INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

A Γ-convergence approach for the detection of points in 2-D images.

Daniele Graziani, Laure BLANC-FÉRAUD, Gilles Aubert

N° 6978

2009

Thème COG

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--6978--FR+ENG

INRIA SOPHIA ANTIPOLIS

A Γ-convergence approach for the detection of points in 2-D images.

Daniele Graziani*, Laure BLANC-FÉRAUD*, Gilles Aubert†

Thème COG — Systèmes cognitifs
Projet ARIANA

Rapport de recherche n° 6978 — 2009 — xxii pages

Abstract: In this report we propose a new variational method to isolate points in a 2-dimensional image. To this purpose we introduce a suitable functional whose minimizers are given by the points we want to detect. Then in order to provide numerical experiments we approximate this energy by means of a sequence of more treatable functionals by using a Γ-convergence approach.

Key-words: points detection, curvature-depending functionals, divergence-measure fields, Γ-convergence, biological 2-D images.

* ARIANA Project-team, CNRS/INRIA/UNSA, 2004 Route des lucioles-BP93, 06902 Sophia-Antipolis Cedex, France
† LABORATOIRE J.A. DIEUDONNÉ Université de Nice SOPHIA ANTIPOLIS, parc valrose 06108 Nice CEDEX 2, FRANCE.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

A Γ-convergence approach for the detection of points in 2-D images.

Résumé :

Mots-clés :

Contents

1 Introduction

Detecting fine structures, like points or curves in two or three dimensional images respectively, is an important issue in image analysis. In biological images a point may represent a viral particle whose visibility is compromised by the presence of other structures like cell membranes or some noise.

From a variational point of view the problem of isolating points is a difficult task, since it is not clear how these singularities must be classified in term of a some differential operator. Indeed, whereas these are usually defined as discontinuity without jump, we cannot use the gradient operator as in the classical problem of detecting contours. As a consequence the functional framework we have to deal with may be not clear.

One possible strategy to overcome this obstacle is considering these kind of pathology as a k-codimension object, meaning that they should be regarded as a singularity of a map $U : \mathbb{R}^{k+m} \to \mathbb{R}^k$, with $k \geq 2$ and $m \geq 0$ (see [7] for a complete survey on this subject). So that the detecting point case corresponds to the case $k = 2$ and $m = 0$.

In this direction, in [5], the authors have suggested a variational approach based on the theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are regarded as the topological singularities of a map $U : \mathbb{R}^2 \to \mathbb{S}^1$, where $\mathbb{S}^1$ is a unit sphere of $\mathbb{R}^2$. So that it is crucial to construct, starting from the initial image $I : \mathbb{R}^2 \to \mathbb{R}$, an initial vector field $U_0 : \mathbb{R}^2 \to \mathbb{S}^1$ with a topological singularity of degree 1, where there is a point in the initial image I. How to do this in a rigorous way, it is a subject of a current investigation.

Here our purpose is to provide a lighter variational formulation, in which the singularity in the image is directly given in term of of a proper differential operator defined on vector fields. Another important difference is that in [5] points and curves are detected both as singularities, while in the present paper our aim is to isolate points and at same time remove any other singularities, like curves precisely, from the initial image.

In our model $\Omega \subset \mathbb{R}^2$ is an open set (the image domain), $\mathbb{B}(\Omega)$ is the σ-algebra of the all Borel sets of Ω, and the initial image $I : \mathbb{B}(\Omega) \to \mathbb{R}$ is a Radon measure.

In order to detect the singularities of the image, we have to find a functional space whose elements generate, in a suitable sense, a measure concentrated on points. Such a space is $\mathcal{DM}^p(\Omega)$, the space of the vector fields $U : \Omega \to \mathbb{R}^2$ whose distributional divergence is a Radon measure introduced in [4]. (see Subsection 2.2 for definitions and examples).

Unfortunately even if we are capable of fabricating an initial vector field U_0 (see below for such a fabrication) belonging to the space $\mathcal{DM}^p(\Omega)$, its singular set could contains several structures, we want to remove from the original image like for instance curve or points generated by some noise. Hence, after the initialization we have to clear away all the structures we are not interested in by building up, starting from the initial data U_0, a new vector field U whose singularities are given by the points of the image I we want to isolate.

Thus, from one hand we have to force the concentration set of the divergence measure of U_0 to contain only the points we want to catch, and on the other hand we have to regularize the initial data U_0 outside the points of singularities. To this end we propose to minimize an energy involving a competition between a divergence term and the counting Hausdorff measure $\mathcal{H}^0$. More precisely the energy is the following

$$\int_{\Omega\setminus P} |\mathrm{div} U|^2 + \lambda \int_{\Omega} |U - U_0|^p + \mathcal{H}^0(P). \tag{1}$$

where $U \in L^{p,2}(\mathrm{div};\Omega\setminus P)$ the space of the L^p vector field whose distributional divergence belongs to $L^2(\Omega\setminus P)$, P is the atomic set we want to target and λ is a positive weight. The first integral forces U to be regular outside P, while the term $\mathcal{H}^0(P)$ penalizes the presence of the singular curves and the noise in the image.

From a practical point of view, this choice allows us to handle with a first order differential operator and permits us to formulate the minimization problem in a common functional framework.

Clearly to initialize the minimization process we need of a vector field U_0 linked to the initial image I, belonging to the space $\mathcal{DM}^p(\Omega)$. Such a vector field can be provided by the gradient of weak solution of the classical Dirichlet problem with measure data.

$$\begin{cases} \Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In section 3 we recall the basic results of the theory of the elliptic equations with data measure.

In order to provide computer examples, we must approximate the functional (1) by means of a sequence of more convenient functionals from a numerical point of view.

The approximation, we suggest in this paper, is based on the so called Γ-convergence, the notion of variational convergence introduced by De Giorgi (see [13, 14]). This theory is designed to approximate a variational problem by a sequence of different variational problems. The most important feature of the Γ-convergence relies on the fact that it implies the convergence of minimizers of the approximating functionals to those of the limiting functional. In this work we provide a possible Γ-convergence approach for the detection of points.

Classically, in image analysis, the Γ-convergence has been used to deal with 1-dimensional objects like smooth boundaries. For instance in ([2, 3]) Ambrosio and Tortorelli have proven that the Mumford-Shah functional can be approximated by a sequence of elliptic functionals that are numerically more treatable.

The main difficulty here is related to the presence of a codimension 2 object, which is not a contour: the set P. In order to obtain a variational approximation close to the one provided in ([2, 3]), the crucial step is then to replace the term $H^0(P)$ of a functional (1) by a more handy, from a variational point of view, functional involving a smooth boundary and his perimeter measure given by the 1-dimensional Hausdorff measure $\mathcal{H}^1$. Following some suggestion from the paper of Braides and Malchiodi and Braides and March (see [9, 10]) such a functional is given by:

$$G_{\beta_\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \big(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(x)\big) d\mathcal{H}^1(x);$$

where D is a proper regular set containing the atomic set P, κ is the curvature of its boundary, the constant $\frac{1}{4\pi}$ is a normalization factor, and β_ε infinitesimal as $\varepsilon \to 0$. Roughly speaking the minimum of this functionals are achieved on the union of balls of small radius, so that when $\beta_\varepsilon \to 0$ the functionals shrinks to the atomic measure $\mathcal{H}^0(P)$. On the other hand this approach requires a non trivial and convenient, for a numerical point of view, approximation a curvature-dependent functional. Such an approximation is based on a celebrated conjecture due to De Giorgi (see [12]. By means of this argument it is possible to substitute the curvature-depending functional with an integral functional involving the Laplacian operator of smooth functions. Then it remains to approximate the $\mathcal{H}^1$ measure and this can be done by retrieving a classical gradient approach used in [15, 16]. This strategy allows to deal with a functional whose Euler-Lagrange equations can be discretized. A simple and intuitive explanation of the construction of the complete approximating functionals will be given in section 3.

The paper is organized as follows: section 1 is intended to remind the reader of mathematical tools useful in the following. In section 2 we recall the notion of Γ-convergence and we illustrate the two Γ-convergence results we need in the sequel. In section 3 we address the construction of the approximating functionals. In section 4 we present the discrete model. In section 5 we explain the detection strategy. Finally in the last section we give some computer examples.

2 Preliminaries

2.1 Measure theory

We start by classical definitions of measure theory. For a general survey on measure theory we refer, among others, to [1].

Definition 2.1. *Let (X, Σ) be a measure space and $\mu : \Sigma \to [0, \infty)$. We say that μ is a positive measure if $\mu(\emptyset) = 0$ and for any sequence $\{E_n\}$ of pairwise disjoints elements of Σ*

$$\mu(\bigcup_{n=0}^{\infty} E_n) = \sum_{n=0}^{\infty} \mu(E_n).$$

We say that μ is finite if $\mu(X) < +\infty$ and that μ is σ-finite if X is the union of increasing sequence of sets with finite measures.

Definition 2.2. *Let (X, Σ) be a measure space and let $N \in \mathbb{N}$, $N \geq 1$.*

(*i*) *We say that $\mu : \Sigma \to \mathbb{R}^N$ is a measure, if $\mu(\emptyset) = 0$ and for any sequence $\{E_n\}$ of pairwise disjoints elements of the σ-algebra Σ*

$$\mu(\bigcup_{n=0}^{\infty} E_n) = \sum_{n=0}^{\infty} \mu(E_n).$$

If $N = 1$ we say that μ is a real measure, if $N > 1$ we say that μ is a vector measure.

(*ii*) *If μ is a measure, we define its total variation $|\mu|$ as follows*

$$\forall E \in \Sigma \quad |\mu|(E) := \sup\{\sum_{n=0}^{\infty} |\mu(E_n)| : \ E_n \in \Sigma \text{ pairwise disjoints, } E = \bigcup_{n=0}^{\infty} E_n\}.$$

(*iii*) *We say that μ is finite is $|\mu|(X) < +\infty$.*

It can be proven that the total variation $|\mu|$ is a positive measure, more precisely, it is the smallest positive measure such that $|\mu|(E) \geq |\mu(E)|$ for every $E \in \Sigma$. Clearly, if μ is a positive measure, then $|\mu| = \mu$.

An important class of measures is the Radon measures. In order to introduce such a concept, the notion of Borel set is necessary.

Definition 2.3. *Let X be a locally compact and separable metric space. Let $\mathcal{A}(X)$ be the collection of all open sets of X. We call the Borel σ-algebra on X the smallest σ-algebra containing $\mathcal{A}(X)$ and we denoted it by $\mathbb{B}(X)$. We call its elements the Borel sets of X.*

Definition 2.4. *Let X be a locally compact and separable metric space and $\mathbb{B}(X)$ be the Borel σ-algebra of X. Consider the measure space $(X, \mathbb{B}(X))$.*

(*i*) *A positive measure on* $(X, \mathbb{B}(X))$ *is a Borel measure.*

(*ii*) *A positive measure on each compact subset* K *of* X *will be called a positive Radon measure. If* μ *is an* $\mathbb{R}^N$*-valued measure defined on all the Borel subset of* X *s.t.* $|\mu|$ *is a Radon measure and* $|\mu|(X) < +\infty$*, we say that* μ *is an* $\mathbb{R}^N$*-valued finite Radon measure. We denote by* $\mathcal{M}(X;\mathbb{R}^N)$ *the space of the all* $\mathbb{R}^N$*-valued finite Radon measures.*

2.2 Distributional divergence

In this subsection we recall the definition of the space $L^{p,q}(\mathrm{div};\Omega)$ and $\mathcal{DM}^p(\Omega)$, introduced in [4].

Let $\Omega \subset \mathbb{R}^2$ be an open set and let $U : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field.

Definition 2.5. *We say that* $U \in L^{p,q}(\mathrm{div};\Omega)$ *if* $U \in L^p(\Omega;\mathbb{R}^2)$ *and if its distributional divergence* $\mathrm{div}U \in L^q(\Omega)$*. If* $p = q$ *the space* $L^{p,q}(\mathrm{div};\Omega)$ *will be denoted by* $L^p(\mathrm{div};\Omega)$*.*

If $p = q = 2$ *the space* $L^2(\mathrm{div};\Omega)$ *equipped with the scalar product*

$$(U, V) = \int_\Omega U \cdot V dxdy + \int_\Omega \mathrm{div}U \mathrm{div}V dxdy;$$

is an Hilbert space.

We say that $U \in L^{p,q}_{loc}(\mathrm{div};\Omega)$ *if* $U \in L^{p,q}(\mathrm{div};A)$ *for every open set* $A \subset\subset \Omega$*.*

Definition 2.6. *For* $U \in L^p(\Omega;\mathbb{R}^2)$*,* $1 \le p \le +\infty$*, set*

$$|\mathrm{divU}|(\Omega) := \sup\{\langle U, \nabla\varphi\rangle \ : \ \varphi \in C^1_0(\Omega), \ |\varphi| \le 1\}.$$

We say that U *is an* L^p*-divergence measure field, i.e.* $U \in \mathcal{DM}^p(\Omega)$ *if*

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega;\mathbb{R}^2)} + |\mathrm{divU}|(\Omega) < +\infty.$$

We say that $U \in \mathcal{DM}^p_{loc}(\Omega)$ *if* $U \in \mathcal{DM}^p(A)$ *for every open set* $A \subset\subset \Omega$*.*

Remark 2.1. *If* $U \in \mathcal{DM}^p(\Omega)$ *then via Riesz Theorem it is possible to represent the distributional divergence of* U *by a Radon measure. More precisely there exists a Radon measure* μ *such that for every* $\varphi \in C^1_0(\Omega)$ *the following equality holds:*

$$\int_\Omega U \cdot \nabla\varphi dxdy = -\int_\Omega \varphi d\mu.$$

For instance the field $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ *belongs to* $\mathcal{DM}^1_{loc}(\mathbb{R}^2)$ *and its divergence measure is given by* $-2\pi\delta_0$*, where* δ_0 *is the the Dirac mass.*

Such a result can be proven by approximation. Let us define the following map:

$$U_\varepsilon(x, y) := \begin{cases} U(x, y) & \text{if } |x| \ge \varepsilon \\ (\frac{x}{\varepsilon^2}, \frac{y}{\varepsilon^2}) & \text{if } |x| < \varepsilon. \end{cases}$$

It is not difficult to check that u_ε is Lipschitz-map with divergence given by $\frac{2}{\varepsilon^2}\chi_{B(0,\varepsilon)}$.

Then for every test function $\varphi \in C_0^1(\mathbb{R}^2)$ we have

$$\int U_\varepsilon \cdot \nabla\varphi dxdy = -\int \frac{2}{\varepsilon^2}\chi_{B(0,\varepsilon)}\varphi dxdy.$$

By applying the changing variable $x = \frac{x_1}{\varepsilon}$, $y = \frac{y_1}{\varepsilon}$ we obtain

$$\int U_\varepsilon \cdot \nabla\varphi dxdy = -2\int \chi_{B(0,1)}\varphi(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon})dx_1dy_1,$$

so that, letting $\varepsilon \to 0$, by dominated convergence theorem we obtain

$$\int_\Omega U \cdot \nabla\varphi dxdy = -2\pi\varphi(0,0) = -2\pi\int_\Omega \varphi d\delta_0$$

3 The Dirichlet problem with measure data

For the initialization of our algorithm we must built a vector field U_0 which should be such that its divergence is singular on points of the image I. Therefore we will use the gradient of the solution of the following Dirichlet problem (applied with $\mu = I$)

$$\begin{cases} \Delta f = \mu & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where μ is a Radon measure

The study of a PDE with data measure is a very classical research topic in Analysis and there is a vast literature about this subject. In what follows we recall just some basic definitions and crucial results for our purpose.

Definition 3.1. *We say that f is a weak solution of the (3) if $f \in W_0^{1,1}(\Omega)$ and the following equality holds for every test $\varphi \in C_0^1(\Omega)$:*

$$\int_\Omega \nabla f \cdot \nabla \varphi = - \int_\Omega \varphi d\mu.$$

Note that if f is a weak solution, then its weak gradient is a divergence measure field, i.e. $U = \nabla f \in \mathcal{DM}^p(\Omega)$, and its divergence is equal to the measure data μ.

A classical example of a weak solution is provided by a fundamental solution of the Laplace equation. Let us consider the problem:

$$\begin{cases} \Delta f = \delta_0 & \text{on } \Omega \\ f = 0 & \partial\Omega. \end{cases} \tag{4}$$

Then, by the same argument used in Remark 2.1, one can easily check that the distributional divergence of ∇f is given by δ_0.

More generally classical results (see [17]) guarantee the existence of a unique solution of the problem (3). Concerning the regularity it is known then $f \in W^{1,p}$ with $p < 2$.

4 Γ-convergence

The key point of our strategy is the approximation of the functional (1) by means of more regular functionals via Γ-convergence.

Therefore this section is devoted to a very simple presentation of the notion of Γ convergence and the two crucial results for our goal: the Modica-Mortola's theorem (see [15, 16]) concerning the approximation of the perimeter measure and the De Giorgi conjecture (see [12]) about the approximation of curvature depending functionals.

We start by recalling the definition of the Γ-convergence just in the special case we need of. For a complete survey on this subject we refer to [8, 11] and references therein. In what follows X stands for a metric space (X, d)

Definition 4.1. *We say that a sequence $F_n : X \to [-\infty, +\infty]$ Γ-converges to a functional $F : X \to [-\infty, +\infty]$ as $(n \to +\infty)$, with respect to the X-distance, if the following two properties hold:*
for every $u \in X$ and every $\{u_n\} \in X$, such that $u_n \to u$ with respect to X-distance,

$$F(u) \leq \liminf_{n \to +\infty} F_n(u_n), \tag{5}$$

for every $u \in X$ there exists $\{\overline{u}_n\} \subset X$, such that $\overline{u}_n \to u$ with respect to X-distance

$$F(u) = \lim_{n \to +\infty} F_n(\overline{u}_n). \tag{6}$$

The functional F is called the Γ-limit of F (with respect to the X-distance) and we write $F = \Gamma - \lim_n F_n$.

The relevance of this notion is explained by the following fundamental theorem

Theorem 4.1. *Let us suppose that $F = \Gamma - \lim_n F_n$ and let a compact set $K \subset X$ exist such that $\inf_X F_n = \inf_K F_n$ for all n. Then*

$$\min_X F = \lim_n \inf_X F_n$$

.

Moreover, if u_n is a converging sequence such that $\lim_{n \to +\infty} F_\varepsilon(u_n) = \lim_{n \to \infty} \inf_X F_n$, then its limit is a minimum point for F.

It is possible to extend the definition of Γ-convergence to a family depending on a real a parameter. So that we can deal with Γ-limits of families $\{F_\varepsilon\}$ as $\varepsilon \to 0$.

Definition 4.2. *We say that a sequence $F_\varepsilon : X \to [-\infty, +\infty]$ Γ-converges to a functional $F : X \to [-\infty, +\infty]$ as $(\varepsilon \to 0)$, if for every sequence $\{\varepsilon_n\}$ converging to 0, as $n \to +\infty$, the sequence F_{ε_n} Γ-converges to F.*

4.1 Modica Mortola's approach

The Modica-Mortola theorem states that it is possible to approximate, in the Γ-convergence sense, a perimeter measure by means of the following sequence of functionals

$$F^1_\varepsilon(u) := \begin{cases} \int_\Omega \left(\varepsilon|\nabla u|^2 + \frac{V(u)}{\varepsilon}\right)dx & \text{if } u \in W^{1,2}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $V(u) = u^2(1-u)^2$ is a double well potential. Besides in order to avoid that the minimizers of the functional F^1_ε are trivial, some constraint on the functions u_ε must be added. In the original version of the Modica-Mortola's paper a volume constrained of the type $\int_\Omega u dx dy = m$, was assumed.

Let us give an intuitive explanation of such a result. Since V has two absolute minimizers at $u = 0, 1$, when ε is small, a local minimizer u_ε is closed to 1 on a part of Ω and close to 0 on the other part, making a rapid transition of order ε between 0 and 1. When $\varepsilon \to 0$ the transition set shrinks to a set of dimension 1, so that u_ε goes to a function taking values u into $\{0, 1\}$ and the functionals converge to the measure of the perimeter of the discontinuities set of u.

By the way the Modica-Mortola's theorem still holds even without assuming the volume constrain. It means that in general the Γ-limit could be given by a functional identically equal to 0. The Modica-Mortola's Theorem is the following.

Theorem 4.2. *The functionals $F^1_\varepsilon : L^1(\Omega) \to [0, +\infty]$ $\Gamma-$converge with respect to the L^1-distance to the following functional*

$$F^1(u) = \begin{cases} C_V \mathcal{H}^1(S_u) & \text{if } u \in \{0,1\} \\ +\infty & \text{otherwise} \end{cases}$$

where, as usual, S_u denotes the set of the discontinuities of u and C_V is a suitable constant depending on the potential V.

4.2 De Giorgi's conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending functional of the type:

$$F^2(D) = \int_{\partial D} (1 + \kappa^2) d\mathcal{H}^1;$$

where D is a regular set and k is a curvature of its boundary ∂D.

Since ∂D can be represented as the discontinuity set of the function $u_0 = 1 - \chi_D$, by Modica-Mortola's Theorem it follows that there is a sequence of non constant local minimizers such that $u_\varepsilon \to u_0$ with respect to the L^1-convergence such that

$$\lim_{\varepsilon \to 0} F^1_\varepsilon(u_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

Furthermore looking at the Euler-Lagrange equation associated to a contour length term, yields a contour curvature term κ, while the Euler-Lagrange equations for the functional $F_\varepsilon^1(u)$ contains a term $2\varepsilon\Delta u - \frac{V^{'}(u)}{\varepsilon}$.

Then De Giorgi suggested to approximate the functional F^2 by adding to the Modica-Mortola approximating functionals the following term

$$F_\varepsilon^2(u) = \int_\Omega (2\varepsilon\Delta u - \frac{V^{'}(u)}{\varepsilon})^2 dx.$$

In ([6]) Bellettini and Paolini proved the lim sup inequality (6), while the validity of lim inf inequality (5) remains an open problem.

5 The approximating functionals

In this section we present the energy we deal with and the construction of the approximating sequence.

The energy we are interested in is given by

$$\int_{\Omega\setminus P}|\mathrm{div}U|^2+\lambda\int_\Omega|U-U_0|^p+\mathcal{H}^0(P).$$

where $U\in L^{p,2}(\mathrm{div};\Omega\setminus P)$, $U_0\in\mathcal{DM}^p_{loc}(\mathbb{R}^2)$ and finally P is an atomic set consisting of a finite number N of points, i.e. $P=\{x_1,...,x_N\}$.

As pointed out in the introduction, the first step is to substitute the counting measure $\mathcal{H}^0(P)$ with a more treatable term given by:

$$G_{\beta_\varepsilon}(D)=\frac{1}{4\pi}\int_{\partial D}\big(\frac{1}{\beta_\varepsilon}+\beta_\varepsilon\kappa^2(x)\big)d\mathcal{H}^1(x);$$

where D is an union of regular simply connected sets $\{D_i\}$ with $i=1,..,N$, such that $x_i\in D_i$, $D_i\bigcap D_j=\emptyset$ for $i\neq j$. κ is the curvature of the boundary of the sets D_i, the constant $\frac{1}{4\pi}$ is a normalization factor and β_ε is infinitesimal as $\varepsilon\to 0$.

To understand why we can approximate $\mathcal{H}^0(P)$ with $G_{\beta_\varepsilon}(D)$ one should note that the solution of the following minimum problem

$$\min_{D\supset P}G_{\beta_\varepsilon}(D) \tag{7}$$

is given by $D=\bigcup_i^N B(x_i,\beta_\varepsilon)$, where x_i are the points of P. We sketch the argument in the single point case.

By the Young's inequality we have

$$G_{\beta_\varepsilon}(D)\geq\frac{1}{4\pi}\int_{\partial D}2\kappa d\mathcal{H}^1(x)$$

and by applying the Gauss-Bonnet Theorem

$$G_{\beta_\varepsilon}(D)\geq\frac{1}{4\pi}(2)(2\pi)=1=H^0(P).$$

Finally a simple calculation shows that, if we evaluate the functional G_{β_ε} on $B(x_1,\beta_\varepsilon)$, we obtain the value 1, i.e. the number of points in P, i.e. $\mathcal{H}^0(P)$. The N points case can be recovered with minor changes by the same argument.

For what follows it is convenient to split the functional G_{β_ε} in two terms:

$$G_{\beta_\varepsilon}(D)=G^1_{\beta\varepsilon}(D)+G^2_{\beta\varepsilon}(D)$$

where

$$G^1_{\beta_\varepsilon}(D) = \frac{1}{4\pi}\int_{\partial D}\frac{1}{\beta\varepsilon}d\mathcal{H}^1(x);$$

and

$$G^2_{\beta_\varepsilon}(D) := \frac{1}{4\pi}\int_{\partial D}\beta_\varepsilon\kappa^2(x)d\mathcal{H}^1(x).$$

We can write an intermediate approximation of the energy (1):

$$E_\varepsilon(U,D) = G^1_{\beta_\varepsilon}(D) + G^2_{\beta_\varepsilon}(D) + \int_\Omega(1-\chi_D)|\mathrm{div}(U)|^2 + \lambda\int_\Omega|U-U_0|^p. \tag{8}$$

The advantage of such a formulation is that we know how to provide a variational approximation of the perimeter measure $H^1\lfloor\partial D$. Following the Modica-Mortola's approach such an approximation can be obtained by using the following measure:

$$\mu_\varepsilon(w,\nabla w)dx = \big(\varepsilon|\nabla w|^2 + \frac{V(w)}{\varepsilon}\big)dx,$$

where $V(w) = w^2(1-w)^2$ is a double well functional.

Next step is expressing the curvature term by means of the function w. Thanks to the De Giorgi's conjecture we can replace the term κ by the term $2\varepsilon\Delta w - \frac{V'(w)}{\varepsilon}$.

So that we can formally write the complete approximating functional:

$$\begin{aligned}\Phi_\varepsilon(U,w): &= \int_\Omega w^2|\mathrm{div}(U)|^2dx + \frac{1}{4\pi}\int_\Omega\beta_\varepsilon(2\varepsilon\Delta w - \frac{V'(w)}{\varepsilon})^2dx + \frac{1}{\beta_\varepsilon}\int_\Omega\mu_\varepsilon(w,\nabla w)dx\\ &+ \lambda\int_\Omega|U-U_0|^pdx,\end{aligned} \tag{9}$$

where $U \in L^{p,2}(\mathrm{div};\Omega)$ and w is smooth function equal to 1 on the boundary, i.e. $1-w \in C_0^\infty(\Omega)$. We add the volume constraint $\frac{1}{\mathcal{L}^2(\Omega)}\int_\Omega wdx = 1$. This last condition prevents w from converging to the function constantly equal to 0 as $\varepsilon \to 0$.

Then if $(U_\varepsilon, w_\varepsilon)$ is a minimizing sequence of Φ_ε, then w_ε must be very close to the values 1 when ε goes to 0, since the double well potential is positive except for $w_\varepsilon = 0,1$ and w must be equal to 1 on the $\partial\Omega$. On the other hand, near the points where the divergence is very big w_ε must be close to 0. Besides when $\varepsilon \to 0$, $\beta_\varepsilon \to 0$ goes to 0 as well, so that the singular set D is given by an union of balls of a small radius β_ε.

Therefore, while the functions U_ε approximate a minimizer U of the original functional, the level set $\{w_\varepsilon = 0\}$ approximate the original singular set P.

The first variation of this functional leads to the following gradient flow system

$$\begin{aligned}\frac{\partial U}{\partial t} &= \nabla(w^2\mathrm{div}U) - \lambda p|U-U_0|^{p-2}(U-U_0)\\ \frac{\partial w}{\partial t} &= -4\frac{\Delta h}{\beta_\varepsilon} + \beta_\varepsilon h + \frac{2}{\varepsilon^2}\frac{1}{\beta_\varepsilon}V''(w)h - 2w|\mathrm{div}U|^2\end{aligned} \tag{10}$$

where h is given by the equation

$$h = 2\varepsilon \Delta w - \frac{1}{\varepsilon} V'(w).$$

6 Detection

In our model the image contains an atomic Radon measure. Thus, in order to find an initial vector field which copies the singularities of the initial image, we use the gradient of the solution of the following Dirichlet problem:

$$\begin{cases} \Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

In this way we obtain a vector field whose divergence is singular on a proper set which contains the points we want to detect. In general this set could contain other structures. For instance if the initial image is a Radon measure concentrated both on points and on curves, the divergence of ∇f is singular on points and on curves at the same time. Besides if there is some noise in the image, it could be not clear how to differentiate the singular points due to the noise, from those we want to catch. As a consequence, by solving the problem (11), we get just a predetection, which has to be refined.

To this purpose we search for a minimizer of the energy $\Phi_\varepsilon(U, w)$ via solving equations (10) with initial data U_0 given by ∇f. So that we obtain a vector field U whose divergence is relevant only on the set P and a function w whose zeros are given by the set P.

6.1 Discretization

The image is an $N \times N$ vector. We endowed the space R^{2N} with the standard scalar product and standard norm.

The discrete version of gradient operator is the following.

Let $I \in \mathbb{R}^{2N}$. Then the gradient ∇I is an element of the space $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$ given by:

$$(\nabla I)_{i,j} = ((\nabla I)^1_{i,j}, (\nabla I)^2_{i,j})$$

where

$$(\nabla I)^1_{i,j} = \begin{cases} I_{i+1,j} - I_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N, \end{cases}$$

$$(\nabla I)^2_{i,j} = \begin{cases} I_{i,j+1} - I_{i,j} & \text{if } j < N \\ 0 & \text{if } j = 0. \end{cases}$$

We also introduce the discrete version of the divergence operator simply defined as the adjoint operator of the gradient: $\text{div} = -\nabla^*$. More in details if $v \in \mathbb{R}^{2N}$ we have

$$(\mathrm{div}v)_{i,j} = \begin{cases} v^1_{i,j} + v^2_{i,j} & \text{if } i,j = 1 \\ v^1_{i,j} + v^2_{i,j} - v^2_{i-1,j} & \text{if } i = 1, 1 < j < N \\ v^1_{i,j} - v^1_{i-1,j} + v^2_{i,j} - v^2_{i-1,j} & \text{if } 1 < i < N, 1 < j < N \\ -v^1_{i-1,j} + v^2_{i,j} - v^2_{i-1,j} & \text{if } i = N, 1 < j < N \\ v^1_{i,j} - v^1_{i-1,j} + v^2_{i,j} & \text{if } 1 < i < N, j = 1 \\ v^1_{i,j} - v^1_{i-1,j} - v^2_{i-1,j} & \text{if } 1 < i < N, j = N \\ -(v^1_{i-1,j} + v^2_{i-1,j}) & \text{if } i,j = N. \end{cases}$$

Then we can define the discrete version of the Laplacian operator as $\Delta I = \mathrm{div}(\nabla I)$.

6.2 Discretization in time

We simply replace $\frac{\partial U}{\partial t}$ and $\frac{\partial w}{\partial t}$ by $\frac{U^{n+1}_{i,j} - U^n_{i,j}}{\delta t}$ and $\frac{w^{n+1}_{i,j} - w^n_{i,j}}{\delta t}$ respectively. Then we write the system (10) in the form (for simplicity we omit the dependance on ε)

$$\begin{cases} U_1^{n+1} = & \delta t \Phi_{U_1}(U_n, w_n) \\ U_2^{n+1} = & \delta t \Phi_{U_2}(U_n, w_n) \\ w^{n+1} = & \delta t \Phi_w(U_n, w_n). \end{cases}$$

We initialize our algorithm with $U(0) = \nabla f$, where f is the solution of the problem (11). To do this, we need to solve the Dirichlet problem (11) with measure data I, therefore we regularize the image by convolution with a Gaussian kernel G_σ with very small σ and then we solve, by classical finite differences method, the problem:

$$\begin{cases} \Delta f = I_\sigma & \text{on } \Omega \\ f = 0 & \partial\Omega, \end{cases} \tag{12}$$

where $I_\sigma = I * G_\sigma$.

To initialize our algorithm, we need of an initial guess on w. So we choose $w(0) = 1$.

7 Computer examples

7.1 Parameter settings

Before running our algorithm all the parameters have to be fixed. The most important are ε and β_ε , which govern the set D approximating points we want to detect. Those parameters are related by the condition $\lim\limits_{\varepsilon\to 0} \dfrac{\varepsilon}{\beta_\varepsilon} = 0$. Furthermore, since the mesh gride size is 1 and β_ε gives the radius of a balls centered in the singular point we want to detect, from a discrete point of view the smallest value we can take is $\frac{\sqrt{2}}{2}$. Then we use values ranging from 0.1 and 0.5 for ε, and from 1.2 to 0.7 for β_ε.

Concerning the parameter λ we mainly used the small value $\lambda = 0.1$, in order to force the algorithm to regularize as much as possible the initial data U_0.

Since we deal with small values of ε, in order to have some stability, we must take a small discretization time step, Practically we mainly used the value $\delta t = 1 \times 10^{-8}$.

Concerning the stop criterion we iterate the algorithm until $\max \left\{ \frac{|U_1^{n+1} - U_2^n|}{|U_1^n|}, \frac{|U_2^{n+1} - U_2^n|}{|U_2^n|}, \frac{|w^{n+1} - w^n|}{|w^n|} \right\}$ ≤ 0.001. Finally we set $p = 1.5$.

In all the computer examples the points are detected by means of the function w_ε. We display the level-set $\{w_\varepsilon \simeq 0\}$.

7.2 Commentaries

The figure 1 shows how resistant to the noise our model is. When the noise is larger the parameter ε must be as closer as possible to the ideal value 0. Besides it is possible to see that for small values of ε our detection is finer according to the continuous setting. More in details in the first row we display the initial image obtained by adding a gaussian noise to a binary image of five points. The increasing level of the noise corresponds to the increasing $PSNR$ between the binary image and the noisy image. The second row shows the behavior of w for intermediate value of the parameters ε and β_ε. In the last row one can note that the smaller ε and β_ε are the finer the detection is.

In figure 2 in the first two rows we display the histograms of the gray level of I and w corresponding to the different values of PSNR. One can see that it is easier fixing a threshold value starting from the function w_ε than from the initial image I. In the last row we display the set $\{w_\varepsilon \simeq 0\}$ obtained by plotting the set $\{w_\varepsilon \leq \alpha\}$ with threshold value $\alpha = 0.5$

In figure 3 we test our algorithm on curves and points at the same time. In the first column we have a sequence of points and a curve with no boundary on Ω. In the second column we consider the same curve with boundary in Ω. The second column we display the function w_ε and in the last column the level set $\{w_\varepsilon \cong 0\}$ once again obtained by fixing a threshold value $\alpha = 0.5$. The result is that, as desired, our algorithm is capable of eliminating the curve from the initial image. According to the continuous setting when ε takes values close to 0 the approximating energy (9) behaves similarly to the limit energy (1), so that the presence of the curve is penalized in the minimization process. Then the set $\{w_\varepsilon \cong 0\}$ contains nothing else but points.

Finally in figure 4 we deal with a biological image. Our task is catching the finest structure present in the image. In both cases of figure 4 the isolated point is well detected, while the branch and the dashes of the cellule is not.

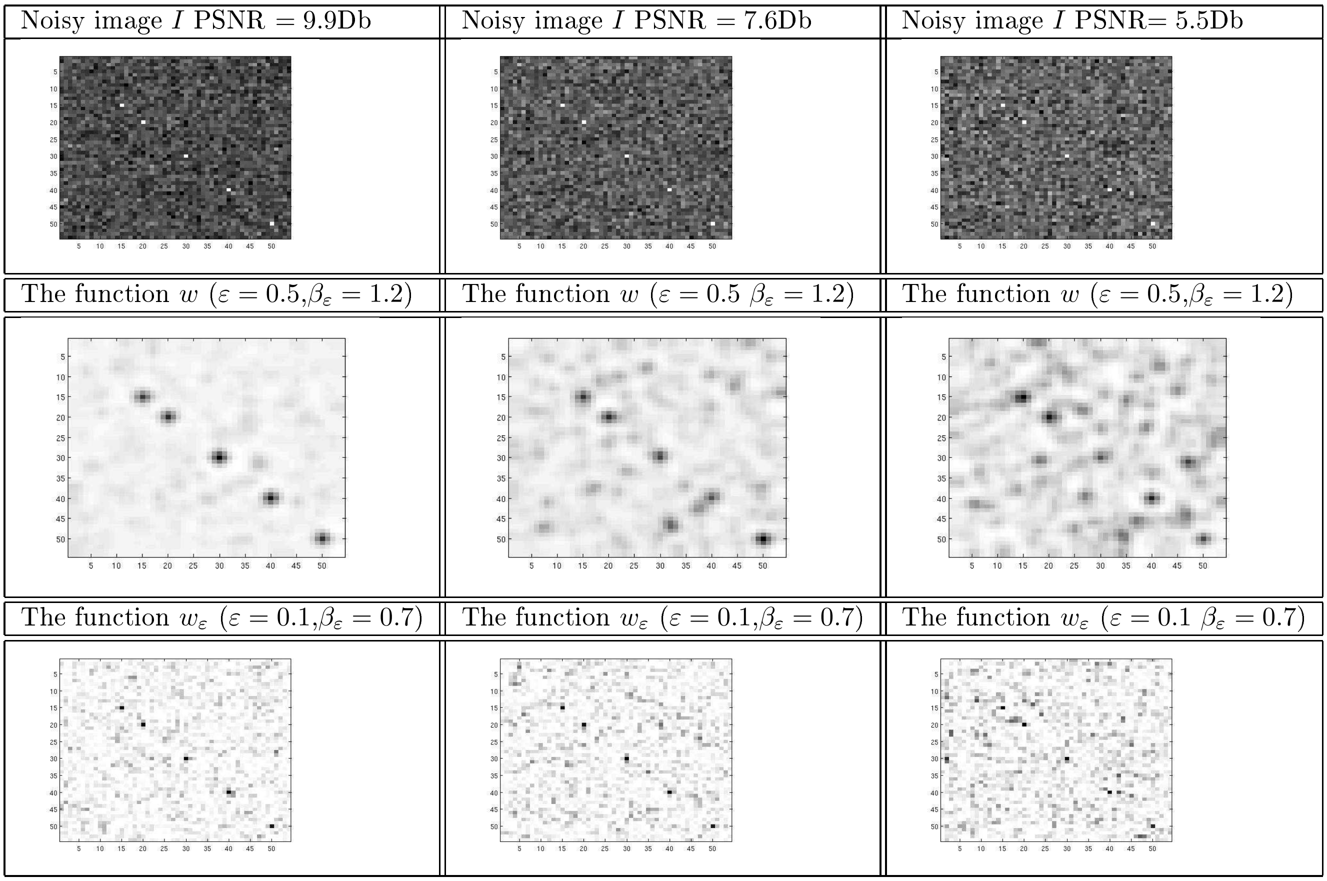

Figure 1: *Synthetic image: we test our algorithm on noisy images. When the parameters ε and β_ε are small as much as possible the detection is finer.*

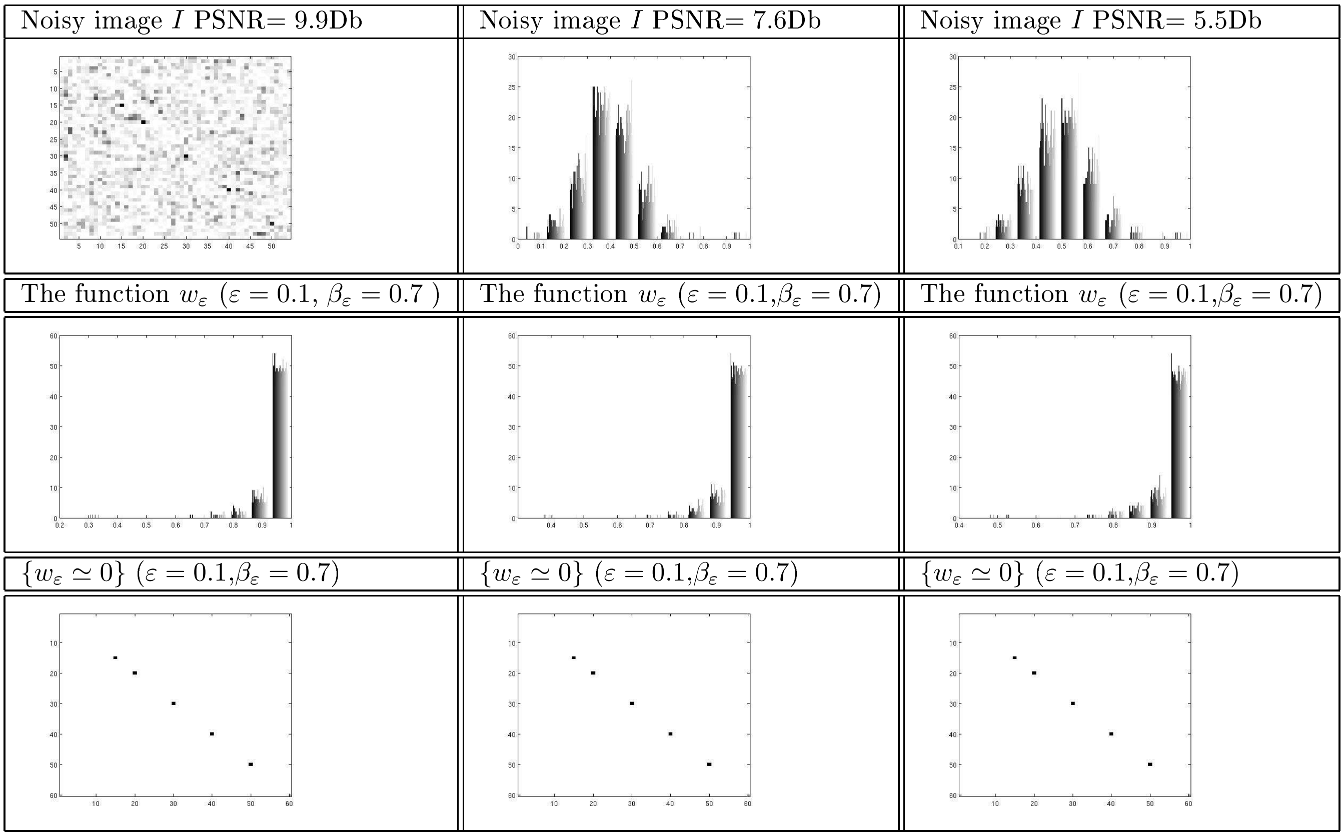

Figure 2: *Synthetic images: The detection is refined by fixing a threshold value $\alpha = 0.5$ for the function w_ε.*

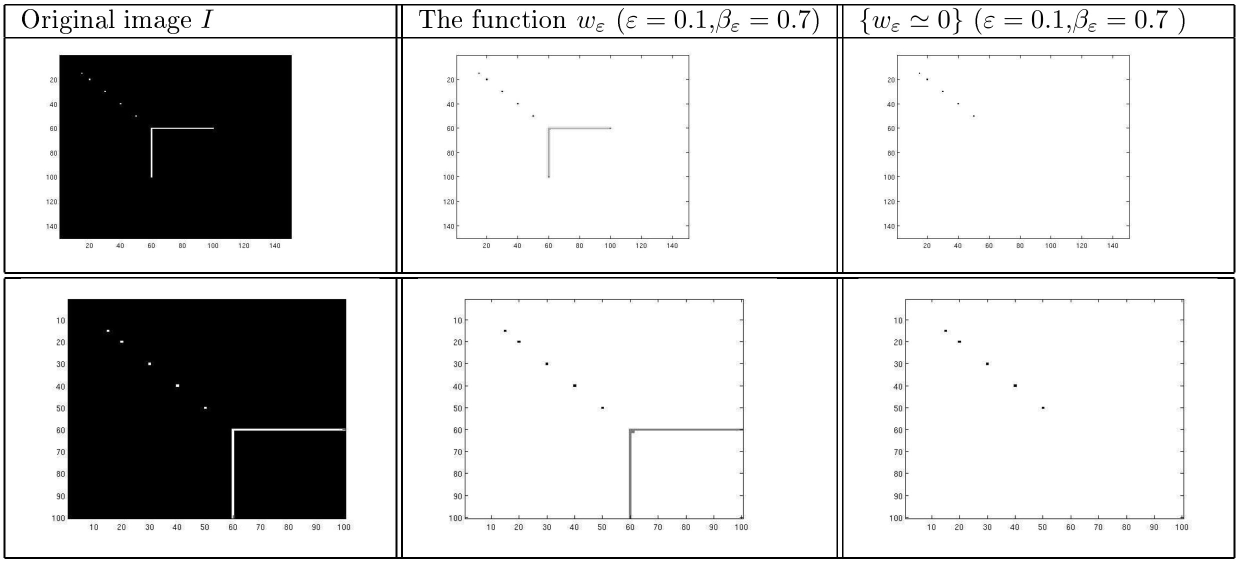

Figure 3: *Synthetic image: curve and points are present in the initial image. As expected our method is capable of removing the curve from the image.*

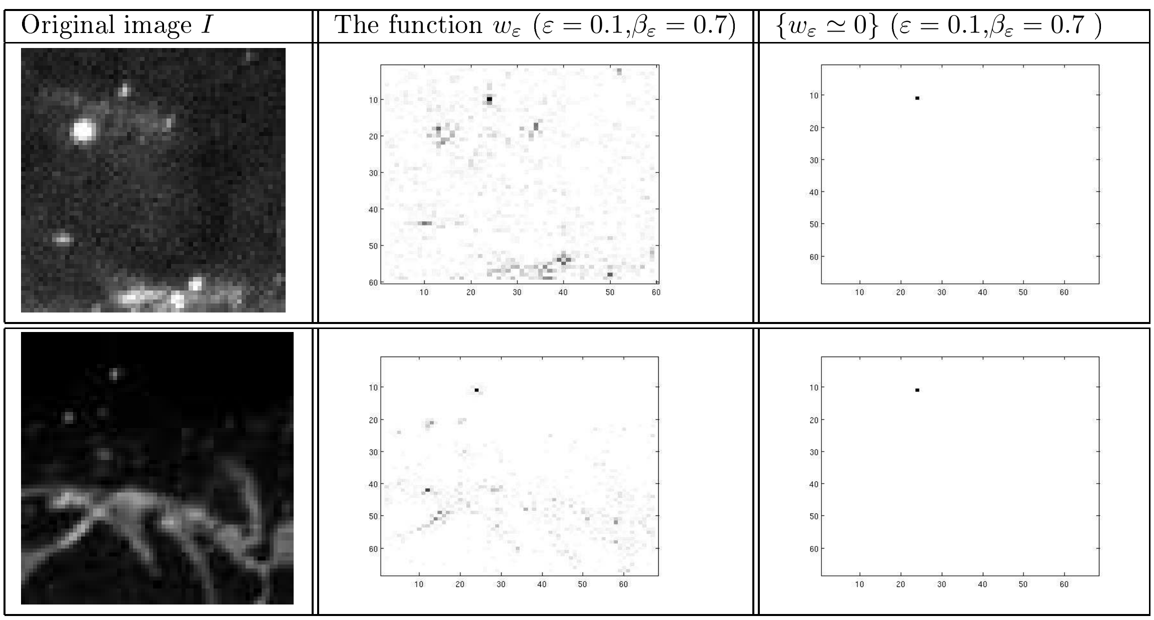

Figure 4: *Biological image: one isolated point, dashes and filament are present in the initial image. After the algorithm process we keep nothing else but the isolated point .*

References

[1] L.Ambrosio, N.Fusco, D.Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford University Press (2000).

[2] L.Ambrosio, V.M.Tortorelli. *Approximation of functionals depending on jumps by elliptic functionals via* Γ*-convergence.* Communic. Pure Appl. Math **43** (1990), 999-1036.

[3] L.Ambrosio, V.M.Tortorelli. *Approximation of functionals depending on jumps by quadratic, elliptic functionals via* Γ*-convergence.* Boll.Un. Mat. Ital. 6-B(1992), 105-123.

[4] G.Anzellotti *Pairings between measures and bounded functions and compensated compactness.* Ann. Mat. Pura Appl. **135** (1983), 293-318.

[5] G.Aubert, J. Aujol, L.Blanc-Feraud *Detecting Codimension-Two Objects in an image with Ginzurbg-Landau Models.* **65** (2005), 29-42.

[6] G.Bellettini, M.Paolini *Approssimazione variazionale di funzionali con curvatura.* Seminario di Analisi Matematica, Dipartimento di Matematica dell'Università di Bologna, 1993.

[7] F. Bethuel, H. Brezis and F. Hélein *G*inzburg-Landau Vortices. Birkäuser, Boston (1994).

[8] A.Braides. Γ*-convergence for beginners.* Oxford University Press, New york (2000).

[9] A.Braides, A.Malchiodi *Curvature Theory of Boundary phases: the two dimensional case.* Interfaces Free. Bound. **4** (2002), 345-370.

[10] A.Braides, R.March. *Approximation by* Γ*-convergence of a curvature-depending functional in Visual Reconstruction* Comm. Pure Appl. Math. **59** (2006), 71-121.

[11] G.Dal Maso. *Introduction to* Γ*-convergence.* Birkhäuser, Boston(1993).

[12] E.De Giorgi *Some Remarks on* Γ*-convergence and least square methods, in* Composite Media and Homogenization Theory (G. Dal Maso and G.F.Dell'Antonio eds.), Birkhauser, Boston, 1991, 135-142.

[13] E.De Giorgi, T.Franzoni. *Su un tipo di convergenza variazionale.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur. **58** (1975), 842-850.

[14] E.De Giorgi, T.Franzoni. *Su un tipo di convergenza variazionale.* Rend. Sem. Mat. Brescia **3** (1979), 63-101.

[15] L.Modica, S.Mortola. *Un esempio di* Γ*-convergenza.* Boll.Un. Mat. Ital. 14-B(1977), $285 - 299$.

[16] L.Modica. *The gradient theory of phase transitions and the minimal interface criterion.* Arch. Rational Mech. Anal. **98** (1987), 123-142.

[17] G.Stampacchia *Le problème de Dirichlet pour les equations elliptiques du seconde ordre à coefficients discontinuous*. Ann. Inst. Fourier (Grenoble), **15** (1965), 180-258.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399