

## $L^p$ -solution of reflected generalized BSDEs with non-Lipschitz coefficients

Auguste Aman

Communicated by Name of Editor

**Abstract.** In this paper, we continue in solving reflected generalized backward stochastic differential equations (RGSDE for short) and fixed terminal time with use some new technical aspects of the stochastic calculus related to the reflected generalized BSDE. Here, existence and uniqueness of solution is proved under the non-Lipschitz condition on the coefficients.

**Key words.** Reflected generalized backward stochastic differential equations;  $p$ -integrable data, non-Lipschitz coefficient.

**AMS classification.** 60F25; 60H20.

### 1. Introduction

The study of nonlinear backward stochastic differential equations (BSDEs, in short) was initiated by Pardoux and Peng [12]. Mainly motivated by financial problems (see e.g. the survey article by El Karoui et al. [8]), stochastic control and stochastic games (see the works by Hamadène and Lepeltier [5] and references therein), the theory of BSDEs was developed at high speed during the 1990. These equations also provide probabilistic interpretation for solutions to both elliptic and parabolic nonlinear partial differential equations (see Pardoux and Peng [13], Peng [15]). Indeed, coupled with a forward SDE, such BSDE's give an extension of the celebrate Feynman-Kac formula to nonlinear case.

In order to provide a probabilistic representation for solution of parabolic or elliptic semi-linear PDEs with Neumann boundary condition, Pardoux and Zhang [14] introduced the so-called generalized BSDEs. This equation involves the integral with respect to an increasing process.

El-Karoui et al. [9] have introduced the notion of reflected BSDEs (RBSDEs, in short). Actually, it is a BSDE, but one of the components of the solution is forced to stay above a given barrier. Since then, many others results on the RBSDEs have been established (see [4, 6] and references therein). In El-Karoui et al. [9], the RBSDEs also provided a probabilistic formula for the viscosity solution of an obstacle problem for a parabolic PDEs.

Following this way, Ren et al [16] have introduced the notion of reflected generalized BSDEs (RGSDE, in short). They connected it to the obstacle problem for PDEs with Neumann boundary condition. More precisely, let consider the following

RGBSDE: for  $0 \leq t \leq T$ ,

$$\begin{aligned}
 (i) \quad & Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dG_s - \int_t^T Z_s dW_s + K_T - K_t \\
 (ii) \quad & Y_t \geq S_t \\
 (iii) \quad & K \text{ is a non-decreasing process such that } K_0 = 0 \text{ and } \int_0^T (Y_t - S_t) dK_t = 0.
 \end{aligned} \tag{1.1}$$

They proved under suitable conditions on the data the existence and uniqueness of the solution  $(Y, Z, K)$ . The increasing process  $K$  is introduced to push the component  $Y$  upwards so that it may remain above the obstacle process  $S$ . In particular, condition (iii) means that the push is minimal and is done only when the constraint is saturated i.e.  $Y_t < S_t$ . In practice (finance market for example), the process  $K$  can be regarded as the subsidy injected by a government in the market to allow the price process  $Y$  of a commodity (coffee, by example) to remain above a threshold price process  $S$ .

In the Markovian framework, the RGBSDE (1.1) is combined with the following reflected forward SDE: for every  $(t, x) \in [0, T] \times \bar{\Theta}$  and  $s \in [t, T]$

$$\begin{cases}
 X_s^{t,x} = x + \int_t^{s \vee t} b(X_r^{t,x}) dr + \int_t^{s \vee t} \sigma(X_r^{t,x}) dW_r + \int_t^{s \vee t} \nabla \psi(X_r^{t,x}) dG_r^{t,x}, & s \geq 0 \\
 X_s^{t,x} \in \bar{\Theta} \text{ and } G_s^{t,x} = \int_t^{s \vee t} 1_{\{X_r^{t,x} \in \partial\Theta\}} dG_r^{t,x},
 \end{cases}$$

where  $G^{t,x}$  is an increasing process and  $\psi \in \mathcal{C}_b^2(\mathbb{R}^d)$  characterize  $\Theta$  and  $\partial\Theta$  as follows:

$$\Theta = \{x \in \mathbb{R}^d : \psi(x) > 0\} \quad \text{and} \quad \partial\Theta = \{x \in \mathbb{R}^d : \psi(x) = 0\}.$$

Assuming the data in the form  $\xi = l(X_T^{t,x})$ ,  $S_s = h(s, X_s^{t,x})$ ,  $f(s, y, z) = f(s, X_s^{t,x}, y, z)$ , and  $g(s, y) = g(s, X_s^{t,x}, y)$ , the RGBSDE (1.1) becomes: for any fixed  $t \in [0, T]$

$$\begin{cases}
 (i) \quad Y_s^{t,x} = l(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}) dG_r^{t,x} \\
 \quad \quad \quad - \int_s^T Z_r^{t,x} dW_r + K_T^{t,x} - K_s^{t,x}, \quad s \in [t, T] \\
 (ii) \quad Y_s^{t,x} \geq h(s, X_s^{t,x}), \quad a.s., \forall s \in [t, T] \\
 (iii) \quad K^{t,x} \text{ is a non-decreasing process such that } K_0^{t,x} = 0 \text{ and } \int_t^T (Y_s^{t,x} - h(s, X_s^{t,x})) dK_s^{t,x} = 0, \quad a.s.,
 \end{cases} \tag{1.2}$$

and gives a probabilistic interpretation of the following type of obstacle problem for a partial differential equation with nonlinear Neumann boundary condition:

$$\left\{ \begin{array}{l} \min \{u(t, x) - h(t, x), \\ -\frac{\partial u}{\partial t}(t, x) - (Lu)(t, x) - f(s, x, u(t, x), (\nabla u(t, x))^* \sigma(t, x))\} = 0, \\ (t, x) \in [0, T] \times \Theta \\ \frac{\partial u}{\partial n}(t, x) + g(t, x, u(t, x)) = 0, (t, x) \in [0, T] \times \partial\Theta \\ u(T, x) = l(x), x \in \bar{\Theta}, \end{array} \right.$$

where  $L$  is the infinitesimal generator corresponding to the diffusion process  $X^x$  and  $\frac{\partial}{\partial n}(\cdot) = \langle \nabla \psi, \nabla(\cdot) \rangle$ .

Apart the work of El Karoui et al. [8] and Briand et al. [3] in the case of standard BSDEs, there has been relatively few papers which deal with the problem of existence and/or uniqueness of the solution for BSDEs and RBSDEs in the case when the coefficients are not square integrable. This limits the scope for several applications (finance, stochastic control, stochastic games, PDEs, etc...). To correct this shortcoming, Hamadène and Popier [7] show that if  $\xi$ ,  $\sup_{0 \leq t \leq T} (S_t^+)$  and  $\int_0^T |f(t, 0, 0)| dt$  belong to  $L^p$  for some  $p \in ]1, 2[$ , then the RBSDEs with one reflecting barrier associated with  $(f, g = 0, \xi, S)$  has a unique solution. They prove existence and uniqueness of the solution in using penalization and Snell envelope of processes methods. In a previous works, Aman [1] give the similar result for a class of RBSDEs (1.1) with Lipschitz condition on the coefficients by used the  $L^\infty$ -approximation. In this paper, we extend the previous result, assuming that in this case coefficients are non-Lipschitz. The rest of the paper is organized as follows. The next section contains all the notations, assumptions and a priori estimates. Section 3 is devoted to existence and uniqueness result in  $L^p$ ,  $p \in (1, 2)$  when the coefficients are non-Lipschitz.

## 2. Preliminaries

### 2.1. Assumptions and basic notations

First of all,  $W = \{W_t\}_{t \geq 0}$  is a standard Brownian motion with values in  $\mathbb{R}^d$  defined on some complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .  $\{\mathcal{F}_t\}_{t \geq 0}$  is the augmented natural filtration of  $W$  which satisfies the usual conditions. In this paper, we will always use this filtration. In most of this work, the stochastic processes will be defined for  $t \in [0, T]$ , where  $T$  is a positive real number, and will take their values in  $\mathbb{R}$ .

For any real  $p > 0$ , let us define the following spaces:

$\mathcal{S}^p(\mathbb{R})$  denotes set of  $\mathbb{R}$ -valued, adapted càdlàg processes  $\{X_t\}_{t \in [0, T]}$  such that

$$\|X\|_{\mathcal{S}^p} = \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t|^p \right)^{1 \wedge \frac{1}{p}} < +\infty,$$

and  $\mathcal{M}^p(\mathbb{R}^d)$  is the set of predictable processes  $\{X_t\}_{t \in [0, T]}$  such that

$$\|X\|_{\mathcal{M}^p} = \mathbb{E} \left[ \left( \int_0^T |X_t|^2 dt \right)^{\frac{p}{2}} \right]^{1 \wedge \frac{1}{p}} < +\infty.$$

If  $p \geq 1$ , then  $\|X\|_{\mathcal{S}^p}$  (resp  $\|X\|_{\mathcal{M}^p}$ ) is a norm on  $\mathcal{S}^p(\mathbb{R})$  (resp.  $\mathcal{M}^p(\mathbb{R}^d)$ ) and these spaces are Banach spaces. But if  $p \in (0, 1)$ ,  $(X, X') \mapsto \|X - X'\|_{\mathcal{S}^p}$  (resp  $\|X - X'\|_{\mathcal{M}^p}$ ) defines a distance on  $\mathcal{S}^p(\mathbb{R})$ , (resp.  $\mathcal{M}^p(\mathbb{R}^d)$ ) and under this metric,  $\mathcal{S}^p(\mathbb{R})$  (resp.  $\mathcal{M}^p(\mathbb{R}^d)$ ) is complete.

Now let us give the following assumptions:

**(A1)**  $(G_t)_{t \geq 0}$  is a continuous real valued increasing  $\mathcal{F}_t$ -progressively measurable process with bounded variation on  $[0, T]$ .

**(A2)** Two functions  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  for some constants  $\beta < 0$ ,  $\lambda > 0$ ,  $\mu \in \mathbb{R}$  and for all  $t \in [0, T]$ ,  $y, y' \in \mathbb{R}$ ,  $z, z' \in \mathbb{R}^d$ :

(i)  $y \mapsto (f(t, y, z), g(t, y))$  is continuous for all  $z$ ,  $(t, \omega)$  a.e.,

(ii)  $f(\cdot, y, z)$  and  $g(\cdot, y)$  are progressively measurable,

(iii)  $|f(t, y, z) - f(t, y, z')| \leq \lambda |z - z'|$ ,

(iv)  $(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu |y - y'|^2$ ,

(v)  $|f(t, y, z)| \leq |f(t, 0, 0)| + M(|y| + |z|)$

(vi)  $(y - y')(g(t, y) - g(t, y')) \leq \beta |y - y'|^2$ ,

(vii)  $|g(t, y)| \leq |g(t, 0)| + M|y|$ ,

(viii)  $\mathbb{E} \left[ \left( \int_0^T |f(s, 0, 0)| ds \right)^p + \left( \int_0^T |g(s, 0)| dG_s \right)^p \right] < \infty$ .

**(A3)** For any  $r > 0$ , we define the process  $\pi_r$  in  $L^p([0, T] \times \Omega, m \otimes \mathbb{P})$  by

$$\pi_r(t) = \sup_{|y| \leq r} |f(t, y, 0) - f(t, 0, 0)|.$$

**(A4)**  $\xi$  is a  $\mathcal{F}_T$ -measurable variable such that  $\mathbb{E}(|\xi|^p) < +\infty$ .

**(A5)** There exists a barrier  $(S_t)_{t \geq 0}$  which is a continuous, progressively measurable, real-valued process satisfying:

- (i)  $\mathbb{E} \left( \sup_{0 \leq t \leq T} (S_t^+)^p \right) < +\infty$ ,
- (ii)  $S_T \leq \xi \quad \mathbb{P}$ - a.s.

Before of all, let us recall what we mean by a  $L^p$ -solution of RGBSDEs.

**Definition 2.1.** A  $L^p$ -solution of RGBSDE associated to the data  $(\xi, f, g, S)$  is a triplet  $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$  of progressively measurable processes taking values in  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$  and satisfying:

- (i)  $Y$  is a continuous process,
- (ii)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dG_s - \int_t^T Z_s dW_s + K_T - K_t, (2.1)$$

- (iii)  $Y_t \geq S_t \quad \text{a.s.}$ ,

- (iv)  $\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t|^p + \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right) < +\infty$ ,

- (v)  $K$  is a non-decreasing process such that  $K_0 = 0$  and  $\int_0^T (Y_s - S_s) dK_s = 0, \quad \text{a.s.}$

## 2.2. A priori estimates

In this paragraph, we state some estimates for solution of RGBSDE associated to  $(\xi, f, g, S)$  in  $L^p$  when  $p > 1$  like in [1]. But the difficulty here comes from the facts the function  $f$  is not supposed to be Lipschitz continuous. Let us give the notation  $\hat{x} = |x|^{-1} x \mathbf{1}_{\{x \neq 0\}}$  introduced in [3] that will play an important role in the sequel.

**Lemma 2.2.** Assume that  $(Y, Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{M}^p(\mathbb{R}^d)$  is a solution of the following BSDE:

$$Y_t = \xi + \int_t^T \tilde{f}(s, Y_s, Z_s) ds + \int_t^T \tilde{g}(s, Y_s) dG_s - \int_t^T Z_s dW_s + A_T - A_t, \quad 0 \leq t \leq T, (2.2)$$

where

- (i)  $\tilde{f}$  and  $\tilde{g}$  are functions which satisfy assumptions **(A2)**,
- (ii)  $\mathbb{P}$  a.s., the process  $(A_t)_{0 \leq t \leq T}$  is of bounded variation type.

Then for any  $0 \leq t \leq T$  we have:

$$\begin{aligned} & |Y_t|^p + c(p) \int_t^T |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \\ & \leq |\xi| + p \int_t^T |Y_s|^{p-1} \widehat{Y}_s \tilde{f}(s, Y_s, Z_s) ds + p \int_t^T |Y_s|^{p-1} \widehat{Y}_s \tilde{g}(s, Y_s) dG_s \\ & \quad + p \int_t^T |Y_s|^{p-1} \widehat{Y}_s dA_s - p \int_t^T |Y_s|^{p-1} \widehat{Y}_s Z_s dW_s. \end{aligned}$$

with  $c(p) = p[(p-1) \wedge 1]/2$ .

We now show how to control the process  $Z$  in terms of the data and the process  $Y$ .

**Lemma 2.3.** *Let assume (A1)-(A4) hold and let  $(Y, Z, K)$  be the solution of RGB-SDE associated to  $(\xi, f, g, S)$ . If  $Y \in \mathcal{S}^p$  then  $Z$  belong to  $\mathcal{M}^p$  and there exists a real constant  $C_{p,\lambda}$  depending only on  $p$  and  $\lambda$  such that,*

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^T |Z_r|^2 dr \right)^{p/2} \right] & \leq C_{p,\lambda} \mathbb{E} \left\{ \sup_{0 \leq t \leq T} |Y_t|^p + \left( \int_0^T f_r^0 dr \right)^p \right. \\ & \quad \left. + \left( \int_0^T g_r^0 dG_r \right)^p + \sup_{0 \leq t \leq T} |S_t^+|^p \right\}, \end{aligned}$$

where  $f_r^0 = |f(r, 0, 0)|$  and  $g_r^0 = |g(r, 0)|$ .

*Proof.* For each integer  $n \geq 1$  let introduce

$$\tau_n = \inf \left\{ t \in [0, T], \int_0^t |Z_r|^2 dr \geq n \right\} \wedge T.$$

The sequence  $(\tau_n)_{n \geq 0}$  is of stationary type since the process  $Z$  belongs to  $\mathcal{M}^p$  and then  $\int_0^T |Z_s|^2 ds < \infty$ ,  $\mathbb{P}$ - a.s.. Next, for any  $\alpha > 0$ , using Itô's formula and assumption (A2), we get

$$\begin{aligned} & |Y_0|^2 + \int_0^{\tau_n} e^{\alpha r} |Z_r|^2 dr + |\beta| \int_0^{\tau_n} e^{\alpha r} |Y_r|^2 dG_r \\ & \leq e^{\alpha \tau_n} |Y_{\tau_n}|^2 + 2 \sup_{0 \leq t \leq T} e^{\alpha t} |Y_t| \times \left[ \int_0^{\tau_n} (f_r^0 dr + g_r^0 dG_r) \right] + (2\lambda + \varepsilon^{-1}\lambda - \alpha) \int_0^{\tau_n} e^{\alpha r} |Y_r|^2 dr \\ & \quad + \varepsilon \int_0^{\tau_n} e^{\alpha r} |Z_r|^2 dr + \frac{1}{\varepsilon} \sup_{0 \leq t \leq \tau_n} e^{2\alpha t} |Y_t|^2 + \varepsilon |K_{\tau_n}|^2 - 2 \int_0^{\tau_n} e^{\alpha r} Y_r Z_r dW_r, \end{aligned}$$

in virtue of the standard inequality  $2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2$  for any  $\varepsilon > 0$  and since  $\beta < 0$ .

But

$$\begin{aligned}
|K_{\tau_n}|^2 \leq C_\lambda & \left\{ |Y_0|^2 + |Y_{\tau_n}|^2 + \left( \int_0^{\tau_n} f_r^0 dr \right)^2 + \int_0^{\tau_n} |Y_r|^2 dr + \int_0^{\tau_n} |Y_r|^2 dG_r \right. \\
& \left. + \left( \int_0^{\tau_n} g_r^0 dG_r \right)^2 + \int_0^{\tau_n} |Z_r|^2 dr + \left| \int_0^{\tau_n} Z_r dW_r \right| \right\} \quad (2.3)
\end{aligned}$$

so that we have:

$$\begin{aligned}
& (1 - \varepsilon C_\lambda) |Y_0|^2 + (1 - \varepsilon - \varepsilon C_\lambda) \int_0^{\tau_n} e^{\alpha r} |Z_r|^2 dr \\
& \leq (\varepsilon C_\lambda + e^{\alpha \tau_n}) |Y_{\tau_n}|^2 + (1 + \varepsilon C_\lambda) \left[ \left( \int_0^{\tau_n} f_r^0 dr \right)^2 + \left( \int_0^{\tau_n} g_r^0 dG_r \right)^2 \right] \\
& + (2\lambda + \varepsilon^{-1} \lambda - \alpha) \int_0^{\tau_n} e^{\alpha r} |Y_r|^2 dr + \left(1 + \frac{1}{\varepsilon}\right) \sup_{0 \leq t \leq \tau_n} e^{2\alpha t} |Y_t|^2 \\
& + \varepsilon C_\lambda \left| \int_0^{\tau_n} Z_r dW_r \right| + 2 \left| \int_0^{\tau_n} e^{\alpha r} Y_r Z_r dW_r \right|.
\end{aligned}$$

Choosing now  $\varepsilon$  small enough and  $\alpha$  such that  $2\lambda + \varepsilon^{-1} \lambda - \alpha < 0$ , we obtain:

$$\begin{aligned}
\left( \int_0^{\tau_n} |Z_r|^2 dr \right)^{p/2} & \leq C_{p,\lambda} \left\{ \sup_{0 \leq t \leq \tau_n} Y_t^p + \left( \int_0^{\tau_n} f_r^0 dr \right)^p \right. \\
& \left. + \left( \int_0^{\tau_n} g_r^0 dG_r \right)^p + \left| \int_0^{\tau_n} e^{\alpha r} Y_r Z_r dW_r \right|^{p/2} \right\}.
\end{aligned}$$

Next thanks to BDG's inequality it follows:

$$\begin{aligned}
\mathbb{E} \left( \left| \int_0^{\tau_n} e^{\alpha r} Y_r Z_r dW_r \right|^{p/2} \right) & \leq d_p \mathbb{E} \left[ \left( \int_0^{\tau_n} |Y_r|^2 |Z_r|^2 dr \right)^{p/4} \right] \\
& \leq \bar{C}_p \mathbb{E} \left[ \sup_{0 \leq t \leq \tau_n} |Y_t|^{p/2} \left( \int_0^{\tau_n} |Z_r|^2 dr \right)^{p/4} \right] \\
& \leq \frac{\bar{C}_p^2}{\eta} \mathbb{E} \left( \sup_{0 \leq t \leq \tau_n} |Y_t|^p \right) + \eta \mathbb{E} \left( \int_0^{\tau_n} |Z_r|^2 dr \right)^{p/2}.
\end{aligned}$$

Finally plugging the last inequality in the previous one, choosing  $\eta$  small enough and finally using Fatou's lemma we obtain the desired result.  $\square$

We will now establish an estimate for the processes  $Y$  and  $Z$ . The difficulty comes from the fact that the function  $y \mapsto |y|^p$  is not  $C^2$  since we work with  $p \in (1, 2)$ . Actually we have:

**Lemma 2.4.** Assume (A1)-(A4). Let  $(Y, Z, K)$  be a solution of the RGBDSE associated to the data  $(\xi, f, g, S)$  where  $Y$  belong to  $\mathcal{S}^p$ . Then there exists a constant  $C_{p,\lambda}$  depending only on  $p$  and  $\lambda$  such that

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} |Y_t|^p + \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right\} \leq C_{p,\lambda} \mathbb{E} \left\{ |\xi|^p + \left( \int_0^T f_s^0 ds \right)^p + \left( \int_0^T g_s^0 dG_s \right)^p + \sup_{0 \leq t \leq T} (S_t^+)^p \right\}.$$

*Proof.* For any  $\alpha > 0$ , it from Lemma 2.2, together with assumption (A2) that

$$\begin{aligned} & e^{p\alpha t} |Y_t|^p + c(p) \int_t^u e^{p\alpha s} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \\ & \leq e^{p\alpha u} |Y_u|^p + p(\lambda - \alpha) \int_u^T e^{p\alpha s} |Y_s|^p ds + p \int_t^u e^{p\alpha s} |Y_s|^{p-1} f_s^0 ds \\ & \quad + p \int_t^u e^{p\alpha s} |Y_s|^{p-1} g_s^0 dG_s + p\lambda \int_u^T e^{p\alpha s} |Y_s|^{p-1} |Z_s| ds \\ & \quad + p \int_t^u e^{p\alpha s} |Y_s|^{p-1} \widehat{Y}_s dK_s - p \int_t^u e^{p\alpha s} |Y_s|^{p-1} \widehat{Y}_s Z_s dW_s. \end{aligned}$$

We have by Young's inequality

$$p\lambda |Y_s|^{p-1} |Z_s| \leq \frac{p\lambda^2}{p-1} |Y_s|^p + \frac{c(p)}{2} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2,$$

and

$$\begin{aligned} p \int_t^u e^{p\alpha s} |Y_s|^{p-1} (f_s^0 ds + g_s^0 dG_s) & \leq (p-1) \gamma^{\frac{p}{p-1}} \sup_{0 \leq s \leq u} |Y_s|^p \\ & \quad + \gamma^{-p} \left[ \left( \int_t^u e^{p\alpha s} f_s^0 ds \right)^p + \left( \int_t^u e^{p\alpha s} g_s^0 dG_s \right)^p \right] \end{aligned}$$

for any  $\gamma > 0$ . Then plug the two last inequalities in the previous one, we obtain:

$$\begin{aligned} & e^{p\alpha t} |Y_t|^p + \frac{c(p)}{2} \int_t^u e^{p\alpha s} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \\ & \leq e^{p\alpha u} |Y_u|^p + (p-1) \gamma^{\frac{p}{p-1}} \sup_{0 \leq s \leq u} |Y_s|^p \\ & \quad + \gamma^{-p} \left[ \left( \int_t^u e^{p\alpha s} f_s^0 ds \right)^p + \left( \int_t^u e^{p\alpha s} g_s^0 dG_s \right)^p \right] \\ & \quad + p \left( \lambda + \frac{\lambda^2}{p-1} - \alpha \right) \int_t^u e^{p\alpha s} |Y_s|^p ds \\ & \quad + p \int_t^u e^{p\alpha s} |Y_s|^{p-1} \widehat{Y}_s dK_s - p \int_t^u e^{p\alpha s} |Y_s|^{p-1} \widehat{Y}_s Z_s dW_s. \end{aligned}$$

Next, the hypothesis related to increments of  $K$  and  $Y - S$  implies that

$$\begin{aligned} \int_t^u e^{p\alpha s} |Y_s|^{p-1} \widehat{Y}_s dK_s &\leq \int_t^u e^{p\alpha s} |S_s|^{p-1} \widehat{S}_s dK_s \\ &\leq \int_t^u e^{p\alpha s} (S_s^+)^{p-1} dK_s \\ &\leq \frac{p-1}{p} \frac{1}{\varepsilon^{\frac{p}{p-1}}} \left( \sup_{0 \leq t \leq u} |S_t^+|^p \right) + \frac{1}{p} \varepsilon^p \left( \int_t^u e^{p\alpha s} dK_s \right) \end{aligned}$$

for any  $\varepsilon > 0$ , so that choosing  $\alpha$  such that  $\lambda + \frac{\lambda^2}{p-1} \leq \alpha$  and put  $u = T$ , we get:

$$\begin{aligned} &\mathbb{E} \left( e^{p\alpha T} |Y_t|^p \right) + \frac{c(p)}{2} \mathbb{E} \left( \int_t^T e^{p\alpha s} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right) \\ &\leq \mathbb{E} \left( e^{p\alpha T} |\xi|^p \right) + (p-1) \gamma^{\frac{p}{p-1}} \mathbb{E} \left( \sup_{0 \leq s \leq T} |Y_s|^p \right) \\ &\quad + \gamma^{-p} \mathbb{E} \left[ \left( \int_t^T e^{p\alpha s} f_s^0 ds \right)^p + \left( \int_t^T e^{p\alpha s} g_s^0 dG_s \right)^p \right] \quad (2.4) \\ &\quad + (p-1) \frac{1}{\varepsilon^{\frac{p}{p-1}}} \mathbb{E} \left( \sup_{0 \leq t \leq T} |S_t^+|^p \right) + \frac{1}{p} \varepsilon^p \mathbb{E} \left( \int_t^T e^{p\alpha s} dK_s \right). \end{aligned}$$

On the other hand the predictable dual projection, Jensen's conditional inequality and together with Lemma 2.3 provide

$$\mathbb{E}[(K_T - K_t)^p] \leq C_{\lambda,p} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y_s|^p + \left( \int_t^T f_s^0 ds \right)^p + \left( \int_t^T g_s^0 dG_s \right)^p \right], \quad (2.5)$$

where  $C_{\lambda,p}$  is a constant which depend on  $p$ ,  $\lambda$  and possibly  $T$  which may change from line to another.

Coming back to inequality (2.4) and using BDG inequality we have

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} e^{p\alpha t} |Y_t|^p &\leq \mathbb{E}(e^{p\alpha T} |\xi|^p) + (p-1) \frac{1}{\varepsilon^{\frac{p}{p-1}}} \mathbb{E} \left( \sup_{0 \leq t \leq T} |S_t^+|^p \right) \\
&\quad + \{C_{\lambda,p}(\gamma^{\frac{p}{p-1}} + \varepsilon^p) + p\eta\} \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t|^p \right) \\
&\quad + C_{\lambda,p} \left( \frac{1}{\gamma^p} + \varepsilon^p \right) \mathbb{E} \left[ \left( \int_0^T e^{p\alpha s} f_s^0 ds \right)^p + \left( \int_0^T e^{p\alpha s} g_s^0 dG_s \right)^p \right] \\
&\quad + \frac{p}{\eta} \mathbb{E} \left( \int_0^T e^{p\alpha s} |Y_s|^{p-2} \widehat{Y}_s \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right) \\
&\leq \left( 1 + \frac{2p}{c(p)\eta} \right) \mathbb{E}(e^{p\alpha T} |\xi|^p) + \left( 1 + \frac{2p}{c(p)\eta} \right) (p-1) \frac{1}{\varepsilon^{\frac{p}{p-1}}} \mathbb{E} \left( \sup_{0 \leq t \leq T} |S_t^+|^p \right) \\
&\quad + \left( 1 + \frac{2p}{c(p)\eta} \right) C_{\lambda,p} \left( \frac{1}{\gamma^p} + \varepsilon^p \right) \mathbb{E} \left[ \left( \int_0^T e^{p\alpha s} f_s^0 ds \right)^p + \left( \int_0^T e^{p\alpha s} g_s^0 dG_s \right)^p \right] \\
&\quad + \left\{ C_{\lambda,p} \left( 1 + \frac{2p}{c(p)\eta} \right) (\gamma^{\frac{p}{p-1}} + \varepsilon^p) + p\eta \right\} \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t|^p \right)
\end{aligned}$$

Finally it is enough to chose  $\eta = \frac{1}{2p}$  and  $\gamma, \varepsilon$  small enough to obtain the desired result.  $\square$

**Lemma 2.5.** Assume that  $(f, g, \xi, S)$  and  $(f', g', \xi', S')$  are two quadruplets satisfying assumptions (A1)-(A4). Suppose that  $(Y, Z, K)$  is a solution of RGBSDE  $(f, g, \xi, S)$  and  $(Y', Z', K')$  is a solution of RGBSDE  $(f', g', \xi', S')$ . Let us set:

$$\begin{aligned}
\Delta f &= f - f', \quad \Delta \xi = \xi - \xi', \quad \Delta S = S - S' \\
\Delta Y &= Y - Y', \quad \Delta Z = Z - Z', \quad \Delta K = K - K'
\end{aligned}$$

and assume that  $\Delta S \in L^p(dt \times \mathbb{P})$ . Then there exists a constant  $C$  such that

$$\begin{aligned}
\mathbb{E} \left( \sup_{t \in [0, T]} |\Delta Y_t|^p \right) &\leq C \mathbb{E} \left[ |\Delta \xi|^p + \left( \int_0^T |\Delta f(s, Y_s, Z_s)| ds \right)^p \right] \\
&\quad + \left( \int_0^T |\Delta g(s, Y_s)| dG_s \right)^p + C(\Psi(T))^{1/p} \mathbb{E} \left[ \sup_{t \in [0, T]} |\Delta S_t|^p \right]^{\frac{p-1}{p}},
\end{aligned}$$

with

$$\begin{aligned} \Psi(T) = & \mathbb{E} \left[ |\xi|^p + \left( \int_0^T f_s^0 ds \right)^p + \left( \int_0^T g_s^0 dG_s \right)^p + \sup_{t \in [0, T]} (S_t^+)^p \right. \\ & \left. + |\xi'|^p + \left( \int_0^T f_s'^0 ds \right)^p + \left( \int_0^T g_s'^0 dG_s \right)^p + \sup_{t \in [0, T]} (S_t'^+)^p \right]. \end{aligned}$$

*Proof.* Using Lemma 2.1 and (A2) we have for all  $0 \leq t \leq T$ :

$$\begin{aligned} & |\Delta Y_t|^p + c(p) \int_t^T |\Delta Y_s|^{p-2} \mathbf{1}_{\{\Delta Y_s \neq 0\}} |\Delta Z_s|^2 ds \\ \leq & |\Delta \xi|^p + p\lambda \int_t^T |\Delta Y_s|^{p-1} \widehat{\Delta Y}_s |\Delta Z_s| ds \\ & + p\lambda \int_t^T |\Delta Y_s|^p ds + p \int_t^T |\Delta Y_s|^{p-1} \widehat{\Delta Y}_s |\Delta f(s, Y_s, Z_s)| ds \quad (2.6) \\ & + p\beta \int_t^T |\Delta Y_s|^p dG_s + p \int_t^T |\Delta Y_s|^{p-1} \widehat{\Delta Y}_s |\Delta g(s, Y_s)| dG_s \\ & + p \int_t^T |\Delta Y_s|^{p-1} \widehat{\Delta Y}_s d(\Delta K_s) - p \int_t^T |\Delta Y_s|^{p-1} \widehat{\Delta Y}_s \Delta Z_s dW_s. \end{aligned}$$

Moreover

$$\begin{aligned} \int_t^T |\Delta Y_s|^{p-1} \widehat{\Delta Y}_s d(\Delta K_s) & \leq \int_t^T |\Delta S_s|^{p-2} (\Delta S_s) \mathbf{1}_{\{\Delta S_s \neq 0\}} dK_s \\ & \quad - \int_t^T |\Delta S_s|^{p-2} (\Delta S_s) \mathbf{1}_{\{\Delta S_s \neq 0\}} dK'_s \\ & \leq \int_t^T |\Delta S_s|^{p-1} d(\Delta K_s) \end{aligned}$$

Thus coming back to (2.6) and thanks to the Burkholder-Davis-Gundy and Young inequalities, we get with  $t = 0$

$$\begin{aligned} & \frac{c(p)}{2} \mathbb{E} \int_0^T |\Delta Y_s|^{p-2} \mathbf{1}_{\{\Delta Y_s \neq 0\}} |\Delta Z_s|^2 ds \\ \leq & \mathbb{E} |\Delta \xi|^p + \left( \frac{p\lambda^2}{p-1} + p\lambda \right) \mathbb{E} \int_0^T |\Delta Y_s|^p ds \\ & + p \mathbb{E} \int_0^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds + p \mathbb{E} \int_0^T |\Delta Y_s|^{p-1} |\Delta g(s, Y_s)| dG_s \\ & + p \mathbb{E} \int_0^T |\Delta S_s|^{p-1} d(\Delta K_s) \quad (2.7) \end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}|\Delta Y_t|^p &\leq \mathbb{E}|\Delta \xi|^p + \left(\frac{p\lambda^2}{p-1} + p\lambda\right)\mathbb{E}\int_0^T |\Delta Y_s|^p ds \\
&\quad + p\mathbb{E}\int_0^T |\Delta Y_s|^{p-1}|\Delta f(s, Y_s, Z_s)|ds + p\mathbb{E}\int_0^T |\Delta Y_s|^{p-1}|\Delta g(s, Y_s)|dG_s \\
&\quad + p\mathbb{E}\int_0^T |\Delta S_s|^{p-1}d(\Delta K_s), \tag{2.8}
\end{aligned}$$

since we recall again  $\beta < 0$ .

We have by holder's inequality

$$\mathbb{E}\int_0^T |\Delta S_s|^{p-1}d(\Delta K_s) \leq \left(\mathbb{E}\sup_{0 \leq t \leq T} |\Delta S_t|^p\right)^{\frac{p}{p-1}} (\Psi_T)^{1/p}$$

and

$$\begin{aligned}
&p\mathbb{E}\int_0^T |\Delta Y_s|^{p-1}|\Delta f(s, Y_s, Z_s)|ds + p\mathbb{E}\int_0^T |\Delta Y_s|^{p-1}|\Delta g(s, Y_s)|dG_s \\
&\leq \gamma\mathbb{E}\sup_{0 \leq t \leq T} |\Delta Y_t|^p + \frac{1}{\gamma}\mathbb{E}\left[\left(\int_0^T |\Delta f(s, Y_s, Z_s)|ds\right)^p + \left(\int_0^T |\Delta g(s, Y_s)|dG_s\right)^p\right]
\end{aligned}$$

for any  $\gamma > 0$ . Finally, return again to (2.6) and use again Burkholder-Davis-Gundy together with inequalities (2.7) and (2.8), it follows after choosing  $\gamma$  small enough:

$$\begin{aligned}
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t|^p\right) &\leq CE\left[|\Delta \xi|^p + \left(\int_0^T |\Delta f(s, Y_s, Z_s)|ds\right)^p + \left(\int_0^T |\Delta g(s, Y_s)|dG_s\right)^p\right] \\
&\quad + \left(\mathbb{E}\sup_{0 \leq t \leq T} |\Delta S_t|^p\right)^{\frac{p}{p-1}} (\Psi_T)^{1/p},
\end{aligned}$$

which ends the proof.  $\square$

### 3. Existence and uniqueness of a solution

With the help of the above a priori estimates, we can obtain an existence and uniqueness result by the use of  $L^\infty$ -approximation.

Firstly, let us give this result which is a slightly extension of Theorem 3.1 of Ren and Xia [16].

**Theorem 3.1.** *Assume (A1)-(A4). Then RGBSDE with data  $(\xi, f, g, S)$  has a unique solution  $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{S}^2$ .*

To prove this theorem, we need an important result which gives an approximation of continuous functions by Lipschitz functions (see Lepeltier and San Martin [10] to appear for the proof).

**Lemma 3.2.** *Let  $f : \mathbb{R}^p \rightarrow \mathbb{R}$  be a continuous function with linear growth, that is, there exists a constant  $K < \infty$  such that  $\forall x \in \mathbb{R}^p, |f(x)| \leq C(1 + |x|)$ . Then the sequence of functions  $f_n(x) = \inf_{y \in \mathbb{Q}^p} \{f(y) + n|x - y|\}$  is well defined for  $n \geq K$  and satisfies*

- (a) *Linear growth:*  $\forall x \in \mathbb{R}^p, |f_n(x)| \leq M(1 + |x|)$ ,
- (b) *Monotonicity:*  $\forall x \in \mathbb{R}^p, f_n(x) \nearrow$ ,
- (c) *Lipschitz condition:*  $\forall x, y \in \mathbb{R}^p, |f_n(x) - f_n(y)| \leq n|x - y|$ ,
- (d) *Strong convergence:* if  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , then  $f_n(x_n) \rightarrow f(x)$  as  $n \rightarrow \infty$ .

*Proof of Theorem 3.1* Consider, for fixed  $(t, \omega)$ , the sequence  $(f_n(t, \omega, y, z), g_n(t, \omega, y))$  associated to  $(f, g)$  by Lemma 3.2. Then,  $f_n, g_n$  are measurable functions as well as Lipschitz functions. Moreover, since  $\xi$  satisfy **(A4)** and  $\{S_t, 0 \leq t \leq T\}$  satisfy **(A5)**, we get from Ren and Xia [16] that there is a unique triple  $\{(Y_t^n, Z_t^n, K_t^n), 0 \leq t \leq T\}$  of  $\mathcal{F}_t$ -progressively measurable processes taking values in  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$  and satisfying

- (i)  $Y^n$  is a continuous process,
- (ii)  $Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds + \int_t^T g_n(s, Y_s^n) dG_s - \int_t^T Z_s^n dW_s + K_T^n - K_t^n$ ,
- (iii)  $Y_t^n \geq S_t$  a.s.,
- (iv)  $\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^n|^p + \int_0^T |Z_s^n|^2 ds \right) < +\infty$ ,
- (v)  $K^n$  is a non-decreasing process such that  $K_0^n = 0$  and  $\int_0^T (Y_s^n - S_s^n) dK_s^n = 0$ , a.s.

Using the comparison theorem of BSDE's in El Karoui et al. [9], we obtain that

$$\forall n \geq m \geq M, Y^n \geq Y^m, dt \otimes d\mathbb{P}\text{-a.s.} \quad (3.1)$$

The idea of the proof of Theorem 3.1 is to establish that the limit of the sequence  $(Y^n, Z^n, K^n)$  is a solution of the RGBSDE (1.1) with parameters  $(\xi, f, g, S)$ . It follows by the same step and technics as in [11], hence we will outline.

First, there exists a constant  $C$  depending only on  $M, T, \mathbb{E}(\xi^2)$  and  $\mathbb{E}(\sup_{0 \leq t \leq T} (S_t^+)^2)$ , such that

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds \right) \leq C. \quad (3.2)$$

Now, we have from (3.1) and (3.2) respectively, the existence of the process  $Y$  such that  $Y_t^n \nearrow Y_t$ ,  $0 \leq t \leq T$ ,  $\mathbb{P}$ -a.s. and from Fatou's lemma, together with the dominated convergence theorem provide respectively

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^n|^2 \right) \leq C \quad \text{and} \quad \int_0^T |Y_s^n - Y_s|^2 (ds + dG_s) \rightarrow 0 \quad (3.3)$$

as  $n \rightarrow \infty$ .

Now, we should prove that the sequence of processes  $Z^n$  converge in  $\mathcal{M}^2$ . For all  $n \geq m \geq n_0 \geq M$ , from Itô's formula for  $t = 0$

$$\begin{aligned} \mathbb{E}|Y_0^n - Y_0^m|^2 + \mathbb{E} \int_0^T |Z_s^n - Z_s^m|^2 ds &= 2\mathbb{E} \int_0^T (Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)) ds \\ &\quad + 2\mathbb{E} \int_0^T (Y_s^n - Y_s^m)(g_n(s, Y_s^n) - g_m(s, Y_s^m)) dG_s \\ &\quad + 2\mathbb{E} \int_0^T (Y_s^n - Y_s^m)(dK_s^n - dK_s^m). \end{aligned}$$

Using the fact that for all  $n$ ,  $Y_t^n \geq S_t$ ,  $0 \leq t \leq T$ , and from the identity  $\int_0^T (Y_t^n - S_t) dK_t^n = 0$ , we have

$$\begin{aligned} \mathbb{E} \int_0^T |Z_s^n - Z_s^m|^2 ds &\leq 2 \left( \mathbb{E} \int_0^T |Y_s^n - Y_s^m|^2 ds \right)^{1/2} \mathbb{E} \left( \int_0^T |f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)|^2 ds \right)^{1/2} \\ &\quad + 2 \left( \mathbb{E} \int_0^T |Y_s^n - Y_s^m|^2 dG_s \right)^{1/2} \mathbb{E} \left( \int_0^T |g_n(s, Y_s^n) - g_m(s, Y_s^m)|^2 dG_s \right)^{1/2}, \end{aligned}$$

where we have used the Hölder inequality. By the uniform linear growth condition on the sequence  $(f_n, g_n)$  and in virtue of (3.2), we obtain the existence of a constant  $C$  such that

$$\forall n, m \geq n_0, \quad \mathbb{E} \int_0^T |Z_s^n - Z_s^m|^2 ds \leq C \mathbb{E} \left( \int_0^T |Y_s^n - Y_s^m|^2 (ds + dG_s) \right).$$

Then from (3.3),  $(Z^n)$  is a Cauchy sequence in  $\mathcal{M}$ , and there exists a  $\mathcal{F}_t$ -progressively measurable process  $Z$  such that  $Z^n \rightarrow Z$  in  $\mathcal{M}^2$ , as  $n \rightarrow \infty$ .

Similarly by Itô's formula and Davis-Burkholder-Gundy inequality, it follows that

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_s^n - Y_s^m|^2 \right) \rightarrow 0$$

as  $n, m \rightarrow \infty$ , from which we deduce that  $\mathbb{P}$ -almost surely,  $Y^n$  converges uniformly in  $t$  to  $Y$  and that  $Y$  is a continuous process.

Now according to RGBSDE (ii), and use the same argument as [11], we have for all  $n, m \geq n_0 \geq M$ , we have

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |K_s^n - K_s^m|^2 \right) \rightarrow 0$$

as  $n, m \rightarrow \infty$ . Consequently, there exists a progressively measurable, increasing (with  $K_0 = 0$ ) and a continuous process  $K$  with value in  $\mathbb{R}_+$  such

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |K_s^n - K_s|^2 \right) \rightarrow 0$$

as  $n \rightarrow \infty$ .

Finally, taking limits in the RGBSDE (ii) we obtain that the triple  $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$  is a solution of the RGBSDE (2.1) and satisfy

- (1)  $Y_t \geq S_t$  a.s.,
- (2)  $\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_s|^2 ds \right) < +\infty$ ,
- (3)  $\int_0^T (Y_s - S_s) dK_s = 0$ , a.s.

□

We now prove our existence and uniqueness result.

**Theorem 3.3.** *Assume (A1)-(A4). Then RGBSDE with data  $(\xi, f, g, S)$  has a unique solution  $(Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{S}^p$ .*

*Proof. Uniqueness*

Let us consider  $(Y, Z, K)$  and  $(Y', Z', K')$  two solutions of RGBSDE with data  $(\xi, f, g, S)$  in the appropriate space. Using Lemma 2.4 (since  $\Delta S = 0 \in L^p$ ,  $\Delta \xi = \Delta f = \Delta g = 0$ ), we obtain immediately  $Y = Y'$ . Therefore we have also  $Z = Z'$  and finally  $K = K'$ , whence uniqueness follows.

Let us turn to the existence part. In order to simplify the calculations, we will always assume that condition (A2-iv) is satisfied with  $\mu \leq 0$ . If it is not true, the change of variables  $\tilde{Y}_t = e^{\mu t} Y_t$ ,  $\tilde{Z}_t = e^{\mu t} Z_t$ ,  $\tilde{K}_t = e^{\mu t} K_t$  reduces to this case

**Existence** Since, the function  $f$  is non-Lipschitz, the proof will be split into two steps

**Step 1.** In this part  $\xi$ ,  $\sup f_t^0$ ,  $\sup g_t^0$ ,  $\sup S_t^+$  are supposed bounded random variables and  $r$  a positive real such that

$$\sqrt{e^{(1+\lambda^2)T}} (\|\xi\|_\infty + T\|f^0\|_\infty + \|G_T\|_\infty \|g^0\|_\infty + \|S^+\|_\infty) < r.$$

Let  $\theta_r$  be a smooth function such that  $0 \leq \theta_r \leq 1$  and

$$\theta_r(y) = \begin{cases} 1 & \text{for } |y| \leq r \\ 0 & \text{for } |y| \geq r + 1. \end{cases}$$

For each  $n \in \mathbb{N}^*$ , we denote  $q_n(z) = z \frac{n}{|z| \vee n}$  and set

$$h_n(t, y, z) = \theta_r(y)(f(t, y, q_n(z)) - f_t^0) \frac{n}{\pi_{r+1}(t) \vee n} + f_t^0.$$

According to the same reason as in [3], this function still satisfies quadratic condition **(A2-iv)** but with a positive constant i.e there exists  $\kappa > 0$  depending on  $n$  such that

$$(y - y')(h_n(t, y, z) - h_n(t, y', z)) \leq \kappa |y - y'|^2.$$

Then  $(\xi, h_n, g, S)$  satisfies assumptions of Theorem 3.1. Hence, for each  $n \in \mathbb{N}$ , the reflected generalized BSDE associated to  $(\xi, h_n, g, S)$  has a unique solution  $(Y^n, Z^n, K^n)$  belong in space  $\mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{S}^2$ .

Since

$$y h_n(t, y, z) \leq |y| \|f^0\|_\infty + \lambda |y| |z|$$

and  $\xi, S$  and  $G$  are bounded, the similar computation of Lemma 2.2 in [2] provide that the process  $Y^n$  satisfies the inequality  $\|Y^n\|_\infty \leq r$ . In addition, from Lemma 2.2,  $\|Z^n\|_{\mathcal{M}^2} \leq r'$  where  $r'$  is another constant. As a byproduct  $(Y^n, Z^n, K^n)$  is a solution to the reflected generalized BSDE associated to  $(\xi, f_n, g, S)$  where

$$f_n(t, y, z) = (f(t, y, q_n(z)) - f_t^0) \frac{n}{\pi_{r+1}(t) \vee n} + f_t^0$$

which satisfied assumption **(A2-iv)** with  $\mu \leq 0$ .

We now have, for  $i \in \mathbb{N}$ , setting  $\bar{Y}^{n,i} = Y^{n+i} - Y^n$ ,  $\bar{Z}^{n,i} = Z^{n+i} - Z^n$ ,  $\bar{K}^{n,i} = K^{n+i} - K^n$ , applying the similar argument as Lemme 2.3, we obtain

$$\begin{aligned} & \Phi(t) |\bar{Y}_t^{n,i}|^2 + \frac{1}{2} \int_t^T \Phi(s) |\bar{Z}_s^{n,i}|^2 ds \\ & \leq 2 \int_t^T \Phi(s) \bar{Y}_s^{n,i} (f_{n+i}(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)) ds \\ & \quad + 2 \int_t^T \Phi(s) \bar{Y}_s^{n,i} d\bar{K}_s^n - 2 \int_t^T \Phi(s) \bar{Y}_s^{n,i} \bar{Z}_s^{n,i} dW_s, \end{aligned}$$

where for  $\alpha > 0$ ,  $\Phi(s) = \exp(2\lambda^2 s)$ . But  $\|\bar{Y}^{n,i}\|_\infty \leq 2r$  so that

$$\begin{aligned} & \Phi(t)|\bar{Y}_t^{n,i}|^2 + \frac{1}{2} \int_t^T \Phi(s)|\bar{Z}_s^{n,i}|^2 ds \\ & \leq 4r \int_t^T \Phi(s)|f_{n+i}(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| ds \\ & \quad + 2 \int_t^T \Phi(s)\bar{Y}_s^{n,i} d\bar{K}_s^{n,i} - 2 \int_t^T \Phi(s)\bar{Y}_s^{n,i} \bar{Z}_s^{n,i} dW_s \end{aligned}$$

and using the BDG inequality, we get, for a constant  $C$  depending only on  $\lambda$ ,  $\mu$  and  $T$ ,

$$\begin{aligned} & \mathbb{E} \left( \sup_{0 \leq t \leq T} |\bar{Y}_t^{n,i}|^2 + \int_0^T |\bar{Z}_s^{n,i}|^2 ds \right) \\ & \leq C r \mathbb{E} \left\{ \int_0^T |f_{n+i}(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| ds \right\}. \end{aligned} \quad (3.4)$$

On the other hand, since  $\|Y^n\|_\infty \leq r$ , we get

$$\begin{aligned} |f_{n+i}(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| & \leq 2\lambda |Z_s^n| \mathbf{1}_{\{|Z_s^n| > n\}} + 2\lambda |Z_s^n| \mathbf{1}_{\{\pi_{r+1}(s) > n\}} \\ & \quad + 2\pi_{r+1}(s) \mathbf{1}_{\{\pi_{r+1}(s) > n\}} \end{aligned}$$

from which we deduce, according assumption **(A3)** and inequality (3.4) that  $(Y^n, Z^n)$  is a cauchy sequence in the Banach space  $\mathcal{S}^2 \times \mathcal{M}^2$ . Let  $(Y, Z)$  its limit in  $\mathcal{S}^2 \times \mathcal{M}^2$ , then for all  $0 \leq t \leq T$ ,  $Y_t \geq S_t$  a.s..

Next, let us define

$$K_t^n = Y_0^n - Y_t^n - \int_0^t f_n(s, Y_s^n, Z_s^n) ds - \int_0^t g(s, Y_s^n) dG_s + \int_0^t Z_s^n dW_s. \quad (3.5)$$

By the convergence of  $Y^n$ , (for a subsequence), the fact that  $f, g$  are continuous and

- $\sup_{n \geq 0} |f(s, Y_s^n, Z_s)| \leq f_s + K \{(\sup_{n \geq 0} |Y_s^n|) + |Z_s|\}$ ,
- $\sup_{n \geq 0} |g(s, Y_s^n)| \leq g_s + K \{(\sup_{n \geq 0} |Y_s^n|)\}$
- $\mathbb{E} \int_0^T |f(s, Y_s^n, q_n(Z_s^n)) - f(s, Y_s^n, Z_s)|^2 ds \leq C \mathbb{E} \int_0^T |q_n(Z_s^n) - Z_s|^2 ds$

we get the existence of a process  $K$  which verifies for all  $t \in [0, T]$

$$\mathbb{E} |K_t^n - K_t|^2 \longrightarrow 0.$$

Moreover

$$\int_0^T (Y_s - S_s) dK_s = 0, \text{ for every } T \geq 0.$$

It is easy to pass to the limit in the approximating equation associated to  $(\xi, f_n, g, S)$ , yielding  $(Y, Z, K)$  as a solution of reflected generalized BSDE associated to data  $(\xi, f, g, S)$ .

**Step 2.** We now treat the general case.

For each  $n \in \mathbb{N}^*$ , let us denote

$$\begin{aligned} \xi_n &= q_n(\xi), \quad f_n(t, y, z) = f(t, y, z) - f_t^0 + q_n(f_t^0), \\ g_n(t, y) &= g(t, y) - g_t^0 + q_n(g_t^0), \quad S_t^n = q_n(S_t). \end{aligned}$$

For each  $n \in \mathbb{N}^*$ , RGBSDE associated with  $(\xi_n, f_n, g_n, S^n)$  has a unique solution  $(Y^n, Z^n, K^n) \in L^2$  thanks to the first step of this proof, but in fact also in  $L^p$ ,  $p > 1$  according the Lemma 2.3. Now from Lemma 2.4, for  $(i, n) \in \mathbb{N} \times \mathbb{N}^*$ ,

$$\begin{aligned} & \mathbb{E} \left\{ \sup_{0 \leq t \leq T} |Y_t^{n+i} - Y_t^n|^p + \left( \int_0^T |Z_s^{n+i} - Z_s^n|^2 ds \right)^{p/2} \right\} \\ & \leq C \mathbb{E} \left\{ |\xi_{n+i} - \xi_n|^p + \int_0^T |q_{n+i}(f_s^0) - q_n(f_s^0)|^p ds \right. \\ & \quad \left. + \int_0^T |q_{n+i}(g_s^0) - q_n(g_s^0)|^p dG_s + \sup_{0 \leq t \leq T} |q_{n+i}(S_t) - q_n(S_t)|^p \right\}, \end{aligned}$$

where  $C$  depends on  $T$  and  $\lambda$ . The right-hand side of the last inequality clearly tends to 0 as  $n \rightarrow \infty$ , uniformly on  $i$  so that  $(Y^n, Z^n)$  is again a cauchy sequence in  $\mathcal{S}^p \times \mathcal{M}^p$ . Let us denote by  $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$  its limit. Then it follows from identical computation as previous that, there exists a non-decreasing process  $K (K_0 = 0)$  such that

$$\mathbb{E} (|K_t^n - K_t|^p) \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

and

$$\int_0^T (Y_s - S_s) dK_s = 0, \text{ for every } T \geq 0.$$

It is easy to pass to the limit in the approximating equation, yielding that the triplet  $(Y, Z, K)$  is a  $L^p$ -solution of RGBSDEs with determinist time associated to  $(\xi, f, g, S)$ .  $\square$

**Acknowledgments.** I would like to thank X, Y and Z for their valuable comments on earlier drafts of this paper.

## References

1. Aman, Auguste.  $L^p$ -solution of generalized backward stochastic differential equations with barrier. *Afr. Diaspora J. Math.* **8** (2009), no. 1 68 – 80
2. Briand, Philippe; Carmona, René. BSDE with polynomial growth generators. *J. Appl. Math. Stochastic. Anal.* **13** (2000), no. 3, 207 – 238.
3. Briand PH.; Deylon D.; Hu Y.; Pardoux E.; Stoica L.  $L^p$ -solutions of backward stochastic differential equations. *Stochastic Process. Appl.* **108** (2003), no. 1, 109 – 129.
4. Hamadène, S. Reflected BSDE's with discontinuous barrier and application. *Stoch. Stoch. Rep.* **74** (2002), no. 3-4, 571 – 596.
5. Hamadène, S.; Lepeltier, J.-P. Zero-sum stochastic differential games and backward equations. *Systems Control Lett.* **24** (1995), no. 4, 259 – 263
6. Hamadène, S.; Lepeltier, J.-P.; Matoussi, A. Double barrier backward SDEs with continuous coefficient. *Backward stochastic differential equations (Paris, 1995-1996)*, 161-175, Pitman Res. Notes Math. Ser., 364, Longman, Harlow, 1997
7. Hamadène, S., Popier A. Lp-solutions for reflected backward stochastic differential equations. *preprint* (2008), (<http://arxiv.org/abs/0807.1846>)
8. El Karoui, N.; Peng S.; Quenez, M. C. Backward stochastic differential equation in finance. *Math. Finance.* **7** (1997), no. 1, 1 – 71.
9. El Karoui N.; Kapoudjian, C.; Peng, S.; Quenez, M. C. Reflected solutions of backward SDE's and related obstacle problem for PDE's. *Ann. Probab.* **25** (1997), no. 2, 702 – 737.
10. Lepeltier, J. P.; San Martin, J. Backward stochastic differential equations with continuous coefficient. *Statist. Probab. Lett.* **32** (1997), no. 4, 425 – 430.
11. Matoussi, Anis. Reflected solutions of backward stochastic differential equations with continuous coefficient. *Statist. Probab. Lett.* **34** (1997), no. 4, 347-354.
12. Pardoux, E; Peng, S. Adapted solution of backward stochastic differential equations. *System Control Lett.* **4** (1990), 55 – 61.
13. Pardoux, E., Peng, S. Backward stochastic differential equations and quasilinear parabolic partial differential equations. *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, 200–217, Lecture Notes in Control and Inform. Sci., 176, Springer, Berlin, 1992.
14. Pardoux, E; Zhang, S. Generalized BSDEs and nonlinear Neumann boundary value problems. *Probab. Theory Related Fields.* **110** (1998), no. 4, 535 – 558.
15. Peng, Shi Ge. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.* **37** (1991), no. 1-2, 61 – 74.
16. Ren, Y.; Xia, N. Generalized reflected BSDE and obstacle problem for PDE with nonlinear Neumann boundary condition. *Stoch. Anal. Appl.* **24** (2006), no. 5, 1013 – 1033.

Received 12 April, 2007; revised 13 July, 2007

**Author information**

Auguste Aman, UFR Mathématique Informatique  
Université de Cocody, 22 BP 582, Abidjan (Côte d'Ivoire).  
Email: [augusteaman5@yahoo.fr](mailto:augusteaman5@yahoo.fr)