ON THE DISCRETIZATION OF BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we are dealing with the approximation of the process (X_t, Y_t, Z_t) solution to the backward doubly stochastic differential equation (BDSDE)

$$X_s = x + \int_0^s b(X_r)\,dr + \int_0^s \sigma(X_r) dW_r,$$

$$Y_s = \phi(X_T) + \int_s^T f(r, X_r, Y_r, Z_r)\,dr + \int_s^T g(r, X_r, Y_r, Z_r)\,d\overleftarrow{B}_r - \int_s^T Z_r dW_r.$$

After proving the L^2-regularity of Z, we use the Euler scheme to discretize X and the Zhang approach in order to give a discretization scheme of the process (Y, Z).

1. INTRODUCTION

Since the pioneering work of E. Pardoux and S. Peng [PP92], backward stochastic differential equations (BSDEs) have been intensively studied during the two last decades. Indeed, this notion has been a very useful tool to study problems in many areas, such as mathematical finance, stochastic control, partial differential equations; see e.g. [MY99] where many applications are described. Discretization schemes for BSDEs have been introduced and studied by several authors. The first papers on this topic are that of V.Bally [Ba97] and D.Chevance [Ch97]. In his thesis, Zhang made an interesting contribution which was the starting point of intense study among which the works of B. Bouchard and N.Touzi [BT04], E.Gobet, J.P. Lemor and X. Warin[GLW05],... The notion of BSDE has been generalized by E. Pardoux and S. Peng [PP94] to that of Backward Doubly Stochastic Differential Equation (BDSDE) as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, T denote some fixed terminal time which will be used throughout the paper, $(W_t)_{0\le t\le T}$ and $(B_t)_{0\le t\le T}$ be two independent standard Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in $\mathbb{R}^d$, and $\mathbb{R}$ respectively. On this space we will deal with two families of σ-algebras:

$$\mathcal{F}_t := \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{N}, \quad \widehat{\mathcal{F}}_t := \mathcal{F}_{0,t}^W \vee \mathcal{F}_{0,T}^B \vee \mathcal{N}, \quad \mathcal{H}_t = \mathcal{F}_{0,T}^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{N}, \tag{1.1}$$

where $\mathcal{F}_{t,T}^B := \sigma(B_r - B_t; t \le r \le T)$, $\mathcal{F}_{0,t}^W := \sigma(W_r - W_0; 0 \le r \le t)$ and $\mathcal{N}$ denotes the class of $\mathbb{P}$ null sets. We remark that $(\widehat{\mathcal{F}}_t)$ is a filtration, $(\mathcal{H}_t)$ is a decreasing family of σ-albegras, while $(\mathcal{F}_t)$ is neither increasing nor decreasing. Given an initial condition $x \in \mathbb{R}^d$, let (X_t) be the d-dimensional diffusion process defined by

$$X_t = x + \int_0^t b(X_r)\,dr + \int_0^s \sigma(X_r)\,dW_r. \tag{1.2}$$

Let $\xi \in L^2(\Omega)$ be an $\mathbb{R}^d$-valued, $\mathcal{F}_T$-measurable random variable, f and g be regular enough coefficients; consider the BDSDE defined as follows:

$$\begin{aligned} Y_s &= \xi + \int_s^T f(r, X_r, Y_r, Z_r)\,dr \\ &\quad + \int_s^T g(r, X_r, Y_r, Z_r)\,d\overleftarrow{B}_r - \int_s^T Z_r dW_r. \end{aligned} \tag{1.3}$$

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In this equation, dW is the forward integral and $d\overleftarrow{B}$ is the backward integral (we send the reader to [NP88] for more details on backward integration). A solution to (1.3) is a pair of real-valued process (Y_t, Z_t), such that X_t and Y_t are $(\mathcal{F}_t)$ for every $t \in [0,T]$, such that (1.3) is satisfied and

$$\mathbb{E}\Big(\sup_{0\leq s\leq T}|Y_s|^2\Big) + \mathbb{E}\int_0^T |Z_s|^2 ds < +\infty. \tag{1.4}$$

In [PP94] Pardoux and Peng have proved that under some Lipschitz property on f and g which will be stated more precisely in section 2, (1.3) has a unique solution (Y, Z).

The aim of this paper is to study the discretization of a Backward Doubly Stochastic Differential Equation For the sake of simplicity, as in Zhang's paper [Z04], we assume that Y and Z are real-valued processes. The extension to higher dimension is cumbersome and without theoretical problems. This discretization scheme of (Y, Z) is motivated by the link between (1.3) and the following backward stochastic partial differential equation when $\xi = \phi(X_T)$ for a regular function ϕ:

$$\begin{aligned} u(t,x) &= \phi(x) + \int_t^T \Big(\mathcal{L}u(s,x) + f\left(s,x,u(s,x),\nabla u(s,x)\sigma(x)\right)\Big)ds \\ &\quad + \int_t^T g\left(s,x,u(s,x),\nabla u(s,x)\sigma(x)\right) d\overleftarrow{B}_s, \end{aligned} \tag{1.5}$$

where $\mathcal{L}$ is the differential operator defined by:

$$\mathcal{L}u(t,x) = \frac{1}{2}\sum_{i,j=1}^{d}\left(\sigma\sigma^*\right)_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(t,x) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}u(t,x).$$

The paper is organized as follows: first we prove the L^2-regularity of Z in section 2. This a crucial step in order to the scheme using Zhang's method, which is done in section 3. Finally, a numerical scheme is described in the last section. To ease notations, we set $\Theta_r := (X_r, Y_r, Z_r)$ for $r \in [0,T]$. As usual, we denote by C_p a constant which depends on some parameter p, and which can change from on line to the next one. Finally, for some function $h(t,x,y,z)$ defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$, we let $\partial_y h(t,x,y,z)$ (resp. $\partial_z h(t,x,y,z)$) the partial derivatives of h with respect to the real variable y (resp. z), while $\partial_x h(t,x,y,z)$ will denote the vector $(\partial_{x_i} h(t,x,y,z), i = 1,\cdots,d)$.

2. Regularity properties

In this section we give some regularity properties of the process X, Y and Z.

The following assumptions which ensure existence and uniqueness of the solution will be in force throughout the paper. For every integer $n \geq 1$, let $M^2([0,T],\mathbb{R}^n)$ denote the set of $\mathbb{R}^n$-valued jointly measurable processes $(\varphi_t, t\in[0,T])$ such that φ_t is $\mathcal{F}_t$-measurable for almost every t and $\mathbb{E}\int_0^T |\varphi_t|^2 dt < +\infty$.

Assumption 1 (for the forward process X)**.** *The maps $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are of class $\mathcal{C}_b^3$.*

Assumption 2 (for the backward process (Y,Z))**.** *Let $f : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be such that f and g are jointly measurable, for every $(x,y,z)\in\mathbb{R}^{d+2}$, $f(.,x,y,z)$ and $g(.,x,y,z)$ belong to $M^2([0,T],\mathbb{R})$, and such that:*

(i) *There exist some nonnegative constants L_f, L_g and a constant $\alpha\in[0,1)$ such that for every $\omega\in\Omega$, $t,t'\in[0,T]$, $x,x'\in\mathbb{R}^d$, $y,y'\in\mathbb{R}$ and $z,z'\in\mathbb{R}$*

$$\begin{aligned} \left|f(t,x,y,z) - f(t',x',y',z')\right|^2 &\leq L_f\left(|t-t'| + |x-x'|^2 + |y-y'|^2 + |z-z'|^2\right), \\ \left|g(t,x,y,z) - g(t',x',y',z')\right|^2 &\leq L_g\left(|t-t'| + |x-x'|^2 + |y-y'|^2\right) + \alpha\,|z-z'|^2, \end{aligned}$$

(ii) *For all $s\in[0,T]$ $f(s,.)$ and $g(s,.)$ are of class $\mathcal{C}^3$ with bounded partial derivatives up to order 3, uniformly in time.*

(iii) *For a function $h(t,x,y,z)$, set $h(t,0) := h(t,0,0,0)$. Then*

$$\sup_{r\in[0,T]} |f(r,0)| + \sup_{r\in[0,T]} |g(r,0)| < \infty.$$

Assumption 3. *Suppose that $\xi := \phi(X_T)$ for some function $\phi : \mathbb{R}^d \to \mathbb{R}$ of class $\mathcal{C}_b^2$ and that for every $\omega \in \Omega$,*

$$\sup_{t,x,y,z} |\partial_z g(t,x,y,z)| < 1.$$

2.1. **Some classical properties of the forward process X.** We at first recall without proof the following well known results on diffusion processes. Define the $\mathbb{R}^{d\times d}$-valued process $(\nabla X_t)_{0\le t\le T}$ by:

$$\nabla X_t := \Big(\frac{\partial}{\partial x_j} X_t^i, i,j = 1,\cdots,d\Big).$$

Then ∇X_t is an invertible $d\times d$ matrix, solution to a linear stochastic differential equation with coefficients depending on X_t. Furthermore, the assumptions on the coefficients σ and b yield the following classical result:

Proposition 2.1. *(i) For all $p \ge 1$, there exist a constant $C_p > 0$ such that for all $t,s \in [0,T]$:*

$$\mathbb{E}\,|X_t - X_s|^{2p} + \mathbb{E}\left|(\nabla X_t)^{-1} - (\nabla X_s)^{-1}\right|^{2p} \le C_p|t-s|^p.$$

(ii) For all $p \in [1,+\infty[$, there exist a constant $C_p > 0$ such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |X_t|^{2p} + \sup_{t\in[0,T]} \left|(\nabla X_t)^{-1}\right|^p\Big) \le C_p.$$

2.2. **Time increments of Y and L^2-regularity of Z.** The following lemma provides upper bounds for time increments of Y.

Lemma 2.2. *Set $\xi = \phi(X_T)$ for some function $\phi : \mathbb{R}^d \to \mathbb{R}$ be of class $\mathcal{C}_b^1$. Then we have*
(i) For all $p \ge 2$, there exist a constant $C_p > 0$ depending on T such that for all $t,s \in [0,T]$

$$\mathbb{E}\,|Y_t - Y_s|^p \le C_p|t-s|^{\frac{p}{2}}. \tag{2.1}$$

(ii) For all $p \ge 1$, there exist a constant $C > 0$ such that

$$\sup_{0\le r\le T} \mathbb{E}\,|Z_r|^{2p} \le C. \tag{2.2}$$

Notice that the inequality (2.1) is different from equation (2.11) in [Z04].

Proof. We at first prove (ii). Let $(\nabla Y_t)_{0\le t\le T} = (\partial_x Y_t)_{0\le t\le T}$ denote the real-valued process defined by differentiation of Y as function of the initial condition x of the diffusion process (X_t). We recall the following representation of Z (see [PP94] Proposition 2.3):

$$Z_t = \nabla Y_t\,(\nabla X_t)^{-1}\,\sigma\,(X_t)\,. \tag{2.3}$$

where $(\nabla Y_t, \nabla Z_t)$ satisfies the linear BDSDE with the forward process $(X_t, \nabla X_t)$ and the evolution equation:

$$\begin{aligned}\nabla Y_t =&\phi'(X_T)\nabla X_T + \int_t^T \Big(f_x(r,\Theta_r)\nabla X_r + f_y(r,\Theta_r)\nabla Y_r + f_z(r,\Theta_r)\nabla Z_r\Big)dr\\ &+ \int_t^T \Big(g_x(r,\Theta_r)\nabla X_r + g_y(r,\Theta_r)\nabla Y_r + g_z(r,\Theta_r)\nabla Z_r\Big)d\overleftarrow{B}_r - \int_t^T \nabla Z_r dW_r.\end{aligned} \tag{2.4}$$

By E.Pardoux and S.Peng [PP94] page 217, we deduce

$$\mathbb{E}\Big(\sup_{0\le t\le T} |\nabla Y_t|^p\Big) < \infty. \tag{2.5}$$

Then Hölder's inequality and Proposition 2.1 yield

$$\mathbb{E}\,|Z_t|^{2p} \le \Big(\mathbb{E}\,|\nabla Y_t|^{6p}\Big)^{\frac13}\Big(\mathbb{E}\left|(\nabla X_t)^{-1}\right|^{6p}\Big)^{\frac13}\Big(\mathbb{E}\,|\sigma\,(X_t)|^{6p}\Big)^{\frac13}.$$

This concludes the proof of (ii).

(i) Suppose that $s<t$, then using (1.3), we deduce that

$$\begin{aligned}|Y_t-Y_s|^p &\leq C_p\left|\int_s^t f(r,\Theta_r)-f(r,X_r,Y_r,0)\,dr\right|^p + C_p\left|\int_s^t f(r,X_r,Y_r,0)\,dr\right|^p\\ &\quad +C_p\left|\int_s^t g(r,\Theta_r)d\overleftarrow{B}_r\right|^p + C_p\left|\int_s^t Z_r dW_r\right|^p.\end{aligned}$$

Recall that $\widehat{\mathcal{F}}_t$ and $\mathcal{H}_t$ have been defined in (1.1). The process $(\int_0^t Z_r dW_r, 0\le t\le T)$ is a $(\widehat{\mathcal{F}}_t)$-martingale, while the process $(\int_t^T g(r,\Theta_r)d\overleftarrow{B}_r, 0\le t\le T)$ is a backward martingale for $(\mathcal{H}_t)$. Hence, the Burkholder-Davies-Gundy and Hölder inequalities yield

$$\begin{aligned}\mathbb{E}\,|Y_t-Y_s|^p &\leq C_p|t-s|^{p-1}\mathbb{E}\int_s^t |f(r,X_r,Y_r,0)|^p\,dr + C_p\mathbb{E}\left|\int_s^t |Z_r|\,dr\right|^p\\ &\quad +C_p\mathbb{E}\left(\int_s^t |g(r,\Theta_r)|^2\,dr\right)^{\frac{p}{2}} + C_p\mathbb{E}\left(\int_s^t |Z_r|^2\,dr\right)^{\frac{p}{2}}. \end{aligned}\tag{2.6}$$

Assumption 2 (i) and (ii), Proposition 2.1 and (1.4) yield

$$\begin{aligned}\mathbb{E}\Big(\int_s^t |g(r,\Theta_r)|^2\,dr\Big)^{\frac{p}{2}} &\leq C_p\mathbb{E}\left(\int_s^t |g(r,\Theta_r)-g(r,0)|^2\,dr\right)^{\frac{p}{2}} + C_p\mathbb{E}\left(\int_s^t |g(r,0)|^2\,dr\right)^{\frac{p}{2}}\\ &\leq C_p|t-s|^{\frac{p}{2}} + C_p\mathbb{E}\left(\int_s^t \left(|X_r|^2+|Y_r|^2\right)dr\right)^{\frac{p}{2}} + C_p\mathbb{E}\left(\int_s^t |Z_r|^2\,dr\right)^{\frac{p}{2}}\\ &\leq C_p|t-s|^{\frac{p}{2}} + C_p\mathbb{E}\left(\int_s^t |Z_r|^2\,dr\right)^{\frac{p}{2}}. \end{aligned}\tag{2.7}$$

Similarly,

$$\begin{aligned}\mathbb{E}\int_s^t |f(r,X_r,Y_r,0)|^p dr &\leq C_p\mathbb{E}\int_s^t |f(r,0)|^p\,dr + C_p\mathbb{E}\int_s^t |f(r,X_r,Y_r,0)-f(r,0)|^p\,dr\\ &\leq C_p\mathbb{E}\int_s^t |f(r,0)|^p\,dr + C_p\int_s^t \mathbb{E}\left(|X_r|^p+|Y_r|^p\right)dr \leq C_p\,|t-s|. \end{aligned}\tag{2.8}$$

Hence, the inequalities (2.6)-(2.8) imply

$$\mathbb{E}\,|Y_t-Y_s|^p \leq C_p|t-s|^{\frac{p}{2}} + C_p\mathbb{E}\left(\int_s^t |Z_r|^2\,dr\right)^{\frac{p}{2}}.$$

Using Hölder's inequality and (2.2) we conclude the proof of (2.1). □

Since equation (2.4) proves that the pair $(\nabla Y,\nabla Z)$ is the solution of a BDSDE with forward process $(X,\nabla X)\in L^p$ for every $p\in[1,+\infty[$, we deduce from (2.1) that for every function $\phi:\mathbb{R}^d\to\mathbb{R}$ of class $\mathcal{C}_b^2$, we have for $0\le s<t\le T$ and $p\in[1,+\infty[$:

$$\mathbb{E}|\nabla Y_t-\nabla Y_s|^p \leq C_p|t-s|^{\frac{p}{2}}, \tag{2.9}$$

for some constant $C_p>0$. We now establish some control of time increments of the process Z, following the idea of J.Zhang [Z04].

Theorem 2.3 (L^2-regularity of Z)**.** *There exists a non negative constant C such that for every subdivision $\pi=\{t_0=0<t_1\cdots<t_n=T\}$ with mesh $|\pi|$, one has*

$$\sum_{1\le i\le n}\mathbb{E}\int_{t_{i-1}}^{t_i}\left(\left|Z_t-Z_{t_{i-1}}\right|^2+|Z_t-Z_{t_i}|^2\right)dt \leq C|\pi|. \tag{2.10}$$

Proof. Using the representation of Z as a product, we deduce (2.3),

$$Z_t-Z_{t_i} = \nabla Y_t\left(\nabla X_t\right)^{-1}\sigma\left(X_t\right) - \nabla Y_{t_i}\left(\nabla X_{t_i}\right)^{-1}\sigma\left(X_{t_i}\right).$$

Then,

$$|Z_t - Z_{t_i}|^2 \leq 3\left|\nabla Y_t - \nabla Y_{t_i}\right|^2 \left|(\nabla X_t)^{-1}\right|^2 \left|\sigma(X_t)\right|^2 + 3\left|\nabla Y_{t_i}\right|^2 \left|(\nabla X_t)^{-1} - (\nabla X_{t_i})^{-1}\right|^2 \left|\sigma(X_t)\right|^2 + 3\left|\nabla Y_{t_i}\right|^2 \left|(\nabla X_{t_i})^{-1}\right|^2 \left|\sigma(X_t) - \sigma(X_{t_i})\right|^2.$$

To conclude the proof, we use Hölder's inequality, Proposition 2.1 and (2.9). □

Theorem 2.3 immediatly yields the following

Corollary 2.4.

$$\sum_{1\leq i\leq n-1} \mathbb{E}\int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr \leq C|\pi|.$$

3. The discretization of (X, Y, Z)

3.1. **Discretization of the process X: The Euler scheme.** We briefly recall the Euler scheme and send the reader to [KP99] for more details. Let $\pi := \{t_0 = 0 < t_1 < ... < t_n = T\}$ be a subdivision of $[0, T]$. We define the process X_t^π, called the Euler scheme, by

$$X_t^\pi = X_{t_0}^\pi + \int_{t_0}^t b\left(X_{s_\pi}^\pi\right) ds + \int_{t_0}^t \sigma(X_{s_\pi}^\pi) dW_s,$$

where $s_\pi := \max\{t_i \leq s\}$. The following result is well known:

Proposition 3.1. *There exists a constant $C > 0$ such that for every subdivision π,*

$$\max_i \mathbb{E}\left|X_{t_i} - X_{t_i}^\pi\right|^2 \leq C|\pi|, \qquad \mathbb{E}\int_{t_{i-1}}^{t_i} \left|X_r - X_{t_i}^\pi\right|^2 dr \leq C|\pi|^2.$$

3.2. **Discretization of the process (Y, Z): The step process.** In this section, we construct an approximation of (Y, Z) using Zhang's approach.
Let $\pi : t_0 = 0 < ... < t_n = T$ be any subdivision on $[0, T]$. Set $\mathcal{G}_t = \mathcal{G}_t^i$ for $t_{i-1} \leq t < t_i$, where we let

$$\mathcal{G}_t^i := \sigma\left(W_r - W_0; 0 \leq r \leq t\right) \vee \sigma\left(B_r - B_{t_{i-1}}; t_{i-1} \leq r \leq T\right), \quad t_{i-1} \leq t \leq t_i,$$

and define the $(\mathcal{G}_t)$-adapted process $\left(Y_t^\pi, Z_t^\pi\right)_{0\leq t\leq T}$ recursively (in a backward manner), as follows:
Set $Y_{t_n}^\pi = \phi(X_{t_n}^\pi)$, $Z_{t_n}^{\pi,1} = 0$; for $i = n-1, ..., 0$, let

$$Z_{t_i}^{\pi,1} := \frac{1}{\Delta t_{i+1}} \mathbb{E}\left(\int_{t_i}^{t_{i+1}} Z_r^\pi dr \middle| \mathcal{F}_{t_i}\right),$$

and for $i = n, ..., 1$, let

$$\Delta t_i = t_i - t_{i-1}, \Delta B_{t_i} = B_{t_i} - B_{t_{i-1}}, \Theta_{t_i}^{\pi,1} := \left(X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^{\pi,1}\right),$$

$$Y_t^\pi = Y_{t_i}^\pi + f\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta t_i + g\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta B_{t_i} - \int_t^{t_i} Z_r^\pi dW_r, \quad \forall t \in [t_{i-1}, t_i). \tag{3.1}$$

Note that the equation (3.1) is not a BDSDE in the sense of [PP94]; however, we have the following:

Proposition 3.2. *For every $i = 1, ..., n$, there exists a process $(Y_t^\pi, Z_t^\pi)_{t\in[t_{i-1},t_i)}$ adapted to the filtration $(\mathcal{G}_t, t_{i-1} \leq t < t_i)$, such that (3.1) holds. Furthermore, $Y_{t_i}^\pi \in \mathcal{F}_{t_i}$.*

Proof. The proof is similar to that in [PP94] page 212 and relies on the martingale representation theorem. Fix an integer $i > 0$ and suppose that the processes (Y_t^π) and (Z_t^π) have been defined for $t \geq t_i$, $(\mathcal{G}_t)$-adapted, and that $Y_{t_k}^\pi$ is $\mathcal{F}_{t_k}$-measurable for $k = i, \cdots, n$. We denote by $\left(M_t^i\right)_{t\in[t_{i-1},t_i]}$ the process defined by

$$M_t^i := \mathbb{E}\left(Y_{t_i}^\pi + f\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta t_i + g\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta B_{t_i} \middle| \mathcal{G}_t^i\right), \quad t_{i-1} \leq t \leq t_i.$$

By the martingale representation theorem, there exists a $\left(\mathcal{G}^i_t, t_{i-1} \le t \le t_i\right)$-adapted and square integrable process $\left(N^i_t, t_{i-1} \le t \le t_i\right)$ such that for $t_{i-1} \le t \le t_i$, $M^i_t = M^i_{t_{i-1}} + \int_{t_{i-1}}^t N^i_s dW_s$. Therefore, $M^i_t = M^i_{t_i} - \int_t^{t_i} N^i_s dW_s$. Clearly, $\mathcal{G}^i_{t_i}$ contains $\mathcal{F}_{t_i}$, $X^\pi_{t_i}$ is $\mathcal{F}^W_{t_i} \subset \mathcal{F}_{t_i}$ measurable and $\Theta^{\pi,1}_{t_i}$ is $\mathcal{F}_{t_i}$-measurable; hence

$$M^i_{t_i} = Y^\pi_{t_i} + f\left(t_i, \Theta^{\pi,1}_{t_i}\right)\Delta t_i + g\left(t_i, \Theta^{\pi,1}_{t_i}\right)\Delta B_{t_i}.$$

Furthermore, note that $\mathcal{G}^i_{t_{i-1}} = \mathcal{F}_{t_{i-1}}$, so that $M^i_{t_{i-1}}$ is $\mathcal{F}_{t_i}$-measurable. This completes the proof by setting: $Y^\pi_t = M^i_t$, $Z^\pi_t = N^i_t$ for $t_{i-1} \le t < t_i$. □

Before stating the main theorem of this section, we introduce the following

Definition 3.3. *Let $\kappa \ge 1$ be a constant. The subdivision π is said to be κ-uniform if $\kappa \Delta t_i \ge |\pi|$ for every $i \in \{1, ..., n\}$.*

The main example of a κ-uniform subdivision is a uniform subdivision (i.e. for all i, $\Delta t_i = |\pi|$) where $\kappa = 1$. The following lemma gives an upper estimate of $Z_{t_i} - Z^{\pi,1}_{t_i}$.

Lemma 3.4. *For any $i = 0, ..., n-1$, any κ-uniform subdivision π and $\beta > 0$ we have:*

$$\Delta t_i \mathbb{E}\left|Z_{t_i} - Z^{\pi,1}_{t_i}\right|^2 \le \kappa(1+\beta)\int_{t_i}^{t_{i+1}} |Z^\pi_r - Z_r|^2 dr + \kappa\left(1+\beta^{-1}\right)\int_{t_i}^{t_{i+1}} |Z_r - Z_{t_i}|^2 dr.$$

Proof. For any $i = 0, ..., n-1$, Z_{t_i} is $\mathcal{F}_{t_i}$-measurable, and $\Delta t_i \le |\pi| \le \kappa \Delta t_{i+1}$; thus

$$\begin{aligned}
\Delta t_i \mathbb{E}\left|Z_{t_i} - Z^{\pi,1}_{t_i}\right|^2 &= \Delta t_i \mathbb{E}\left|Z_{t_i} - \frac{1}{\Delta t_{i+1}}\mathbb{E}\left(\int_{t_i}^{t_{i+1}} Z^\pi_r dr \middle| \mathcal{F}_{t_i}\right)\right|^2 \\
&= \frac{\Delta t_i}{(\Delta t_{i+1})^2}\mathbb{E}\left|\mathbb{E}\left(\int_{t_i}^{t_{i+1}} (Z_{t_i} - Z^\pi_r)\, dr \middle| \mathcal{F}_{t_i}\right)\right|^2 \\
&\le \frac{\kappa}{\Delta t_{i+1}}\mathbb{E}\left|\int_{t_i}^{t_{i+1}} (Z_{t_i} - Z^\pi_r)\, dr\right|^2 \le \kappa \mathbb{E}\int_{t_i}^{t_{i+1}} |Z_{t_i} - Z^\pi_r|^2 dr.
\end{aligned}$$

where the last step is deduced from Schwarz's inequality. Using the usual estimate $|Z_{t_i} - Z^\pi_r|^2 \le (1+\beta)|Z^\pi_r - Z_r|^2 + (1+\beta^{-1})|Z_r - Z_{t_i}|^2$, we conclude the proof. □

The following theorem is the main result of this section. It proves that as $|\pi| \to 0$, (Y^π, Z^π) converges to (Y, Z).

Theorem 3.5. *Let π be a κ-uniform subdivision with sufficiently small mesh $|\pi|$, $\alpha < \frac{1}{\kappa}$, let $\phi \in \mathcal{C}^2$ and $\xi = \phi(X_T)$. Then we have*

$$\max_{0 \le i \le n} \mathbb{E}\left|Y_{t_i} - Y^\pi_{t_i}\right|^2 + \mathbb{E}\int_0^T |Z_r - Z^\pi_r|^2 dr \le C|\pi|. \tag{3.2}$$

Proof. Set $I_n = \mathbb{E}|\phi(X_T) - \phi(X^\pi_T)|^2$ and for $i = 1, ..., n$, let

$$I_{i-1} := \mathbb{E}\left|Y_{t_{i-1}} - Y^\pi_{t_{i-1}}\right|^2 + \mathbb{E}\int_{t_{i-1}}^{t_i} |Z_r - Z^\pi_r|^2 dr.$$

Using (1.3) with $\xi = \phi(X_T)$ and (3.1), we deduce

$$\begin{aligned}
Y_{t_{i-1}} - Y^\pi_{t_{i-1}} + \int_{t_{i-1}}^{t_i} (Z_r - Z^\pi_r)\, dW_r =& Y_{t_i} - Y^\pi_{t_i} + \int_{t_{i-1}}^{t_i} \left(f(r, \Theta_r) - f\left(t_i, \Theta^{\pi,1}_{t_i}\right)\right) dr \\
&+ \int_{t_{i-1}}^{t_i} \left(g(r, \Theta_r) - g\left(t_i, \Theta^{\pi,1}_{t_i}\right)\right) d\overleftarrow{B}_r.
\end{aligned} \tag{3.3}$$

By construction, $Y_{t_{i-1}} - Y^\pi_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}} = \mathcal{G}^i_{t_{i-1}}$ measurable while for $r \in [t_{i-1}, t_i)$, $Z_r - Z^\pi_r$ is $(\mathcal{G}_r)$-adapted. Hence, $Y_{t_{i-1}} - Y^\pi_{t_{i-1}}$ is orthogonal to $\int_{t_{i-1}}^{t_i} (Z_r - Z^\pi_r)\, dW_r$. Therefore,

$$I_{i-1} = \mathbb{E}\left| Y_{t_{i-1}} - Y^\pi_{t_{i-1}} + \int_{t_{i-1}}^{t_i} (Z_r - Z^\pi_r)\, dW_r \right|^2.$$

Since $g(r, \Theta_r)$ (resp. $g(t_i, \Theta^{\pi,1}_{t_i})$) is $\mathcal{F}_r$ (resp. $\mathcal{F}_{t_i}$)-measurable, the random variables $Y_{t_i} - Y^\pi_{t_i}$ and $\int_{t_{i-1}}^{t_i} \left(g(r, X_r, Y_r) - g\left(t_i, X^\pi_{t_i}, Y^\pi_{t_i}\right)\right) d\overleftarrow{B}_r$ are orthogonal. Hence for every $\epsilon > 0$, using assumption 2, the L^2-isometry of backward stochastic integrals, Schwarz's inequality and (3.3), we deduce

$$\begin{aligned}
I_{i-1} \leq& \left(1 + \frac{\Delta t_i}{\epsilon}\right) \mathbb{E}\left|Y_{t_i} - Y^\pi_{t_i}\right|^2 + \left(1 + 2\frac{\epsilon}{\Delta t_i}\right) \mathbb{E}\left| \int_{t_{i-1}}^{t_i} \left(f(r, \Theta_r) - f\left(t_i, \Theta^{\pi,1}_{t_i}\right)\right) dr \right|^2 \\
&+ \left(1 + \frac{\Delta t_i}{\epsilon}\right) \mathbb{E}\left| \int_{t_{i-1}}^{t_i} \left(g(r, \Theta_r) - g\left(t_i, \Theta^{\pi,1}_{t_i}\right)\right) d\overleftarrow{B}_r \right|^2 \\
\leq& \left(1 + \Delta t_i \epsilon^{-1}\right) \mathbb{E}\left|Y_{t_i} - Y^\pi_{t_i}\right|^2 + (\Delta t_i + 2\epsilon)\, \mathbb{E}\int_{t_{i-1}}^{t_i} \left| f(r, \Theta_r) - f\left(t_i, \Theta^{\pi,1}_{t_i}\right)\right|^2 dr \\
&+ \left(1 + \Delta t_i \epsilon^{-1}\right) \mathbb{E}\int_{t_{i-1}}^{t_i} \left| g(r, \Theta_r) - g\left(t_i, \Theta^{\pi,1}_{t_i}\right)\right|^2 dr \\
\leq& \Big[1 + \Delta t_i \epsilon^{-1} + 2L_f\left(\Delta t_i^2 + 2\epsilon \Delta t_i\right) + 2L_g\left(\Delta t_i + \Delta t_i^2 \epsilon^{-1}\right)\Big] \mathbb{E}\left|Y_{t_i} - Y^\pi_{t_i}\right|^2 \\
&+ \Big[L_f(\Delta t_i + 2\epsilon) + L_g\left(1 + \Delta t_i \epsilon^{-1}\right)\Big] \mathbb{E}\int_{t_{i-1}}^{t_i} \left(|\pi| + \left|X_r - X^\pi_{t_i}\right|^2 + 2\left|Y_r - Y_{t_i}\right|^2\right) dr \\
&+ \Big[L_f(\Delta t_i + 2\epsilon) + \alpha\left(1 + \Delta t_i \epsilon^{-1}\right)\Big] \mathbb{E}\int_{t_{i-1}}^{t_i} \left|Z_r - Z^{\pi,1}_{t_i}\right|^2 dr.
\end{aligned}$$

For $|\pi| \leq 1$, $\Delta t_i^2 \leq \Delta t_i$; using Proposition 2.1 with $p = 2$ and Proposition 3.1, we deduce

$$\mathbb{E}\int_{t_{i-1}}^{t_i} \left(|\pi| + \left|X_r - X^\pi_{t_i}\right|^2 + 2\left|Y_r - Y_{t_i}\right|^2\right) dr \leq C|\pi|^2,$$

for some constant $C > 0$. Hence for any $\gamma > 0$

$$\begin{aligned}
I_{i-1} \leq& \Big[1 + \left(\epsilon^{-1} + 2L_f(1 + 2\epsilon) + 2L_g\left(1 + \epsilon^{-1}\right)\right)\Delta t_i\Big] \mathbb{E}\left|Y_{t_i} - Y^\pi_{t_i}\right|^2 \\
&+ C\Big[L_f(\Delta t_i + \epsilon) + L_g\left(1 + \Delta t_i \epsilon^{-1}\right)\Big] |\pi|^2 \\
&+ \left(1 + \gamma^{-1}\right)\left[L_f(\Delta t_i + 2\epsilon) + \alpha\left(1 + \Delta t_i \epsilon^{-1}\right)\right] \mathbb{E}\int_{t_{i-1}}^{t_i} \left|Z_r - Z_{t_i}\right|^2 dr \\
&+ (1 + \gamma)\left[L_f(\Delta t_i + 2\epsilon) + \alpha\left(1 + \Delta t_i \epsilon^{-1}\right)\right] \Delta t_i \mathbb{E}\left|Z_{t_i} - Z^{\pi,1}_{t_i}\right|^2.
\end{aligned}$$

Lemma 3.4 yields for some positive constants C_ϵ, $C_{\epsilon,\gamma}$ and $C_{\epsilon,\gamma,\beta}$, we have:

$$\begin{aligned}
I_{i-1} \leq & (1+C_\epsilon \Delta t_i)\,\mathbb{E}\left|Y_{t_i}-Y^\pi_{t_i}\right|^2 + C_\epsilon\,|\pi|^2 + C_{\epsilon,\gamma}\,\mathbb{E}\int_{t_{i-1}}^{t_i} |Z_r - Z_{t_i}|^2\,dr \\
& + \kappa\,(1+\gamma)\,(1+\beta)\Big[L_f\,(\Delta t_i + 2\epsilon) + \alpha\left(1+\Delta t_i \epsilon^{-1}\right)\Big]\mathbb{E}\int_{t_i}^{t_{i+1}} |Z^\pi_r - Z_r|^2\,dr \\
& + \kappa\,(1+\gamma)\left(1+\beta^{-1}\right)\Big[L_f\,(\Delta t_i + 2\epsilon) + \alpha\left(1+\Delta t_i \epsilon^{-1}\right)\Big]\mathbb{E}\int_{t_i}^{t_{i+1}} |Z_r - Z_{t_i}|^2\,dr \\
\leq & (1+C_\epsilon\,\Delta t_i)\,\mathbb{E}\left|Y_{t_i}-Y^\pi_{t_i}\right|^2 + C_\epsilon|\pi|^2 + C_{\epsilon,\gamma,\beta}\,\mathbb{E}\int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2\,dr \\
& + \kappa\,(1+\gamma)\,(1+\beta)\Big[L_f\,(\Delta t_i + 2\epsilon) + \alpha\left(1+\Delta t_i \epsilon^{-1}\right)\Big]\int_{t_i}^{t_{i+1}} |Z^\pi_r - Z_r|^2\,dr. \qquad (3.4)
\end{aligned}$$

Recall that $\alpha < \frac{1}{\kappa}$ and let $0 < \delta < 1-\kappa\alpha$. Then choose positive constants β and γ small enough to ensure $\kappa\,(1+\gamma)\,(1+\beta)\,\alpha < 1 - \frac{2\delta}{3}$. Finally, let $\epsilon > 0$ small enough to ensure that $2\kappa\,(1+\gamma)\,(1+\beta)\,L_f\epsilon < \frac{\delta}{6}$. Then (3.4) implies the existence of $C > 0$ such that for every $i = 1, ..., n-1$,

$$I_{i-1} + \frac{\delta}{3}\mathbb{E}\int_{t_i}^{t_{i+1}} |Z^\pi_r - Z_r|^2\,dr \leq (1+C\Delta t_i)\,I_i + C|\pi|^2 + C\mathbb{E}\int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2\,dr. \qquad (3.5)$$

Using the discrete Gronwall lemma in [Z04] (Lemma 5.4 page 479), we deduce

$$\begin{aligned}
\max_{0\leq i\leq n} I_i \leq & C\,e^{CT}\mathbb{E}\Big(I_n + \sum_{1\leq i\leq n-1}\int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2\,dr + |\pi|\Big) \\
\leq & C\mathbb{E}\Big(\,|\phi\,(X_T) - \phi\,(X^\pi_T)|^2 + \sum_{1\leq i\leq n}\int_{t_{i-1}}^{t_i}\Big(\left|Z_r - Z_{t_{i-1}}\right|^2 + |Z_r - Z_{t_i}|^2\Big)\,dr + |\pi|\Big).
\end{aligned}$$

Since ϕ is Lipschitz, Proposition 3.1 implies that $\mathbb{E}|\phi(X_T) - \phi(X^\pi_T)|^2 \leq C|\pi|$; thus Theorem 2.3 implies

$$\max_{0\leq i\leq n} \mathbb{E}\left|Y_{t_i} - Y^\pi_{t_i}\right|^2 \leq C|\pi|. \qquad (3.6)$$

Moreover, summing both sides of (3.5) over i from 1 to $n-1$ and using Corollary 2.4 we obtain:

$$\begin{aligned}
\sum_{0\leq i\leq n-2} I_i + \frac{\delta}{3}\,\mathbb{E}\int_{t_1}^{T} |Z^\pi_r - Z_r|^2\,dr \leq & \sum_{1\leq i<n} (1+C\Delta t_i)\,I_i + C|\pi| \\
& + C\sum_{1\leq i<n}\mathbb{E}\int_{t_{i-1}}^{t_{i+1}} |Z_r - Z_{t_i}|^2\,dr, \\
\leq & C|\pi| + \sum_{1\leq i\leq n-1} (1+C\Delta t_i)\,I_i.
\end{aligned}$$

Therefore,

$$I_0 + \frac{\delta}{3}\,\mathbb{E}\int_{t_1}^{T} |Z^\pi_r - Z_r|^2\,dr \leq C|\pi| + I_{n-1} + C\sum_{1\leq i\leq n-1} \Delta t_i I_i$$

Since $\delta < 1-\kappa\alpha < 3$, using (3.6) we deduce

$$\frac{\delta}{3}\,\mathbb{E}\int_0^{T} |Z^\pi_r - Z_r|^2\,dr \leq C|\pi| + \mathbb{E}\int_{t_{n-1}}^{t_n} |Z^\pi_r - Z_r|^2 dr + C|\pi|\mathbb{E}\int_0^{T} |Z^\pi_r - Z_r|^2 dr. \qquad (3.7)$$

The equations (1.3) and (3.1) imply

$$\int_{t_{n-1}}^{t_n} (Z_r^\pi - Z_r) dW_r = (Y_{t_n}^\pi - Y_{t_n}) - (Y_{t_{n-1}}^\pi - Y_{t_{n-1}}) + \int_{t_{n-1}}^{t_n} \left(f\left(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0\right) - f\left(r, X_r, Y_r, Z_r\right) \right) dr + \int_{t_{n-1}}^{t_n} \left(g\left(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0\right) - g\left(r, X_r, Y_r, Z_r\right) \right) d\overleftarrow{B}_r.$$

The L^2-isometry, Schwarz's inequality, (3.6), Lemma 2.2, Propositions 2.1 and 3.1

$$\begin{aligned}\mathbb{E}\int_{t_{n-1}}^{t_n} \left|Z_r^\pi - Z_r\right|^2 dr \leq & 4\mathbb{E}\left|Y_{t_n}^\pi - Y_{t_n}\right|^2 + 4\mathbb{E}\left|Y_{t_{n-1}}^\pi - Y_{t_{n-1}}\right|^2 \\ & + 4|\pi|\mathbb{E}\int_{t_{n-1}}^{t_n} \left|f\left(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0\right) - f\left(r, X_r, Y_r, Z_r\right)\right|^2 dr \\ & + 4\mathbb{E}\int_{t_{n-1}}^{t_n} \left|g\left(t_n, X_{t_n}^\pi, Y_{t_n}^\pi, 0\right) - g\left(r, X_r, Y_r, Z_r\right)\right|^2 dr \\ \leq & C|\pi| + C|\pi| \sup_{t_{n-1}\leq r\leq t_n} \mathbb{E}\left(|X_r - X_T|^2 + |X_{t_n}^\pi - X_T|^2\right) \\ & + C|\pi| \sup_{t_{n-1}\leq r\leq t_n} \mathbb{E}\left(|Y_r - Y_T|^2 + |Y_{t_n}^\pi - Y_T|^2 + |Z_r|^2\right) \\ \leq & C|\pi|. \end{aligned} \tag{3.8}$$

For $|\pi|$ small enough, we have $C|\pi| \leq \delta/6$; thus (3.7) and (3.8) conclude the proof. □

4. A numerical scheme

In this section we propose a numerical scheme based on the results of the previous sections. First of all, given $x \in \mathbb{R}^d$, $s < t$ we set:

$$X_t(s,x) := x + (t-s)\, b(x) + \sigma(x)\,(W_t - W_s).$$

We clearly have $X_{t_i}^\pi = X_{t_i}\left(t_{i-1}, X_{t_{i-1}}^\pi\right)$ for every $i = 1, \dots, n$. Then, given a vector $(x_0, \dots, x_i; x_{i+1}, \dots, x_n) \in \mathbb{R}^{(i+1)d} \times \mathbb{R}^{n-i}$, set $\mathbf{x}_{n+1} = \emptyset$ and for $i = 0, \dots, n-1$, let

$$\mathbf{x}^i := (x_0, ..., x_i), \quad \mathbf{x}_{i+1} := (x_{i+1}, ..., x_n).$$

Define by induction, the functions $u_i^\pi,\ v_i^\pi : \mathbb{R}^{(i+1)d} \times \mathbb{R}^{n-i} \to \mathbb{R}$ (resp. the random variables $U_i^\pi, V_i^\pi : \mathbb{R}^{(i+1)d} \times \Omega \times \mathbb{R}^{n-i-1} \to \mathbb{R}$) as follows:

$$u_n^\pi(x_0, ..., x_n) := \phi(x_n), \quad v_n^\pi(x_0, ..., x_n) := 0,$$

and for $i = 0, \dots, n-1$ let

$$\begin{aligned} U_i^\pi\left(\mathbf{x}^i, \omega, \mathbf{x}_{i+2}\right) := & u_{i+1}^\pi\left(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}\right) \\ & + f\Big(t_{i+1}, X_{t_{i+1}}(t_i, x_i), u_{i+1}^\pi\left(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}\right), \\ & \quad v_{i+1}^\pi\left(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}\right)\Big)\Delta t_{i+1}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} V_i^\pi\left(\mathbf{x}^i, \omega, \mathbf{x}_{i+2}\right) := & g\Big(t_{i+1}, X_{t_{i+1}}(t_i, x_i), u_{i+1}^\pi\left(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}\right), \\ & \quad v_{i+1}^\pi\left(\mathbf{x}^i, X_{t_{i+1}}(t_i, x_i), \mathbf{x}_{i+2}\right)\Big), \end{aligned} \tag{4.2}$$

$$u_i^\pi\left(\mathbf{x}^i; \mathbf{x}_{i+1}\right) := \mathbb{E}U_i^\pi\left(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}\right) + x_{i+1}\mathbb{E}V_i^\pi\left(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}\right), \tag{4.3}$$

$$\begin{aligned} v_i^\pi\left(\mathbf{x}^i; \mathbf{x}_{i+1}\right) := & \frac{1}{\Delta t_{i+1}}\mathbb{E}\left(U_i^\pi\left(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}\right)\Delta W_{t_{i+1}}\right) \\ & + \frac{x_{i+1}}{\Delta t_{i+1}}\mathbb{E}\left(V_i^\pi\left(\mathbf{x}^i, \omega, \mathbf{x}_{i+1}\right)\Delta W_{t_{i+1}}\right). \end{aligned} \tag{4.4}$$

Theorem 4.1. *We have for all $i = 0, ..., n$*

$$Y_{t_i}^{\pi} = u_i^{\pi}\left(X_{t_0}^{\pi}, ..., X_{t_i}^{\pi}; \Delta B_{t_{i+1}}, ..., \Delta B_{t_n}\right), \tag{4.5}$$

$$Z_{t_i}^{\pi,1} = v_i^{\pi}\left(X_{t_0}^{\pi}, ..., X_{t_i}^{\pi}; \Delta B_{t_{i+1}}, ..., \Delta B_{t_n}\right). \tag{4.6}$$

Proof. We proceed by backward induction. For $i = n$, by definition $Y_{t_n}^{\pi} = \phi\left(X_{t_n}^{\pi}\right)$, so (4.5) and (4.6) hold trivially.
Suppose that the result is true for $j = n, n-1, \cdots, i$. The scheme described in (3.1) implies that

$$Y_{t_{i-1}}^{\pi} = Y_{t_i}^{\pi} + f\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta t_i + g\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta B_{t_i} - \int_{t_{i-1}}^{t_i} Z_r^{\pi} dW_r. \tag{4.7}$$

To prove (4.5), we take the conditional expectation of (4.7) with respect to $\widehat{\mathcal{F}}_{t_{i-1}} = \mathcal{F}_{0,t_{i-1}}^W \vee \mathcal{F}_{0,T}^B$; this yields

$$\begin{aligned}\mathbb{E}\left(Y_{t_{i-1}}^{\pi}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) =& \mathbb{E}\left(Y_{t_i}^{\pi}|\widehat{\mathcal{F}}_{t_{i-1}}\right) + \mathbb{E}\left(f\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta t_i\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) \\ &+ \mathbb{E}\left(g\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta B_{t_i}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) - \mathbb{E}\left(\int_{t_{i-1}}^{t_i} Z_r^{\pi} dW_r\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right).\end{aligned}$$

Using the fact that $\int_{t_{i-1}}^{t_i} Z_r^{\pi} dW_r$ is orthogonal to any $\widehat{\mathcal{F}}_{t_{i-1}}$-measurable random variable, and the induction hypothesis we deduce:

$$\begin{aligned}Y_{t_{i-1}}^{\pi} =& \mathbb{E}\left(Y_{t_i}^{\pi}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) + \mathbb{E}\left(f\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta t_i\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) + \mathbb{E}\left(g\left(t_i, , \Theta_{t_i}^{\pi,1}\right)\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right)\Delta B_{t_i}. \\ =& \mathbb{E}\left(u_i^{\pi}\left(X_{t_0}^{\pi}, ..., X_{t_{i-1}}^{\pi}, X_{t_i}(t_{i-1}, X_{t_{i-1}}^{\pi}), \Delta B_{t_{i+1}}, ..., \Delta B_{t_n}\right)\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) \\ &+ \Delta t_i\, \mathbb{E}\Big(f\Big(t_i, X_{t_i}(t_{i-1}, X_{t_{i-1}}^{\pi}), u_i^{\pi}\left(X_{t_0}^{\pi}, ..., X_{t_{i-1}}^{\pi}, X_{t_i}(t_{i-1}, X_{t_{i-1}}^{\pi}), \Delta B_{t_{i+1}}, ..., \Delta B_{t_n}\right), \\ &\qquad v_i^{\pi}\left(X_{t_0}^{\pi}, ..., X_{t_{i-1}}^{\pi}, X_{t_i}(t_{i-1}, X_{t_{i-1}}^{\pi}), \Delta B_{t_{i+1}}, ..., \Delta B_{t_n}\right)\Big)\Big|\widehat{\mathcal{F}}_{t_{i-1}}\Big) \\ &+ \Delta B_i\, \mathbb{E}\Big(g\Big(t_i, X_{t_i}(t_{i-1}, X_{t_{i-1}}^{\pi}), u_i^{\pi}\Big(X_{t_0}^{\pi}, ..., X_{t_{i-1}}^{\pi}, X_{t_i}(t_{i-1}, X_{t_{i-1}}^{\pi}), \Delta B_{t_{i+1}}, ..., \Delta B_{t_n}\Big), \\ &\qquad v_i^{\pi}\left(X_{t_0}^{\pi}, ..., X_{t_{i-1}}^{\pi}, X_{t_i}(t_{i-1}, X_{t_{i-1}}^{\pi}), \Delta B_{t_{i+1}}, ..., \Delta B_{t_n}\right)\Big)\Big|\widehat{\mathcal{F}}_{t_{i-1}}\Big).\end{aligned}$$

Since all ΔB_{t_l}, $l = 1, \ldots, n$ and $X_{t_k}^{\pi}, k = 0, \ldots, i-1$ are $\widehat{\mathcal{F}}_{t_{i-1}}$ measurable while $W_{t_i} - W_{t_{i-1}}$ is independent of $\widehat{\mathcal{F}}_{t_{i-1}}$; we deduce (4.5).
To prove (4.6), multiply (4.7) by $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$ and take the conditional expectation with respect to $\widehat{\mathcal{F}}_{t_{i-1}}$, this yields

$$\begin{aligned}\mathbb{E}\Big(Y_{t_{i-1}}^{\pi}\Delta W_{t_i}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\Big) =& \mathbb{E}\Big(Y_{t_i}^{\pi}\Delta W_{t_i}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\Big) + \mathbb{E}\left(f\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta t_i \Delta W_{t_i}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) \\ &+ \mathbb{E}\left(g\left(t_i, \Theta_{t_i}^{\pi,1}\right)\Delta B_{t_i}\Delta W_{t_i}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) - \mathbb{E}\left(\Delta W_{t_i}\int_{t_{i-1}}^{t_i} Z_r^{\pi} dW_r\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right).\end{aligned}$$

Since $Y_{t_{i-1}}^{\pi} \in \widehat{\mathcal{F}}_{t_{i-1}}$ and ΔW_{t_i} is independent of $\widehat{\mathcal{F}}_{t_{i-1}}$ and centered we deduce

$$\mathbb{E}\left(Y_{t_{i-1}}^{\pi}\Delta W_{t_i}\Big|\widehat{\mathcal{F}}_{t_{i-1}}\right) = 0.$$

Furthermore,

$$\begin{aligned}\mathbb{E}\Big(\Delta W_{t_i}\int_{t_{i-1}}^{t_i} Z_r^{\pi} dW_r\Big|\widehat{\mathcal{F}}_{t_{i-1}}\Big) =& \mathbb{E}\Big(\int_{t_{i-1}}^{t_i} Z_r^{\pi} dr\Big|\widehat{\mathcal{F}}_{t_{i-1}}\Big) \\ =& \mathbb{E}\Big(\int_{t_{i-1}}^{t_i} Z_r^{\pi} dr\Big|\mathcal{F}_{t_{i-1}}\Big) = \Delta t_i Z_{t_i}^{\pi,1}.\end{aligned}$$

this completes the proof of (4.6). □

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References

[Ba97] Bally V., Approximation scheme for solutions of BSDE *Backward stochastic differential equations (Paris, 1995–1996), 177–191, Pitman Res. Notes Math. Ser.*, 364, Longman, Harlow, 1997.

[BT04] Bouchard B., Touzi N., Discrete time approximation and Monte Carlo simulation of backward stochastic differential equations *Stochastic process and applications* **111** (2004) 175-206

[Ch97] Chevance D., Numerical methods for backward stochastic differential equations *Numerical methods in finance, 232–244, Publ. Newton Inst., Cambridge Univ. Press, Cambridge,* 1997.

[GLW05] Gobet E., Lemor J.P., Warin X., A regression-based Monte Carlo method to solve Backward stochastic differential equations *The Annals of Applied Probability* (2005), Vol. 15, No. 3, 2172–2202

[KP99] Kloeden P.E., Platen E., *Numerical solution of stochastic differential equations* Springer (1999)

[MY99] Ma J., Yong J., *Forward-Backward Stochastic Differential Equations and their Applications* Lecture Note in Math. 1702 Springer

[MT09] Milstein G.N., Tretyakov M.V., Solving parabolic stochastic partial differential equations via averaging over characteristics *Mathematics of computation* (2009)

[NP88] Nualart D., Pardoux E., Stochastic calculus with anticipatng integrands *Probability theory related fields* 78, 535-581 (1988)

[PP92] Pardoux E., Peng S., Backward stochastic differential equation and quasilinear parabolic partial differential equations. In: *B. L.Rozovski, R. B. Sowers (eds). Stochastic partial equations and their applications. Lect. Notes control Inf. Sci.* **176**, $200 - 217$, Springer, Berlin, (1992).

[PP94] Pardoux E., Peng S., Backward doubly stochastic differential equations and systems of quasilinear SPDEs *Probability theory related fields* (1994) 209-227

[SYY08] Shi Y., Yang W., Yuan J., Numerical computation for backward doubly SDEs and SPDEs arXiv:0805.4662

[Z04] Zhang J., A numerical scheme for BSDEs *The Annals of Applied Probability* Vol. 14 No. 1 (2004) 459-488

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