



1 **RESEARCH ARTICLE**

2 **Synchronous and Asynchronous**
3 **Evaluation of Dynamic Neural Fields**

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7 In [26], we've introduced a dynamic model of visual attention based on the Continuum Neural
8 Field Theory [29] that explained attention as being an emergent property of a dynamic neural
9 field. The fundamental property of the model is its facility to select a single stimulus out
10 of several perfectly identical input stimuli by applying asynchronous computation. In the
11 absence of external noise and with a zero initial state, the theoretical mathematical solution
12 of the field equation predicts the final equilibrium state to equally represent all of the input
13 stimuli. This finding is valid for synchronous numerical computation of the system dynamics
14 where elements of the spatial field are computed all together at each time point. However,
15 asynchronous computation, where elements of the spatial field are iterated in time one after
16 the other yields different results leading the field to move towards a single stable input pattern.
17 This behavior is in fact quite similar to the effect of noise on dynamic fields. The present work
18 aims at studying this phenomenon in some details and characterizes the relation between
19 noise, synchronous evaluation (the "regular" mathematical integration) and asynchronous
20 evaluation in the case of a simple dual particle system. More generally, we aim at explaining
21 the behavior of a general differential equation system when it is considered as a set of particles
22 that may or may not iterated by synchronous computations.

23 **Keywords:** Synchronous computation, Asynchronous computation, Local update, Dynamic
24 Neural Fields

25 **AMS Subject Classification:** PACS: 02.30.Hq, 07.05.Mh, 84.35.+i

26 **1. Introduction**

27 Most computational paradigms linked to artificial neural networks (using rate code)
28 or cellular automata use implicitly what is called synchronous evaluation of activity.
29 This means that information at time $t + dt$ is evaluated exclusively on informa-
30 tion available at time t . The explicit numerical procedure of performing such a
31 synchronized update is to implement a temporary buffer at the unit level where
32 activity computed at time $t + \Delta t$ is stored. Once all units have evaluated their
33 activity at time $t + \Delta t$, the current activity is replaced by the content of the
34 buffer. We point out that other update procedures have been developed [22] but
35 the basic idea remains the same, namely not to mix information between time t
36 and time $t + \Delta t$. To perform such a synchronization, there is thus a need for a
37 global signal that basically tell units that evaluation is over and they can replace
38 their previous activity with the newly computed one. At the computational level,
39 this synchronization is rather expensive and is mostly justified by the difficulty of
40 handling asynchronous models. For example, cellular automata have been exten-
41 sively studied during the past decades for the synchronous case and mathematical

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42 studies have been performed. However, recent theoretical works on asynchronous
 43 computation in distributed computational networks [3, 4] and cellular automata
 44 [14] showed that the behavior of these same models and associated properties may
 45 be of a radical different nature depending on the level of synchrony of the model
 46 (you can asynchronously evaluate only a subpart of all the available automata).
 47 In the framework of computational neuroscience we may then wonder what is the
 48 relevance of synchronous evaluation since most generally, the system of equations
 49 is supposed to give account of a population of neurons that have no reason to be
 50 synchronized (if they are not provided with an explicit synchronization signal).
 51 We would like in this article to shed some light on such phenomenon and the
 52 consequences on modelling, especially in the framework of dynamic neural fields.
 53 After defining what we call synchronous and asynchronous evaluation of a system
 54 of differential equation, we introduced some results relative to dynamic neural field
 55 that underline clearly (and numerically) the difference between synchronous and
 56 asynchronous evaluation. To study this phenomenon, we then consider a degener-
 57 ated system made of only two potentials that will help us to understand what is
 58 going on. Finally, we make a conjecture regarding the link between synchronous
 59 and asynchronous evaluation.

60 **2. Synchronous and Asynchronous Evaluation**

61 In order to define what we called asynchronous evaluation of a differential system,
 62 we need first to define properly synchronous evaluation. Let us consider a generic
 63 discrete set of n first order differential equations:

$$\forall i \in [1, n], x_i : \mathbb{R}^+ \rightarrow \mathbb{R} \tag{1}$$

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \dots, x_n(t)) \tag{2}$$

64 with a set of initial conditions:

$$[x_1(0), \dots, x_n(0)] \in \mathbb{R}^n \tag{3}$$

65 When symbolic resolution is not possible, one can approximate the evolution
 66 of such a system using numerical integration, i.e. low-order methods as the
 67 Euler-forward or methods of higher order such as the Runge-Kutta methods [32].
 68 For sake of notation, we will use the Euler-forward method in the following but
 69 the same definitions apply to other methods as well.

70
 71 The Euler method provides us with an approximation for first order differential
 72 equations using the approximation

$$\Delta x_i(t) = \Delta t f_i(x_1(t), \dots, x_n(t)) \tag{4}$$

73 or

$$\begin{aligned} \Delta x_i(t) &= \Delta t f_i(x_1(t), \dots, x_n(t)) , i \in \mathcal{S} \\ \Delta x_j(t) &= 0 , j \in \bar{\mathcal{S}} \end{aligned} \tag{5}$$

74 where \mathcal{S} is a set of integers between 1 and n and $\bar{\mathcal{S}}$ represents its complement.
 75 Interestingly, Eq. (4) reveals that the systems fixed points are independent from
 76 the choice of \mathcal{S} since $\Delta x_i(t) = 0$ stipulates $f_i(x_1(t), \dots, x_n(t)) = 0$.
 77 The following paragraphs distinguish different choices of the set \mathcal{S} yielding different
 78 evaluation types.

79 **2.1 Synchronous evaluation**

80 The conventional update rule evaluates all elements synchronously, i.e. \mathcal{S} is the set
 81 of all integers between 1 and n , i.e. $\mathcal{S} = \{1, \dots, n\}$. Consequently (5) read

$$x_i(t + \Delta t) = x_i(t) + \Delta t f_i(x_1(t), \dots, x_n(t)) , \forall i = 1, \dots, n \quad (6)$$

82 This approximation is most commonly iterated over time until the desired state
 83 is reached, e.g. a given final time t_{final} . The pseudo-code for this computation type
 84 reads

85 *Algorithm 1.* Computational synchronous evaluation

```

86  $t = 0$ 
87 repeat
88   for all  $\bar{x}_i$  do
89      $\bar{x}_i = x_i + \Delta t f_i(x_1, \dots, x_n)$ 
90   end for
91   for all  $x_i$  do
92      $x_i = \bar{x}_i$ 
93   end for
94    $t = t + \Delta t$ 
95 until  $t \geq t_{final}$ 
    
```

96 This algorithm computes n updates in each time interval Δt .
 97 From a mathematical perspective, this is what corresponds to the conventional
 98 definition of the Euler-forward approximation. From a more physical perspective,
 99 this also makes sense if we consider t as the common or unified time for all the
 100 different variables $x_i(t)$.

101
 102 We point out that the evaluation scheme (6) is a multi-dimensional map of the
 103 type $\mathbf{x}_{i+1} = \mathbf{g}(\mathbf{x}_i)$ with vectors \mathbf{x}_i , $\mathbf{g} \in \mathbb{R}^n$ and obeys the mathematical rules of
 104 differential equations for $\Delta t \rightarrow 0$.

105 **2.2 Asynchronous evaluation**

106 However, as we underlined in the introduction, this unification of time is not that
 107 straightforward if we consider those equations to represent neuron potentials that
 108 can now be considered largely as independent biological elements, even if they are
 109 linked to other neurons, e.g. through synapses. Consequently, each element might
 110 have its own time and hence its own update time. To give a mathematical formula-
 111 tion of this situation, the set \mathcal{S} in (5) is chosen to $\mathcal{S} = rand(n)$ containing the single
 112 integer chosen randomly from the interval $[1; n]$. Hence each element x_i is updated
 113 separately and the evaluation is asynchronous. This evaluation procedure is also
 114 called local update [9, 12, 23]. In other words at each time point the asynchronous
 115 procedure updates a single element i only and this element is chosen randomly.

116 In mathematical terms, the numerical evaluation scheme can be formulated by

$$x_i(t + \Delta t/n) = x_i(t) + \delta_{ij} \xi(j) (\Delta t/n) f_i(x_1(t), \dots, x_n(t)) , \forall i = 1, \dots, n . \quad (7)$$

117 The term $\xi(j) \in [1; n]$ represents a random process, which fills the interval $[1; n]$
 118 with integers in random order. If the interval is filled, the interval is emptied and the
 119 filling process starts again. This process is used in physical chemistry and biology
 120 and is known as random sequential adsorption, see e.g. [8]. We conclude here that
 121 by virtue of the random nature of the update rule, the asynchronously updated
 122 systems do not obey the mathematical rules of differential equations and hence
 123 novel effects may occur. However we will see in the following sections that the limit
 124 $\Delta t \rightarrow 0$ diminishes the random effects and the dynamics obtained by asynchronous
 125 evaluation approach the synchronous results, i.e. the analytical results gained for
 126 differential equations.

127 Two different ways to implement such an asynchronous procedure are given in
 128 the following algorithms, which ensure n computations in each interval Δt .

129 *Algorithm 2.* Computational asynchronous evaluation (non-uniform)

130

```

131   t = 0
132   repeat
133     i = rand(n)
134     x_i = x_i + Δt f_i(x_1, ..., x_n)
135     t = t + Δt/n
136   until t ≥ t_final
    
```

137 Here $rand(n)$ denotes a random integer taken from the interval $[1; n]$. In each
 138 time interval $\Delta t/n$, we update only one $x_i(t)$. In statistical terms, this evaluation
 139 resembles draws in an urn model with return while n elements are drawn from the
 140 urn in each time interval.

141 We may also define a more uniform asynchronous evaluation which guarantees that
 142 each of the x_i is evaluated only once in the time interval:

143 *Algorithm 3.* Computational asynchronous evaluation (uniform)

144

```

145   t = 0
146   repeat
147     index = shuffle([1..n])
148     for i = 1 to n do
149       x_index[i] = x_index[i] + Δt f_index[i](x_1, ..., x_n)
150     end for
151     t = t + Δt
152   until t ≥ t_final
    
```

153 Here, $shuffle([1..n])$ denotes the sequence randomization of integers in the in-
 154 terval $[1; n]$. This evaluation scheme corresponds to an homogenous system where
 155 all the x_i evolve along a common time axis. In a statistical sense, this evaluation
 156 resembles the urn model without return and a complete return of all elements after
 157 the time Δt .

158 In addition we mention that the asynchronous evaluation scheme is not restricted
 159 to explicit evaluation schemes such as the Euler method and may be formulated
 160 for semi-implicit and implicit scheme as well.

161 The natural question concerning the differences between synchronous and asyn-
 162 chronous evaluation is to know whether they approximate the same system or if

163 they are different in nature. To do so, we would like first to illustrate these two
 164 evaluation types using a model of dynamic neural field.

165 3. Dynamic Neural Fields

166 Biological neural networks exhibit multiple spatial and temporal scales and thus
 167 it is a difficult task to model their spatio-temporal dynamics in all scales. Never-
 168 theless to explain various phenomena found experimentally, previous studies have
 169 focussed on specific spatial and temporal scales. A well-studied description level is
 170 the neural population level which considers the population firing rate of the neu-
 171 ral ensemble, the spatial scale of hundreds of micrometers and the temporal scale
 172 of few milliseconds. This model type, called neural field, allowed for the mathe-
 173 matical description of experimental phenomena, such as visual hallucinations [13],
 174 spiral waves in the cortex [18], the power spectrum in anesthesia [6, 28] and sleep
 175 cycles [27]. Moreover neural fields are supposed to model the storage of patterns
 176 in neural populations, such as breathers [15] or static bumps [21, 25]. Such phe-
 177 nomena are self-stabilizing in the absence of external stimuli, while some recent
 178 studies investigated the effect of external inputs on waves [16] and static localized
 179 activity [31].

180 We have been studying the Continuum Neural Field Theory (CNFT) [1, 2, 10, 20,
 181 29, 33, 34] extensively in [26] where we have introduced a dynamic model of visual
 182 attention that explains attention as being an emergent property of such dynamic
 183 neural field. The fundamental property of the model is its facility to select a single
 184 stimulus out of several perfectly identical input stimuli at the presence of spatial
 185 input noise. In other words, the model is able to make a choice by selecting an input
 186 among those available. Moreover the previous study [26] considers asynchronous
 187 numerical computation.

188 However in the absence of spatial input noise, the mathematical solution of the
 189 field equation predicts the final equilibrium state to equally represent all of the
 190 input stimuli. The reason for the selection to occur as shown in [26] is indeed
 191 the asynchronous evaluation that introduces the necessary asymmetry that lead
 192 the system to reach an equilibrium state reflecting just a single input stimulus.
 193 Moreover, we point out that this selection can not be predicted by neural field
 194 theory, since asynchronous evaluation implies a random process and thus does not
 195 obey the analysis rules of differential equations.

196 The following paragraphs illustrate these results in the CNFT for synchronous and
 197 asynchronous evaluation, two different inputs and various time intervals Δt .

198 3.1 Continuum Neural Field Theory

199 Using notations introduced by [1], a neural position is labelled by a vector \mathbf{x} on a
 200 manifold \mathcal{M} . The field variable represents the membrane potential of a neuronal
 201 population at the point \mathbf{x} at time t and is denoted by $u(\mathbf{x}, \mathbf{t})$. It is assumed that
 202 there are lateral connections with the weight function $w_M(\mathbf{x} - \mathbf{x}')$ which is in
 203 our case a difference of Gaussian function as a function of the distance $|\mathbf{x} - \mathbf{x}'|$.
 204 The model also considers an afferent connection weight function $s(\mathbf{x}, \mathbf{y})$ from the
 205 position \mathbf{y} in the manifold M' to the point \mathbf{x} in M . This function weights the input
 206 into the spatial field under study and thus reflects receptive field connections. The

207 membrane potential $u(\mathbf{x}, t)$ satisfies the following equation (8):

$$\begin{aligned} \tau \frac{\partial u(\mathbf{x}, t)}{\partial t} = & -u(\mathbf{x}, t) + \int_M w_M(\mathbf{x} - \mathbf{x}') f[u(\mathbf{x}', t)] d\mathbf{x}' \\ & + \int_{M'} s(\mathbf{x}, \mathbf{y}) I(\mathbf{y}, t) d\mathbf{y} + h \end{aligned} \quad (8)$$

208 where τ denotes the synaptic time constant, f represents the mean firing rate as
 209 the function of the membrane potential u of the population, $I(\mathbf{y}, t)$ is the input
 210 from position \mathbf{y} at time t in M' and h is the mean neuron threshold. In detail, the
 211 firing rate function f is chosen as the piece-wise linear function

$$f[u] = \begin{cases} 0 & \text{if } u \leq 0, \\ u & \text{if } 0 < u < 1, \\ 1 & \text{if } u \geq 1, \end{cases} \quad (9)$$

212 the lateral connectivity function w_M reads

$$w_M(\mathbf{x} - \mathbf{x}') = Ae^{-\frac{|\mathbf{x} - \mathbf{x}'|^2}{a^2}} - Be^{-\frac{|\mathbf{x} - \mathbf{x}'|^2}{b^2}} \text{ with } A, B, a, b \in \mathfrak{R}^{*+} \quad (10)$$

213 and the afferent connections are described by

$$s(\mathbf{x}, \mathbf{y}) = Ce^{-\frac{|\mathbf{x} - \mathbf{y}|^2}{c^2}} \text{ with } C, c \in \mathfrak{R}^{*+} \quad (11)$$

214 In the following, the spatial domain is $[-0.5, 0.5]^2$ on both manifold \mathcal{M} , \mathcal{M}'
 215 involving periodic boundary conditions.

216 3.2 Symmetric input

217 We consider the case where there are two distinct gaussian inputs within the \mathcal{M}
 218 manifold, one centered at $(\frac{1}{3}, \frac{1}{3})$ and one centered at $(-\frac{1}{3}, -\frac{1}{3})$ such that:

$$\begin{aligned} G(x, y, \sigma) &= e^{-\frac{x^2 + y^2}{2\sigma^2}} \\ \mathcal{M}(x, y) &= G(x - \frac{1}{3}, y - \frac{1}{3}, .1) + G(x + \frac{1}{3}, y + \frac{1}{3}, .1) \end{aligned} \quad (12)$$

219 The manifolds \mathcal{M} and \mathcal{M}' have been respectively discretized into a set of
 220 30×30 units and each of the unit of \mathcal{M}' receives the corresponding input from \mathcal{M}
 221 (function s from Eq. (11) is degenerated into a single afferent point).

222

223 Starting from a perfectly null state in the output and using equations introduced
 224 in the previous section and synchronous evaluation (see algo. 1), we ran simulation
 225 for 10 seconds using $\Delta t = 1000ms$ and $\Delta t = 10ms$ (see figure 1). As predicted,
 226 resulting output patterns represent both input stimuli. Aynchronous evaluation
 227 (see algo. 3) yields different results. For a large $\Delta t = 1000ms$, only one of the
 228 input is fully represented in the output while the other vanished (see fig. 2). To
 229 make sure that one bump survives only, we examined numerically the neural field
 230 activity at the location of the expected second bump and found vanishing activity.
 231 This result indicates that the second bump vanished indeed.

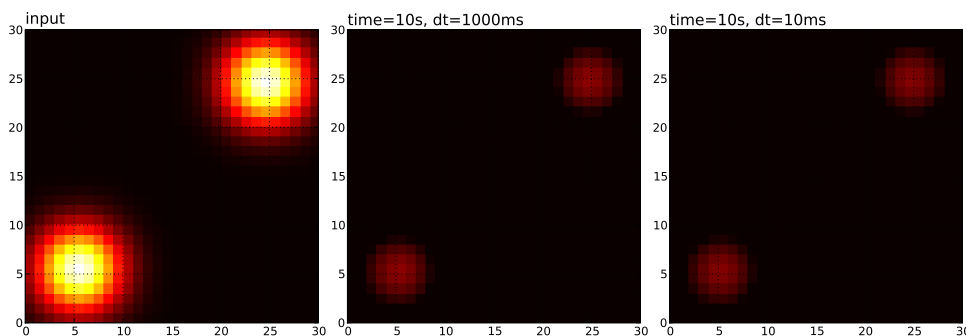


Figure 1. Symetric input, synchronous evaluation

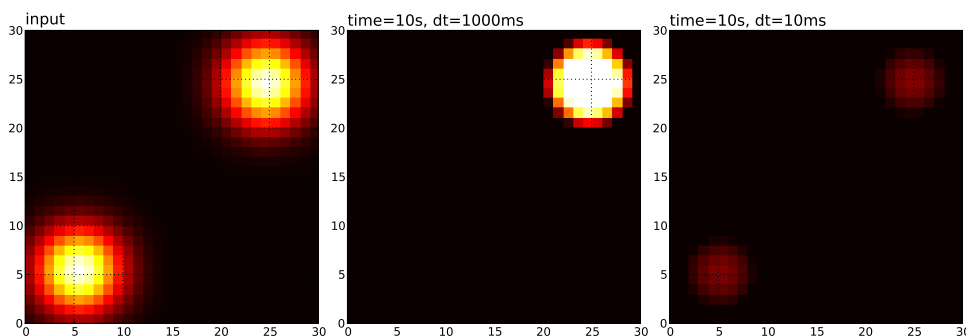


Figure 2. Symetric input, uniform asynchronous evaluation

232 Only when we reduce Δt to $10ms$, we then observe results comparable to the
 233 synchronous case. Note that we've also tested algorithm 2 (not represented) and
 234 obtained the same results.

235 At a first glance, the disappearance of one bump and thus the symmetry break-
 236 ing with $\Delta t = 1000ms$ in Fig. 2 is surprising and can not be understood by neural
 237 field theory. Since the two bumps re-occur for the smaller time step $\Delta t = 10ms$,
 238 we argue that the disappearance of one bump results from the asynchronous eval-
 239 uation scheme, which implies random processes (section2.2) and hence can not be
 240 understood by mathematical analysis based on the Eq. (8).

241 **3.3 Asymmetric input**

242 We also consider asymmetric input where input is given by:

$$G(x, y, \sigma) = e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{13}$$

$$\mathcal{M}(x, y) = \frac{1}{2}G(x - \frac{1}{3}, y - \frac{1}{3}, .1) + G(x + \frac{1}{3}, y + \frac{1}{3}, .1)$$

243 and ran simulations as in a similar way as of the previous subsection. Since the
 244 input is not symmetric anymore, we observe in the output that the most salient
 245 stimulus is fully represented (see fig. 3).

246 In the case of asynchronous evaluation (either algo. 2 or 3), we obtained exactly
 247 the same results (see fig. 4), whatever the Δt . This lead us to consider the nature
 248 of the final states and to make the link between stability of the state and the
 249 probability to reach such a state in case of asynchronous evaluation. Since the
 250 CNFT may be too complex for a thorough analysis, we considered a reduced model
 251 to explain the underlying dynamics.

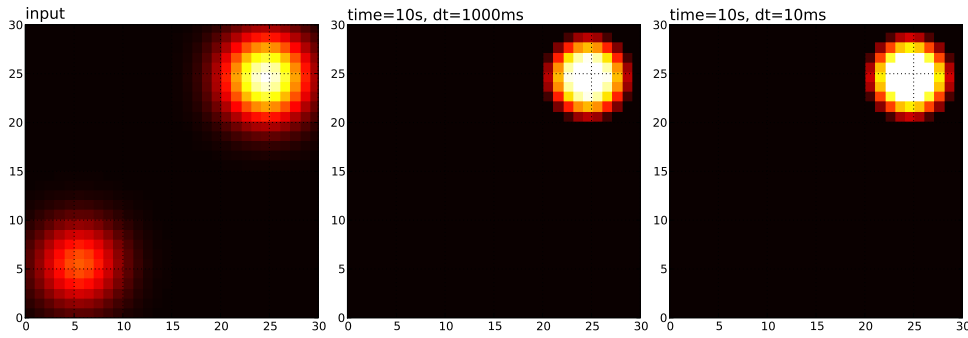


Figure 3. Asymmetric input, synchronous evaluation

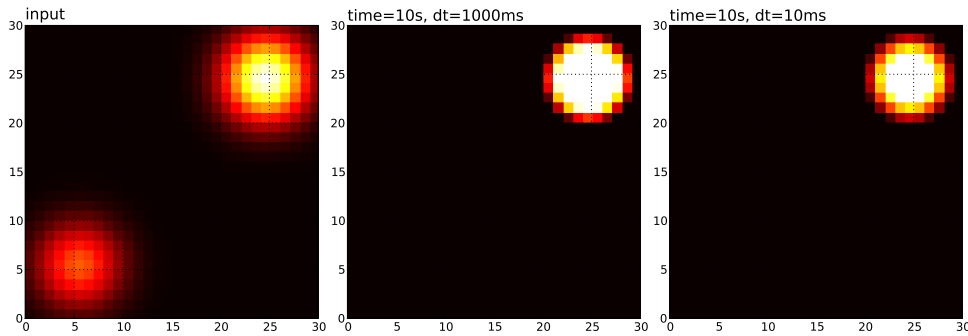


Figure 4. Asymmetric input, uniform asynchronous evaluation

252 **4. The reduced model**

253 To explain in detail the spatio-temporal behavior in section 3, we introduce a low-
 254 dimensional model, whose behavior reflects the major phenomena observed in the
 255 CNFT. The model equations read

$$\begin{aligned} \dot{y} &= -\alpha y + (y - z)(1 - y) + \alpha I_y \\ \dot{z} &= -\alpha z + (z - y)(1 - z) + \alpha I_z \end{aligned} \tag{14}$$

256 with the absorbing boundary conditions $y(t_0) = 0 \rightarrow y(t > t_0) = 0$, $y(t_0) = 1 \rightarrow$
 257 $y(t > t_0) = 1$, $z(t_0) = 0 \rightarrow z(t > t_0) = 0$, $z(t_0) = 1 \rightarrow z(t > t_0) = 1$. Here I_y , I_z are
 258 the external inputs which are specified to $I_y = 1$, $I_z = I$ in the following discussion.
 259 Further we choose $0 < \alpha < 2$ and the parameter I is the constant external input
 260 with $0 < I \leq 1$.

261 Since dynamic neural fields are mainly concerned with competition among units,
 262 we build this model in order to benefit from a very simple competition mechanism
 263 where the growing of one variable is conditioned to both its difference from the
 264 other variable and to how far it is from the input. For example, if at a given time
 265 y is greater than z , then the term $y - z$ is positive and lead y to reach the input
 266 value I_y . At the same time, the variable z tends to decrease since the term $z - y$ is
 267 now negative. The greater this difference is, the faster the two variables will reach
 268 their respective final state. If at any time the two variables are equal, then they do
 269 not influence each other and can reach their respective inputs.

270 Although no direct derivation of the model (14) from the neural field equations (8)
 271 exist, we may relate parameters of both models. For instance the parameter α in
 272 (14) defines the susceptibility of the system to the external input and reflects the
 273 rate of convergence to fixed points, i.e. its stability. This can be seen at $(x = 0, y =$

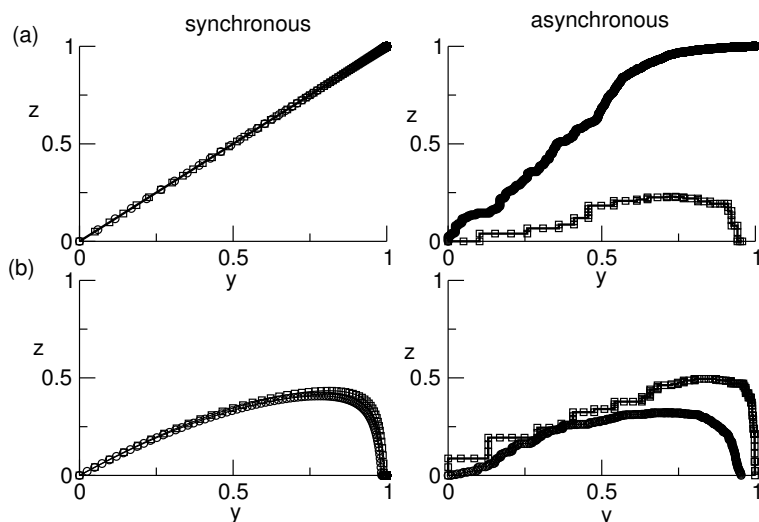


Figure 5. Example trajectories based on the reduced model for the synchronous and the uniform asynchronous evaluation, two different inputs I and two values of Δt . (a) $I = 1$, (b) $I = 0.85$. The values of Δt are chosen to $\Delta t = 0.1$ (squares) and $\Delta t = 0.01$ (circles). Further $\alpha = 0.5$ and the initial conditions are $x(0) = y(0) = 0$.

274 0) where $(\dot{y}, \dot{z}) = \alpha(I_y, I_z)$ and at $(x = 0, y = 0)$ where we find $(\dot{y}, \dot{z}) = -\alpha(y, z)$. In
 275 the neural field model it is well-known that the nonlinear gain, i.e. the steepness of
 276 the transfer function, defines the excitability of the system, i.e. the susceptibility
 277 to external input, and the stability of the field [7, 19, 30]. Hence, the steepness of
 278 the transfer function, i.e. the mean firing rate function, in (8) and α in (14) are
 279 strongly related.

280 Figure 5 presents some numerical solutions of (14). In the case of a symmetric
 281 input, i.e. $I = 1$, the synchronous evaluation yields the final state $(1, 1)$ for
 282 both values of Δt (Fig. 5(a), left panel). Consequently the input $(1, 1)$ yields the
 283 equilibrium $(1, 1)$ and thus resembles the CNFT-result shown in Fig. 1. Applying
 284 the uniform asynchronous evaluation scheme introduced in section 2.2 the system
 285 reaches the state $(1, 1)$ for small Δt , but approaches the state $(1, 0)$ for large Δt , cf.
 286 Fig. (5)(a), right panel. This behavior shows good accordance to the corresponding
 287 CNFT-case observed in Fig. 2. Moreover considering the different input stimulus
 288 $I < 1$ (Fig. 5(b)), the synchronous and asynchronous computation yield the same
 289 final stationary state irrespective the value of Δt . This result also shows good ac-
 290 cordance to the findings in the CNFT-model, cf. Figs. (3), (4). Summarizing, the
 291 low-dimensional model (14) shares the major dynamical properties of the CNFT-
 292 model (8) and replaces it in good approximation. Consequently the detailed study
 293 of the low-dimensional model allows for deeper insight into the understanding of
 294 the CNFT-model.

295

296 4.1 Dynamical properties

297 To better understand the dynamical behavior observed in Fig. 5, let us study to
 298 the stationary states of the model (14) subjected to the external input I . We find
 299 a critical input $I_c = 1 - \alpha/4$, which allows to distinct two cases for $0 < y < 1, 0 <$
 300 $z < 1$:

- 301 • for $0 < I < I_c$ a single fixed point FP exists at

$$y_0 = (I + 1)/2 - \alpha, z_0 = (I + 1)/2$$

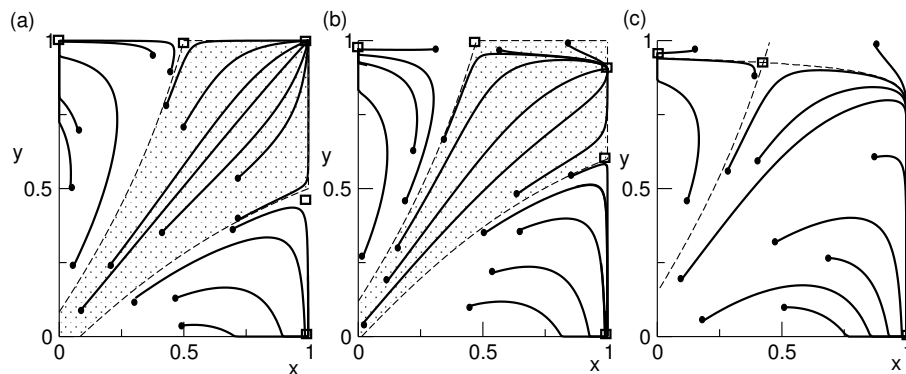


Figure 6. The topography of the low-dimensional model (14) for different inputs I . (a) $I = 1 > I_c$, (b) $I = 0.92 > I_c$ and (c) $I = 0.85 < I_c$. Here $\alpha = 0.5$ which leads to $I_c = 0.845$. The solid lines in the panels represent the trajectories with initial points denoted by filled dots. Further the dashed lines represent the separatrix, the dotted domain in (a) and (b) denote the basin of attraction of FP1 and the open squares mark the positions of fixed points calculated analytically.

302 • for $I_c \leq I \leq 1$ three fixed points exist at

$$FP1 : y_0 = 1 \quad , \quad z_0 = (\alpha - 2) \left(1 + \sqrt{1 + 4(\alpha I - 1)/(\alpha - 2)^2} \right) / 2$$

$$FP2 : y_0 = 1 \quad , \quad z_0 = (\alpha - 2) \left(1 - \sqrt{1 + 4(\alpha I - 1)/(\alpha - 2)^2} \right) / 2$$

$$FP3 : y_0 = (I + 1)/2 - \alpha \quad , \quad z_0 = (I + 1)/2 .$$

303 In the specific case $I = 1$, the fixed points read

$$FP1 : y_0 = 1 \quad , \quad z_0 = 1$$

$$FP2 : y_0 = 1 \quad , \quad z_0 = 1 - \alpha$$

$$FP3 : y_0 = 1 - \alpha \quad , \quad z_0 = 1.$$

304 To gain the linear stability conditions of the corresponding fixed points, we linearize
 305 Eqs. (14) about the corresponding fixed points and find two real-valued Lyapunov
 306 exponents λ_1, λ_2 for each fixed point:

307 • for $0 < I < I_c$, the single fixed point FP is a saddle node with $\lambda_1 < 0, \lambda_2 > 0$.

308 • for $I_c \leq I \leq 1$, FP1 is a stable node and FP2 and FP3 are saddle nodes.

309 Moreover, the system evolves on the boundary and a linear stability analysis reveals
 310 fixed points

$$FPBy : y_0 = (1 - \alpha)(1 + \sqrt{1 + 4\alpha I/(1 - \alpha)^2})/2$$

$$FPBz : z_0 = (1 - \alpha)(1 + \sqrt{1 + 4\alpha/(1 - \alpha)^2})/2 ,$$

311 which are stable irrespective to the choice of $0 < I < 1$.

312 Figure 6 summarizes the latter analytical results and reveals a basin of attraction
 313 of FP1 for $I_c \leq I \leq 1$ which vanishes for smaller values $I < I_c$. In general we
 314 observe that $1 > I$, i.e. $I_y > I_z$ and the input into y is stronger than into z , yields
 315 an increase of the basin of attraction of $FPBy$.

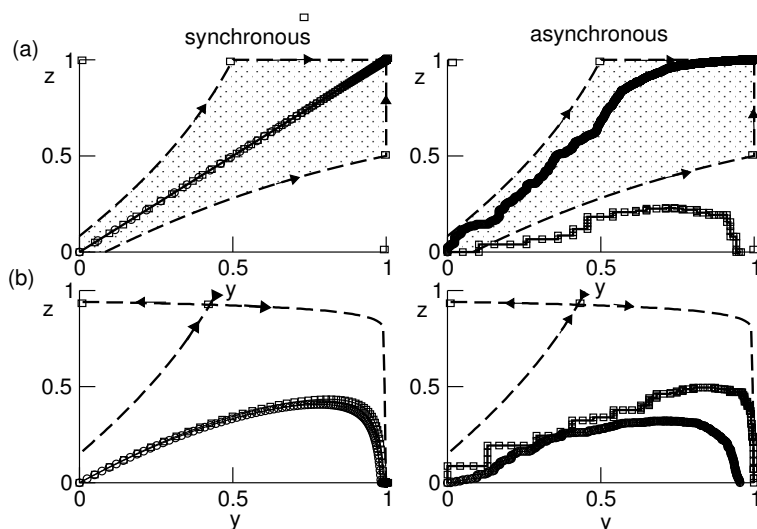


Figure 7. Trajectories and topology overlaid in single plots. (a) $I = 1 > I_c$, (b) $I = 0.85 < I_c$. Other parameters and symbols are taken from Figs. 5 and 6.

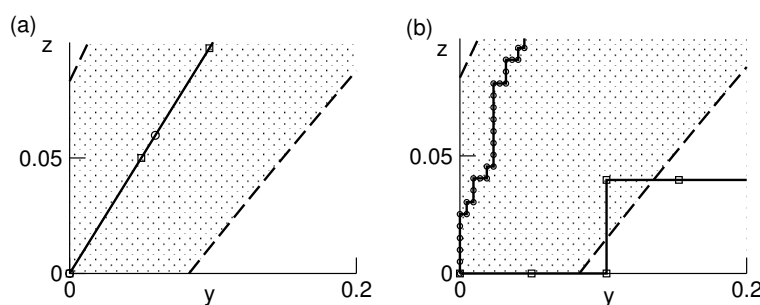


Figure 8. Focus on panels in Fig. 7(a). (a) synchronous computation, (b) asynchronous computation. Circles and squares encode $\Delta t = 0.01$ and $\Delta t = 0.1$.

316 4.2 Effect of the external input

317 To further investigate the systems evolution for different inputs, Fig. 7 overlays
 318 the trajectories from Fig. 5 and the systems topology shown in Fig. 6. For $I = 1$,
 319 we observe that the trajectories computed synchronously start in the basin of
 320 attraction of FP1 and stay there until they reach $(1, 1)$ (Fig. 7(a), left panel),
 321 while trajectories computed asynchronously may leave the basin of attraction for
 322 large Δt , see Fig. 7(a), right panel. Moreover, $I = 0.85$ destroys FP1 and its
 323 basin of attraction and puts the initial point $(0, 0)$ into the basin of attraction
 324 of $(1, 0)$, (Fig. 7(b)). Consequently all trajectories shown approach the stationary
 325 point $(1, 0)$. In general decreasing I diminishes the input into z and increases the
 326 basin of attraction of $(1, 0)$. This behavior resembles the results in neural fields for
 327 large enough Δt , where the stronger input is preferred.

328 In addition we observe that $I = 1$ allows the trajectories to approach the final
 329 states $(1, 0)$, $(0, 1)$ and $(1, 1)$, while $I = 0.85 < I_c$ yields either $(1, 0)$ or $(0, 1)$.
 330 Hence input stimuli $I_c < I = I_z < 1$ are different from $I_y = 1$, but may not be
 331 detected as different since the systems trajectory may approach $(1, 1)$. In turn the
 332 larger I_c , the better the system can distinguish different stimuli I_y and I_z .

333 4.3 Effect of the computation type and Δt

334 To understand the different effects of synchronous and asynchronous computation,
 335 Fig. 8 presents a focus of the panels in Fig. 7(a). In the case of synchronous com-

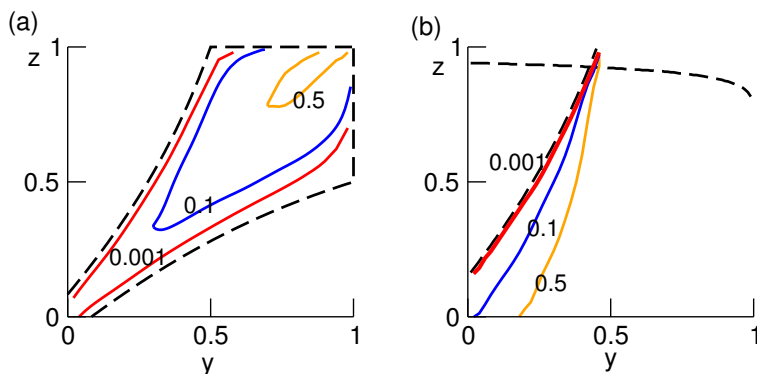


Figure 9. Contour lines of the probability of trajectories to reach a fixed point for different Δt . (a) $I = 1$, (b) $I = 0.85$. The solid lines give the initial locations where 90 of 100 trajectories approach the fixed point (a) $(1, 1)$ (b) $(1, 0)$. Hence these lines are contour lines of the probability distribution to reach a stationary state with the fixed probability 0.9. The numbers in both panels are values of Δt of the corresponding contour lines and the dashed lines represent the separatrix. Other parameters are taken from Fig. 6.

336 putation (Fig. 8(a)), the variables y and z are changed at the same time and thus
 337 the trajectory obeys the vectorfield (\dot{y}, \dot{z}) , i.e. stays in the basin of attraction. Fur-
 338 further in the shown example the vectorfield points to the fixed point FP1 and the
 339 length of the change vector $(\Delta y, \Delta z) \sim \Delta t$ does not point to locations outside the
 340 basin of attraction for both Δt . In contrast, the size of Δt matters in the case of
 341 asynchronous computation (Fig. 8(b)). This evaluation type changes either y or
 342 z and thus the trajectory does not obey the vectorfield (\dot{y}, \dot{z}) . Consequently it is
 343 possible that one variable changes in a way that the new trajectory point is located
 344 outside the basin of attraction. This probability to leave the basin of attraction
 345 is small for small Δt since the length of the change vector is small, cf. (Fig. 8(b),
 346 line with circles). However larger Δt yield a higher probability to leave the basin of
 347 attraction. As shown in Fig. 8(b), the trajectory might leave the basin and re-enter
 348 it.

349 In the previous paragraph we have discussed that trajectories computed syn-
 350 chronously are much less susceptible to Δt than asynchronous trajectories since
 351 the latter does not obey the vectorfield (\dot{y}, \dot{z}) in each time step. To clarify this
 352 interplay between asynchronous evaluation and the size of Δt , Fig. 9 plots the
 353 initial locations of trajectories which approach the point $(1, 1)$ (Fig. 9(a)) or $(1, 0)$
 354 (Fig. 9(b)) with the probability 0.9. We observe that the basin of attraction of
 355 the asynchronous trajectories depends on Δt and increases with decreasing Δt .
 356 Further this asynchronous basin of attraction approaches the basin of attraction of
 357 the model (14), i.e. the synchronous basin of attraction. Consequently, the asyn-
 358 chronous computation is equivalent to synchronous computation for $\Delta t \rightarrow 0$.

359 **5. Conclusion and Future Directions**

360 This work distinguishes the synchronous and asynchronous evaluation scheme in
 361 dynamical systems and illustrates their different effects by numerical simulations
 362 in continuum neural fields. To gain deeper insight into the phenomena observed, we
 363 introduce a low-dimensional model which exhibits similar behavior and allows to re-
 364 place the CNFT-model in a first approximation. For this new model, the detailed
 365 analysis reveals the systems topology and uncovers subsequently the underlying
 366 differences of synchronous and asynchronous evaluation. At first, we observe that
 367 the system feels the presence of its fixed points for both evaluation schemes and
 368 hence obeys the systems topology. Consequently the system may approach its sta-
 369 ble fixed points for both evaluation schemes. The only difference between the two

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370 schemes is the system trajectories, which do not necessarily obey the vector field of
371 the dynamical system in the case of asynchronous evaluation and exhibits jumps
372 in along a single coordinate axis due to its random nature. The strength of this
373 random element in the asynchronous evaluation scheme depends strongly on the
374 implementation time step. For very small time steps the random effects are reduced
375 and the asynchronous evaluation resembles the synchronous evaluation. From the
376 broader perspective of differential systems, we can make the conjecture that asyn-
377 chronous evaluation with an infinitesimal Δt is identical to synchronous evaluation
378 with same Δt .

379 The results from the reduced model may give explanations for the behavior of the
380 neural field dynamics using the asynchronous evaluation. For instance, according
381 to neural field theory, i.e. theory of integral-differential equations, a single bump in
382 neural fields does not exist in the presence of two bumps in the input, but may ex-
383 ist in numerical simulations applying the asynchronous computation scheme. The
384 reason for this difference is the random nature of the asynchronous computation
385 scheme, which allows the system to leave the basin of attraction of the stable fixed
386 point representing two bumps and approach the stable fixed point representing a
387 single bump, cf. Fig. 7 and 8. Moreover, the selection of the bumps in the asyn-
388 chronous evaluation scheme is biased by the input as illustrated in Fig. 7: the
389 element subjected to the stronger input is approached. This may explain the se-
390 lection mechanism for both synchronous and asynchronous evaluation as observed
391 in Fig. 3 and 4.

392 To learn more about the the neural field dynamics, we recall the relation of the
393 nonlinear gain of the population firing rate function and the parameter α in the
394 reduced model. The increase of α , i.e. the increase of excitability, decreases the
395 critical input I_c (cf. section 4.1) and thus facilitates the preference of either $(1, 0)$
396 or $(0, 1)$ as the final state. In other words we argue that increasing the excitability
397 in neural fields may improve the distinction of different input patterns and thus
398 changes the visual attention. Indeed the relation of neural excitability and visual
399 attention has been found experimentally [5, 24].

400 In realistic situations, one finds visual stimuli with different saliencies. Consid-
401 ering a neural population in the visual system and assuming an underlying asyn-
402 chronous evaluation scheme, the visual system may choose the most salient stimulus
403 and one may explain the stimulus selection by a stronger basin of attraction of the
404 resulting pattern. In other words, the visual system may select the stronger bump
405 with a higher probability than the other ones. However, the visual system may also
406 select a bump with a lower saliency due to random nature of the systems trajectory,
407 which however is much less probable (cf. Fig. 8).

408 Future work may study various model systems typically applied in computational
409 neuroscience, such as a recurrent network of McCulloch-Pitts neurons, coupled
410 FitzHugh-Nagumo or Hodgkin-Huxley models [11], or a network of spike-response
411 neurons [17]. Especially the last model attracted much attention in the last years
412 to analyse spiking neural networks. Even if such networks may benefit from a
413 deterministic timing of spike emissions, they may be nonetheless considered in the
414 light of asynchronous evaluation in their computational implementation.

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