$(d, 1)$-total labelling of sparse graphs

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Abstract

The $(d, 1)$-total number $\lambda_d^T(G)$ of a graph G is the width of the smallest range of integers that suffices to label the vertices and the edges of G so that no two adjacent vertices have the same color, no two incident edges have the same color and the distance between the color of a vertex and the color of any incident edge is at least d. This notion was introduced by Havet and Yu in [6]. In this paper, we study the $(d, 1)$-total number of sparse graphs and prove that for any $0 < \varepsilon < \frac{1}{2}$, and any positive integer d, there exists a constant $C_{d,\varepsilon}$ such that for any $\varepsilon\Delta$-sparse graph G with maximum degree Δ, we have $\lambda_d^T(G) \leq \Delta + C_{d,\varepsilon}$.

1 Introduction

In the channel assignment problem, we need to assign frequency bands to transmitters. If two transmitters are too close, interferences will occur if they attempt to transmit on close frequencies. In order to avoid this situation, the channels assigned must be sufficiently far. Moreover, if two transmitters are close but not too close, the channels assigned must still be different. This problem is known under the $L(p, q)$-labelling problem of a graph G, where a $L(p, q)$-labelling is an integer assignment L to the vertices of G such that $\forall(u, v) \in V(G)^2, d_G(u, v) = 1 \Rightarrow |L(u) - L(v)| \geq p$ and $\forall(u, v) \in V(G)^2, d_G(u, v) = 2 \Rightarrow |L(u) - L(v)| \geq q$. In 1992, Griggs and Yeh introduced this labelling with $p = 2$ and $q = 1$ in [4]. Since, this notion has been widely studied and gives many challenging problems. In particular, in 1995, Whittlesey, Georges and Mauro [3] studied the $L(2, 1)$-labelling of the incidence graph obtained from G. The incidence graph of G is the graph obtained from G by replacing each edge by a path of length 2. The $L(2, 1)$-labelling of the incidence graph of G is equivalent to an assignment of integers to each element of $V(G) \cup E(G)$ such that :

1. the edge-coloring is proper, i.e. no two incident edges receive the same integer ;
2. the vertex-coloring is proper, i.e. no two adjacent vertices receive the same integer ;
3. the difference between the integer assigned to a vertex and those assigned to its incident edges is at least 2.

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This labelling is called a $(2,1)$-total labelling. It was introduced by Havet and Yu in 2002 [6, 5] and generalized to the $(d,1)$-total labelling of a graph G.

More formally, a $(d,1)$-total labelling of a graph $G = (V,E)$ is a function $c : V \cup E \to \mathbb{N}$ verifying:

(i) $\forall (u,v) \in V^2 : uv \in E \Rightarrow c(u) \neq c(v)$
(ii) $\forall (u,v,w) \in V^3 : uv \in E, uw \in E \Rightarrow c(uv) \neq c(uw)$
(iii) $\forall (u,v) \in V^2 : uv \in E \Rightarrow |c(u) - c(uv)| \geq d$

The span of a $(d,1)$-total labelling is the maximum difference between two assigned integers. The $(d,1)$-*total number* of a graph G, denoted by $\lambda_d^T(G)$, is the minimum span of a $(d,1)$-total labelling of G. Figure 1 gives an example of a $(2,1)$-total labelling with 6 colors (we use integers belonging to an interval beginning by zero).

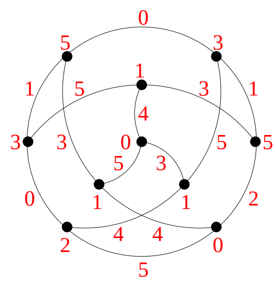

Figure 1: $(2,1)$-total labelling of the Petersen's graph.

Notice that the $(1,1)$-total labelling is the traditional total coloring.

We recall some bounds (without proof) and a conjecture for the $(d,1)$-total number:

Theorem 1 ([6]) *Let G be a graph with maximum degree Δ, then:*

(i) $\lambda_d^T(G) \geq \Delta + d - 1$.
(ii) *If G is Δ-regular,* $\lambda_d^T(G) \geq \Delta + d$.
(iii) *If* $d \geq \Delta$, $\lambda_d^T(G) \geq \Delta + d$.

Let $\chi(G)$ (resp. $\chi'(G)$) be the chromatic number (resp. index) of G. Observe that if we color the vertices with colors belonging to an interval I_v containing $\chi(G)$ colors and the edges with colors belonging to an interval I_e containing $\chi'(G)$ colors, I_v and I_e being separated by an interval of size $d-1$, we obtain a $(d,1)$-total labelling of the graph. Theorem 2 is deduced from this observation :

Theorem 2 ([6]) *Let G be a graph, then*

(i) $\lambda_d^T(G) \leq \chi(G) + \chi'(G) + d - 2$
(ii) $\lambda_d^T(G) \leq 2\Delta + d - 1$

Theorem 3 ([10]) *Let G be a connected graph with maximum degree Δ, $d \geq 2$, then $\lambda_d^T(G) \leq \Delta + 2d - 2$ in the following cases :*

(i) $\Delta \geq 2d+1$ *and* $Mad(G) < \frac{5}{2}$
(ii) $\Delta \geq 2d+2$ *and* $Mad(G) < 3$
(iii) $\Delta \geq 2d+3$ *and* $Mad(G) < \frac{10}{3}$

where $Mad(G)$ *is the maximum average degree of* G*, i.e.* $Mad(G) = \max\{2|E(H)|/|V(H)|, H \subsetneq G\}$.

Conjecture 1 ([6]) *Let* G *be a graph with maximum degree* Δ*, then* $\lambda_d^T(G) \leq \min\{\Delta + 2d - 1, 2\Delta + d - 1\}$.

Finally, the best known upper bound for general graphs is due to Esperet and Havet [2] who proved :

Theorem 4 *Let* G *be a graph with maximum degree* Δ*, then* $\lambda_d^T(G) \leq \Delta + O(\log \Delta)$.

In [7], Molloy and Reed proved that the total chromatic number of any graph with maximum degree Δ is at most Δ plus an absolute constant. Moreover, in [9], they gave a simpler proof of this result for sparse graphs. In this paper, we generalize their approach to the $(d,1)$-total number of sparse graphs.

A vertex v is called *α-sparse* iff $|E(N(v))| \leq \binom{\Delta}{2} - \alpha\Delta$. An α-sparse graph is a graph in which all the vertices are α-sparse.

Our main result is the following :

Theorem 5 *For any* $0 < \varepsilon < \frac{1}{2}$*, and any positive integer* d*, there exists a constant* $C_{d,\varepsilon}$ *such that for any* $\varepsilon\Delta$*-sparse graph* G *with maximum degree* Δ*, we have* $\lambda_d^T(G) \leq \Delta + C_{d,\varepsilon}$.

The proof of Theorem 5 is based on a probabilistic approach due to Molloy and Reed. It uses intensively concentration inequalities as well as Lovász Local Lemma. Moreover, we conjecture:

Conjecture 2 *For any positive integer* d*, there exists a constant* C_d*, such that for any graph* G *with maximum degree* Δ*, we have* $\lambda_d^T(G) \leq \Delta + C_d$.

In Section 2, we present the procedure used to prove Theorem 5. In Section 3, we analyse this procedure. In the following, we will need some probabilistic tools (see Appendix A and [9] for more details).

2 Proof of Theorem 5

Since $\lambda_d^T(G) \leq 2\Delta + d - 1$, if we prove that for some $\Delta_0(d,\varepsilon)$ and some $C_{d,\varepsilon}$, any $\varepsilon\Delta$-sparse graph G of maximum degree $\Delta \geq \Delta_0$ verifies $\lambda_d^T(G) \leq \Delta + C_{d,\varepsilon}$, then Theorem 5 will be proved.

Let ϕ be a full or partial coloring of G. Any edge $e = uv$ such that $|\phi(u) - \phi(e)| < d$ or/and $|\phi(v) - \phi(e)| < d$ is called a *reject edge*. The graph R induced by the reject edges is called the *reject graph*. It will be convenient for us to consider the *reject degree* of a vertex v, which is the number of edges $e = uv$ such that $|\phi(u) - \phi(e)| < d$. Observe that $deg_R(v)$ is at most the reject degree of v plus $2d - 1$.

2.1 Sketch of Proof

To prove Theorem 5, we apply the following steps :

Step 1. First, we will color the edges by Vizing's Theorem using the colors $\{1, \dots, \Delta\}$.

Step 2. Then we will use the Naive Coloring Procedure to color the vertices with colors $\{1, \dots, \Delta + 2d - 1\}$. This procedure creates reject edges. However, we can prove that after the procedure, the maximum degree of the reject graph R is a constant $D_{d,\varepsilon}$ which does not depend on Δ.

Step 3. Finally, we erase the color of the vertices of R and recolor these vertices greedily with the colors $\{\Delta + 3d - 2, \dots, \Delta + 3d - 1 + D_{d,\varepsilon}\}$. Taking $C_{d,\varepsilon} = D_{d,\varepsilon} + 3d - 2$, this proves that $\lambda_d^T(G) \leq \Delta + C_{d,\varepsilon}$.

We now present the Naive Coloring Procedure.

2.2 The Naive Coloring Procedure

For each vertex v, we maintain two lists of colors : L_v and F_v. L_v is the set of colors which do not appear in the neighborhood of v. Initially, $L_v = \{1, \dots, \Delta + 2d - 1\}$. After iteration I (specified later), F_v will be a set of forbidden colors. Until iteration I, $F_v = \emptyset$.

During the Naive Coloring procedure, we will perform i^* (specified later) iterations of the following procedure :

Step 1. Assign to each uncolored vertex v a color choosen uniformly at random in L_v.

Step 2. Uncolor any vertex which receives the same color as a neighbor in this iteration.

Step 3. **Iteration $i \leq I$.** Let v be a vertex having more than T (specified later) neighbors u which are assigned a color $c(u)$ such that $|c(uv) - c(u)| < d$ in this iteration. For any v, we uncolor all such neighbors.

Iteration $i > I$.

- (a) Uncolor any vertex v which receives a color from F_v in this iteration.
- (b) Let v be a vertex having more than one neighbor u which is assigned a color such that $|c(uv) - c(u)| < d$ in this iteration. For any v, we uncolor all such neighbor.
- (c) Let v be a vertex having at least one neighbor u such that $|c(uv) - c(u)| < d$ in this iteration. For any v, we place $\{c(vw) - d + 1, \dots, c(vw), \dots, c(vw) + d - 1\}$ in F_w for every $w \in N(v)$.

Step 4. For any vertex v which retained its color c, we remove c from L_u for any $u \in N(v)$.

After i^* iterations of this procedure, we have have a partial coloring of G. We then complete this coloring in order to obtain a reject graph R with a bounded maximum degree which does not depend on Δ (Section 4.3).

3 Analysis of the procedure

3.1 The first iteration

Let $\zeta = \frac{\varepsilon}{2e^3}$. In this subsection, we prove that:

Claim 1 *The first iteration produces a partial coloring with bounded reject degree for which every vertex has at least $\frac{\zeta}{2}\Delta$ repeated colors in its neighborhood.*

We recall that $\mathcal{C} = \Delta + 2d - 1$ is the initial size of each color list L_v. Let A_v be the number of colors c such that at least two neighbors of v receive the color c and all such vertices retain their color during Step 2. Let B_v be the number of neighbors of v which are uncolored at Step 3. Notice that vertices are uncolored at Step 3 regardless of what happened at Step 2. Let X_v be the event that "$A_v < \zeta\Delta$". Let Y_v be the event that "$B_v \geq \frac{\zeta}{2}\Delta$". If no type X event occurs, every vertex has at least $\zeta\Delta$ repeated colors in its neighborhood at the end of Step 2. If no type Y event occurs, less than $\frac{\zeta}{2}\Delta$ vertices are uncolored in each neighborhood. As a consequence, if we show that with positive probability, no type X or Y event occurs, Claim 1 will be proved.

Claim 2 $\mathbf{Pr}(X_v) < e^{-\alpha \log^2 \Delta}$*, for a particular constant* $\alpha > 0$.

Proof. We first bound the expected value of A_v. Let A'_v be the number of colors c such that exactly two neighbors of v receive the color c and are not uncolored during Step 2. Notice that $A_v \geq A'_v$, and thus $\mathbf{E}(A_v) \geq \mathbf{E}(A'_v)$. Let u and w be two non adjacent neighbors of v. The probability that u and w are colored with c, while no other neighbor of v is colored with c, and while no neighbor of u or w is colored with α is exactly $\left(\frac{1}{\mathcal{C}}\right)^2\left(1-\frac{1}{\mathcal{C}}\right)^{3\Delta-3} > \left(\frac{1}{\mathcal{C}}\right)^2\left(1-\frac{1}{\mathcal{C}}\right)^{3\Delta}$. Since G is $\varepsilon\Delta$-sparse, $|E(N(v))| \leq \binom{\Delta}{2} - \varepsilon\Delta^2$, which implies that there are at least $\varepsilon\Delta^2$ pairs of non adjacent vertices among the neighbors of v. There are $\mathcal{C}$ choices for the color c, thus

$$\mathbf{E}(A'_v) > \mathcal{C}\varepsilon\Delta^2\left(\frac{1}{\mathcal{C}}\right)^2\left(1-\frac{1}{\mathcal{C}}\right)^{3\Delta} = \frac{\varepsilon\Delta^2}{\mathcal{C}}\left(1-\frac{1}{\mathcal{C}}\right)^{3\Delta}$$

For $\Delta > 2$, we have $\ln(1-\frac{1}{\mathcal{C}}) \geq -\frac{1}{\mathcal{C}} - \frac{1}{\mathcal{C}^2}$, and thus $\left(1-\frac{1}{\mathcal{C}}\right)^{3\Delta} \geq e^{-3}e^{-\frac{3}{\mathcal{C}}}$. For Δ large enough, $\Delta/\mathcal{C} > \sqrt{3}/2$ and $e^{-\frac{3}{\mathcal{C}}} > \sqrt{3}/2$, so:

$$\mathbf{E}(A'_v) > \frac{3\varepsilon\Delta}{4e^3} = \frac{3}{2}\zeta\Delta$$

Since $\mathbf{E}(A_v) \geq \mathbf{E}(A'_v)$, we also have $\mathbf{E}(A_v) > \frac{3}{2}\zeta\Delta$. Let AT_v be the number of colors assigned to at least two neighbors of v, and let Del_v be the number of colors assigned to at least two neighbors of v and not retained by at least one of them. Note that $A_v = AT_v - \mathrm{Del}_v$, and by linearity of expectation, $\mathbf{E}(A_v) = \mathbf{E}(AT_v) - \mathbf{E}(\mathrm{Del}_v)$. The random variable AT_v only depends on the Δ colors assigned to the neighbors of v. Moreover, changing one of these colors can only affect AT_v by at most 1. Using the Simple Concentration bound, we obtain:

$$\mathbf{Pr}\left(|AT_v - \mathbf{E}(AT_v)| > t\right) < 2e^{-\frac{t^2}{2\Delta}}. \tag{1}$$

The random variable Del_v only depends on the nearly Δ^2 colors assigned to the vertices at distance at most 2 from v. As previously, changing one of these colors can only affect Del_v by at most 1. Furthermore, if $\mathrm{Del}_v \geq s$, we can find at most $3s$ vertices, which colors certify that $\mathrm{Del}_v \geq s$ (for each color α counted by $\mathrm{Del}_v \geq s$, we take two neighbors x and y of v colored with α and a neighbor z of x or y also colored with α). Applying Talagrand's Inequality with $c = 1$ and $r = 3$, we obtain for all $t \geq \sqrt{\Delta\log\Delta}$

$$\mathbf{Pr}\left(|\mathrm{Del}_v - \mathbf{E}(\mathrm{Del}_v)| > t\right) < 4e^{-\frac{\left(t-60\sqrt{3\mathbf{E}(Del_v)}\right)^2}{24\mathbf{E}(Del_v)}} < 4e^{-\frac{t^2}{25\Delta}}, \tag{2}$$

since $\mathbf{E}(\mathrm{Del}_v) \leq \Delta$. Recall that $\mathbf{E}(A_v) = \mathbf{E}(AT_v) - \mathbf{E}(\mathrm{Del}_v)$. Let $t = \frac{1}{2}\log\Delta\sqrt{\mathbf{E}(A_v)}$. If $|A_v - \mathbf{E}(A_v)| > \log\Delta\sqrt{\mathbf{E}(A_v)}$ we have either $|AT_v - \mathbf{E}(AT_v)| > t$ or $|\mathrm{Del}_v - \mathbf{E}(\mathrm{Del}_v)| > t$. Using (1) and (2), the probability that this happens is at most

$$2e^{-\frac{t^2}{2\Delta}} + 4e^{-\frac{t^2}{25\Delta}} < 2e^{-\frac{3}{16}\zeta\log^2\Delta} + 4e^{-\frac{3}{200}\zeta\log^2\Delta} < e^{-\frac{\zeta}{100}\log^2\Delta}$$

So, for Δ large enough, $\mathbf{Pr}\left(|A_v - \mathbf{E}(A_v)| > \log\Delta\sqrt{\mathbf{E}(A_v)}\right) < e^{-\frac{\zeta}{100}\log^2\Delta}$.

$$\begin{aligned}\mathbf{Pr}\left(|A_v - \mathbf{E}(A_v)| > \log\Delta\sqrt{\mathbf{E}(A_v)}\right) &\geq \mathbf{Pr}\left(A_v < \mathbf{E}(A_v) - \log\Delta\sqrt{\mathbf{E}(A_v)}\right)\\ &\geq \mathbf{Pr}\left(A_v < \frac{3}{2}\zeta\Delta - \log\Delta\sqrt{\Delta}\right)\\ &\geq \mathbf{Pr}\left(A_v < \zeta\Delta\right)\end{aligned}$$

Since $\mathbf{Pr}(X_v) = \mathbf{Pr}(A_v < \zeta\Delta)$, we proved that $\mathbf{Pr}(X_v) < e^{-\frac{\zeta}{100}\log^2\Delta}$. □

Claim 3 $\mathbf{Pr}(Y_v) < e^{-\beta\Delta}$, *for a particular constant* $\beta > 0$.

Proof. Let u be a neighbor of v. The vertex u will be uncolored in Step 3 if for some neighbor w of u, u and T other neighbors $x_1, \ldots, x_T$ of w are each assigned a color $c(x_i)$ such that $|c(u) - c(wu)| < d$ and $|c(x_i) - c(wx_i)| < d$ for all $1 \leq i \leq T$. The probability that this happens is at most

$$\Delta\binom{\Delta-1}{T}\left(\frac{2d-1}{\mathcal{C}}\right)^{T+1} < \frac{(2d-1)^T}{T!}$$

For T large enough, $(2d-1)^T/T! < \zeta/4$, and thus $\mathbf{E}(B_v) < \frac{\zeta\Delta}{4}$. The random variable B_v only depends on the nearly Δ^3 colors assigned to the vertices at distance at most 3 from v. Changing one of these colors can affect B_v by at most $T+1$. Moreover, if $B_v \geq s$ there is a set of at most $(T+1)s$ vertices which colors certify that $B_v \geq s$ (for each uncolored neighbor u of v, take u and T other neighbors $x_1, \ldots, x_T$ of some neighbor w of u, such that $|c(u) - c(wu)| < d$ and $|c(x_i) - c(wx_i)| < d$ for all $1 \leq i \leq T$). Applying Talagrand's Inequality to B_v with $c = T+1$ and $r = T+1$, we obtain for all $t \geq \sqrt{\Delta\log\Delta}$

$$\mathbf{Pr}\left(|B_v - \mathbf{E}(B_v)| > t\right) < 4e^{-\frac{\left(t-60(T+1)\sqrt{(T+1)\mathbf{E}(B_v)}\right)^2}{8(T+1)^3\mathbf{E}(B_v)}} < 4e^{-\frac{t^2}{9(T+1)^3\Delta}}.$$

Taking $t = \frac{\zeta\Delta}{8}$, we obtain $\mathbf{Pr}\left(|B_v - \mathbf{E}(B_v)| > \frac{\zeta\Delta}{8}\right) < 4e^{-\frac{\zeta^2\Delta}{576(T+1)^3}} < e^{-\frac{\zeta^2\Delta}{577(T+1)^3}}$. Now, since

$$\begin{aligned}\mathbf{Pr}\left(|B_v - \mathbf{E}(B_v)| > \frac{\zeta\Delta}{8}\right) &\geq \mathbf{Pr}\left(B_v > \mathbf{E}(B_v) + \frac{\zeta\Delta}{8}\right)\\ &\geq \mathbf{Pr}\left(B_v > \frac{3}{8}\zeta\Delta\right)\\ &\geq \mathbf{Pr}\left(B_v \geq \frac{\zeta\Delta}{2}\right)\end{aligned}$$

we have $\mathbf{Pr}(Y_v) < e^{-\frac{\zeta^2}{577(T+1)^3}\Delta}$. □

We now use Lovász Local Lemma to prove Claim 1. Each event X_v only depends on the colors assigned to the vertices at distance at most 2 from v, and each event Y_v depends on the colors assigned to the vertices at distance at most 3 from v. Hence, each event is mutually independent of all but at most $2\Delta^6$ other events. For Δ sufficiently large, $\mathbf{Pr}(X_v) < \frac{1}{8\Delta^6}$ and $\mathbf{Pr}(Y_v) < \frac{1}{8\Delta^6}$. Using Lovász Local Lemma, this proves that with positive probability no type X or Y event happens. Thus with positive probability, the first iteration produces a partial coloring with bounded reject degree, such that each vertex has at least $\frac{\zeta\Delta}{2}$ repeated colors in its neighborhood.

3.2 The next iterations

Let $d_i = \left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right)^i \Delta$ and $f_i = \frac{4(2d-1)}{\zeta}\sum_{j=I+1}^{i-1} D_j$. Let i^* be the smallest integer i such that $d_i \leq \sqrt{\Delta}$. Observe that for any $i \leq i^*$, we have $d_i \geq (1 - \frac{1}{4}e^{-\frac{2}{\zeta}})\sqrt{\Delta}$.

Claim 4 *At the end of each iteration $1 \leq i \leq i^*$, with positive probability every vertex has at most d_i uncolored neighbors, and each list F_v has size at most f_i.*

Proof. We prove Claim 4 by induction on i. At the end of the first iteration, every vertex has at least $\frac{\zeta\Delta}{2}$ repeated colors in its neighborhood. So the number of uncolored vertices in the neighborhood of any vertex is at most $(1-\zeta)\Delta$, which is less than $d_1 = \left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right)\Delta$. Morever, for any vertex v, the list F_v is still empty at the end of the first iteration, thus $|L_v| \leq 0 = f_1$.

Suppose $i > 1$. By induction, there are at most d_{i-1} uncolored vertices in each neighborhood at the beginning of iteration i, and each F_v has size at most f_{i-1}. We define the random variable D_v^i as the number of uncolored neighbors of v after iteration i, and the random variable F_v^i as the size of the list F_v after iteration i. To complete the induction, we show that with positive probability, $D_v^i \leq d_i$ and $F_v^i \leq f_i$ for any vertex v. Since every vertex v has at least $\frac{\zeta\Delta}{2}$ repeated colors in its neighborhood, every list L_v has size at least $\frac{\zeta\Delta}{2}$. Thus, the probability that a newly colored vertex is not uncolored during Step 2 is at least $\left(1 - \frac{2}{\zeta\Delta}\right)^\Delta$. So the probability that a newly colored vertex is uncolored during Step 2 is at most:

$$1 - \left(1 - \frac{2}{\zeta\Delta}\right)^\Delta \leq 1 - \frac{3}{4}e^{-\frac{2}{\zeta}}$$

For $i \leq I$, the probability that the newly colored vertex v is uncolored during Step 3 is at most:

$$\Delta\binom{d_{i-1}}{T}\left(\frac{2d-1}{\zeta\Delta/2}\right)^{T+1} \leq \left(\frac{2(2d-1)}{\zeta\Delta}\right)^{T+1}\frac{1}{T!} \leq \frac{1}{4}e^{-\frac{2}{\zeta}}$$

Observe that for I sufficiently large in terms of ζ and d, we have

$$\begin{aligned} f_i = \frac{4(2d-1)\Delta}{\zeta}\sum_{j=I+1}^{i-1}\left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right)^j &\leq \frac{4(2d-1)\Delta}{\zeta} \times 4e^{\frac{2}{\zeta}}\left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right)^{I+1} \\ &< \frac{\zeta\Delta}{16}e^{-\frac{2}{\zeta}}. \end{aligned}$$

Thus, for $i > I$, the probability that the vertex v is uncolored during Step $3(a)$ is at most:

$$\frac{|F_v|}{|L_v|} \leq \frac{2}{\zeta\Delta}f_{i-1} < \frac{1}{8}e^{-\frac{2}{\zeta}}$$

And the probability that v is uncolored during Step $3(b)$ is at most:

$$\Delta d_{i-1}\left(\frac{2(2d-1)}{\zeta\Delta}\right)^2 \leq \left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right)^I\left(\frac{2(2d-1)}{\zeta}\right)^2 \leq \frac{1}{8}e^{-\frac{2}{\zeta}}$$

Combining these results, the probability that a newly colored vertex is uncolored during Step 2 or Step 3 is at most $1 - \frac{3}{4}e^{-\frac{2}{\zeta}} + \frac{1}{4}e^{-\frac{2}{\zeta}} = 1 - \frac{1}{2}e^{-\frac{2}{\zeta}}$. As a consequence,

$$\mathbf{E}(D_v^i) \leq \left(1 - \frac{1}{2}e^{-\frac{2}{\zeta}}\right)d_{i-1}$$

Let X_v^i be the event that $D_v^i > \left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right) d_{i-1}$. We define the random variable NF_v^i as the number of colors added to F_v during iteration i. Let Y_v^i be the event that $NF_v^i > \frac{4(2d-1)}{\zeta} d_{i-1}$. Using Lovász Local Lemma, we prove that with positive probability none of the type X or Y events occurs.

Claim 5 $\mathbf{Pr}(X_v^i) < e^{-\delta \log^2 d_{i-1}}$, *for a particular constant* $\delta > 0$.

Proof. Let v be a vertex of G. Let A be the number of neighbors of v that are uncolored during Step 2. For $i \leq I$ we define B as the number of neighbors of v that are uncolored during Step 3. For $i > I$ we define C (resp. D) as the number of neighbors of v that are uncolored during Step 3.(a) (resp. 3.(b)). Using the Simple Concentration Bound on A, Talagrand's Inequality on B and D, and Chernoff Bound on C, combined with $\mathbf{E}(D_v^i) \leq (1 - \frac{1}{2}e^{-\frac{2}{\zeta}})d_{i-1}$, we prove the following inequalities:

$$\mathbf{Pr}\left(|A - \mathbf{E}(A)| > \frac{1}{2}\log d_{i-1}\sqrt{\mathbf{E}(A+B)}\right) < 2e^{-\frac{e^{-\frac{2}{\zeta}}}{64}\log^2 d_{i-1}} \tag{3}$$

$$\mathbf{Pr}\left(|B - \mathbf{E}(B)| > \frac{1}{2}\log d_{i-1}\sqrt{\mathbf{E}(A+B)}\right) < 4e^{-\frac{e^{-\frac{2}{\zeta}}}{64(T+1)^3}\log^2 d_{i-1}} \tag{4}$$

$$\mathbf{Pr}\left(|A - \mathbf{E}(A)| > \frac{1}{3}\log d_{i-1}\sqrt{\mathbf{E}(A+C+D)}\right) < 2e^{-\frac{e^{-\frac{2}{\zeta}}}{144}\log^2 d_{i-1}} \tag{5}$$

$$\mathbf{Pr}\left(|C - \mathbf{E}(C)| > \frac{1}{3}\log d_{i-1}\sqrt{\mathbf{E}(A+C+D)}\right) < 2e^{-\frac{1}{144}\log^2 d_{i-1}} \tag{6}$$

$$\mathbf{Pr}\left(|D - \mathbf{E}(D)| > \frac{1}{3}\log d_{i-1}\sqrt{\mathbf{E}(A+C+D)}\right) < 2e^{-\frac{e^{-\frac{2}{\zeta}}}{1152}\log^2 d_{i-1}} \tag{7}$$

The proof of these results is very close from the proofs of Claims 2 and 3. Combining (3), (4), (5), (6) and (7), we obtain for T and Δ large enough :

$$\mathbf{Pr}(X_v^i) < e^{-\frac{e^{-\frac{2}{\zeta}}}{65(T+1)^3}\log^2 d_{i-1}}$$

□

Claim 6 $\mathbf{Pr}(Y_v^i) < e^{-\gamma d_{i-1}}$, *for a particular constant* $\gamma > 0$.

Proof. The probability that a neighbor u of v is assigned a color $c(u)$ such that $|c(u) - c(uv)| < d$ is $\frac{2d-1}{|L_u|} \leq \frac{2(2d-1)}{\zeta\Delta}$. Thus $\mathbf{E}(NF_v) \leq \frac{2(2d-1)}{\zeta\Delta} d_{i-1}$. Applying Talagrand's Inequality to the random variable NF_v with $c = (2d-1)^2$ and $r = 1$, we obtain :

$$\mathbf{Pr}\left(|NF_v - \mathbf{E}(NF_v)| > t\right) < 4e^{-\frac{\zeta t^2}{16(2d-1)^5 d_{i-1}}}$$

for any $t > \log d_{i-1}\sqrt{d_{i-1}}$. Taking $t = \frac{2d-1}{\zeta} d_{i-1}$, we obtain :

$$\mathbf{Pr}\left(NF_v > \frac{4(2d-1)}{\zeta} d_{i-1}\right) \leq \mathbf{Pr}\left(|NF_v - \mathbf{E}(NF_v)| > \frac{2d-1}{\zeta} d_{i-1}\right) < 4e^{-\frac{d_{i-1}}{2\zeta(2d-1)^3}}$$

□

The variable X_v^i only depends on the colors assigned to the vertices at distance at most 3 from v during iteration i, while the variable Y_v^i depends on the colors assigned to the vertices at distance at most 2 from v during iteration i. Thus, each type X or Y event is mutually independant from all but at most $2d_{i-1}^6$ other events. Using Claims 5 and 6, we have $\mathbf{Pr}(X_v^i) < \frac{1}{8d_{i-1}^6}$ and $\mathbf{Pr}(Y_v^i) < \frac{1}{8d_{i-1}^6}$ for Δ large enough (recall that according to our choice of i^* we always have $d_i \geq (1 - \frac{1}{4}e^{-\frac{2}{\zeta}})\sqrt{\Delta}$). Lovász Local Lemma completes the induction. □

3.3 The final phase

At this point, we have a partial coloring such that:

- each vertex v has at most $\sqrt{\Delta}$ uncolored neighbors;
- the reject degree of each vertex is at most $IT + 1$;
- each vertex has a list of at least $\frac{\zeta\Delta}{2}$ available colors.

It will be more convenient to use lists of equal sizes. So we arbitrarily delete colors from each list, so that for every uncolored vertex v, we have $|L_v| = \frac{\zeta\Delta}{2}$. For each uncolored vertex, we choose a subset of colors from L_v which will be *candidates* for v and we prove that with positive probability, there exists a candidate for each uncolored vertex, such that we can complete our partial coloring of G.

A candidate a for v is said to be *good* if:

Condition 1 for every neighbor u of v, a is not candidate for u;
Condition 2 for every neighbor u of v, and every neighbor w of u, there is no candidate b of w such that $|c(uv) - a| < d$ and $|c(uw) - b| < d$.

If we find a good candidate for every uncolored vertex, Condition 1 ensures that the vertex coloring obtained is proper, and Condition 2 ensures that no reject degree increases by more than one.

Claim 7 *There exists a set of candidates S_v for each uncolored vertex v, such that each set contains at least one good candidate.*

Proof. For each uncolored vertex v, we choose a random permutation of L_v, and take the first twenty colors of the list as set of candidates for v. Let C_v be the event that none of the candidates for v is a good candidate. Each event C_v depends from at most Δ^4 other events. We now show that $\mathbf{Pr}(C_v) < \frac{1}{4\Delta^4}$. Lovasz Local Lemma will complete the proof.

Let v be an uncolored vertex of G. We define:

$Bad_1 = \{c \in L_v : c \text{ is candidate for some neighbor of } v\}$
$Bad_2 = \{c \in L_v : \text{choosing } c \text{ for } v \text{ violates Condition 2}\}$
$Bad = Bad_1 \cup Bad_2$

Let D be the event that $|Bad| \leq 60(2d-1)^2\sqrt{\Delta}$. A candidate for v is good if and only if it does not belong to Bad. Observe that :

$$\mathbf{Pr}(C_v|D) \leq \left(\frac{|Bad|}{|L_v|}\right)^{20} \leq \left(\frac{60(2d-1)^2\sqrt{\Delta}}{\frac{\zeta\Delta}{2}}\right)^{20} \leq \frac{120^{20}(2d-1)^{40}}{\zeta^{20}\Delta^{10}}$$

So for Δ sufficiently large, $\mathbf{Pr}(C_v|D) < \frac{1}{8\Delta^4}$.

Each vertex has at most $\sqrt{\Delta}$ uncolored neighbors, thus $|Bad_1| \leq 20\sqrt{\Delta} \leq 20(2d-1)^2\sqrt{\Delta}$. We now show that with very high probability, the size of Bad_2 is at most $40(2d-1)^2\sqrt{\Delta}$. A color c belongs to Bad_2 if for some neighbor u of v such that $|c(uv) - c| < d$, there is a neighbor w of u and a candidate a for w such that $|c(uw) - a| < d$. Thus we obtain:

$$\mathbf{Pr}(c \in Bad_2) \leq (2d-1) \times 20\sqrt{\Delta} \times \frac{2d-1}{\frac{\zeta\Delta}{2}} \leq \frac{40(2d-1)^2}{\zeta\sqrt{\Delta}}$$

$$\mathbf{E}(|Bad_2|) \leq \frac{\zeta\Delta}{2} \times \frac{40(2d-1)^2}{\zeta\sqrt{\Delta}} \leq 20(2d-1)^2\sqrt{\Delta}$$

The random variable $|Bad_2|$ only depends on at most Δ^2 permutations of color lists of uncolored vertices at distance at most 2 from v. Moreover, exchanging two members of one of the permutations can affect $|Bad_2|$ by at most $2d-1$. If $|Bad_2| \geq s$, we can certify this by giving, for each color $\alpha \in Bad_2$, a neighbor u of v such that $|c(uv)-\alpha| < d$, as well as a neighbor w of u having a candidate a such that $|c(uw)-a| < d$. Recall that a is a candidate for w if it belongs to the first twenty positions of the permutation of L_w. So we only need to give s choices of candidates to certify that $|Bad_2| \geq s$. We apply McDiarmid's Inequality to $X = |Bad_2|$ with $n = 0$, $m = \Delta^2$, $c = 2d-1$, $r = 1$, and $t = 10(2d-1)^2\sqrt{\Delta}$:

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > 10(2d-1)^2\sqrt{\Delta} + 60(2d-1)\sqrt{\mathbf{E}(X)}\right) < 4e^{-\frac{100(2d-1)^4\Delta}{8(2d-1)^2\mathbf{E}(X)}}$$

Since $\mathbf{E}(X) \leq 20(2d-1)^2\sqrt{\Delta}$, this implies for Δ sufficiently large:

$$\mathbf{Pr}\left(|Bad_2| > 40(2d-1)^2\sqrt{\Delta}\right) < 4e^{-\frac{5}{8}\sqrt{\Delta}}$$

So for Δ large enough, $\mathbf{Pr}\left(\overline{D}\right) < \frac{1}{8\Delta^4}$. We can express the probability of C_v as $\mathbf{Pr}(C_v) = \mathbf{Pr}(C_v|D)\mathbf{Pr}(D) + \mathbf{Pr}(C_v|\overline{D})\mathbf{Pr}(\overline{D})$. And so,

$$\mathbf{Pr}(C_v) \leq \mathbf{Pr}(C_v|D) + \mathbf{Pr}(\overline{D}) < \frac{1}{4\Delta^4}$$

□

We obtain a coloring of G with maximum reject degree at most $IT+2$. So the reject graph R obtained has maximum degree at most $IT+2d+1$. We uncolor the vertices of R and recolor them greedily with the colors $\{\Delta+3d-2,\ldots,\Delta+IT+5d\}$. This final coloring is a $(d,1)$-total labelling of G. Since I and T are independant of Δ, we proved that $\lambda_d^T(G) \leq \Delta + C_{d,\varepsilon}$.

Remark 1 *By looking carefully at each inequality during the procedure, we can replace* $\Delta + C_{d,\varepsilon}$ *by* $\Delta + C_\varepsilon d\log d$*, where* C_ε *is a constant that does not depend on* d*.*

4 Further work

Theorem 5 can be transformed into a randomized algorithm, using a powerful technique introduced by Beck [1] and extended to a wide range of applications of the symmetric form of the Local Lemma by Molloy and Reed [8].

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A Probalistic tools

Simple Concentration Bound. *Let X be a random variable determined by n independent trials $T_1,\dots,T_n$ and satisfying:*

1. *Changing the outcome of any one trial can affect X by at most c.*

Then,

$$\mathbf{Pr}(|X-\mathbf{E}(X)|>t)\le 2e^{-\frac{t^2}{2c^2n}}$$

Talagrand's Inequality. *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $T_1,\dots,T_n$, and satisfying the following for some $c,r>0$:*

1. *Changing the outcome of any one trial can affect X by at most c.*
2. *For any s, if $X\ge s$ then there is a set of at most rs trials whose outcomes certify that $X\ge s$.*

Then for any $0\le t\le \mathbf{E}(X)$,

$$\mathbf{Pr}\left(|X-\mathbf{E}(X)|>t+60c\sqrt{r\mathbf{E}(X)}\right)\le 4e^{-\frac{t^2}{8c^2r\mathbf{E}(X)}}$$

McDiarmid's Inequality. *Let X be a non-negative random variable, not identically 0, which is determined by n indenpendent trials $T_1,\dots,T_n$ and m independent permutations $\Pi_1,\dots,\Pi_m$ and satisfying the following for some $c,r>0$:*

1. *Changing the outcome of any trial can affect X by at most c.*
2. *Interchanging two elements in any one permutation can affect X by at most c.*
3. *For any s, if $X\ge s$ then there is a set of at most rs choices whose outcomes certify that $X\ge s$.*

Then for any $0\le t\le \mathbf{E}(X)$,

$$\mathbf{Pr}\left(|X-\mathbf{E}(X)|>t+60c\sqrt{r\mathbf{E}(X)}\right)\le 4e^{-\frac{t^2}{8c^2r\mathbf{E}(X)}}$$

Lovász Local Lemma. *Consider a set $\mathcal{E}$ of (typically bad) events such that for each $A\in\mathcal{E}$*

1. $\mathbf{Pr}(A)\le p<1$, *and*
2. *A is mutually independent of a set of all but at most d of the other events.*

If $4pd\le 1$ then with positive probability, none of the events in $\mathcal{E}$ occur.