Ornstein-Uhlenbeck semi-groups on stratified groups

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Abstract

We consider, in the setting of stratified groups G, two analogues of the Ornstein-Uhlenbeck semi-group, namely Markovian diffusion semi-groups acting on $L^q(p(\gamma)d\gamma)$, whose invariant density p is a heat kernel at time 1 on G.

The first one is symmetric on $L^2(pd\gamma)$, its generator is $\sum_{i=1}^n X_i^* X_i$, where $(X_i)_{i=1}^n$ is a basis of the first layer of the Lie algebra of G.

The second one, denoted by $T_t = e^{-tN}, t > 0$, is non symmetric on $L^2(pd\gamma)$ and the formal real part of N is $\sum_{i=1}^n X_i^* X_i$. The operators e^{-tN} are compact on $L^q(pd\gamma)$, $1 < q < \infty$. The spectrum of N on this space is the set of integers $\mathbb{N}$ if polynomials are dense in $L^2(p(\gamma)d\gamma)$, i.e if G has at most 4 layers; and we determine in this case its eigenspaces. When G is step 2, we give another description of these eigenspaces, very similar to the classical definition of "Hermite polynomials" by their generating function.

Keywords: stratified groups, sub Laplacian, heat kernel measure, Ornstein-Uhlenbeck semi-groups.

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1 Introduction and notation

Let G be a stratified Lie group equipped with its (biinvariant) Haar measure dg and dilations $(\delta_t)_{t\geq 0}$. Let Q be the homogeneous dimension of G. We denote by $\mathcal{D}(G)$ the space of $\mathcal{C}^\infty$ compactly supported functions on G, by $\mathcal{S}(G)$ the space of Schwartz functions, by $\mathcal{S}'(G)$ its dual, and $L^q(\varphi dg) = L^q(G, \varphi dg)$ for a measurable non negative function φ.

As usual, elements Z of the Lie algebra $\mathcal{G}$ are identified with left invariant vector fields by

$$(Zf)(g) = \frac{d}{dt} |_{t=0} f(g \exp tZ).$$

Let L be a subLaplacian on G, i.e. an operator on $\mathcal{S}(G)$ defined by

$$L = -\sum_1^n X_i^2 \tag{1}$$

where $(X_i)_{1\leq i\leq n}$ is a linear basis of the first layer of $\mathcal{G}$. Obviously L commutes with left translations and satisfies

$$\delta_{t^{-1}} L \delta_t = t^2 L, \ t > 0. \tag{2}$$

The following facts are well known, see e.g. [FS, propositions 1.68, 1.70, 1.74]: $-\frac{L}{2}$ generates a strongly continuous semi-group $e^{-\frac{t}{2}L}$ of convolution operators which are contractions on $L^q(dg)$, $1 \leq q \leq \infty$. The kernel p_t of $e^{-\frac{t}{2}L}$ is a positive function such that $p_t(g) = p_t(g^{-1})$, it lies in $\mathcal{S}(G)$ and has norm one in $L^1(dg)$. Denoting $p_1 = p$,

$$p_t(g) = t^{-\frac{Q}{2}} p \circ \delta_{\frac{1}{\sqrt{t}}}(g).$$

Equivalently, for $f \in L^q(dg)$,

$$e^{-\frac{t}{2}L}(f)(\gamma) = f * p_t(\gamma) = \int_G f(\gamma g^{-1}) p_t(g) dg = \int_G f(\gamma \delta_{\sqrt{t}} g^{-1}) p(g) dg. \tag{3}$$

The aim of this paper is to generalize the Ornstein-Uhlenbeck semi-group in the setting of stratified groups, namely to consider Markovian semi-groups acting on $L^q(p(\gamma)d\gamma), 1 \leq q \leq \infty$, for which $p(\gamma)d\gamma$ is an invariant measure, whose generators are related to the first layer gradient

$$\nabla = (X_1, .., X_n).$$

The classical Ornstein-Uhlenbeck semi-group is defined on $\mathcal{S}(\mathbb{R}^n)$ by Mehler formula

$$e^{-tN_0}(f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) p(y) dy, \ t \geq 0,$$

where the gaussian density $p(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|y|^2}$ is the kernel of $e^{-\frac{\Delta}{2}}$, and Δ is the (positive) Laplacian on $\mathbb{R}^n$. The O-U semi-group is contracting on $L^q(\mathbb{R}^n, pdx)$, $1 \leq q \leq \infty$, compact if $1 < q < \infty$, but not compact on $L^1(\mathbb{R}, pdx)$ [D, theorem 4.3.5], and p is an invariant measure. The generator $-N_0$ satisfies

$$N_0 = \sum_{j=1}^{n} (\frac{\partial}{\partial x_j})^* \frac{\partial}{\partial x_j} = \Delta - \sum_{j=1}^{n} \frac{\frac{\partial p}{\partial x_j}}{p} \frac{\partial}{\partial x_j} = \Delta + \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} = \Delta + A$$

where $(\frac{\partial}{\partial x_j})^*$ denotes the adjoint on $L^2(\mathbb{R}^n, pdx)$ and A is the generator of dilations on $\mathbb{R}^n$. On $L^q(\mathbb{R}^n, pdx), 1 < q < \infty$, the spectrum of N_0 is $\mathbb{N}$, and the Hermite polynomials on $\mathbb{R}^n$ form an orthogonal basis of eigenvectors of e^{-tN_0} in $L^2(\mathbb{R}^n, pdx)$.

The generator N_0 has a fruitful generalization in (commutative or non commutative) analysis on deformed or q-Fock spaces, namely the number operator N, i.e. the second differential quantization of identity. A substitute of Mehler formula holds and $(e^{-tN})_{t>0}$ is the compression of a one parameter group of unitary dilations, see e.g. [LP$_2$].

Our motivation in this paper is to exploit Mehler formula in another direction: in the setting of stratified groups Mehler formula still defines a semi-group $(e^{-tN})_{t>0}$ and we study which properties of the classical O-U semi-group remain valid. We also hope that this semi-group might throw some light on properties of the heat density p.

Results and organization of the paper

In section 2 we recall some properties of the self-adjoint semi-group on $L^2(pd\gamma)$ whose generator is $-\nabla^*\nabla = -\sum_{i=1}^{n} X_i^* X_i$, X_i^* being the formal adjoint of X_i with respect to $L^2(pd\gamma)$. We give in passing a simple proof of the known Poincaré inequality in $L^2(pd\gamma)$.

In the main section 3 we consider another generalization, the Mehler semi-group, which is defined for $t \geq 0$ by (theorem 3)

$$T_t(f)(\gamma) = \int_G f(\delta_{e^{-t}} \gamma \delta_{\sqrt{1-e^{-2t}}} g) p(g) dg = e^{-tN}(f)(\gamma).$$

Some properties are described in 3.2, in particular $pd\gamma$ is an invariant measure. This semi-group is not selfadjoint on $L^2(pd\gamma)$, but formally the real part of its generator $-N$ is $-\nabla^*\nabla$ and $N = L + A$ where A is the generator of the group $(\delta_{e^t})_{t\in\mathbb{R}}$ of dilations, studied in 3.3.

We show in 3.4 that every $T_t, t > 0$, is compact on $L^q(pd\gamma), 1 < q < \infty$, (proposition 6), with common spectrum $e^{-t\mathbb{N}}$ on the closed subspace spanned by polynomials (theorem 7), which coincides with the whole space only if the number of layers of $\mathcal{G}$ is ≤ 4 (proposition 8). We describe the eigenspaces in this case.

In 3.5 we give another description of these eigenspaces if G is step two, similar to the usual definition of one variable Hermite polynomials by their generating function.

More notation

We denote $\mathcal{G} = V_1 \oplus .. \oplus V_k$, where $V_1, .., V_k$ are the layers of the Lie algebra $\mathcal{G}$ of G, $V_k = \mathcal{Z}$ being the central layer, so that [FS, p. 5]

$$[V_j, V_h] \subset V_{j+h},\ [V_1, V_h] = V_{h+1}, 1 \leq h < k$$

The homogeneous dimension of G is

$$Q = \sum_{j=1}^{k} j \dim V_j.$$

Generic elements of the layers are denoted respectively by $X, Y..., U$, and respective basis of the layers are denoted by $(X_1, .., X_n)$, $(Y_1, .., Y_m)$, ..., $(U_1, .., U_k)$. Such a basis is also denoted by $(Z_j)_{1 \leq j \leq N}$. We denote accordingly

$$\begin{aligned} g &= \exp(\sum x_i X_i + \sum y_i Y_i + .. + \sum u_i U_i) = \exp(X + Y + .. + U) \\ &= (x, y, .., u) = \exp(\sum_{j=1}^{N} z_j Z_j) = (z_j)_{j=1}^{N}, \end{aligned}$$

since the mapping $(z_j)_{j=1}^N \to g$ is a diffeomorphism: $\mathbb{R}^N \to G$.

We denote by $\mathcal{P}$ the space of polynomials on G, as defined in [FS, chapter I-C] for the fixed basis $(Z_j)_{j=1}^N$: they are polynomials w.r. to the coordinates $z_j, 1 \leq j \leq N$.

The dilation $\delta_t, t \geq 0$, are defined on $\mathcal{G}$ and G by

$$\delta_t(X + Y + .. + U) = tX + t^2 Y + .. + t^k U,\ \ \delta_t(\exp Z) = \exp \delta_t(Z),\ Z \in \mathcal{G}.$$

For a function f on G,

$$\delta_t(f) = f \circ \delta_t.$$

The generator A of the one parameter group $(\delta_{e^s})_{s\in\mathbb{R}}$ of dilations on G satisfies: for $f \in \mathcal{S}(G)$ and $s > 0$

$$\frac{d}{dt} |_{t=1} f \circ \delta_t = A(f) = -tt^A \frac{d}{dt} t^{-A}(f) = -t\delta_t \frac{d}{dt}(f \circ \delta_{\frac{1}{t}}). \tag{4}$$

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2 The semi-group $e^{-t\nabla^*\nabla}$ on $L^2(pdg)$

This semi-group has already been introduced in [BHT], under a probabilistic point of view, in connection with some Markov processes on Lie groups. We use instead an analytic point of view as in [O]. We consider this semi-group firstly because it is a natural generalization of the classical O-U semi-group, secondly because its generator $\nabla^*\nabla$ is the real part of the generator N we shall study in part 3, see theorem 3.

2.1 Definition and some properties

We consider the (closed) accretive sesquilinear form

$$a(f, h) = \int_G (\nabla f.\nabla h) pdg = \int_G \sum_{i=1}^n X_i f \overline{X_i h} pdg$$

whose (dense) domain in $L^2(pdg)$ is the Hilbert space

$$H^1(p) = \{f \in L^2(pdg) \mid X_i f \in L^2(pdg), 1 \leq i \leq n\}$$

equipped with the norm $\|f\|^2_{H^1(p)} = \|f\|^2_{L^2(p)} + \||\nabla f|\|^2_{L^2(p)}$; this form is continuous on $H^1(p) \times H^1(p)$.

Hence [O, proposition 1.51, theorem 1.53] it defines an operator, which we denote by $\nabla^*\nabla$, such that $-\nabla^*\nabla$ is the generator of a strongly continuous semi-group of contractions on $L^2(pdg)$; moreover this semi-group is holomorphic on the sector $\Sigma_{\frac{\pi}{2}} = \{|\arg z| < \frac{\pi}{2}, z \neq 0\}$, and $e^{-z\nabla^*\nabla}$ is a contraction on $L^2(pdg)$ for $z \in \Sigma_{\frac{\pi}{2}}$. Obviously, on $\mathcal{S}(G)$,

$$\nabla^*\nabla = \sum_{i=1}^{n} X_i^* X_i = L - \sum_{i=1}^{n} \frac{X_i p}{p} X_i = L - B. \tag{5}$$

Since X_i is a derivation, the chain rule holds, hence $X_i(f^+) = (X_i f)1_{\{f>0\}}$ by the same proof as for usual derivations on $\mathbb{R}^N$ [O, proposition 4.4], and $a(f^+, f^-) = 0$; since the form a also preserves real valued functions, the semi-group $e^{-t\nabla^*\nabla}$ is positivity preserving [O, theorem 2.6]. Since $e^{-t\nabla^*\nabla}(1) = 1$, the semi-group is thus contracting on $L^\infty(pdg)$. Since moreover $\nabla^*\nabla$ is self-adjoint, $e^{-t\nabla^*\nabla}$ is measure preserving, i.e.

$$\int_G e^{-t\nabla^*\nabla}(f) pdg = \int_G f pdg, t > 0,$$

so it extends as a contraction semi-group on $L^1(pdg)$ hence on $L^q(pdg), 1 < q < \infty$ by interpolation.

2.2 Poincaré inequality in $L^2(pdg)$

Poincaré inequality [DM, theorem 4.2] means that the spectrum of $\nabla^*\nabla$ on $L^2(pdg)$ lies in $\{0\} \cup [C^{-1}, \infty[$: there exists $C > 0$ such that, for $f \in \mathcal{S}(G)$,

$$\left\| f - \int_G f pdg \right\|_{L^2(pdg)}^2 \leq C \int_G |\nabla f|^2 pdg = C \int_G f(\nabla^*\nabla f) pdg. \tag{6}$$

(6) follows from the inequality (used for $q = 2$) [DM, theorem 4.1]

$$\left|\nabla(e^{-tL} f)\right|^q \leq C_q e^{-tL}(|\nabla f|^q),\ 1 < q < \infty, \tag{7}$$

which B. Driver and T. Melcher proved, first for $\mathbb{H}_1$, then for nilpotent groups G (see T. Melcher's thesis), using Malliavin calculus. See also [BHT] for some extensions.

We shall show in proposition 1 that (7) also follows easily from gaussian estimates of p and ∇p.

Using the explicit formula for the Carnot-Caratheodory distance, H.Q. Li [Li, corollary 1.2] obtained (7) for $q = 1$, on the 3-dimensional Heisenberg group $G = \mathbb{H}_1$. As well known [A, théorème 5.4.7], this implies Log-Sobolev inequality for the measure pdg on $\mathbb{H}_1$ and (6). Another proof of this Log-Sobolev inequality for $\mathbb{H}_1$, hence for $\mathbb{H}_k$, is given in [HZ, theorem 7.3].

Proposition 1 *[DM] Let G be a stratified group. Then (7) and Poincaré inequality (6) hold true.*

Proof: By [DM, theorem 4.2, proposition 2.6, lemma 2.3] it is enough to prove (7) for $t = \frac{1}{2}$, at $\gamma = 0$. Hence, it is enough to prove, for an element X of the basis of V_1, and $f \in \mathcal{S}(G)$,

$$\left|X(e^{-\frac{1}{2}L}f)(0)\right| = |X(f*p)(0)| = \left|\int_G (\widehat{X}f)(g)p(g)dg\right| \le C_{q,X}\,\|\nabla f\|_{L^q(pdg)}\,;$$

here [FS, p. 22 and proposition 1.29]

$$(\widehat{X}f)(g) = \frac{d}{dt}\mid_{t=0} f((\exp tX)g), \quad \widehat{X} = X + \sum_{j>n} Q_{X,j}Z_j$$

where $(Z_j)_{j=1}^N$ is a basis of $\mathcal{G}$ respecting the layers and $Q_{X,j}$ is a polynomial (with homogeneous degree $h-1$ if $Z_j \in V_h, 2 \le h \le k$).
Since $[V_1, V_{h-1}] = V_h, 2 \le h \le k$, we may choose $Z_j \in V_h$ such that $Z_j = [Y, A]$, where Y is an element of the basis of V_1 and $A \in V_{h-1}$.Then

$$\left|\int_G Z_jf(g)Q_{X,j}(g)p(g)dg\right| \le \left|\int_G Yf\ A(Q_{X,j}p)dg\right| + \left|\int_G Af\ Y(Q_{X,j}p)dg\right|.$$

Iterating for $A \in V_1 + .. + V_{k-1}$ and so on, $\left|\int_G (\widehat{X}f)(g)p(g)dg\right|$ is finally less than a finite number (which does not depend on f) of terms $\left|\int_G Yf\ Z(Qp)dg\right|$ where Y is an element of the basis of V_1, $Z \in \mathcal{G}$, and Q is a polynomial. Each of these terms can be estimated by

$$\left|\int_G Yf\ Z(Qp)dg\right| \le \|\,|\nabla f|\,\|_{L^q(pdg)}\,(\|ZQ\|_{L^{q'}(pdg)} + \left\|Q\frac{Zp}{p}\right\|_{L^{q'}(pdg)})$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Then $\|ZQ\|_{L^{q'}(pdg)}$ is finite since ZQ is a polynomial and $p \in \mathcal{S}(G)$. The main point is that $\left\|Q\frac{Zp}{p}\right\|_{L^{q'}(pdg)}$ is finite. Indeed, denoting $d(g) = d(0, g)$ where d is the Carnot-Caratheodory distance on G, one uses [CSV, theorem IV.4.2 and Comments on chapter IV]: for $0 < \varepsilon < 1$,

$$C_\varepsilon e^{-\frac{1}{2-2\varepsilon}d^2(g)} \le p(g) \le K_\varepsilon e^{-\frac{1}{2+2\varepsilon}d^2(g)}. \tag{8}$$

and, for $Z \in \mathcal{G}$,

$$(Zp)(g) \le K_{\varepsilon,Z}\ e^{-\frac{1}{2+2\varepsilon}d^2(g)}. \tag{9}$$

Hence $Q\frac{Zp}{p}$ lies in $L^r(pdg), 1 \le r < \infty$, which ends the proof.■

3 Definition and properties of the Mehler semi-group

3.1 Preliminaries

The next proposition extends a classical property of independant gaussian variables and will imply the semi-group property of our family of operators.

Proposition 2 *Let γ, g be independant $G-$valued random variables with law pdg. Then the r.v.*

$$\delta_{\cos\theta}\gamma\delta_{\sin\theta}g,\ 0 \leq \theta \leq \frac{\pi}{2}$$

has the same law, i.e. for any bounded borelian function f on G,

$$\int_{G^2} f(\delta_{\cos\theta}\gamma\delta_{\sin\theta}g)p(\gamma)p(g)d\gamma dg = \int_G f(g)p(g)dg.$$

More generally, if $g_1, ..., g_n$ are G-valued i.i.d r.v. with law pdg and $\sum\limits_{1\leq j\leq n} a_j^2 = 1$, $(a_j \geq 0)$, the law of $\prod\limits_{j=1}^{j=n} \delta_{a_j} g_j$ is pdg.

Proof: By two changes of variables, denoting $C = \sin\theta\cos\theta$,

$$\begin{aligned}
\int_{G^2} f(\delta_{\cos\theta}\gamma\delta_{\sin\theta}g)p(g)p(\gamma)d\gamma dg &= \frac{1}{C^Q}\int_{G^2} f(\gamma' g')p(\delta_{\frac{1}{\cos\theta}}\gamma')p(\delta_{\frac{1}{\sin\theta}}g')d\gamma' dg' \\
&= \frac{1}{C^Q}\int_{G^2} f(g)p(\delta_{\frac{1}{\cos\theta}}\gamma')p(\delta_{\frac{1}{\sin\theta}}(\gamma'^{-1}g))d\gamma' dg \\
&= \int_G f(g)(p_{\cos^2\theta} * p_{\sin^2\theta})(g)dg \\
&= \int_G f(g)p(g)dg.
\end{aligned}$$

The second assertion follows by iteration.

Remark 1: A central limit theorem for i.i.d centered random variables with values in a stratified group G and law μ with order 2 moments is proved in [CR, theorem 3.1]. The density p of the limit law is the kernel at time 1 of a diffusion semi-group whose generator satisfies (2).

Remark 2: If X, Y are i.i.d standard gaussian vectors with values in $\mathbb{R}^n$, the couple $(X\cos\theta + Y\sin\theta,\ \frac{d}{d\theta}(X\cos\theta + Y\sin\theta))$ has the same joint law as

(X, Y). This fact implies, in the O-U case, that $\cos^{N_0}\theta$ is the compression of the isometry R_θ of $L^2(\mathbb{R}^n \times \mathbb{R}^n, p(x)p(y)dxdy)$ defined by

$$R_\theta(F)(x, y) = F(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$

and $(R_\theta)_{\theta\in\mathbb{R}}$ is a one parameter group preserving the measure $p(x)p(y)dxdy$. This point of view was exploited e.g. in [P, theorem 2.2] in order to get a concentration inequality for the gaussian measure .

In the stratified setting we were not able to exhibit explicit unitary dilations for the Mehler operators T_t defined below.

3.2 The Mehler semi-group

We now define the Mehler semi-group on $L^q(G, pdg)$.

Theorem 3 *Let L, defined by (1), be a subLaplacian on a stratified group G, and let p be the kernel of $e^{-\frac{L}{2}}$.*

a) The family of operators $(T_t)_{t\geq 0}$ defined on $\mathcal{S}(G)$ by

$$T_t(f)(\gamma) = \int_G f(\delta_{e^{-t}}\gamma\delta_{\sqrt{1-e^{-2t}}}g)p(g)dg = e^{-\frac{L}{2}(1-e^{-2t})}(f)(\delta_{e^{-t}}\gamma) \tag{10}$$

is a semi-group whose generator $-N$ is defined on $\mathcal{S}(G)$ by

$$N = L + A. \tag{11}$$

b)The probability measure $pd\gamma$ is invariant by $(T_t)_{t\geq 0}$ i.e.

$$\int_G T_t(f)(\gamma)p(\gamma)d\gamma = \int_G f(\gamma)p(\gamma)d\gamma \tag{12}$$

and, for $f \in \mathcal{S}(G)$, $\int_G (Nf)pdg = 0$.

c) $(T_t)_{t\geq 0}$ extends as a Markovian semi-group of contractions on $L^q(G, pd\gamma), 1 \leq q \leq \infty$, strongly continuous if $q \neq \infty$.

d) If $f \in L^q(pd\gamma), 1 \leq q < \infty$,

$$\left\| T_t(f) - \int_G fpdg \right\|_{L^q(pd\gamma)} \to_{t\to\infty} 0.$$

e) $(T_t)_{t>0}$ is not self-adjoint on $L^2(G, pd\gamma)$ as soon as G is not abelian. Formally $\nabla^*\nabla$ is the real part of N, i.e., for $f, h \in \mathcal{S}(G)$,

$$\langle Nf, h\rangle_{L^2(p)} = \langle(\nabla^*\nabla + iC)f, h\rangle_{L^2(p)}$$

where C is a non zero first order differential operator satisfying $\langle Cf, h\rangle = \langle f, Ch\rangle$. In particular, for $f \in \mathcal{S}(G)$,

$$\Re\int_G (Nf)fpd\gamma = \int_G |\nabla f|^2\, pd\gamma = \int_G (\nabla^*\nabla f)fpd\gamma.$$

If moreover f is real valued, the left integral is real.

By the change of notation $e^{-t} = \cos\theta$, $< \theta < \frac{\pi}{2}$, (10) can be rewritten as

$$\cos^N\theta(f)(\gamma) = \int_G f(\delta_{\cos\theta}\gamma\delta_{\sin\theta}g)p(g)dg = \delta_{\cos\theta}\circ e^{-\frac{1}{2}\sin^2\theta L}(f)(\gamma). \qquad (13)$$

Proof: a) Let $\varphi(g') = T_t(f)(g')$; we compute

$$\begin{aligned} T_s(\varphi)(\gamma) &= \int_G \varphi(\delta_{e^{-s}}\gamma\ \delta_{\sqrt{1-e^{-2s}}}h)p(h)dh \\ &= \int_{G^2} f(\delta_{e^{-t}}[\delta_{e^{-s}}\gamma\delta_{\sqrt{1-e^{-2s}}}h]\ \delta_{\sqrt{1-e^{-2t}}}g)p(g)p(h)dgdh \\ &= \int_G f(\delta_{e^{-(t+s)}}\gamma\delta_{\sqrt{1-e^{-2(s+t)}}}k)p(k)dk = T_{s+t}(f)(\gamma) \end{aligned}$$

where the third equality comes from proposition 2 applied to (h, g).
By the chain rule applied to (10),

$$Nf = -\frac{d}{dt}\mid_{t=0} T_t(f) = Lf + A(f).$$

b) Proposition 2 gives (12). Differentiating (12) at $t = 0$ for $f \in \mathcal{S}(G)$ implies

$$\int_G (Nf)pdg = 0.$$

Another proof will be given in Remark 3.

c) T_t is contracting both on $L^1(G, pd\gamma)$, since it is positivity and measure preserving, and on $L^\infty(G, pd\gamma)$, since it is positivity preserving and $T_t(1) = 1$. Hence T_t is contracting on $L^q(G, pd\gamma), 1 \leq q \leq \infty$ by interpolation.

Since $\mathcal{D}(G)$ is norm dense in $L^q(G)$, it is norm dense in $L^q(pd\gamma), 1 \leq q < \infty$: indeed, if $F \in L^{q'}(pd\gamma)$ $(\frac{1}{q} + \frac{1}{q'} = 1)$ and $\int_G fFpd\gamma = 0$ for every $f \in \mathcal{D}(G)$, then $Fp \in L^{q'}(G)$ hence $Fp = 0$ $d\gamma$ a.s.. Writing $e^{-t} = \cos\theta$, one has, for $f \in \mathcal{D}(G)$,

$$\begin{aligned}\|T_t(f) - f\|^q_{L^q(pd\gamma)} &= \left\| \int_G [f(\delta_{\cos\theta}\gamma\delta_{\sin\theta}g) - f(\gamma)]p(g)dg \right\|^q_{L^q(pd\gamma)} \\ &\leq \int_{G^2} |f(\delta_{\cos\theta}\gamma\delta_{\sin\theta}g) - f(\gamma)|^q\, p(\gamma)p(g)d\gamma dg,\end{aligned}$$

which converges to 0 as $\theta \to 0$ by the dominated convergence theorem. Since T_t is contracting, the strong continuity on $L^q(pd\gamma)$ follows by density.

d) Similarly, if f is bounded and continuous on G,

$$f(\delta_{e^{-t}}\gamma\ \delta_{\sqrt{1-e^{-2t}}}g) \rightarrow_{t\rightarrow\infty} f(g);$$

by dominated convergence theorem $T_t(f) \rightarrow_{t\rightarrow\infty} \int_G f(g)p(g)dg$ pointwise and in the norm of $L^q(pd\gamma)$. The claim follows by density.

e) By (11), (5) and lemma 4 below, for $f \in \mathcal{S}(G)$,

$$(N - \nabla^*\nabla)f = A(f) + \sum_{1\leq j\leq n} \frac{X_j p}{p} X_j f = \sum_{1\leq j\leq N} b_j Z_j f$$

where the functions b_j are not all zero if $j > n = \dim V_1$. Hence for $h \in \mathcal{S}(G)$,

$$\int_G (N - \nabla^*\nabla)(f)\overline{h}pdg = -\int_G f[\sum_{1\leq j\leq N} b_j(g)(Z_j\overline{h})p + \overline{h}Z_j(b_jp)]dg.$$

By b), the left hand side is zero for $h = 1$, hence $\sum\limits_{1\leq j\leq N} Z_j(b_jp) = 0$. Since T_t preserves real valued functions, so does N, hence

$$\int_G (N - \nabla^*\nabla)(f)\overline{h}pdg = -\int_G f(N - \nabla^*\nabla)(\overline{h})pdg = -\int_G f\overline{(N - \nabla^*\nabla)(h)}pdg,$$

which proves $(iC)^* = -iC$, where $iC = N - \nabla^*\nabla = A + B$. The remaining assertions are obvious.

Remark 3: We now give another instructive proof of $\int_G (Nf)pdg = 0, f \in \mathcal{S}(G)$, hence of (12). We claim that, for $f, h \in \mathcal{S}(G)$,

$$\int_G (Nf)hdg = \int_G f[L(h) - Qh + \frac{d}{ds} \mid_{s=1} h \circ \delta_{\frac{1}{s}}]dg = \int_G f(L - QId - A)(h)dg.$$

Indeed, $N = L + A$, L is formally selfadjoint on $L^2(dg)$ and the claim follows by differentiating at $s = 1$ the right hand side of

$$\int_G f(\delta_s\gamma)h(\gamma)d\gamma = s^{-Q} \int_G f(\gamma')h(\delta_{\frac{1}{s}}\gamma)d\gamma'.$$

By (4) and [LP, lemma 2], p may be precisely defined as the unique solution in $L^1(G)$, satisfying $\int_G p(g)dg = 1$, of

$$(L - QId - A)(p) = Lp - Qp + s\delta_s \frac{d}{ds}(p \circ \delta_{\frac{1}{s}}) = 0.\blacksquare$$

Remark 4: As already mentioned in section 2.2, Log-Sobolev inequality for $pd\gamma$ is known for $\mathbb{H}_k$. It is equivalent both to hypercontractivity of e^{-tN} and to hypercontractivity of $e^{-t\nabla^*\nabla}$ on $\mathbb{H}_k$, since p is an invariant measure for these markovian semigroups and $N, \nabla^*\nabla$ are diffusion operators [A, theorem 2.8.2].

3.3 The generator of dilations

We may identify G with a group of finite matrices [V, theorem 3.6.6]. The derivation formula for an exponential of a matrix valued function, see e.g. [H, theorem 69], applied to a smooth function $Z(s)$: $\mathbb{R} \to \mathcal{G}$, where $\mathcal{G}$ has k layers, gives

$$\begin{aligned} \frac{d}{ds} \exp Z(s) &= \lim_{h \to 0} \frac{\exp Z(s+h) - \exp Z(s)}{h} \\ &= \lim_{h \to 0} \frac{\exp(Z(s) + hZ'(s)) - \exp Z(s)}{h} \\ &= [\exp Z(s)]V(Z(s)), \qquad (14) \end{aligned}$$

where

$$V(Z(s)) = (d\exp)_{Z(s)}(Z'(s)) = Z'(s) + \sum_{l=1}^{k-1} \frac{(-1)^l}{(l+1)!}(AdZ(s))^l(Z'(s)). \quad (15)$$

Hence

$$\exp Z(s+h) = \exp Z(s) \exp h[V(Z(s)) + o(1)],$$

which entails for $f \in \mathcal{C}^\infty(G)$

$$\frac{d}{ds} f(\exp Z(s)) = V(Z(s))(f)(\exp Z(s)). \quad (16)$$

Lemma 4 *Let A be the generator of the group of dilations $(\delta_{e^t})_{t\in\mathbb{R}}$. Then*

$$A(f)(g) = \sum_{1\le j\le N} a_j(g) Z_j f(g)$$

where the functions a_j are polynomials w.r. to the coordinates of g, and are not all zero for $j > n = \dim V_1$.

Proof: Assume that $\mathcal{G}$ has k layers, $k \ge 2$. Let

$$\delta_s g = \exp(sX + s^2 Y + .. + s^k U) = \exp Z(s).$$

By (16) $A = V(Z(1))$. Noting that $Z' - Z \in V_2 + .. + V_k$, we get $(AdZ(1))^l(Z'(1)) \in V_3 + .. + V_k, l \ge 1$. So $V(Z(1)) - (X + 2Y)$ lies in $V_3 + .. + V_k$.■

Notation: We denote by $\mathcal{P}_n$ the (finite dimensional) space of homogeneous polynomials on G with homogeneous degree $n, n \in \mathbb{N}$, i.e. satisfying

$$\delta_s(P) = s^n P, \ P \in \mathcal{P}_n; \quad (17)$$

equivalently, $\mathcal{P}_n$ is the eigensubspace of A on $\mathcal{P}$ associated to n. The finite dimensional subspaces $B_n = \mathcal{P}_0 + .. + \mathcal{P}_n$ are stable under L and dilations, hence under $e^{-\frac{tL}{2}}$ and $\cos^N \theta$ by (10), these operators being naturally extended on $\mathcal{S}'(G)$. In particular $e^{\frac{L}{2}}$ is well defined on B_n and is the inverse of $e^{-\frac{L}{2}}$, which is thus one to one on every B_n hence on $\mathcal{P} = \cup_{n\ge 0} B_n$.

The next lemma is the key for the computation of the spectrum of $\cos^N \theta$. It will be exploited again in section 3.5.

Lemma 5 *a) The generator A of dilations on G satisfies $[L, A] = 2L$ on $\mathcal{C}^\infty(G)$.*

b) $e^{-\frac{L}{2}} \circ \cos^N \theta = \delta_{\cos\theta} e^{-\frac{L}{2}}$ on $\mathcal{S}'(G)$.

c) The set of polynomials $e^{\frac{L}{2}}(\mathcal{P}_n)$ is a space of eigenvectors of $\cos^N \theta$ associated to the eigenvalue $\cos^n \theta, n \geq 0$.

Proof: a) We rewrite (2) as

$$Le^{tA} = e^{2t} e^{tA} L,\ t \in \mathbb{R},$$

and a) follows by differentiating at $t = 0$.
b) By (3), on $\mathcal{S}(G)$, hence on $\mathcal{S}'(G)$, for $t > 0$,

$$e^{-\frac{t^2}{2}L} = \delta_{\frac{1}{t}} \circ e^{-\frac{L}{2}} \circ \delta_t. \tag{18}$$

Hence, on $\mathcal{S}'(G))$, by (10) and (18) applied to $t = \cos\theta$,

$$e^{-\frac{L}{2}} \circ \cos^N \theta = e^{-\frac{L}{2}} \circ \delta_{\cos\theta} \circ e^{-\frac{\sin^2\theta}{2}L} = \delta_{\cos\theta} \circ e^{-\frac{L}{2}}.$$

c) Since $e^{-\frac{L}{2}}$ is invertible on $\mathcal{P}$, and $\mathcal{P}$ is stable under $cos^N\theta$, b) implies on $\mathcal{P}$

$$\cos^N \theta \circ e^{\frac{L}{2}} = e^{\frac{L}{2}} \circ \delta_{\cos\theta}.$$

Applying this to $\mathcal{P}_n$ proves the result.

3.4 Compacity and spectrum of $\cos^N \theta$ on $L^q(pd\gamma)$

Proposition 6 *Let $\cos^N \theta$ be defined by (13). Then*

a) $\cos^N \theta$ is a Hilbert-Schmidt operator on $L^2(pd\gamma)$.

b) $\cos^N \theta$ is compact on $L^q(pd\gamma), 1 < q < \infty$; its non zero eigenvalues and corresponding eigenspaces are the same on $L^2(pd\gamma)$ and $L^q(pd\gamma)$. In particular its spectrum $\sigma(\cos^N \theta)$ does not depend on q and

$$\sigma(\cos^N \theta) = (\cos\theta)^{\sigma(N)} \cup \{0\}.$$

Actually, $\cos^N \theta$ is a trace class operator on $L^2(pd\gamma)$ by a) and the semi-group property of $(e^{-tN})_{t>0}$.
Proof: a) We must show that the kernel of $\cos^N \theta$ lies in $L^2(G \times G, pd\gamma \otimes pdg)$. For fixed γ and $\theta, 0 < \theta < \frac{\pi}{2}$,

$$\int_G f(\delta_{\cos\theta}\gamma\delta_{\sin\theta}g)p(g)dg = \frac{1}{\sin^Q\theta}\int_G f(z)p(\delta_{\frac{\cos\theta}{\sin\theta}}\gamma^{-1}\delta_{\frac{1}{\sin\theta}}z)dz,$$

so we must prove the convergence of the integral

$$I(\theta) = \int_{G^2} p^2(\delta_{\frac{\cos\theta}{\sin\theta}}\gamma^{-1}\delta_{\frac{1}{\sin\theta}}z)\frac{p(\gamma)}{p(z)}dzd\gamma.$$

By the gaussian estimates (8)

$$\frac{C_\varepsilon}{K_\varepsilon^3}p^2(\delta_{\frac{\cos\theta}{\sin\theta}}\gamma^{-1}\delta_{\frac{1}{\sin\theta}}z)\frac{p(\gamma)}{p(z)} \le \exp(\frac{d^2(z)}{2-2\varepsilon} - \frac{d^2(\gamma)}{2+2\varepsilon} - \frac{d^2(\delta_{\frac{\cos\theta}{\sin\theta}}\gamma^{-1}\delta_{\frac{1}{\sin\theta}}z)}{1+\varepsilon}) = \exp\beta.$$

The Carnot distance d satisfies

$$d(g) \le d(\gamma^{-1}g) + d(\gamma) \; and \; d(\delta_t g) = td(g).$$

Hence

$$\begin{aligned}(1+\varepsilon)\beta &\le \frac{d^2(z)}{2(1-\varepsilon)^2} - \frac{d^2(\gamma)}{2} - (\frac{1}{\sin\theta}d(z) - \frac{\cos\theta}{\sin\theta}d(\gamma))^2 \\ &\le d^2(z)(\frac{1}{2-4\varepsilon} - \frac{1-\cos\theta}{\sin^2\theta}) + d^2(\gamma)(\frac{\cos\theta-\cos^2\theta}{\sin^2\theta} - \frac{1}{2}).\end{aligned}$$

Since $\frac{1-\cos\theta}{\sin^2\theta} > \frac{1}{2}$ on $]0, \frac{\pi}{2}]$, the coefficient of $d^2(\gamma)$ is strictly negative, and so is the coefficient of $d^2(z)$ for small enough $\varepsilon > 0$. Hence, for some $c, C > 0$,

$$I(\theta) \le C\int\int_{G^2} e^{-c(d^2(z)+d^2(\gamma))}dzd\gamma = C(\int_G e^{-cd^2(z)}dz)^2.$$

By the left hand side of (8), for small ε,

$$C_\varepsilon\int_G e^{-cd^2(z)}dz \le \int_G p^{2c(1-\varepsilon)}(z)dz,$$

and the last integral is finite since $p \in \mathcal{S}(G)$. This proves a).

b) By interpolation, since $cos^N\theta$ is compact on $L^2(p(g)dg)$ and bounded on $L^\infty(pdg)$ and $L^1(pdg)$, it is compact on $L^q(pdg), 1 < q < \infty$, with the same spectrum and the same eigenspaces associated to non zero eigenvalues [D, theorems 1.6.1 and 1.6.2].

By the compacity on $L^q(pdg)$, the set of these eigenvalues is $\{\cos^\lambda\theta \mid \lambda \in \sigma_q(N)\}$ where $\sigma_q(N)$ denotes the spectrum of N on $L^q(pdg)$ [L, chap. 34.5, theorem 13]. Hence $\sigma_q(N) = \sigma_2(N)$ is discrete and lies in $\{\lambda \in \mathbb{C} \mid \Re\lambda \ge 0\}$ since $cos^N\theta$ is contracting on $L^2(pdg)$ (or since $\Re\langle Nf, f\rangle \ge 0$).■

Theorem 7 *Let G be a step k stratified group.*

1) If $k \leq 4$

a) the spectrum of $\cos^N \theta$ on $L^2(pdg)$ is $\sigma(\cos^N \theta) = (\cos\theta)^{\mathbb{N}} \cup \{0\}$ and $\sigma(N) = \mathbb{N}$.

b) the corresponding eigenspaces $E_n, n \geq 0$, (which are not pairwise orthogonal in $L^2(pdg)$) are

$$E_n = e^{\frac{1}{2}L}(\mathcal{P}_n).$$

2) If $k > 4$, assertions a) b) remain true for the restriction of $\cos^{\mathbb{N}} \theta$ to the closed subspace $L^2_{\mathcal{P}}(pdg)$ spanned by polynomials.

If $k = 1$ polynomials in E_n are the Hermite polynomials with degree n.
Proof: 1) follows from 2) and proposition 8 below.
2) We first define E_n by $E_n = e^{\frac{L}{2}}(\mathcal{P}_n)$. By lemma 5, E_n lies in the eigenspace of $\cos^N \theta$ associated to the eigenvalue $\cos^n \theta$. By proposition 6, $\cos^N \theta$ is compact on $L^2_{\mathcal{P}}(pdg)$. The claim then follows from the following facts:

Let $T : E \rightarrow E$ be a compact operator on an infinite dimensional Banach space E; let Λ be a set of eigenvalues of T and let E_λ, $\lambda \in \Lambda$, be eigensubspaces whose union is total in E. Then

a) the spectrum of T is $\Lambda \cup \{0\}$

b) for $\lambda \in \Lambda, E_\lambda$ is the whole eigenspace associated to λ.

Indeed, assume that T has an eigenvalue $\lambda_0 \notin \Lambda$. Then $T - \lambda_0 I$ has a closed range with non zero finite codimension (see e.g. [L, chap. 21.1, theorems 3, 4]). But this range contains the linear span of the E_λ's, $\lambda \in \Lambda$, hence is the whole of E. This is a contradiction, which proves a).

Let $\lambda_0 \in \Lambda$; since E_{λ_0} is stable under T, T acts on the quotient space E/E_{λ_0} and is still compact. The E_λ's, $\lambda \in \Lambda\backslash\{\lambda_0\}$ span a dense subspace of E/E_{λ_0}. Applying a) to E/E_{λ_0}, λ_0 cannot belong to the spectrum of T on the quotient space, which proves b).■

The proof of the next proposition is essentially due to W. Hebisch (private communication).

Proposition 8 *Let G be a stratified group. Then the polynomials are dense in $L^2(pdg)$ if and only if G is step k with $k \leq 4$.*

Proof: 1) We recall that polynomials are dense in $L^2(\mathbb{R}, e^{-c|x|^\alpha} dx)$ if and only if $\alpha \geq \frac{1}{2}$: obviously, this does not depend on c and is equivalent to the density of polynomials in $L^2(\mathbb{R}^+, e^{-x^\alpha} dx)$. If $0 < \alpha < \frac{1}{2}$, [PS, Part III, problem 153] produces a non zero bounded function g_α which is orthogonal to polynomials in $L^2(\mathbb{R}^+, e^{-\cos(\alpha\pi)x^\alpha}dx)$. If $\alpha \geq \frac{1}{2}$, the result follows from the trick of [Ham, p 197-198]. Indeed, if $\psi \in L^2(\mathbb{R}^+, e^{-x^\alpha}dx)$ and $\alpha \geq \frac{1}{2}$, the function

$$F(z) = \int_{\mathbb{R}^+} \psi(x) e^{\sqrt{x}z} e^{-x^\alpha} dx = \int_{\mathbb{R}^+} \psi(y^2) e^{yz} e^{-y^{2\alpha}} y dy$$

is bounded and holomorphic on $\{\Re z < \beta\}$ for some $\beta > 0$, by Cauchy-Schwarz inequality.
Expanding $z \to e^{\sqrt{x}z}$ in power series, one gets $F(-z) = -F(z)$ if ψ is orthogonal to polynomials in $L^2(\mathbb{R}^+, e^{-x^\alpha}dx)$. Thus F extends as a bounded entire function, which must be zero by Liouville theorem since $F(0) = 0$. Hence the Fourier transform of $y \to \psi(y^2)e^{-y^{2\alpha}}y$ is zero, i.e. $\psi = 0$ a.s..

2) We identify $g = \exp Z \in G$ with the coordinates $(x, y, .., w)$ of Z w.r. to a basis respecting the layers and denote

$$\eta(g) = \sum_{i \leq l} |x_i|^2 + \sum_{i \leq m} |y_i|^{\frac{2}{2}} + ... + \sum_{i \leq r} |w_i|^{\frac{2}{k}}.$$

Obviously $\eta(\delta_s g) = s^2 \eta(g)$, in particular $\eta(g) = d^2(g)\eta(\delta_{\frac{1}{d(g)}} g)$, d denoting the Carnot distance. Since η is strictly positive and bounded on the d-unit sphere of G, there exist constants $c', C' > 0$ such that

$$c'\eta(g) \leq d^2(g) \leq C'\eta(g).$$

By (8) there exist constants $c, C > 0$ such that the following embeddings

$$L^2(e^{-C\eta(g)}dg) \to L^2(pdg) \to L^2(e^{-c\eta(g)}dg)$$

are continuous, with dense ranges since $\mathcal{D}(G)$ is dense in the three spaces.

3) The algebraic tensor product

$$\mathcal{E} = \otimes_{i \leq l} L^2(e^{-Cx_i^2}dx_i) \otimes ... \otimes_{i \leq p} L^2(e^{-C|w_i|^{\frac{2}{k}}}dw_i),$$

is dense in $L^2(e^{-C\eta(g)}dg)$. For $k \leq 4$, one variable polynomials are dense in every factor of $\mathcal{E}$ by step 1), hence polynomials are dense in $L^2(e^{-C\eta(g)}dg)$ and in $L^2(pdg)$.

Let $k \geq 5$. By 1) there exists a non zero function $g \in L^2(e^{-c|w_r|^{\frac{2}{k}}}dw_r)$ which is orthogonal to polynomials w.r. to w_r. Then $1 \otimes .... \otimes 1 \otimes g \in L^2(e^{-c\eta(g)}dg)$ is orthogonal to all polynomials, so polynomials are neither dense in $L^2(e^{-c\eta(g)}dg)$, nor in $L^2(pdg)$.$\blacksquare$

3.5 Generating functions of polynomial eigenvectors of N

The usual Hermite polynomials on $\mathbb{R}$, denoted by $H_n, n \in \mathbb{N}$, are the eigenvectors of the Ornstein-Uhlenbeck operator N_0, and have the generating function

$$e^{ixt+\frac{1}{2}t^2} = \sum_{n\geq 0} \frac{(it)^n}{n!} H_n(x) = e^{\frac{1}{2}\Delta}(e^{ixt}) = e^{\frac{1}{2}\Delta} \circ \delta_t(e^{ix}),$$

noting that $x \to e^{ix}$ is a bounded eigenvector of Δ. In particular

$$i^n H_n(x) = \frac{d^n}{dt^n} |_{t=0}\, e^{\frac{1}{2}\Delta} \circ \delta_t(e^{ix}).$$

We shall verify (proposition 11) that a similar formula gives polynomial eigenvectors of N. When G is step two, these vectors are total in $L^q(pdg), 1 \leq q < \infty$, see theorem 12 below. More precisely we give in 3.5.1 a technical lemma producing eigenvectors of N out of eigenvectors of L. In 3.5.3 we use this lemma when φ is both an eigenvector of L and a coefficient function of a representation of G (proposition 11). We shall first gather in 3.5.2 well known facts about these functions.

3.5.1 Candidates for generating functions of eigenvectors of N

In the next lemma 9 we state technical assumptions ensuring the validity of the computation of some eigenvectors of N. Using lemma 5 b), the point is to define "$e^{\frac{L}{2}}\varphi$" for suitable functions φ : in lemma 5 c), we choose $\varphi \in \mathcal{P}$, here we choose eigenvectors of L.

Lemma 9 *Let G be a stratified group and let $\varphi \in \mathcal{S}'(G) \cap \mathcal{C}^\infty(G)$ be an eigenvector of L such that $L\varphi = \lambda\varphi$. We assume that, for $n \geq 1$,*

(i) $\frac{d^n}{dt^n} |_{t=0} \int_G \delta_t(\varphi)(\gamma g^{-1}) p(g) dg = \int_G \frac{d^n}{dt^n} |_{t=0}\, \delta_t(\varphi)(\gamma g^{-1}) p(g) dg$

(ii) $\frac{d^n}{dt^n} |_{t=0}\, \delta_t(\varphi)$ is a polynomial on G.

Let

$$f_t = e^{\frac{t^2\lambda}{2}} \delta_t(\varphi),\ t > 0;\ h_n = \frac{d^n}{dt^n} |_{t=0}\, f_t.$$

Then h_n is a polynomial on G and

$$\cos^N \theta(h_n) = \cos^n \theta\; h_n.$$

Proof: Since $\varphi \in \mathcal{C}^\infty(G)$, $t \rightarrow f_t$ is $\mathcal{C}^\infty$ on $\mathbb{R}^+$. By (2) $L \circ \delta_t(\varphi) = t^2 \lambda \delta_t(\varphi)$, so that $\delta_t(\varphi) = e^{-\frac{L}{2}} f_t$. By lemma 5 b)

$$e^{-\frac{L}{2}} \cos^N \theta(f_t) = \delta_{\cos\theta} e^{-\frac{L}{2}} f_t = \delta_{\cos\theta}\delta_t(\varphi) = \delta_{t\cos\theta}(\varphi) = e^{-\frac{L}{2}} f_{t\cos\theta}. \quad (19)$$

We claim that

$$\frac{d^n}{dt^n} \mid_{t=0} e^{-\frac{L}{2}} \cos^N \theta(f_t) = e^{-\frac{L}{2}} \cos^N \theta(\frac{d^n}{dt^n} \mid_{t=0} f_t) = e^{-\frac{L}{2}} \cos^N \theta(h_n). \quad (20)$$

In particular, applying (20) with $\theta = 0$, $\frac{d^n}{dt^n} \mid_{t=0} e^{-\frac{L}{2}}(f_t) = e^{-\frac{L}{2}}(h_n)$.
Hence, by (20) and (19),

$$e^{-\frac{L}{2}} \cos^N \theta \ (h_n) = \frac{d^n}{dt^n} \mid_{t=0} e^{-\frac{L}{2}} f_{t\cos\theta} = e^{-\frac{L}{2}} \cos^n \theta \ h_n. \quad (21)$$

By Leibnitz rule, it is enough to prove the claim for $\delta_t(\varphi)$ instead of f_t. By lemma 5 b) we may replace $e^{-\frac{L}{2}} \cos^N \theta$ in the claim by $\delta_{\cos\theta} e^{-\frac{L}{2}}$. The claim now follows from assumption (i).

By Leibnitz rule and assumption (ii), h_n is a polynomial. So is $\cos^N \theta(h_n)$ and the result follows from (21) since $e^{-\frac{L}{2}}$ is one to one on $\mathcal{P}$.

Remark 5: φ and $\varphi \circ \delta_\beta, \beta > 0$, give colinear h_n's, since

$$\frac{d^n}{dt^n} \mid_{t=0} e^{\frac{1}{2}t^2\beta^2\lambda} \delta_{t\beta}(\varphi) = \beta^n \frac{d^n}{dt^n} \mid_{t=0} e^{\frac{1}{2}t^2\lambda} \delta_t(\varphi) = \beta^n h_n.$$

3.5.2 A total set of eigenvectors of L in $L^q(pdg), 1 \leq q < \infty$.

Let $\Pi : G \rightarrow B(L^2(\mathbb{R}^k, d\xi))$ be a non trivial unitary irreducible representation of G. By definition, $F \in L^2(\mathbb{R}^k)$ is a $\mathcal{C}^\infty$ vector for Π if the vector valued function: $g \rightarrow \Pi(g)(F)$ is $\mathcal{C}^\infty$ on G. We still denote by Π the associated differential representation, defined for a $\mathcal{C}^\infty$ vector F and $X \in \mathcal{G}$ by

$$X\Pi(g)(F) = \frac{d}{dt} \mid_{t=0} \Pi(g \exp tX)(F) = \Pi(g)\Pi(X)(F), \ g \in G, \quad (22)$$

and $\Pi(X^m) = \Pi(X)^m$, see e.g. [CG, p.227]; by definition, $\Pi(X^m)(F)$ still lies in $L^2(\mathbb{R}^k)$ and is still a $\mathcal{C}^\infty$ vector for Π.
Π extends as a representation of the convolution algebra $M(G)$ by

$$\Pi(\mu) = \int_G \Pi(g) d\mu(g).$$

In particular $(\Pi(p_t dg))_{t\geq 0}$ is a semigroup of operators on $L^2(\mathbb{R}^k)$, whose generator is $-\Pi(L)$. Indeed, for a $\mathcal{C}^\infty$ vector F, by (22),

$$\begin{aligned} -\frac{d}{dt}\int_G \Pi(g)(F)p_t(g)dg &= \int_G \Pi(g)(F)(Lp_t)(g)dg = \int_G L\circ\Pi(g)(F)p_t(g)dg \\ &= \int_G \Pi(g)\circ\Pi(L)(F)p_t(g)dg \to_{t\to 0^+} \Pi(L)(F). \end{aligned}$$

Since $p\in\mathcal{S}(G)$, $\Pi(pdg) = e^{-\frac{1}{2}\Pi(L)}$ is a trace class operator [CG, theorem 4.2.1]; in particular its non zero eigenvalues are $\{e^{-\frac{1}{2}\lambda}, \lambda\in\sigma_2(\Pi(L))\}$, where λ runs through the eigenvalues of $\Pi(L)$ on $L^2(\mathbb{R}^k)$. Moreover, for $F\in L^2(\mathbb{R}^k)$, the function $\Pi(pdg)(F)$ is a $\mathcal{C}^\infty$ vector for Π [CG, theorem A.2.7 p. 241].

Let $\mathcal{U}$ be a set of non trivial unitary irreducible representations of G whose equivalence classes support the Plancherel measure for G. By Kirillov theory, there exists an integer k, which does not depend on $\Pi\in\mathcal{U}$, such that $\Pi : G\to B(L^2(\mathbb{R}^k))$, see more details in 3.5.4 below.

Proposition 10 *Let G be a stratified group and let $\mathcal{F}$ be the set of coefficient functions*

$$\mathcal{F} =\{\varphi^{\Pi,\mu,\mu'} = \langle\Pi(.)(F_\mu), F_{\mu'}\rangle \mid \Pi\in\mathcal{U}, F_\mu, F_{\mu'}\in\mathcal{B}_\Pi\}\subset L^\infty(dg)$$

where $\mathcal{B}_\Pi$ is an orthogonal basis of $L^2(\mathbb{R}^k)$ chosen among eigenvectors of $e^{-\frac{1}{2}\Pi(L)}$. Then $\mathcal{F}$, which lies in $\mathcal{C}^\infty(G)$, is a set of eigenvectors of L which is total in $L^q(p(g)dg), 1\leq q<\infty$.

For fixed Π,μ the functions $\{\varphi^{\Pi,\mu,\mu'} \mid F_{\mu'}\in\mathcal{B}_\Pi\}$ are independent and belong to the same eigenspace of L.

Proof: a) For every non trivial unitary irreducible representation Π of G, since $\Pi(pdg)(F_\mu) = e^{-\frac{1}{2}\Pi(L)}(F_\mu) = e^{-\frac{1}{2}\lambda_\mu}F_\mu$, F_μ is a $\mathcal{C}^\infty$ vector for Π, hence $\varphi^{\Pi,\mu,\mu'}\in\mathcal{C}^\infty(G)$; $\varphi^{\Pi,\mu,\mu'}$ is an eigenvector of L with eigenvalue λ_μ by (22). Since Π is irreducible, the closed invariant subspace

$$\{F\in L^2(\mathbb{R}^k) \mid \forall g\in G \ \ \langle\Pi(g)(F_\mu), F\rangle = 0\}$$

is reduced to $\{0\}$, which implies the independence of the $\varphi^{\Pi,\mu,\mu'}$'s. (In the Heisenberg case, see [T, p. 19, 51]).

b) Let $\psi \in L^{q'}(pdg)$, $\frac{1}{q}+\frac{1}{q'}=1$, be orthogonal to $\mathcal{F}$, i.e. for $\Pi \in \mathcal{U}$,

$$0=\int_G \langle \Pi(g)(F_\mu), F_{\mu'}\rangle\, \psi(g)p(g)dg = \left\langle (\int_G \Pi(g)\psi(g)p(g)dg)(F_\mu), F_{\mu'}\right\rangle .$$

Equivalently $\Pi(\psi p)=\widehat{\psi p}(\Pi)=0$ for $\Pi\in\mathcal{U}$. Then Plancherel formula for G (see e.g. [CG, theorem 4.3.10]) implies that $\psi p = 0$ dg a.s.. Indeed, this is clear if $\psi p \in L^2(dg)$, in particular if $q' \geq 2$. In general, $\psi p \in L^1(dg)$, $\|(\psi p)*p_t-\psi p\|_{L^1(dg)} \to_{t\to 0} 0$ and $(\psi p)*p_t \in L^2(dg)$; moreover $(\psi p)*p_t=0$ a.s. since, for every $\Pi\in\mathcal{U}$,

$$\Pi((\psi p)*p_t)=\Pi(\psi p)\Pi(p_t)=0.\blacksquare$$

3.5.3 Polynomial eigenvectors of N built from coefficients of representations

We now consider the functions $e^{\frac{1}{2}t^2\lambda_\mu}\varphi^{\Pi,\mu,\mu'}\circ\delta_t$ as generating functions of polynomial eigenvectors of N.

Proposition 11 *Let* $\varphi^{\Pi,\mu,\mu'}=\langle\Pi(.)(F_\mu),F_{\mu'}\rangle\in\mathcal{F}$ *be as in proposition 10. For* $n\geq 1$, *let*

$$h_n^{\Pi,\mu,\mu'}=\frac{d^n}{dt^n}\,|_{t=0}\; e^{\frac{1}{2}t^2\lambda_\mu}\varphi^{\Pi,\mu,\mu'}\circ\delta_t.$$

Then $h_n^{\Pi,\mu,\mu'}$ *is a polynomial eigenvector of* $\cos^N\theta$ *with eigenvalue* $\cos^n\theta$.

Proof: By proposition 10 and lemma 9, it is enough to prove assumptions (i) and (ii) in lemma 9. We claim the existence of a polynomial ψ_n, $n\geq 1$, which does not depend on t, such that, for $0\leq t\leq 1$ and $n\geq 0$,

$$\left|\frac{d^n}{dt^n}\,\varphi^{\Pi,\mu,\mu'}\circ\delta_t\right|\leq\psi_n.$$

Since $g\to\psi_n(\gamma g^{-1})$ is still a polynomial, it lies in $L^1(pdg)$ for every $\gamma\in G$, and this will prove assumption (i). We now verify the claim.

Case 1: The computation of derivatives being easier if G is step two, we first consider this setting.

By Schur lemma, the restriction of Π to the center $\exp \mathcal{Z}$ of G is given by a character $u \to e^{i\langle l,u\rangle}$ where l is some linear form on $\mathcal{Z}$, see e.g. [CG, p. 184]. If $g = (x,u)$ and $X = \sum_{j=1}^{n} x_j X_j \in V_1$,

$$\varphi^{\Pi,\mu,\mu'}(\delta_t g) = e^{it^2\langle l,u\rangle} \left\langle \Pi(\exp tX)(F_\mu), F_{\mu'} \right\rangle = e^{it^2\langle l,u\rangle} \Phi_t^{\Pi,\mu,\mu'}(x)$$

and, by (22),

$$\frac{d^m}{dt^m} \Phi_t^{\Pi,\mu,\mu'}(x) = \left\langle \Pi(\exp tX)\Pi(X)^m(F_\mu), F_{\mu'} \right\rangle . \tag{23}$$

Since $\Pi(X)^m(F_\mu)$ lies in $L^2(\mathbb{R}^k, d\xi)$, $\langle \Pi(X)^m(F_\mu), F_{\mu'}\rangle$ and $\|\Pi(X)^m(F_\mu)\|_{L^2(d\xi)}$ are polynomials w.r. to x, $\frac{d^m}{dt^m}\ |_{\alpha=0}\ \delta_t(\varphi^{\Pi,\mu,\mu'})$ is a polynomial w.r. to x, u, and $\left|\frac{d^m}{dt^m} e^{it^2\langle l,u\rangle}\Phi_t^{\Pi,\mu,\mu'}(x)\right|$ is, for $0 \le t \le 1$, less than a polynomial ψ_n which does not depend on t. This proves (i) and (ii) in this case.

General case: As in (14) and (15), for $g = \exp Z = \exp(X + Y + .. + U)$ and $t > 0$, since $V(\Pi(\delta_t Z)) = \Pi(V(\delta_t Z))$,

$$\frac{d}{dt}\varphi^{\Pi,\mu,\mu'}(\delta_t g) = \frac{d}{dt} \left\langle \exp \Pi(\delta_t Z)(F_\mu), F_{\mu'} \right\rangle = \left\langle \Pi(V(\delta_t Z))(F_\mu), \exp -\Pi(\delta_t Z)(F_{\mu'}) \right\rangle .$$

At $t = 0$ this reduces to the polynomial $\langle \Pi(X)(F_\mu), F_{\mu'}\rangle$. Since $\Pi(V(\delta_t Z)$ has polynomial coefficients w.r. to t and the coordinates of g, so does $\|\Pi(V(\delta_t Z))(F_\mu)\|_{L^2(d\xi)}$. Hence there is a polynomial ψ_1 w.r. to the coordinates of g such that

$$sup_{0\le t\le 1} \|\Pi(V(\delta_t Z))(F_\mu)\|_{L^2(d\xi)} \le \psi_1.$$

This proves the claim for $n = 1$. Clearly this can be iterated for upper derivatives, which proves (i) and (ii).■

3.5.4 The step two setting: generalized Hermite polynomials

In this case, the key facts are the extension of the explicit functions $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$ as entire functions on the complexification of G and the explicit expression of p. Theorem 12 gives another proof of theorem 7 a) in this setting, with another description of the eigenspaces of N by generating functions.

Theorem 12 *Let G be a step two stratified group. Then* *a) every* $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$ *lies in the closed subspace of* $L^q(pdg), 1 \leq q < \infty$, *spanned by constants and the polynomials* $\{h_n^{\Pi,\mu,\mu'}, n \geq 1\}$ *defined in proposition 11.*

b) The set of generalized Hermite polynomials

$$\cup_{\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}} \{h_n^{\Pi,\mu,\mu'}, n \geq 1\}$$

together with the constants is a set of eigenvectors of N *which is total in* $L^q(pdg), 1 \leq q < \infty$.

c) For fixed $n \geq 1$, $\cup_{\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}} \{h_n^{\Pi,\mu,\mu'}\}$ *spans the eigenspace of* N *associated to* n *in* $L^q(pdg), 1 < q < \infty$.

In contrast, if G has more than 4 layers, assertion b) is false by proposition 8, hence a) is false for some $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$, by proposition 10. If G has 3 or 4 layers, we do not know if the conclusions of theorem 12 hold true.

Proof of theorem 12: a) implies b) by propositions 10 and 11.

b) implies c) as recalled in the proof of theorem 7.

a) The proof is given in three steps. In step 1 we state two standard sufficient conditions ensuring statement a); in step 2 we verify these conditions when G is a Heisenberg group; in step 3 we show how the general step 2 case mimicks the Heisenberg case.

Step 1: Let $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$ and assume that

(i) for every $g \in G$, the function $t \to \varphi^{\Pi,\mu,\mu'}(\delta_t g)$ extends as a holomorphic function $z \to \varphi_z^{\Pi,\mu,\mu'}(g)$ on $\mathbb{C}$.

(ii) for some connected neighborhood Ω of the real axis, for every compact $K \subset \Omega$, there exists $w_K \in L^q(pdg)$, $1 \leq q < \infty$, such that

$$\left|\varphi_z^{\Pi,\mu,\mu'}\right| \leq w_K,\ z \in K.$$

We claim that $\varphi^{\Pi,\mu,\mu'} = \varphi$ then lies in the closed subspace of $L^q(pdg)$ spanned by $h_n^{\Pi,\mu,\mu'}, n \geq 1$. Indeed, let $\psi \in L^{q'}(pdg), \frac{1}{q} + \frac{1}{q'} = 1$, and let

$$m(t) = \int_G \varphi(\delta_t g)\psi(g)p(g)dg.$$

By the assumptions, m extends as a holomorphic function on Ω and

$$\frac{d^n}{dz^n} m = \int_G (\frac{d^m}{dz^m}\varphi_z)\psi pdg,\ m \geq 0.$$

By proposition 10, $L(\varphi) = \lambda\varphi$ for some $\lambda = \lambda_\mu$. Hence $t \to e^{\frac{1}{2}t^2\lambda}m(t)$ also extends as a holomorphic function on Ω and

$$\frac{d^n}{dz^n}\,|_{z=0}\; e^{\frac{1}{2}z^2\lambda}m = \int_G [\frac{d^n}{dz^n}\,|_{z=0}\; e^{\frac{1}{2}z^2\lambda}\varphi_z]\psi p dg = \int_G h_n^{\Pi,\mu,\mu'}\psi p dg,\ n \geq 0.$$

If ψ is orthogonal to $\{h_n^{\Pi,\mu,\mu'}, n \geq 0\}$, these derivatives are zero, hence $e^{\frac{1}{2}z^2\lambda}m$ is zero on Ω. In particular $m(1) = 0$, i.e. ψ is orthogonal to φ, which proves the claim.

Step 2: The Heisenberg groups $\mathbb{H}_k$

A basis of the first layer of the Lie algebra is $X_1, Y_1, .., X_k, Y_k$ where $[X_j, Y_j] = -4U$, U spans the center, and the other commutators are zero. By the Campbell-Hausdorff formula,

$$g = \exp(\sum_{j=1}^k x_j X_j + y_j Y_j + uU) = \exp uU \prod_{j=1}^k \exp(-2x_j y_j U)\exp y_j Y_j \exp x_j X_j.$$

We first consider the Schrödinger (unitary irreducible) representation $\Pi_S : \mathbb{H}_k \to B(L^2(\mathbb{R}^k))$, defined on the Lie algebra by

$$\Pi_S(X_j) = \frac{\partial}{\partial\xi_j},\ \Pi_S(Y_j) = i\xi_j,\ \Pi_S(U) = -\frac{1}{4}[\frac{\partial}{\partial\xi_j}, i\xi_j] = -\frac{i}{4}I.$$

For $F \in L^2(\mathbb{R}^k)$, this implies

$$\Pi_S(g)(F)(\xi) = e^{-i\frac{u}{4}} e^{\frac{i}{2}\sum_{j=1}^k x_j y_j} e^{i\sum_{j=1}^k y_j\xi_j} F(\xi + x), \tag{24}$$

and

$$\Pi_S(L) = H = \sum_{j=1}^k (-\frac{\partial^2}{\partial\xi_j^2} + \xi_j^2)$$

is the harmonic oscillator. If $k = 1$, an o.n. basis of eigenvectors of H in $L^2(\mathbb{R})$ is the sequence of Hermite functions $F_\mu, \mu \in \mathbb{N}$. The so called special Hermite functions [T, p. 18-19] are, for $\mu, \mu' \in \mathbb{N}$ and $\varepsilon_{\mu,\mu'} = sgn(\mu' - \mu)$,

$$\begin{aligned} \langle \Pi_S(x, y, 0)(F_\mu), F_{\mu'}\rangle &= \Phi_{\mu,\mu'}(x, y) = \int_{\mathbb{R}} e^{iy\xi} F_\mu(\xi + \frac{x}{2}) F_{\mu'}(\xi - \frac{x}{2}) d\xi \\ &= r_{\mu,\mu'}(x^2 + y^2) e^{-\frac{1}{2}(x^2+y^2)} (x + i\varepsilon_{\mu,\mu'} y)^{|\mu-\mu'|}, \quad (25) \end{aligned}$$

where $r_{\mu,\mu'} = r_{\mu',\mu}$ is a one variable polynomial with real coefficients.
An o.n basis of eigenvectors of H in $L^2(\mathbb{R}^k)$ is the sequence $\left(\prod_{j=1}^{k} F_{\mu_j}(\xi_j)\right)_{\mu\in\mathbb{N}^k}$, which gives, for $\mu, \mu' \in \mathbb{N}^k$ and $g = (x, y, u)$,

$$\varphi^{\Pi_S,\mu,\mu'}(g) = \langle \Pi(g)(F_\mu), F_{\mu'}\rangle = e^{-i\frac{u}{4}} \prod_{j=1}^{k} \Phi_{\mu_j,\mu'_j}(x_j, y_j).$$

By (25) the function $z \to \varphi^{\Pi_S,\mu,\mu'}(zx, zy, z^2u)$ is holomorphic on $\mathbb{C}$. Let

$$R_{a,\delta} = \{\alpha + i\beta \mid |\alpha| < a, |\beta| < \delta\} \subset \mathbb{C}.$$

For some constant $C_{a,\delta}$, and $z \in \overline{R_{a,\delta}}$,

$$\left|\varphi^{\Pi_S,\mu,\mu'}(zx, zy, z^2u)\right| \le C_{a,\delta} e^{\frac{1}{2}a\delta|u|} \prod_{j=1}^{k} e^{\delta^2(x_j^2+y_j^2)}.$$

We now look for conditions on a, δ ensuring that the right hand side lies in $L^q(pdg)$. We recall [Hu] that

$$p(x, y, u) = \int_{\mathbb{R}} e^{i\lambda u} Q(x, y, \lambda) d\lambda = c_k \int_{\mathbb{R}} e^{i\lambda u} \prod_{j=1}^{k} \frac{2\lambda}{sh 2\lambda} e^{-\frac{\lambda}{th 2\lambda}(x_j^2+y_j^2)} d\lambda.$$

Noting that $Q(x, y, \lambda) = \prod_{j=1}^{k} Q_1(x_j, y_j, \lambda)$ is even w.r. to λ, we get, for $q \ge 1$,

$$\frac{1}{2}\int_{\mathbb{R}} e^{\frac{q}{2}a\delta|u|} p(x, y, u) du \le \int_{\mathbb{R}} ch(\frac{q}{2}a\delta u) p(x, y, u) du = Q(x, y, ia\delta\frac{q}{2}).$$

We need the convergence of

$$\int_{\mathbb{R}^{2k}} \prod_{j=1}^{k} e^{q\delta^2(x_j^2+y_j^2)} Q_1(x_j, y_j, i\frac{q}{2}a\delta) dx dy = c \prod_{j=1}^{k} \int_{\mathbb{R}^2} e^{(q\delta^2 - \frac{1}{2}\frac{qa\delta}{tg qa\delta})(x_j^2+y_j^2)} dx_j dy_j,$$

which holds for $qa\delta \le \frac{\pi}{4}$ and $a > 2\delta$. Thus, taking $a = N \in \mathbb{N}$, $\varphi^{\Pi_S,\mu,\mu'}$ satisfies the assumptions of step 1 on

$$\Omega = \cup_{N\geq 2} R_{N,\frac{\pi}{4qN}}.$$

Plancherel formula for $\mathbb{H}_k$ (see e.g. [T, Theorem 1.3.1] or [CG, p.154]) involves the representations

$$\rho_h(x,y,u) = e^{-\frac{i}{4}hu}\Pi_S(x,hy,0).$$

By the Stone-Von Neumann theorem [T, theorem 1.2.1] every irreducible unitary representation Π of $\mathbb{H}_k$ satisfying $\Pi(0,0,u) = e^{-\frac{i}{4}hu}$ for a real $h \neq 0$ is unitarily equivalent to ρ_h. Hence ρ_{β^2} (resp. $\rho_{-\beta^2}$) is unitarily equivalent to $\Pi_S \circ \delta_\beta$, (resp. $\Pi_S \circ \sigma \circ \delta_\beta$), $\beta > 0$, where σ is the automorphism of $\mathbb{H}_k$ defined by $\sigma(x,y,u) = (x,-y,-u)$).

Since $\Pi_S(L) = \Pi_S \circ \sigma(L)$, we get $\varphi^{\Pi_S\circ\sigma,\mu,\mu'} = \varphi^{\Pi_S,\mu,\mu'} \circ \sigma = \overline{\varphi^{\Pi,\mu,\mu'}}$, hence

$$\mathcal{F} = \{\varphi^{\Pi_S,\mu,\mu'} \circ \delta_\beta, \overline{\varphi^{\Pi_S,\mu,\mu'}} \circ \delta_\beta, \beta > 0, \mu,\mu' \in \mathbb{N}^k\}.$$

The conditions of step 1 are satisfied by $\varphi^{\Pi,\mu,\mu'} \circ \delta_\beta$, replacing $R_{a,\delta}$ by $R_{\beta a,\beta\delta}$, which ends the proof of theorem 12 for $\mathbb{H}_k$. Taking remark 5 into account, the set $\cup_{\mu,\mu',n}\{h_n^{\Pi_S,\mu,\mu'}, \overline{h_n^{\Pi_S,\mu,\mu'}}\}$ is total in $L^2(\mathbb{H}_k, pdg)$.

Step 3. We first recall some more facts on representations and compute the set $\mathcal{F}$ for step 2 stratified groups. We shall follow Cygan's scheme [Cy].

Let $l \in \mathcal{G}^*$. Among the Lie subalgebras $\mathcal{M} \subset \mathcal{G}$ satisfying $\langle l, [X,Y]\rangle = 0$ for every $X, Y \in \mathcal{M}$, some have minimal codimension m_l and are denoted by $\mathcal{M}_l$. Then the map

$$Z \in \mathcal{M}_l \to e^{i\langle l,Z\rangle} \tag{26}$$

is a representation of the subgroup $\exp \mathcal{M}_l$ and induces an irreducible unitary representation of G as follows [CG, theorems 1.3.3, 2.2.1 and p 41] : One chooses independent vectors $(X_j)_{i=1}^{m_l}$ such that $\mathcal{G} = \mathcal{M}_l + span\{(X_j)_{i=1}^{m_l}\}$. For $(g,\xi) \in G \times \mathbb{R}^{m_l}$ there exist $(\xi', M) \in \mathbb{R}^{m_l} \times \mathcal{M}_l$ such that

$$\exp(\sum_{i=1}^{m_l} \xi_i X_i).\ g = \exp M\ .\exp(\sum_{i=1}^{m} \xi'_i X_i).$$

Then, for $F \in L^2(\mathbb{R}^{m_l})$,

$$\Pi_l(g)(F)(\xi) = e^{i\langle l,M\rangle}F(\xi'). \tag{27}$$

The set of $\mathcal{C}^\infty$ vectors for Π_l is $\mathcal{S}(\mathbb{R}^k)$ [CG, corollary 4.1.2]. Every irreducible unitary representation of G is equivalent to a representation constructed in this way; different $\mathcal{M}_l, \mathcal{M}'_l$ and different l, l' in the same coadjoint orbit induce equivalent representations [CG, theorems 2.2.2, 2.2.3, 2.2.4].

By Kirillov theory there is an integer k and a set $\mathcal{U}_0 \subset \mathcal{G}^*$ of "generic" orbits with maximal dimension $2k$, such that $m_l = k$ for $l \in \mathcal{U}_0$. The Plancherel measure is supported by $\mathcal{U}_0$ [CG, theorem 4.3.10].

We now compute such a Π_l when G is step 2**.** Let $U_1, .., U_d$ be a basis of the central layer $\mathcal{Z}$ and let $\chi_1, ., \chi_n$ be a basis of the first layer V_1 of $\mathcal{G}$.

Let $l \in \mathcal{G}^*$ and let $\lambda = \sum_{j=1}^d \lambda_j U_j^*$ be its central part, identified with a vector $\lambda \in \mathbb{R}^d$. Let A_λ be the $n \times n$ matrix with coefficients $\langle \lambda, [\chi_j, \chi_h]\rangle$.

By Campbell-Hausdorff formula, for $Y \in \mathcal{G}, X \in V_1, U \in \mathcal{Z}, g = \exp(X + U)$,

$$\exp Adg(Y) = g\ \exp Y\ g^{-1} = e^{[X,Y]} \exp Y = \exp(Y + [X,Y]),$$

hence the coadjoint orbit of l, i.e. $\{l \circ Adg,\ g \in \mathcal{G}\} \subset \mathcal{G}^*$, is $l + range\ A_\lambda$. We now assume that l lies in $\mathcal{U}_0$, so that the range of A_λ has dimension $2k$.There exists an orthogonal matrix Ω_λ such that

$$A_\lambda = \Omega_\lambda A'_\lambda \Omega^*_\lambda$$

where A'_λ is block diagonal, the non zero blocks having the form

$$\nu_j(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \nu_j(\lambda) > 0. \tag{28}$$

The new basis of V_1 (defined by the columns of Ω_λ) is denoted by $X_1, Y_1, ..,$ $X_k, Y_k, S_1, ..., S_{n-2k}$, so that

$$\langle \lambda, [X_j, X_h]\rangle = 0 = \langle \lambda, [Y_j, Y_h]\rangle,\ \langle \lambda, [X_j, Y_h]\rangle = \nu_j(\lambda)\delta_{jh},\ 1 \le j, h \le k. \tag{29}$$

We denote $t = \Omega_\lambda(x, y, s) \in \mathbb{R}^n$, where

$$\sum_{j=1}^n t_j \chi_j = \sum_{j=1}^k x_j X_j + y_j Y_j + \sum_{h=1}^{n-2k} s_h S_h = X + Y + S \in V_1.$$

Choosing $\mathcal{M}_l = \mathcal{Z} + span\{Y_j, S_h\}_{1\le j\le k, 1\le h\le n-2k}$, let us compute Π_l. By definition $\Pi_l(\exp u_j U_j) = e^{iu_j\lambda_j}$. For $g = \exp(X+Z)$ and $Z = Y+S$,

$$\begin{aligned}\exp(\sum_{j=1}^k \xi_j X_j)g &= \exp[\sum_{j=1}^k \xi_j X_j, X+Z]\; g\; \exp(\sum_{j=1}^k \xi_j X_j)\\ &= \exp([\sum_{j=1}^k \xi_j X_j, X+Z] + \frac{1}{2}[X,Z]) \exp Z \exp X \exp(\sum_{j=1}^k \xi_j X_j)\\ &= \exp M \exp(\sum_{j=1}^k (\xi_j + x_j) X_j).\end{aligned}$$

Hence, by (27) and (29), for $F \in L^2(\mathbb{R}^k)$,

$$\Pi_l(g)(F)(\xi) = e^{i\langle l,M\rangle} F(\xi + x) = e^{i\sum_{j=1}^k \nu_j y_j(\xi_j + \frac{1}{2}x_j)} e^{i\langle l, Y+S\rangle} F(\xi+x). \qquad (30)$$

Since we may replace l by l' in the orbit of l, we may suppose $\langle l, Y_j\rangle = 0$, $1 \le j \le k$. In particular, by (30),

$$\Pi_l(X_j) = \frac{\partial}{\partial \xi_j}, \Pi_l(Y_j) = i\nu_j\xi_j,\ 1\le j\le k, \Pi_l(S_h) = i\left\langle l, S_h\right\rangle I,\ 1 \le h \le n-2k.$$

Since Ω_λ is orthogonal, $-L = \sum_{j=1}^k (X_j^2 + Y_j^2) + \sum_{h=1}^{n-2k} S_h^2$, which entails

$$\Pi_l(L) = \sum_{j=1}^k -\frac{\partial^2}{\partial \xi_j^2} + \nu_j^2\xi_j^2 + \sum_{h=1}^{n-2k} \left\langle l, S_h\right\rangle^2 I.$$

A basis of eigenvectors of $\Pi_l(L)$ is thus $\left(\prod_{j=1}^k F_{\mu_j}(\sqrt{\nu_j}\xi_j)\right)_{\mu\in\mathbb{N}^k}$. By (30) and (25), for $g = (x, y, s, u)$,

$$\varphi^{\Pi_l,\mu,\mu'}(g) = e^{i\langle\lambda,u\rangle} e^{i\sum_{h=1}^{n-2k} s_h\langle l,S_h\rangle} \prod_{j=1}^k \frac{1}{\sqrt{\nu_j}} \Phi_{\mu_j,\mu'_j}(\sqrt{\nu_j}x_j, \sqrt{\nu_j}y_j).$$

Hence, for $z \in R_{a,\delta}$ and some constant $C_{a,\delta}$, with $t = \Omega_\lambda(x, y, s)$,

$$\begin{aligned}\left|\varphi^{\Pi_l,\mu,\mu'}(zt,z^2u)\right| &\leq C_{a,\delta}e^{2a\delta|\langle\lambda,u\rangle|}e^{\delta\sum_{h=1}^{n-2k}|s_h\langle l,S_h\rangle|}\prod_{j=1}^{k}\frac{1}{\sqrt{\nu_j}}e^{\delta^2\nu_j(x_j^2+y_j^2)}\\ &= e^{2a\delta|\langle\lambda,u\rangle|}w_{a,\delta,l}(x,y,s).\end{aligned}$$

By [Cy, corollary 5.5] the heat kernel $p(t,u)$ is the Fourier transform of $CQ(t,\lambda)$ w.r. to the central variables, where

$$Q(t,\lambda)=\prod_{h=1}^{n-2k}e^{-\frac{1}{2}s_h^2}\prod_{j=1}^{k}Q_1(x_j,y_j,\frac{\nu_j}{4})=Q(t,-\lambda).$$

Again, we need the convergence of

$$\int_{\mathbb{R}^n}w_{a,\delta,l}^q(x,y,s)\prod_{h=1}^{n-2k}e^{-\frac{1}{2}s_h^2}\prod_{j=1}^{k}Q_1(x_j,y_j,\frac{iqa\delta\nu_j}{2})dxdyds,$$

which holds if $qa\delta\max\nu_j\leq\frac{\pi}{4}$ and $a>2\delta$. This ends the proof of theorem 12.■

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