.5) yields

$$
\begin{aligned}
& l(X_T) - u_n(t,X_t) \\
= \; & \int_t^T \left[\frac{\partial u_n}{\partial s}(s,X_{s^-}) + (a + \int_{|y|\geq 1} y\nu(dy))\sigma(X_{s^-}) \frac{\partial u_n}{\partial x}(s,X_{s^-}) + \int_{\mathbb{R}} u_n^1(s,X_{s^-},y)\nu(dy) \right] ds \\
& + \int_t^T e(X_s) \frac{\partial u_n}{\partial x}(s,X_s) \mathbf{1}_{\{X_s \in \partial\Theta\}} d|\eta|_s \\
& + \int_t^T \left[\int_{\mathbb{R}} u_n^1(s,X_{s^-},y) p_1(y)\nu(dy) + \sigma(X_{s^-}) \frac{\partial u_n}{\partial x}(s,X_{s^-}) \left(\int_{\mathbb{R}} y^2 \nu(dy) \right)^{1/2} \right] dH_s^{(1)} \\
& + \sum_{i=2}^m \int_t^T \left(\int_{\mathbb{R}} u_n^1(s,X_{s^-},y) p_i(y)\nu(dy) \right) dH_s^{(i)}. \\
= \; & - \int_t^T f(s,X_{s^-},u_n(s,X_s),(u_n(s,X_s))_{i=1}^m) ds + n \int_t^T (u_n(s,X_s) - h(s,X_s))^- ds \\
& - \int_t^T g(s,X_{s^-},u_n(s,X_s)) dB_s - \int_t^T \varphi(s,X_{s^-},u_n(s,X_s)) d|\eta|_s \\
& + \int_t^T \left[\int_{\mathbb{R}} u_n^1(s,X_{s^-},y) p_1(y)\nu(dy) + \sigma(X_{s^-}) \frac{\partial u_n}{\partial x}(s,X_{s^-}) \left(\int_{\mathbb{R}} y^2 \nu(dy) \right)^{1/2} \right] dH_s^{(1)} \\
& + \sum_{i=2}^m \int_t^T \left(\int_{\mathbb{R}} u_n^1(s,X_{s^-},y) p_i(y)\nu(dy) \right) dH_s^{(i)}.
\end{aligned}
$$

From which passing in the limit on n, and using the previous section we get the desired result of the Theorem. □

Next, we give a example of reflected stochastic PDIEs with a nonlinear Neumann boundary condition.

Example 5.3. Suppose the Lévy process L has the form of $L_t = at + \sum_{i=1}^{\infty}(N^{(i)} - \alpha_i t)$, where $(N^{(i)})_{i=0}^{\infty}$ is a sequence of independent Poisson processes with parameters $(\alpha^i)_{i=0}^{\infty}, (\alpha_i > 0)$. Its Lévy measure is $\nu(dx) = \sum_{i=1}^{\infty} \alpha_i \delta_{\beta_i}(dx)$, where δ_{β_i} denotes the positive point mass measure at $\beta_i \in \mathbb{R}$ of size 1. Furthermore, we assume that $\sum_{i=1}^{\infty} \alpha_i |\beta_i|^2 < \infty$. Recall that this Lévy process has only one jumps size and no continuous parts so that $H_t^{(1)} = \sum_{i=1}^{\infty} \frac{\beta_i}{\sqrt{\alpha_i}}(N_t^{(i)} - \alpha_i t)$ and $H_t^{(i)} = 0, i \geq 2$ (see [11]). Let (Y,Z,K) be the unique solution of the following reflected GBDSDEs

$$
\begin{aligned}
Y_t \; = \; & l(X_T) + \int_t^T f(s,X_{s^-}Y_{s^-},Z_s) ds + \int_t^T \phi(s,X_{s^-},Y_{s^-}) d|\eta|_s + \int_t^T g(s,X_{s^-},Y_{s^-}) \, dB_s \\
& - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} d(N_s^{(i)} - \alpha_i s) + K_T - K_t, \;\; 0 \leq t \leq T
\end{aligned}
$$

such that the following holds $\mathbb{P}$-a.s

(i) $Y_t \geq h(t,X_t), \;\; 0 \leq t \leq T,$

(ii) $\int_0^T (Y_{t^-} - h(t,X_t)) \, dK_t = 0.$

Then

$$\begin{aligned} Y_t &= u(t,X_t), \\ Z_t^{(1)} &= \alpha_1 u^1(t,X_{t^-},\beta_1)p_1(\beta_1)+\sigma(X_{t^-})\frac{\partial u}{\partial x}(t,X_{t^-})\left(\sum_{i=1}^{\infty}\alpha_i|\beta_i|^2\right)^{1/2} \\ Z_t^{(i)} &= \alpha_i u^1(t,X_{t^-},\beta_i)p_i(\beta),\ i\geq 2, \end{aligned}$$

where u is the solution of the following reflected stochastic PDIEs with a nonlinear Neumann boundary condition:

$$\begin{cases} \min\Big\{u(t,x)-h(t,x),\ \frac{\partial u}{\partial t}(t,x)+a'\sigma(x)\frac{\partial u}{\partial x}(t,x)+f(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x)) \\ \qquad +\sum_{i=1}^{\infty}\alpha_i u^1(t,x,\beta_i)+g(t,x,u(t,x))dB_t\Big\}=0,\ \ (t,x)\in[0,T]\times\Theta \\ e(x)\frac{\partial u}{\partial x}(t,x)+\phi(t,x,u(t,x))=0,\ \ (t,x)\in[0,T]\times\{-\theta,\theta\}, \\ u(T,x)=l(x), \quad x\in\Theta. \end{cases}$$

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