NUMERICAL SCHEME FOR BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

We study a discrete-time approximation for solutions of systems of decoupled forward-backward doubly stochastic differential equations (FBDSDEs). Assuming that the coefficients are Lipschitz-continuous, we prove the convergence of the scheme when the step of time discretization, $|\pi|$ goes to zero. The rate of convergence is exactly equal to $|\pi|^{1/2}$. The proof is based on a generalization of a remarkable result on the ^{2}-regularity of the solution of the backward equation derived by J. Zhang [11].

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1 Introduction

In this paper, we study a discrete time approximation scheme for the solution of a system of the (decoupled) forward-backward doubly stochastic differential equations (FBDSDEs, in short) on the time interval $[0,T]$:

$$\begin{cases} X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t = h(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)\,ds + \int_t^T g(s,X_s,Y_s)d\overleftarrow{B_s} - \int_t^T Z_s dW_s. \end{cases} \tag{1.1}$$

Here W and B are two independent Brownian motion such that, the integral with respect to B_t is a backward Itô integral and the one with respect to W_t is a standard forward Itô integral. Let us note that such equations naturally appear in probabilistic interpretation of stochastic partial differential equations (SPDEs, in short). Indeed, under standard Lipschitz assumptions on the coefficients b,σ, f, g, and h, the existence and uniqueness of the solution (Y,Z) have been proved by Pardoux and Peng [9]. Moreover, they give the link between the classical solution of SPDE in the following. More precisely let consider the SPDE

$$-\frac{\partial}{\partial t}u(t,x) - [Lu(t,x) - f(t,x,u(t,x),\sigma^*(x)\nabla u(t,x))] - g(t,x,u(t,x))\Diamond B_s = 0,$$
$$u(T,x) = h(x), \tag{1.2}$$

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where $\Diamond$ denotes the Wick product and, thus, indicates that the differential is to understand in Itô's sense, and

$$L = \frac{1}{2}\sum_{i,j}^{n}(\sigma\sigma^*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i(x)\frac{\partial}{\partial x_i}.$$

Under more strengthen assumptions (the coefficients f, g and h are $\mathcal{C}^3$ class), the component Y of the solution of (1.1) is related to the classical solution u of SPDE (1.2), in the sense that

$$Y_t = u(t, X_t). \tag{1.3}$$

Furthermore, Buckdahn and Ma relax the assumptions of coefficient to standard Lipschitz one and they proved among other that the relation (1.3) give the stochastic viscosity solution of SPDE (1.2). Thus, solving (1.1) or (1.2) is essentially the same. However it is known that only a limited number of BDSDE can be solved explicitly. In order to solved the large class of BDSDE and of course provide an alternative to classical numerical schemes for a large class of SPDE, the numerical method and numerical algorithm is very helpful.

In the one stochastic case, i.e $g \equiv 0$, the numerical approximation of (1.1) has already been studied in the literature; see e.g. Zhang [11], Bally and Pages [2], Bouchard and Touzi [3] or Gobet et al. [5]. In [3], the authors suggest the following implicit scheme. Given a partition regular grid $\pi : 0 = t_0 < t_1 < .... < t_n = T$ of the interval $[0, T]$, they approximate X by its well-know Euler scheme X^π and (Y, Z), by the discrete-time process $(Y^\pi_{t_i}, Z^\pi_{t_i})_{0 \le i \le n}$ defined backward by

$$\begin{cases} Z^\pi_{t_i} = \frac{1}{\Delta^\pi_{i+1}}\mathbb{E}\left[Y^\pi_{t_{i+1}}\Delta^\pi W_{t_{i+1}}|\mathcal{F}_{t_i}\right] \\ \\ Y^\pi_{t_i} = \mathbb{E}\left[Y^\pi_{t_{i+1}}|\mathcal{F}_{t_i}\right] + \Delta^\pi_{i+1} f(t_i, X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}), \end{cases}$$

where $Y^\pi_{t_n} = h(X^\pi_T)$, $\Delta^\pi W_{i+1} = W_{t_{i+1}} - W_{t_i}$ and $\Delta^\pi_{i+1} = t_{i+1} - t_i$. Then, it turn out that the discretization error

$$Err_\pi(Y, Z) = \left\{\sup_{0 \le t \le T}\mathbb{E}|Y_t - Y^\pi_t|^2 + \int_0^T \mathbb{E}\left[|Z_s - Z^\pi_s|^2\right] ds\right\}^{1/2}$$

is intimately related to the quantity

$$\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[|Z_s - \tilde{Z}_{t_i}|^2\right] ds \quad \text{where} \quad \tilde{Z}_{t_i} = \frac{1}{\Delta^\pi_{i+1}}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s|\mathcal{F}_{t_i}\right].$$

Under Lipschitz continuity conditions on the coefficients, Zhang [11] was able to prove that the latter is of order of $|\pi|$, the partition's mesh. This remarkable result allows them to derive the bound $Err_\pi(Y, Z) \le C|\pi|^{1/2}$. Observe that this rate of convergence cannot be improved in general. Consider, for example, the case where X is equal to the Brownian motion W, h is the identity, and $f = 0$. Then, $Y = W$ and $Y^\pi_{t_i} = W_{t_i}$.

In this paper, we extend the approach of Bouchard and Touzi [3], and approximate the solution of (1.1) by the following backward scheme.

$$\begin{cases} Z^\pi_{t_i} = \frac{1}{\Delta_{i+1}}\mathbb{E}^\pi_i\left[\left(Y^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, Y^\pi_{t_{i+1}})\Delta B_{i+1}\right)\Delta W_{i+1}\right], \\ \\ Y^\pi_{t_i} = \mathbb{E}^\pi_i\left[Y^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, Y^\pi_{t_{i+1}})\Delta B_{i+1}\right] + f(t_i, X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i})\Delta_{i+1} \end{cases}$$

where $Y^\pi_{t_n} = h(X^\pi_T)$ and $\Delta B_{i+1} = B_{t_{i+1}} - B_{t_i}$. By adapting the arguments of Bouchard and Touzi [3], we first prove that our discretization error $Err_\pi(Y,Z)$ converge to 0 as the step of the discretization $|\pi|$ tends to 0. We then provide upper bounds on

$$\max_{i<n} \sup_{0\le t\le t_i} \mathbb{E}|Y_t - Y_{t_i}|^2 + \sum_i^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_s - \tilde{Z}_{t_i}|^2\right] ds.$$

When the coefficients are Lipschitz continuous, we obtain

$$\max_{i<n} \sup_{0\le t\le t_i} \mathbb{E}|Y_t - Y_{t_i}|^2 + \sum_i^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_s - \tilde{Z}_{t_i}|^2\right] ds < C|\pi|.$$

This extends to our framework the remarkable result derived by Zhang [11]. It allows us to show that our discrete-time scheme achieves, under the standard Lipschitz conditions, a rate of convergence exactly equal to $|\pi|^{1/2}$.

Observe that, in opposition to algorithms based on the approximation of the Brownian motion by discrete processes taking a finite number of possible values (see e.g. [10] and the references therein), our scheme does not provide a fully implementable numerical procedure, since it involves the computation of a large number of conditional expectations.

This paper is organized as follows. In Section 2, we introduce some fundamental knowledge and assumptions of BDSDEs and give extension of the remarkable L^2-regularity results derived by Zhang [11] to the doubly stochastic case, which is our first main result. In Section 3, we describe the approximation scheme and state convergence result, our second main result.

Notations. We shall denote by $\mathbb{M}^{n,d}$ the set of all $n\times d$ matrices with real coefficients. We simply denote $\mathbb{R}^n = \mathbb{M}^{n,1}$ and $\mathbb{M}^n = \mathbb{M}^{n,n}$. We shall denote by $\|a\| = (\sum_{i,j} a^2_{i,j})^{1/2}$ the Euclidian norm on $\mathbb{M}^{n,d}$, a^* the transpose of a, a^k the k-th column of a. To simplify, we denote respectively by $|x|$ and a_k, the norm and the the k-th component of $a \in \mathbb{R}^n$. Finally, we denote by $x.y = \sum_i x_i y_i$ the scalar product in $\mathbb{R}^n$.

2 Forward-Backward doubly SDEs

2.1 Preliminaries and Assumptions

Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be two complete probability spaces and $T > 0$ a fixed final time. Throughout this paper we consider $\{W_t, 0 \le t \le T\}$ and $\{B_t, 0 \le t \le T\}$ two mutually independent standard Brownian motions processes, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^\ell$, defined respectively on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. For any process $(\eta_s : 0 \le s \le T)$ defined on $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $(i = 1, 2)$, we denote

$$\mathcal{F}^\eta_{s,t} = \sigma\{\eta_r - \eta_s, s \le r \le t\} \vee \mathcal{N}, \; \mathcal{F}_t{}^\eta = \mathcal{F}^\eta_{0,t}.$$

In the sequel of the paper unless otherwise specified we denote

$$\Omega = \Omega_1 \times \Omega_2,\ \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \text{ and } \mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2.$$

Moreover, we put

$$\mathcal{F}_t = \mathcal{F}_t^W \otimes \mathcal{F}_T^B \vee \mathcal{N}$$

where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets and denote $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$. Further, for random variables $\varepsilon(\omega_1)$, $\omega_1 \in \Omega_1$ and $\beta(\omega_2)$, $\omega_2 \in \Omega_2$, we view them as random variables in Ω by the following identification:

$$\varepsilon(\omega) = \varepsilon(\omega_1);\ \ \beta(\omega) = \beta(\omega_2),\ \omega = (\omega_1, \omega_2).$$

Given $C > 0$, we consider two functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{M}^d$ two functions satisfying the Lipschitz condition

$$(\mathbf{H1})\ |b(x) - b(x')| + \|\sigma(x) - \sigma(x')\| \leq C|x - x'|,\ \forall\, x, x' \in \mathbb{R}^d.$$

Then it is well-known that (see e.g Karatzas and Shreve [6]), for any initial condition $x \in \mathbb{R}^d$, the forward stochastic differential equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad t \in [0,T] \tag{2.1}$$

has a $\mathcal{F}_t$-adapted solution $(X_t)_{0\leq t\leq T}$ satisfying

$$\mathbb{E}(\sup_{0\leq t\leq T} |X_t|^2) < \infty.$$

Before introducing the backward doubly SDE, we need to define some additional notations. Given some real number $p \geq 2$, we denote by $\mathcal{S}^p$ the set of real valued adapted càdlàg processes Y such that

$$\|Y\|_{\mathcal{S}^p} = \mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^p\right] < \infty.$$

$\mathcal{H}^p$ is the set of progressively measurable $\mathbb{R}^d$-valued processes Z such that

$$\|Z\|_{\mathcal{H}^p} = \mathbb{E}\left[\int_0^T |Z_t|^p dt\right]^{1/p} < \infty.$$

The set $\mathcal{B}^p = \mathcal{S}^p \times \mathcal{H}^p$ is endowed with the norm

$$\|(Y,Z)\|_{\mathcal{B}^p} = \left(\|Y\|^p_{\mathcal{S}^p} + \|Z\|^p_{\mathcal{H}^p}\right)^{1/p}.$$

The aim of this paper is to study a discrete-time approximation of the pair (Y,Z) solution on $[0,T]$ of the backward doubly stochastic differential equation

$$Y_t = h(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\, ds + \int_t^T g(s, X_s, Y_s) d\overleftarrow{B}_s - \int_t^T Z_s dW_s,\ \ 0 \leq t \leq T. \tag{2.2}$$

By a solution, we mean a triplet $(Y,Z) \in \mathcal{B}^p$ satisfying (2.2).

In order to ensure the existence and uniqueness of a solution to (2.2), and the convergence of our discrete-time approximation, we assume that the map $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^\ell$ and $h : \mathbb{R}^d \to \mathbb{R}$ satisfied the Lipschitz condition:

(**H2**)

(i) $|f(s,x,y,z)-f(s',x',y',z')|^2 \le C\left(|s-s'|^2+|x-x'|^2+|y-y'|^2+|z-z'|^2\right)$
(ii) $|g(s,x,y)-g(s,x',y')|^2 \le C(|s-s'|^2+|x-x'|^2+|y-y'|^2)$
(iii) $|h(x)-h(x')|^2 \le C|x-x'|^2$

for some constant $C>0$ independent of all the variables.

Remark 2.1. In order to ensure the existence and uniqueness to the solution of (2.2), we need only that f and g are Lipschitz with respect variables y and z. See Pardoux and Peng [9] for more detail.

The following lemmas collect without proof, some standard results in SDE and BDSDE literature. We list them for ready references. For ease of notation, we shall denote by C_p a generic constant depending only on p, the constants $C, b(0), \sigma(0), h(0)$ and T and the functions $f(.,0,0,0)$ and $g(.,0,0)$.

Lemma 2.2. *Assume b and σ satisfy* (**H1**) *and X be the unique solution of forward SDE* (2.1). *Then*

$$\|X\|^p_{\mathcal{S}^p} \le C_p(1+|x|^p)$$

and

$$\mathbb{E}\left[|X_t-X_s|^p\right] \le C_p(1+|x|^p)|t-s|^{p/2}.$$

Lemma 2.3. *Assume* (**H2**) *and (Y,Z) be the unique solution of backward doubly SDE* (2.2). *Then*

$$\|(Y,Z)\|^p_{\mathcal{B}^p} \le C_p(1+|x|^p)$$

and

$$\mathbb{E}\left[|Y_t-Y_s|^p\right] \le C_p\left\{(1+|x|^p)|t-s|^{p-1}+\|Z\|^p_{\mathcal{H}^p}\right\}.$$

2.2 L^2-regularity

In this subsection we establish the first main result of this paper, which we shall call the L^2-regularity. Such a regularity, plays a key role for deriving the rate of convergence of our numerical scheme in Section 4 and, in our mind generalized Theorem 3.4.3 in [11].

To begin with, let $\pi : 0=t_0<\ldots<t_n=T$ be a partition of the time interval $[0,T]$, with $|\pi| = \max_{1\le i\le n}|t_{i-1}-t_i|$, the size of the partition. and X be the solution of the forward SDE (2.1). We denote by (Y,Z) the solution of the following backward SDE

$$Y_t = \phi^\pi(X_{t_0},\ldots,X_{t_n}) + \int_t^T f(s,X_s,Y_s,Z_s)\,ds + \int_t^T g(s,X_s,Y_s)d\overleftarrow{B_s} - \int_t^T Z_s dW_s, \qquad (2.3)$$

the generalized form of BDSDE (2.2). Next, for X^π the well-know Euler scheme of X that will be explicit in Section 3, let (Y^π,Z^π) be the adapted solution to the following BDSDE

$$Y^\pi_t = \phi^\pi(X^\pi_{t_0},\ldots,X^\pi_{t_n}) + \int_t^T f(s,X^\pi_s,Y^\pi_s,Z^\pi_s)\,ds + \int_t^T g(s,X^\pi_s,Y^\pi_s)d\overleftarrow{B_s} - \int_t^T Z^\pi_s dW_s. \qquad (2.4)$$

To simplify presentations, in what follows we assume that $X_t,\ X_t^\pi \in \mathbb{R}^d$, and the other processes are all one-dimensional. But the results can be extended to cases with higher-dimensional on this processes without significant difficulties. For simplicity we also denote by $\Xi = (X,Y)$, $\Theta = (X,Y,Z)$ and $\Xi^\pi = (X^\pi, Y^\pi)$, $\Theta^\pi = (X^\pi, Y^\pi, Z^\pi)$.

Now we have

Lemma 2.4. *Assume the functions* $\phi^\pi : \mathbb{R}^{d(n+1)} \to \mathbb{R}$, $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^2 \to \mathbb{R}$ *and* $g : [0,T]\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ *satisfying assumptions* $(\mathbf{H2})$ *with adequate norm. For each* $1 \le i \le n$, *we define*

$$\tilde{Z}^\pi_{t_{i-1}} \quad = \quad \frac{1}{\triangle_i^\pi}\mathbb{E}^\pi_{i-1}\left[\int_{t_{i-1}}^{t_i} Z_s ds\right],$$

where $\mathbb{E}^\pi_{i-1}(.) = \mathbb{E}(.|\mathcal{F}^W_{t_{i-1}} \vee \mathcal{F}^B_T)$. *Then*

$$\limsup_{\pi\to 0} |\pi|^{-1}\mathbb{E}\left[\max_{1\le i\le n}\sup_{t_{i-1}\le t\le t_i}|Y_t - Y_{t_{i-1}}|^2 + \sum_{i=1}^n\int_{t_{i-1}}^{t_i}|Z_s - \tilde{Z}^\pi_{t_{i-1}}|^2 ds\right] < \infty. \tag{2.5}$$

Before prove this important theorem, we state the following needed result. To this end let us assume the following: $\phi^\pi \in C_b^1(\mathbb{R}^{d(n+1)}),\ f \in C_b^{0,1}([0,T]\times\mathbb{R}^d\times\mathbb{R}^2)$ and $g \in C_b^{0,1}([0,T]\times\mathbb{R}^d\times\mathbb{R})$. Moreover, for all $x = (x_0,\dots,x_n) \in \mathbb{R}^{d(n+1)}$,

$$\sum_{i=0}^n |h^\pi_{x_i}(x)| \le C. \tag{2.6}$$

We also design by φ_u the partial differential of φ which respect the variable u.

Next, we denote by ∇X^π the solution of the following variational equation:,

$$\nabla X^\pi_t = I_d + \int_0^t b_x(X^\pi_r)\nabla X^\pi_r dr + \int_0^t \sigma_x(X^\pi_r)\nabla X^\pi_r dW_r, \tag{2.7}$$

and by $(\nabla^i Y^\pi, \nabla^i Z^\pi)$ the solution of the following BDSDE on $[t_{i-1},T]$:

$$\begin{aligned}\nabla^i Y^\pi_t \quad = \quad & \sum_{j\ge i} h^\pi_{x_j}(X^\pi_{t_0},\dots,X^\pi_{t_n})\nabla X^\pi_{t_j} + \int_t^T [f_x(\Theta^\pi_r)\nabla X^\pi_r + f_y(\Theta^\pi_r)\nabla^i Y^\pi_r + f_z(\Theta^\pi_r)\nabla^i Z^\pi_r]dr \\ & + \int_t^T [g_x(\Xi^\pi_r)\nabla X^\pi_r + g_y(\Xi^\pi_r)\nabla^i Y^\pi_r)]\overleftarrow{dB}_r - \int_t^T \nabla^i Z^\pi_r dW_r,\ \ t\in[t_{i+1},T],\end{aligned}$$

for $i = 1,\dots,n$. (2.8)

On the other hand, we denote by

$$\nabla^\pi Y^\pi_t = \sum_{i=1}^n \nabla^i Y^\pi_t \mathbf{1}_{[t_{i-1},t_i)}(t) + \nabla^n Y^\pi_{T^-}\mathbf{1}_{\{T\}}(t), \quad t\in[0,T]; \tag{2.9}$$

hence $\nabla^\pi Y^\pi$ is a càdlàg process.

For application convenience, we shall rewrite $\nabla^\pi Y^\pi$ in another form. Note that for each i (2.8) is linear. Let (γ^0,ζ^0) and $(\gamma^j,\zeta^j),\ j=1,\dots,n$ be the adapted solutions of the BDSDEs

$$\begin{aligned}\gamma^0_t &= \int_t^T [f_x(\Theta^\pi_r)\nabla X^\pi_r + f_y(\Theta^\pi_r)\gamma^0_r + f_z(\Theta^\pi_r)\zeta^0_r]dr \\ &\quad + \int_t^T [g_x(\Xi^\pi_r)\nabla X^\pi_r + g_y(\Xi^\pi_r)\gamma^0_r]d\overleftarrow{B}_r - \int_t^T \zeta^0_r dW_r, \\ \gamma^j_t &= h^\pi_{x_j}(X^\pi_{t_0},\dots\dots,X^\pi_{t_n})\nabla X^\pi_{t_j} + \int_t^T [f_y(\Theta^\pi_r)\gamma^j_r + f_z(\Theta^\pi_r)\zeta^j_r]dr \\ &\quad + \int_t^T g_y(\Xi^\pi_r)\gamma^j_r d\overleftarrow{B}_r - \int_t^T \zeta^j_r dW_r,\end{aligned} \tag{2.10}$$

respectively, then we have the following decomposition:

$$\nabla^i Y_s = \gamma^0_s + \sum_{j\ge i}\gamma^j_s, \qquad s\in[t_{i-1},t_i). \tag{2.11}$$

We may simplify (2.11) further. Let us define, for any $\eta\in L^1(\mathbf{F},[0,T])$ and $(\Theta_1,\Theta_2)\in L^2(\mathbf{F},[0,T];\mathbb{R})\times L^2(\mathbf{F},[0,T];\mathbb{R})$,

$$\begin{aligned}\Lambda^s_t(\eta) &= \exp\left(\int_s^t \eta(r)dr\right), \quad s,t\in[0,T], \\ {}^1\mathcal{E}^s_t(\Theta_1) &= \exp\left\{\int_s^t \Theta_1(r)dW_r - \frac{1}{2}\int_s^t |\Theta_1(r)|^2dr\right\}, \quad s,t\in[0,T], \\ {}^2\mathcal{E}^s_t(\Theta_2) &= \exp\left\{\int_s^t \Theta_2(r)d\overleftarrow{B}_r - \frac{1}{2}\int_s^t |\Theta_2(r)|^2dr\right\}, \quad s,t\in[0,T].\end{aligned}$$

(${}^1\mathcal{E}^s_t(\Theta_1)$ and ${}^2\mathcal{E}^s_t(\Theta_2)$ are respectively the well known Daléan-Dade stochastic exponential of Θ_1 with respect W and Θ_2 with respect B). Then it is easily checked that, for any $p>0$, one has

$$[{}^i\mathcal{E}^s_t(\Theta_i)]^p = {}^i\mathcal{E}^s_t(p\Theta_i)\Lambda^s_t(\frac{p(p-1)}{2}|\Theta_i|^2), \tag{2.12}$$

and

$$[{}^i\mathcal{E}^s_t(\Theta_i)]^{-1} = {}^i\mathcal{E}^s_t(-\Theta_i)\Lambda^s_t(|\Theta_i|^2), \quad i=1,2. \tag{2.13}$$

In particular, we denote, for $s,t\in[0,T]$,

$$\Lambda^s_t \;=\; \Lambda^s_t(-f_y)\,{}^2\mathcal{E}^s_t(-g_y), \;\; M^s_t = {}^1\mathcal{E}^s_t(f_z), \tag{2.14}$$

and if there is no danger of confusion, we denote $\Lambda_\cdot = \Lambda^0_\cdot$ and $M_\cdot = M^0_\cdot$. Since f_z is uniformly bounded, by Girsanov's Theorem (see, e.g., [6]) we know that M is a $\mathbb{P}$-martingale on $[0,T]$, and $\widetilde{W}_t = W_t - \int_0^t f_z(\Theta^\pi_s)dr$, $t\in[0,T]$ is an $\mathbf{F}$-Brownian motion on the new probability space $(\Omega,\mathcal{F},\widetilde{\mathbb{P}})$, where $\widetilde{\mathbb{P}}$ is defined by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = M_T$. Moreover noting that f_y, f_z and g_y

are uniformly bounded, by virtue of (2.12) and (2.13) one can deduce easily from (2.14) that, for $p \geq 1$, there exists a constant C_p depending only on T, C and p, such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |\Lambda_t|^p + |\Lambda_t^{-1}|^p\right) \leq C_p;\;\; \mathbb{E}\left(\sup_{0\leq t\leq T} [|M_t|^p + |M_t^{-1}|^p]\right) \leq C_p;$$
$$\mathbb{E}\left(|\Lambda_t - \Lambda_s|^p + |\Lambda_t^{-1} - \Lambda_s^{-1}|^p\right) \leq C_p|t-s|^{p/2}; \tag{2.15}$$
$$\mathbb{E}\left(|M_t - M_s|^p + |M_t^{-1} - M_s^{-1}|^p\right) \leq C_p|t-s|^{p/2}.$$

Lemma 2.5. *Assume $\sigma, b \in C_b^1$ and f, g, l satisfy the previous assumptions. Then for all $i = 1, ..., n$*

$$\nabla^i Y_t^\pi = \left(\xi_t^0 + \sum_{j\geq i} \xi_t^j\right) M_t^{-1}\Lambda_t - \int_0^t f_x(\Theta_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} dr \Lambda_t - \int_0^t g_x(\Xi_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} d\overleftarrow{B}_r \Lambda_t,$$

where ξ_t^0 and ξ_t^j, for $j = 1, \cdots, n$ will be explicit in the proof.

Proof. Let us denote the following:

$$\widetilde{\xi}^0 = \textstyle\int_0^T f_x(\Theta_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} dr + \int_0^T g_x(\Xi_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} d\overleftarrow{B}_r,\;\; \widetilde{\zeta}_t^0 = \zeta^0 \Lambda_t^{-1},$$

$$\widetilde{\gamma}_t^0 = \gamma_t^0 \Lambda_t^{-1} + \textstyle\int_0^t f_x(\Theta_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} dr + \int_0^t g_x(\Xi_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} d\overleftarrow{B}_r$$

$$\widetilde{\xi}^i = h^\pi(X_{t_0}^\pi, \cdots, X_{t_n}^\pi)\nabla X_{t_i}^\pi \Lambda_T^{-1},\;\; \widetilde{\zeta}_t^i = \zeta_t^i \Lambda_t^{-1},\;\; \widetilde{\gamma}_t^i = \gamma_t^i \Lambda_t^{-1}.$$

Then, using integration by parts and equation (2.10) we have, for $i = 0, 1, ..., n$,

$$\widetilde{\gamma}_t^i = \widetilde{\xi}^i - \int_t^T \widetilde{\zeta}_r^i d\widetilde{W}_r,\;\; t \in [0, T],$$

so that, $\int_0^t \widetilde{\zeta}_r^i d\widetilde{W}_r$ being a uniformly integrable martingale with in particular zero expectation, we get

$$\widetilde{\gamma}_t^i = \widetilde{\mathbb{E}}(\widetilde{\xi}^i | \mathcal{F}_t).$$

Therefore, by the Bayes rule (see e.g, [6] Lemma 3.5.3) we have for $t \in [0, T]$

$$\begin{aligned}
\gamma_t^0 &= \widetilde{\gamma}_t^0 \Lambda_t - \int_0^t f_x(\Theta_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} dr \Lambda_t - \int_0^t g_x(\Xi_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} d\overleftarrow{B}_r \Lambda_t \\
&= \xi_t^0 M_t^{-1}\Lambda_t - \int_0^t f_x(\Theta_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} dr \Lambda_t - \int_0^t g_x(\Xi_r^\pi)\nabla X_r^\pi \Lambda_r^{-1} d\overleftarrow{B}_r \Lambda_t, \\
\gamma_t^i &= \widetilde{\gamma}_t^i \Lambda_t = \widetilde{\mathbb{E}}\left(\widetilde{\xi}^i | \mathcal{F}_t\right)\Lambda_t = \mathbb{E}\left(M_T \widetilde{\xi}^i | \mathcal{F}_t\right) M_t^{-1}\Lambda_t = \xi_t^i M_t^{-1}\Lambda_t,
\end{aligned}$$

where, for $i = 0, 1, ..., n$,

$$\xi_t^i = \mathbb{E}\left(M_T \widetilde{\xi}^i | \mathcal{F}_t\right) = \mathbb{E}\left(M_T \widetilde{\xi}^i\right) + \int_0^t \chi_s^i dW_s. \tag{2.16}$$

Note that the boundedness of f_z and (2.15) imply that $M_T \in L^p(\Omega)$ and $\nabla X \in L^p(\mathbf{F}, C([0,T];\mathbb{M}^d))$ for all $p \geq 2$. Therefore for each $p \geq 1$, (2.6) leads to

$$\mathbb{E}\left\{\sum_{j=0}^{n} |M_T \widetilde{\xi}^j|\right\} \leq C\mathbb{E}\left\{|M_T|^p \sup_{0\leq t\leq T} |\nabla X_t|^p\right\} \leq C.$$

In particular, for each $j = 0, \cdots, n$, $M_T \widetilde{\xi}^j \in L(\mathcal{F}_T)$. So (2.16) makes sens. Finally the result follows by (2.11). □

Proof of Lemma 2.4. For all $1 \leq i \leq n$ and each $t \in [t_{i-1}, t_i)$, applying Lemma 2.3, we get

$$\mathbb{E}\left(|Y_t - Y_{t_{i-1}}|^2\right) \leq C|\pi|.$$

Then by Burkölder-Davis-Gundy inequality we have

$$\mathbb{E}\left[\max_{1\leq i\leq n} \sup_{t_{i-1}\leq t\leq t_i} |Y_t - Y_{t_{i-1}}|^2\right] \leq C|\pi|. \tag{2.17}$$

The estimate for the second term of the left hand in (2.5) is little involved. First we assume that $b, \sigma, \phi^\pi,\ f,\ g \in C_b^1$ such that ϕ^π satisfied (2.6). Let recall (Y^π, Z^π) denote the adapted solution to the BDSDE (2.4) and X^π the solution of the Euler scheme associated to EDS (2.1). Under the Lipschitz conditions on b and σ, we have

$$\lim_{\pi\to 0} \max_{1\leq i\leq n} \mathbb{E}\left[\sup_{0\leq t\leq T} |X_t^\pi - X_t|^2 + \sup_{t_{i-1}\leq t\leq t_i} |X_t - X_{t_{i-1}}|^2\right] = 0. \tag{2.18}$$

Now by the Lipschitz assumption on ϕ^π and (2.18), applying Lemma 2.2 we know that

$$\lim_{\pi\to 0} \mathbb{E}\left\{\sup_{0\leq t\leq T} |Y_t^\pi - Y_t|^2 + \int_0^T |Z_t^\pi - Z_t|^2 dt\right\} = 0. \tag{2.19}$$

Recalling (2.5) and applying Lemma 3.4.2 of Zhang [11] we have

$$\begin{aligned} \sum_{i=1}^{n} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} |Z_s - \tilde{Z}_{t_{i-1}}^\pi|^2 ds\right] &\leq \sum_{i=1}^{n} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} |Z_s - Z_{t_{i-1}}^\pi|^2 ds\right] \\ &\leq 2\sum_{i=1}^{n} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} (|Z_s - Z_s^\pi|^2 + |Z_s^\pi - Z_{t_{i-1}}^\pi|^2) ds\right]. \end{aligned} \tag{2.20}$$

By (2.19) and (2.20), to estimate the second term and prove the theorem it remain to show that

$$\sum_{i=1}^{n} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} |Z_s^\pi - Z_{t_{i-1}}^\pi|^2 ds\right] \leq C|\pi|, \tag{2.21}$$

where C is independent of π.

To do this, let us recall that from Proposition 2.3 of [9] and its proof, we know that the martingale part Z^π has a continuous version given by

$$Z_t^\pi = \nabla^i Y_t^\pi [\nabla X_t^\pi]^{-1} \sigma(X_t^\pi),\ \ \forall t \in [t_{i-1}, t_i),$$

which together with Lemma 2.5 provide

$$Z_t^\pi = \left[\left(\xi_t^0 + \sum_{j\geq i}\xi_t^j\right)M_t^{-1} - \int_0^t f_x(\Theta_r^\pi)\nabla X_r^\pi \Lambda_r^{-1}dr - \int_0^t g_x(\Xi_r^\pi)\nabla X_r^\pi \Lambda_r^{-1}d\overleftarrow{B}_r\right]\Lambda_t[\nabla X_t^\pi]^{-1}\sigma(X_t^\pi).$$

Therefore,

$$|Z_t^\pi - Z_{t_{i-1}}^\pi| \leq I_t^1 + I_t^2 + I_t^3 + I_t^4 \tag{2.22}$$

where

$$\begin{aligned}
I_t^1 &= \left|[\xi_t^0 + \sum_{j\geq i}\xi_t^j] - [\xi_{t_{i-1}}^0 + \sum_{j\geq t_{i-1}+1}\xi_{t_{i-1}}^j]\right| \times \left|M_{t_{i-1}}^{-1}\Lambda_{t_{i-1}}[\nabla X_{t_{i-1}}^\pi]^{-1}\sigma(X_{t_{i-1}}^\pi\right|,\\
I_t^2 &= \left|\xi_t^0 + \sum_{j\geq i}\xi_t^j\right| \left|M_t^{-1}\Lambda_t[\nabla X_t^\pi]^{-1}\sigma(X_t^\pi) - M_{t_{i-1}}^{-1}\Lambda_{t_{i-1}}[\nabla X_{t_{i-1}}^\pi]^{-1}\sigma(X_{t_{i-1}}^\pi)\right|,\\
I_t^3 &= \left|\int_0^t f_x(r)\nabla X_r^\pi\Lambda_r^{-1}dr\Lambda_t[\nabla X_t^\pi]^{-1}\sigma(X_t^\pi) - \int_0^{t_{i-1}} f_x(r)\nabla X_r^\pi\Lambda_r^{-1}dr\Lambda_{t_{i-1}}[\nabla X_{t_{i-1}}^\pi]^{-1}\sigma(X_{t_{i-1}}^\pi)\right|,\\
I_t^4 &= \left|\int_0^t g_x(r)\nabla X_r^\pi\Lambda_r^{-1}d\overleftarrow{B}_r\Lambda_t[\nabla X_t^\pi]^{-1}\sigma(X_t^\pi) - \int_0^{t_{i-1}} g_x(r)\nabla X_r^\pi\Lambda_r^{-1}d\overleftarrow{B}_r\Lambda_{t_{i-1}}[\nabla X_{t_{i-1}}^\pi]^{-1}\sigma(X_{t_{i-1}}^\pi)\right|.
\end{aligned}$$

Recalling (2.15) and applying Lemma 2.2 and Lemma 2.3, one can easily prove that

$$\mathbb{E}(|I_t^3|^2 + |I_t^4|^2) \leq C|\pi|. \tag{2.23}$$

Recalling (2.16) and (2.6), we have

$$|\xi_t^0 + \sum_{j\geq i}\xi_t^j| \leq C\mathbb{E}(\sup_{0\leq t\leq T}|\nabla X_t^\pi| \mid \mathcal{F}_t).$$

Thus by using again Lemma 2.2 and Lemma 2.3 one can similarly show that

$$\mathbb{E}(|I_t^2|^2) \leq C|\pi|. \tag{2.24}$$

It remains to estimate I_t^1. To this end we denote

$$\Gamma_t = \sup_{0\leq s\leq t}\left\{1 + |X_s^\pi| + |[\nabla X_s^\pi]^{-1}| + |M_s^{-1}|\right\}.$$

Noting that Λ is bounded and that $\Gamma_{t_{i-1}} \in \mathcal{F}_{t_{i-1}}$, by (2.16), we have

$$\begin{aligned}
\mathbb{E}|I_t^1|^2 &\leq C\mathbb{E}\left\{\Gamma_{t_{i-1}}^6\left|[\xi_t^0 + \sum_{j\geq i}\xi_t^j] - [\xi_{t_{i-1}}^0 + \sum_{j\geq i}\xi_{t_{i-1}}^j]\right|^2\right\}\\
&\leq C\mathbb{E}\left\{\Gamma_{t_{i-1}}^6\mathbb{E}\left\{|\xi_t^0 - \xi_{t_{i-1}}^0|^2 + \sum_{j\geq i}|\xi_t^j - \xi_{t_{i-1}}^j|^2|\mathcal{F}_{t_{i-1}}\right\}\right\}\\
&\leq C\mathbb{E}\left\{\Gamma_{t_{i-1}}^6\left[\int_{t_{i-1}}^{t_i}|\chi_r^0|^2dr + \int_{t_{i-1}}^{t_i}\left|\sum_{j\geq i}\chi_r^j\right|^2dr\right]\right\}.
\end{aligned}$$

Therefore, by following the step of [11], we get

$$\sum_{i=1}^{n} \mathbb{E}\left(\int_{t_{i-1}}^{t_i} |I_t^1|^2 dt\right) \leq C|\pi|\mathbb{E}(\Gamma_T^{12}) \leq C|\pi|. \tag{2.25}$$

Combining (2.23), (2.24) and (2.25), we infer (2.21) from (2.22). This, together with (2.20), leads to

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |Z_s - \tilde{Z}^{\pi}_{t_{i-1}}|^2 ds \leq C|\pi|,$$

which ends the estimate of the second term for the smooth case.

In general case, let b^ε, σ^ε, $\phi^{\pi,\varepsilon}$, f^ε and g^ε be molifiers of b, σ, ϕ^π, f and g, respectively, and let $(Y^\varepsilon, Z^\varepsilon)$ solution of BDSDE

$$Y_t^{\varepsilon,\pi} = \phi^{\pi,\varepsilon}(X_{t_0}^{\varepsilon,\pi}, \ldots, X_{t_n}^{\varepsilon,\pi}) + \int_t^T f^\varepsilon(\Theta_s^{\varepsilon,\pi})\, ds + \int_t^T g^\varepsilon(\Xi_s^{\varepsilon,\pi}) d\overleftarrow{B_s} - \int_t^T Z_s^{\varepsilon,\pi} dW_s, \ \ 0 \leq t \leq T,$$

where $X^{\varepsilon,\pi}$ is the well-know Euler approximation of the diffusion X^ε, the solution to the corresponding forward SDE (2.1) modified in an obvious way. Then by the above arguments we have

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |Z_s^\varepsilon - \tilde{Z}^{\pi,\varepsilon}_{t_{i-1}}|^2 ds \leq C|\pi|. \tag{2.26}$$

Therefore using again Lemma 3.4.2 of Zhang, [11], we have

$$\begin{aligned}
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |Z_s - \tilde{Z}^{\pi}_{t_{i-1}}|^2 ds &\leq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |Z_s - \tilde{Z}^{\pi,\varepsilon}_{t_{i-1}}|^2 ds \\
\leq \quad & \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [|Z_s - Z_s^\varepsilon|^2 + |Z_s^\varepsilon - \tilde{Z}^{\pi,\varepsilon}_{t_{i-1}}|^2] ds \\
\leq \quad & C\left\{\mathbb{E}\int_0^T |Z_s - Z_s^\varepsilon|^2 ds + |\pi|\right\}.
\end{aligned} \tag{2.27}$$

Applying Lemma 2.2 we have

$$\lim_{\varepsilon \to 0} \mathbb{E}\int_0^T |Z_s - Z_s^\varepsilon|^2 ds = 0,$$

which, combined with (2.27), proves the estimate of the second term of (3.7) and together with (2.17) prove the theorem. □

3 Discrete-time approximation error

In order to approximate the solution of the above decoupled FBDSDE (1.1), we introduce the following discretized version. Let $\pi : t_0 < t_1 < ..... < t_n = T$ be the partition of the time interval $[0, T]$ with mesh

$$|\pi| = \max_{1 \leq i \leq n} |t_i - t_{i-1}|$$

defined in the previous section. Throughout the rest of the paper, we will use the notations.

$$\triangle_i^\pi = t_i - t_{i-1},\ \triangle^\pi W_i = W_{t_i} - W_{t_{i-1}},\ \text{and}\ \triangle^\pi B_i = B_{t_i} - B_{t_{i-1}}\ \text{ for } i = 1, \ldots, n.$$

The forward component will be approximated by the classical Euler scheme

$$\begin{aligned} X_{t_0}^\pi &= X_{t_0}, \\ X_{t_i}^\pi &= X_{t_{i-1}}^\pi + b(X_{t_{i-1}}^\pi)\triangle_i^\pi + \sigma(X_{t_{i-1}}^\pi)\triangle^\pi W_i \ \text{ for } \ i = 1, \ldots, n \end{aligned} \tag{3.1}$$

and we set

$$X_t^\pi = X_{t_{i-1}}^\pi + b(X_{t_{i-1}}^\pi)(t - t_{i-1}) + \sigma(X_{t_{i-1}}^\pi)(W_t - W_{t_{i-1}}) \ \text{ for } \ t \in (t_{i-1}, t_i).$$

We shall denote by $\{\mathcal{F}_{t_i}^\pi\}_{0 \le i \le n}$ the associated discrete-time filtration define by

$$\mathcal{F}_{t_i}^\pi = \mathcal{F}_{t_i}^W \vee \mathcal{F}_T^B.$$

Under the Lipschitz conditions on b and σ, the following L^p estimate for the error due to the Euler scheme is well known

$$\limsup_{\pi \longrightarrow 0} |\pi|^{-1/2} \max_{1 \le i \le n} \mathbb{E}\left[\sup_{0 \le t \le T} |X_t^\pi - X_t|^p + \sup_{t_{i-1} \le t \le t_i} |X_t - X_{t_{i-1}}|^p\right]^{1/p} < \infty, \tag{3.2}$$

for all $p \ge 1$ (see e.g Kloeden and Platen, [7]). We next consider the following natural discrete-time approximation of the backward component Y:

$$Y_{t_n}^\pi = h(X_T^\pi),\ Z_{t_n}^\pi = 0$$

$$Z_{t_{i-1}}^\pi = \frac{1}{\triangle_i^\pi} \mathbb{E}_{i-1}^\pi[\left(Y_{t_i}^\pi + g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)\triangle^\pi B_i\right)\triangle^\pi W_i], \tag{3.3}$$

$$Y_{t_{i-1}}^\pi = \mathbb{E}_{i-1}^\pi[Y_{t_i}^\pi + g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)\triangle^\pi B_i] + f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)\triangle_i^\pi, \tag{3.4}$$

where $\mathbb{E}_i^\pi[.] = \mathbb{E}[.|\mathcal{F}_{t_i}^\pi]$. The above conditional expectation are well defined at each step of the algorithm. Indeed using the backward induction argument, it easily checked that $Y_{t_i}^\pi \in L^2$ for all i.

Remark 3.1. Using the induction argument, it easily seen that the random variable $Y_{t_i}^\pi$ and $Z_{t_i}^\pi$ are ω_1 deterministic function of $X_{t_i}^\pi$ for each $i = 0, \ldots, n$. Then using the fixed point of Banach argument (3.4) have a unique solution when the mesh of the partition $|\pi|$ is small enough.

For later use, we need a continuous-time approximation of (Y, Z). Since $Y_{t_i}^\pi$ $+g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)\triangle^\pi B_i = \tilde{Y}_{t_i}^\pi$ being in L^2 for all $1 \le i \le n$, an obvious extension of Itô martingale representation theorem yields the existence of the $\mathcal{F}_t$-progressively measurable and square integrable process Z^π satisfying

$$\tilde{Y}_{t_i}^\pi = \mathbb{E}[\tilde{Y}_{t_i}^\pi|\mathcal{F}_{t_{i-1}}^\pi] + \int_{t_{i-1}}^{t_i} Z_s^\pi dW_s. \tag{3.5}$$

Then we define inductively

$$
\begin{aligned}
Y_t^\pi &= Y_{t_{i-1}}^\pi - (t - t_{i-1}) f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi) - g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)(B_t - B_{t_{i-1}}) \\
&\quad + \int_{t_{i-1}}^t Z_s^\pi dW_s, \qquad t_{i-1} < t \le t_i. \qquad (3.6)
\end{aligned}
$$

The following property of the Z^π is needed for the proof of the main result of this section.

Lemma 3.2. *For all* $1 \le i \le n$*, we have*

$$
\triangle_i^\pi Z_{t_{i-1}}^\pi = \mathbb{E}_{i-1}^\pi \left[\int_{t_{i-1}}^{t_i} Z_s^\pi ds \right].
$$

Proof. Since

$$
\triangle_i^\pi Z_{t_{i-1}}^\pi = \frac{1}{\triangle_i^\pi} \mathbb{E}_{i-1}^\pi [\left(Y_{t_i}^\pi + g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi) \triangle^\pi B_i\right) \triangle^\pi W_i],
$$

recalling (3.5), we have

$$
Z_{t_{i-1}}^\pi = \frac{1}{\triangle_i^\pi} \mathbb{E}_{i-1}^\pi \left[\triangle^\pi W_i \int_{t_{i-1}}^{t_i} Z_s^\pi dW_s \right].
$$

The result follows by Itô's isometry. □

We also need the following estimate, which is a particular case of Lemma 2.4.

Lemma 3.3. *For each* $1 \le i \le n$*, we define*

$$
\tilde{Z}_{t_{i-1}}^\pi = \frac{1}{\triangle_i^\pi} \mathbb{E}_{i-1}^\pi \left[\int_{t_{i-1}}^{t_i} Z_s ds \right].
$$

Then

$$
\limsup_{\pi \to 0} |\pi|^{-1} \mathbb{E} \left[\max_{1 \le i \le n} \sup_{t_{i-1} \le t \le t_i} |Y_t - Y_{t_{i-1}}|^2 + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |Z_s - \tilde{Z}_{t_{i-1}}^\pi|^2 ds \right] < \infty. \qquad (3.7)
$$

We are now ready to state our main result of this section, which provides the rate of convergence of the approximation scheme (3.3) and (3.4) of the same order than Bouchard and Touzi [3].

Theorem 3.4.

$$
Err_\pi(Y,Z) = \left\{ \sup_{0 \le t \le T} \mathbb{E}|Y_t - Y_t^\pi|^2 + \mathbb{E}\left[\int_0^T |Z_s - Z_s^\pi|^2 ds \right] \right\}^{1/2} < C|\pi|^{1/2}.
$$

Proof. In the following, $C > 0$ will denote the generic constant independent of i and n that may take values from line to line. Let $i \in \{0, \ldots, n-1\}$ be fixed, and set

$$
\begin{aligned}
\delta^\pi Y_t &= Y_t - Y_t^\pi, \;\; \delta^\pi Z_t = Z_t - Z_t^\pi, \;\; \delta^\pi f(t) = f(t, X_t, Y_t, Z_t) - f(t_i, X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi) \\
&\quad \text{and } \; \delta^\pi g(t) = g(t, X_t, Y_t,) - g(t_{i+1}, X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi),
\end{aligned}
$$

for $t \in [t_i, t_{i+1})$. By Itô's formula, we compute that

$$\begin{aligned} V_t &= \mathbb{E}|\delta^\pi Y_t|^2 + \mathbb{E}\int_t^{t_{i+1}} |\delta^\pi Z_s|^2 ds - |\delta^\pi Y_{t_{i+1}}|^2 \\ &= 2\mathbb{E}\int_t^{t_{i+1}} \langle \delta^\pi Y_s, \delta^\pi f(s)\rangle ds + \int_t^{t_{i+1}} |\delta^\pi g(s)|^2 ds, \quad t_i \le t \le t_{i+1}. \end{aligned}$$

Let $\beta > 0$ be a constant to be chosen later. From Lipschitz property of f, g and h, together with the inequality $ab \le \beta a^2 + b^2/\beta$ this provides

$$\begin{aligned} V_t \le\ & \frac{C}{\beta}\int_t^{t_{i+1}} \mathbb{E}\left\{|\pi|^2 + |X_s - X^\pi_{t_i}|^2 + |Y_s - Y^\pi_{t_i}|^2 + |Z_s - Z^\pi_{t_i}|^2\right\} ds \\ &+ \int_t^{t_{i+1}} C\mathbb{E}\left\{|\pi|^2 + |X_s - X^\pi_{t_{i+1}}|^2 + |Y_s - Y^\pi_{t_{i+1}}|^2\right\} ds \\ &+ \beta\int_t^{t_{i+1}} \mathbb{E}|\delta^\pi Y_s|^2 ds. \end{aligned} \tag{3.8}$$

Now observe that

$$\begin{aligned} &\mathbb{E}|X_s - X^\pi_{t_i}|^2 + \mathbb{E}|X_s - X^\pi_{t_{i+1}}|^2 \le C|\pi|, \\ &\mathbb{E}|Y_s - Y^\pi_{t_i}|^2 \le 2\left(\mathbb{E}|Y_s - Y_{t_i}|^2 + \mathbb{E}|\delta^\pi Y_{t_i}|^2\right) \le C\left(|\pi| + \mathbb{E}|\delta^\pi Y_{t_i}|^2\right) \\ &\mathbb{E}|Y_s - Y^\pi_{t_{i+1}}|^2 \le 2\left(\mathbb{E}|Y_s - Y_{t_{i+1}}|^2 + \mathbb{E}|\delta^\pi Y_{t_{i+1}}|^2\right) \le C\left(|\pi| + \mathbb{E}|\delta^\pi Y_{t_{i+1}}|^2\right) \end{aligned} \tag{3.9}$$

by (3.2) and (3.7). Also, with the notation of Lemma 3.3, it follows from Lemma 3.2 that

$$\begin{aligned} \mathbb{E}|Z_s - Z^\pi_{t_i}|^2 &\le 2\left(\mathbb{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 + \mathbb{E}|\tilde{Z}^\pi_{t_i} - Z^\pi_{t_i}|^2\right) \\ &= 2\left(\mathbb{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 + \mathbb{E}\left|\frac{1}{\Delta^\pi_{i+1}}\int_{t_i}^{t_{i+1}} \mathbb{E}\left(\delta^\pi Z_r|\mathcal{F}_{t_i}\right) dr\right|^2\right) \\ &\le 2\left(\mathbb{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 + \frac{1}{\Delta^\pi_{i+1}}\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta^\pi Z_r|^2 dr\right) \end{aligned} \tag{3.10}$$

by Jensen's inequality.

We now plug (3.9) and (3.10) into (3.8) to obtain

$$\begin{aligned} V_t \le\ & \frac{C}{\beta}\int_t^{t_{i+1}} \mathbb{E}\left\{|\pi| + |\delta^\pi Y_{t_i}|^2 + |Z_s - \tilde{Z}^\pi_{t_i}|^2\right\} ds \\ &+ C\int_t^{t_{i+1}} \mathbb{E}\left\{|\pi| + |\delta^\pi Y_{t_{i+1}}|^2\right\} ds \\ &+ \frac{1}{\Delta^\pi_{i+1}}\frac{C}{\beta}\int_t^{t_{i+1}}\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta^\pi Z_r|^2 dr ds \\ &+ \beta\int_t^{t_{i+1}} \mathbb{E}|\delta^\pi Y_s|^2 ds \\ \le\ & \frac{C}{\beta}\int_t^{t_{i+1}} \mathbb{E}\left\{|\pi| + |\delta^\pi Y_{t_i}|^2 + |Z_s - \tilde{Z}^\pi_{t_i}|^2\right\} ds \\ &+ C\int_t^{t_{i+1}} \mathbb{E}\left\{|\pi| + |\delta^\pi Y_{t_{i+1}}|^2\right\} ds \\ &+ \frac{C}{\beta}\int_t^{t_{i+1}} \mathbb{E}|\delta^\pi Z_s|^2 ds + \beta\int_t^{t_{i+1}} \mathbb{E}|\delta^\pi Y_s|^2 ds. \end{aligned}$$

From the definition of V_t and (3.11), we see that, for $t_i \le t \le t_{i+1}$,

$$\mathbb{E}|\delta^\pi Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E}|\delta^\pi Z_s|^2 ds \le \beta \int_t^{t_{i+1}} \mathbb{E}|\delta^\pi Y_s|^2 ds + A_i \tag{3.11}$$

where

$$\begin{aligned} A_i &= (1+C\pi)\mathbb{E}|\delta^\pi Y_{t_{i+1}}|^2 + \frac{C}{\beta}\left[|\pi|^2 + |\pi|\mathbb{E}|Y^\pi_{t_i}| + \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 ds\right] \\ &\quad + \frac{C}{\beta}\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta^\pi Z_s|^2 ds. \end{aligned}$$

By Gronwall's Lemma, this shows that $\mathbb{E}|\delta^\pi Y_t|^2 \le A_i e^{\beta|\pi|}$ for $t_i \le t < t_{i+1}$, which plugged in the second inequality of (3.11) provides

$$\mathbb{E}|\delta^\pi Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E}|\delta^\pi Z_s|^2 ds \le A_i\left(1 + |\pi|\beta e^{\beta|\pi|}\right) \le A_i(1 + C\beta|\pi|) \tag{3.12}$$

for $|\pi|$ small enough. For $t = t_i$ and β sufficiently large than C, such that $\frac{C}{\beta} < 1$, we deduce from the last inequality that

$$\begin{aligned} &\mathbb{E}|\delta^\pi Y_{t_i}|^2 + (1 - \frac{C}{\beta})\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta^\pi Z_s|^2 ds \\ \le\ & (1 + C|\pi|)\left\{\mathbb{E}|\delta^\pi Y_{t_{i+1}}|^2 + |\pi|^2 + \int_{t_i}^{t_{i+1}} \mathbb{E}[|Z_s - \tilde{Z}^\pi_{t_i}|^2] ds\right\} \end{aligned}$$

for small $|\pi|$.

Iterating the last inequality, we get

$$\begin{aligned} &\mathbb{E}|\delta^\pi Y_{t_i}|^2 + (1 - \frac{C}{\beta})\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta^\pi Z_s|^2 ds \\ \le\ & (1 + C|\pi|)^{T/|\pi|}\left\{\mathbb{E}|\delta^\pi Y_T|^2 + |\pi| + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \mathbb{E}[|Z_s - \tilde{Z}^\pi_{t_{i-1}}|^2] ds\right\}. \end{aligned}$$

Using the estimate (3.7), together with the Lipschitz property of g and (3.2), this provides

$$\begin{aligned} &\mathbb{E}|\delta^\pi Y_{t_i}|^2 + (1 - \frac{C}{\beta})\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta^\pi Z_s|^2 ds \\ \le\ & (1 + C|\pi|)^{T/\pi}\left\{\mathbb{E}|\delta^\pi Y_T|^2 + |\pi| + C|\pi|\right\} \le C|\pi| \end{aligned} \tag{3.13}$$

for small $|\pi|$. Summing up inequality (3.12) with $t = t_i$, we get

$$\begin{aligned} &\left[1 - \frac{C}{\beta}(1 + C\beta|\pi|)\right]\int_0^T \mathbb{E}|\delta^\pi Z_s|^2 ds \\ \le\ & (1 + C\beta|\pi|)\frac{C}{\beta}|\pi| + (1 + C\beta|\pi|)(1 + C|\pi|)\mathbb{E}|\delta^\pi Y_T|^2 \\ & + \left[(1 + C\beta|\pi|)\frac{C}{\beta}|\pi| - 1\right]\mathbb{E}|\delta^\pi Y_0|^2 \\ & + \left[(1 + C\beta|\pi|)((1 + C|\pi|) + \frac{C}{\beta}|\pi|) - 1\right]\sum_{i=1}^{n-1}\mathbb{E}|\delta^\pi Y_{t_i}|^2 \\ & + (1 + C\beta|\pi|)\frac{C}{\beta}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 ds. \end{aligned}$$

For β sufficiently larger that C, this proves that for small $|\pi|$:

$$\int_0^T \mathbb{E}|\delta^\pi Z_s|^2 ds \leq C\Bigg[|\pi| + \mathbb{E}|\delta^\pi Y_T|^2 + |\pi|\sum_{i=1}^{n-1}\mathbb{E}|\delta^\pi Y_{t_i}|^2 + \sum_{i=0}^{n-1}\mathbb{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 ds\Bigg],$$

where we recall that C is a generic constant which changes from line to line. We now use (3.13) and (3.7) to see that

$$\int_0^T \mathbb{E}|\delta^\pi Z_s|^2 ds \leq C|\pi|.$$

Together with Lemma 3.3 and (3.13), this shows that $A_i \leq C|\pi|$, and therefore,

$$\sup_{0\leq t\leq T}|\delta^\pi Y_t|^2 \leq C|\pi|.$$

by taking the supremum over t in (3.12). This end the proof of the theorem. □

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