

Graph selection with GGMselect

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Abstract: Applications on inference of biological networks have raised a strong interest on the problem of graph estimation in high-dimensional Gaussian graphical model. To handle this problem, we propose a two-stage procedure which first builds a family of candidate graphs from the data and then selects one graph among this family according to a dedicated criterion. This estimation procedure is shown to be consistent in a high-dimensional setting and its risk is controlled by a non-asymptotic oracle-like inequality. A nice behavior on numerical experiments corroborates these theoretical results.

The procedure is implemented in the R-package GGMselect available online.

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1. Introduction

We consider the problem of graph estimation in Gaussian graphical models. The graph of a random vector (X_1, \dots, X_p) represents the conditional dependences between the variables X_1, \dots, X_p . More precisely, assume that (X_1, \dots, X_p) is a centered Gaussian vector with covariance matrix Σ . Then, the law \mathbb{P}_Σ of (X_1, \dots, X_p) is a graphical model according to a graph G , if for any nodes a and b that are not neighbours in G , the variables X_a and X_b are independent conditionally on the remaining variables. There exists a unique graph G_Σ which is minimal for the inclusion and such that \mathbb{P}_Σ is a graphical model according to G_Σ . Our aim is to estimate this graph from a n -sample of \mathbb{P}_Σ . We will pay a special attention to the case where $n < p$. In what follows, we shall always assume that Σ is non-singular.

The problem of graph estimation in Gaussian graphical model when the sample size n is smaller (or much smaller) than the number p of variables is motivated by applications in post-genomic and is a current active field of research in statistics. Biotechnological developments in proteomics or transcriptomics enable to produce a huge amount of data. One of the challenges for the statistician

is to infer from these data the regulation network of a family of genes (or proteins). The task is difficult due to the very high-dimensional nature of the data and the small sample size. For example, microarrays measure the expression levels of a few thousand genes and the sample size n is no more than a few tens. The Gaussian graphical modeling has been proposed as a tool to handle this issue, see e.g. the papers of Kishino and Waddell [14], Dobra *et al* [7], Wu and Ye [24]. The gene expression levels are modeled by a Gaussian law \mathbb{P}_Σ and the regulation network of the genes is then depicted by the graph G_Σ of the conditional dependences.

Many estimation procedures have been proposed recently to perform graph estimation in Gaussian graphical model when $n < p$. A first class of procedures is based on multiple testing on empirical partial covariance. Actually, if G_Σ denotes the (minimal) graph of the law \mathbb{P}_Σ , there is an edge in G_Σ between a and b , if and only if the conditional covariance of X_a and X_b given all the other variables is non-zero. When $n < p$, the empirical version of the latter conditional covariance cannot be computed, so several papers suggest to use instead the empirical conditional covariance of X_a and X_b given $\{X_s, s \in S\}$ for some subsets S of $\{1, \dots, p\} \setminus \{a, b\}$ with cardinality less than $n - 2$. A multiple testing procedure is then applied to detect if the conditional covariance $\text{cov}(X_a, X_b | X_s, s \in S)$ is non-zero. Wille and Bühlmann [23] restrict to the sets S of cardinality less than one, Castelo and Roverato consider the sets S with cardinality at most q (for some fixed q) and Spirtes *et al.* [20] (see also Kalisch and Bühlmann [13]) propose a procedure which avoid an exhaustive search over all S . A second class of procedures relies on the fact that the entries $\Omega_{a,b}$ of the inverse covariance matrix $\Omega = \Sigma^{-1}$ are non-zero if and only if there is an edge between a and b in G_Σ . Several papers then suggest to perform a sparse estimation of Ω in order to estimate the graph G_Σ , see Huang *et al.* [12], Yuan and Lin [25], Banerjee *et al.* [1], Friedman *et al.* [10], and Fan *et al.* [9]. They propose to maximise the log-likelihood of Ω under l^1 constraints to enforce sparsity and they design optimisation algorithms to perform this maximisation. Finally, a third class of procedures uses the fact that the coefficients $\theta_{a,b}$ of the regression of X_a on $\{X_b, b \neq a\}$ are non-zeros if and only if there is an edge between a and b in G_Σ . Meinshausen and Bühlmann [17] or Rocha *et al.* [18] perform regressions with l^1 constraints, whereas Giraud [11] (see also Verzelen [21]) proposes an exhaustive search to obtain a sparse estimate of the matrix θ and then detect the graph G_Σ .

In this paper, we propose a new estimation scheme which combines the good properties of these different procedures. Actually, the procedures based on the empirical covariance or on l^1 regularisation share some nice computational properties and they can handle several hundred of variables X_1, \dots, X_p . Nevertheless, the theoretical results assessing their statistical accuracy are either of asymptotic nature or rely on strong assumptions on the covariance [15, 19]. Moreover, their performance heavily depends on one (or several) tuning parameter, which is usually not dimensionless and whose optimal value is unknown. On the other hand, the exhaustive search of [11] has a good statistical accuracy and strong theoretical results have been established, but the computational complexity of

the exhaustive search is huge and it cannot be performed when the number p of variables is larger than a few tens. Our strategy here is to build a data-driven family of candidate graphs with several fast above-mentioned procedures and then to apply the selection criterion presented in [11] to select one graph among them. The resulting estimation procedure can handle several hundred of variables X_1, \dots, X_p and presents good statistical properties. Actually, the procedure is shown to be consistent in a high-dimensional setting and its risk is controlled by a non-asymptotic oracle-like inequality. The assumptions needed to establish these results are weaker than those commonly used in the literature. In addition, numerical experiments show a nice behavior on simulated examples. The procedure is implemented in the R -package *GGMselect* available on <http://w3.jouy.inra.fr/unites/miaj/public/logiciels/GGMselect/>

The remaining of the paper is organized as follows. We describe the estimation procedure in the next section and state some theoretical results on its statistical accuracy in Section 3. In Section 4, we carry out some numerical experiments and Section 5 is devoted to the proofs.

Notations. To estimate the graph G_Σ , we will start from a n -sample $X^{(1)}, \dots, X^{(n)}$ of the law \mathbb{P}_Σ . We denote by \mathbf{X} the $n \times p$ matrix whose rows are given by the vectors $X^{(i)}$, namely $\mathbf{X}_{i,a} = X_a^{(i)}$ for $i = 1, \dots, n$ and $a = 1, \dots, p$. We write \mathbf{X}_a for the a^{th} column of \mathbf{X} . We also set $\Gamma = \{1, \dots, p\}$ and for any graph G with nodes indexed by Γ , we write $d_a(G)$ for the degree of the node a in the graph G (which is the number of edges incident to a) and $\deg(G) = \max_{a \in \Gamma} d_a(G)$ for the degree of G . Moreover, the notation $a \stackrel{G}{\sim} b$ means that the nodes a and b are neighbours in the graph G . Finally, we write Θ for the set of $p \times p$ matrices with 0 on the diagonal, $\|\cdot\|_{q \times p}$ for the Frobenius norm on $q \times p$ matrices

$$\|A\|_{q \times p}^2 = \text{Tr}(A^T A) = \sum_{i=1}^q \sum_{j=1}^p A_{i,j}^2,$$

$\|\cdot\|_n$ for the Euclidean norm on \mathbb{R}^n divided by \sqrt{n} , and for any $\beta \in \mathbb{R}^p$ we defined $\text{supp}(\beta)$ as the set of the labels $a \in \Gamma$ such that $\beta_a \neq 0$.

2. Estimation procedure

GGMselect is a two-stage estimation procedure which first builds a data-driven family $\widehat{\mathcal{G}}$ of candidate graphs and then applies a selection procedure to pick one graph among these. We present the selection procedure in the next paragraph and then describe different possible choices for the family of candidate graphs $\widehat{\mathcal{G}}$.

2.1. Selection procedure

We assume here that we have at hand a family $\widehat{\mathcal{G}}$ of candidate graphs, which all have a degree smaller than $n - 2$. To select a graph \widehat{G} among the family $\widehat{\mathcal{G}}$, we

use the selection criterion introduced in [11]. We briefly present this criterion here and refer to [11] for further details. We write θ for the $p \times p$ matrix such that

$$\mathbb{E}_\Sigma [X_a | X_b, b \neq a] = \sum_{b \neq a} \theta_{a,b} X_b \quad \text{and} \quad \theta_{a,a} = 0 \quad \text{for all } a \in \{1, \dots, p\}.$$

The matrix θ minimizes $\|\Sigma^{1/2}(I - \theta')\|_{p \times p}$ over the set Θ of $p \times p$ matrices θ' with 0 on the diagonal. Since $\mathbf{X}^T \mathbf{X}/n$ is an empirical version of Σ , an empirical version of $\|\Sigma^{1/2}(I - \theta)\|_{p \times p}$ is $\|\mathbf{X}(I - \theta)\|_{n \times p}$ divided by \sqrt{n} . Therefore, for any graph G in $\widehat{\mathcal{G}}$, we associate an estimator $\widehat{\theta}_G$ of θ by setting

$$\widehat{\theta}_G = \operatorname{argmin} \{ \|\mathbf{X}(I - \theta')\|_{n \times p} : \theta' \in \Theta_G \}, \quad (1)$$

where Θ_G is the set of $p \times p$ matrices θ' such that $\theta'_{a,b}$ is non-zero if and only if there is an edge between a and b in G .

Finally, we select a graph \widehat{G} in $\widehat{\mathcal{G}}$ by taking any minimizer over $\widehat{\mathcal{G}}$ of the criterion

$$\operatorname{Crit}(G) = \sum_{a=1}^p \left[\|\mathbf{X}_a - \mathbf{X}[\widehat{\theta}_G]_a\|_n^2 \left(1 + \frac{\operatorname{pen}[d_a(G)]}{n - d_a(G)} \right) \right], \quad (2)$$

where $d_a(G)$ is the degree of the node a in the graph G and the penalty function $\operatorname{pen} : \mathbb{N} \rightarrow \mathbb{R}^+$ is of the form of the penalties introduced in Baraud *et al.* [2]. To compute this penalty, we define for any integers d and N the DKhi function by

$$\operatorname{DKhi}(d, N, x) = \mathbb{P} \left(F_{d+2, N} \geq \frac{x}{d+2} \right) - \frac{x}{d} \mathbb{P} \left(F_{d, N+2} \geq \frac{N+2}{Nd} x \right), \quad x > 0,$$

where $F_{d, N}$ denotes a Fisher random variable with d and N degrees of freedom. The function $x \mapsto \operatorname{DKhi}(d, N, x)$ is decreasing and we write $\operatorname{EDKhi}[d, N, x]$ for its inverse, see [2] Sect. 6.1 for more details. Then, we fix some constant $K > 1$ and set

$$\operatorname{pen}(d) = K \frac{n-d}{n-d-1} \operatorname{EDKhi} \left[d+1, n-d-1, \left(\binom{p-1}{d} (d+1)^2 \right)^{-1} \right]. \quad (3)$$

When d remains small compared to n , the penalty function increases approximately linearly with d . Actually, when $d \leq \gamma n / \left(2(1.1 + \sqrt{\log p})^2 \right)$ for some $\gamma < 1$, we approximately have for large values of p and n

$$\operatorname{pen}(d) \lesssim K \left(1 + e^\gamma \sqrt{2 \log p} \right)^2 (d+1),$$

see Proposition 4 in Baraud *et al.* [2] for an exact bound.

The selection procedure depends on a dimensionless tuning parameter K . A larger value for K yields a procedure more conservative. In theory (and in practice) K has to be larger than one. We propose to set K between 2 and 3 depending on the desired level of false discovery rate of edges.

2.2. Family $\widehat{\mathcal{G}}$ of candidate graphs

The computational complexity of the minimization of the criterion (2) over the family $\widehat{\mathcal{G}}$ is linear with respect to its size. In particular, minimizing (2) over all the graphs with degree smaller than some integer D , as proposed in [11], is untractable when p is larger than a few tens. To overcome this issue, we propose to build a much smaller (data-driven) family $\widehat{\mathcal{G}}$ of candidate graphs, with the help of various fast algorithms dedicated to graph estimation.

We present below four way to build a family of candidate graphs, chosen on the basis of theoretical results and simulation studies. They will be denoted by $\widehat{\mathcal{G}}_{\text{QE}}$, $\widehat{\mathcal{G}}_{\text{C01}}$, $\widehat{\mathcal{G}}_{\text{LA}}$, and $\widehat{\mathcal{G}}_{\text{EW}}$. Even if the procedure applies for any family $\widehat{\mathcal{G}}$, we advise to choose in practice one of these four families or the union of them.

In the following, we describe these four families, provide algorithms to compute them efficiently, and discuss their computational complexity and their size. Each family depends on an integer D , smaller than $n - 2$, which corresponds to the maximal degree of the graphs in this family.

Quasi-exhaustive family $\widehat{\mathcal{G}}_{\text{QE}}$

Description. Roughly, the idea is to break down the minimization of the criterion (2) over all the graphs of degree at most D into p independent problems. For each node $a \in \Gamma$, we estimate the neighborhood of a by

$$\widehat{\text{ne}}(a) = \operatorname{argmin} \left\{ \|\mathbf{X}_a - \operatorname{Proj}_{V_S}(\mathbf{X}_a)\|_n^2 \left(1 + \frac{\text{pen}(|S|)}{n - |S|} \right) : S \subset \Gamma \setminus \{a\} \text{ and } |S| \leq D \right\},$$

where pen is the penalty function (3) and $\operatorname{Proj}_{V_S}$ denotes the orthogonal projection from \mathbb{R}^n onto $V_S = \{\mathbf{X}\beta : \beta \in \mathbb{R}^p \text{ and } \operatorname{supp}(\beta) = S\}$. We know from Verzelen [21] that $\widehat{\text{ne}}(a)$ is a good estimator of the true neighborhood of a , from a non-asymptotic point of view. We then build two nested graphs $\widehat{G}_{K,\text{and}}$ and $\widehat{G}_{K,\text{or}}$ as in Meinshausen and Bühlmann [17]

$$\begin{aligned} a \stackrel{\widehat{G}_{K,\text{and}}}{\sim} b &\iff a \in \widehat{\text{ne}}(b) \text{ \underline{and} } b \in \widehat{\text{ne}}(a), \\ a \stackrel{\widehat{G}_{K,\text{or}}}{\sim} b &\iff a \in \widehat{\text{ne}}(b) \text{ \underline{or} } b \in \widehat{\text{ne}}(a), \end{aligned}$$

and define the family $\widehat{\mathcal{G}}_{\text{QE}}$ as the collection of all the graphs that lie between $\widehat{G}_{K,\text{and}}$ and $\widehat{G}_{K,\text{or}}$

$$\widehat{\mathcal{G}}_{\text{QE}} = \left\{ G, \widehat{G}_{K,\text{and}} \subset G \subset \widehat{G}_{K,\text{or}} \text{ and } \deg(G) \leq D \right\}.$$

It is likely that the graph $\widehat{G}_{\text{exhaustive}}$ which minimizes (2) over all the graphs of degree at most D belongs to the family $\widehat{\mathcal{G}}_{\text{QE}}$. In such a case, the minimizer \widehat{G}_{QE} of the criterion (2) over $\widehat{\mathcal{G}}_{\text{QE}}$ coincides with the estimator $\widehat{G}_{\text{exhaustive}}$ of [11].

QE Algorithm

1. Compute $\widehat{ne}(a)$ for all $a \in \Gamma$.
2. Compute the graphs $\widehat{G}_{K,\text{and}}$ and $\widehat{G}_{K,\text{or}}$.
3. Work out the family $\widehat{\mathcal{G}}_{\text{QE}}$.

Complexity. The complexity of the computation of the collections $\widehat{ne}(a)$ is much smaller than the complexity of the computation of $\widehat{G}_{\text{exhaustive}}$. Nevertheless, it still remains of order $np^{D+1}D^3$ and the size of the family $\widehat{\mathcal{G}}_{\text{QE}}$ can be of order $2^{pD/2}$ in the worst cases. However, for sparse graphs G_Σ , the graphs $\widehat{G}_{K,\text{and}}$ and $\widehat{G}_{K,\text{or}}$ are quite similar in practice, which makes the size of $\widehat{\mathcal{G}}_{\text{QE}}$ much smaller. The procedure then remains tractable for p and D reasonably small. Computational times for some examples are given in Section 4.1.

C01 family $\widehat{\mathcal{G}}_{\text{C01}}$

Description. The family $\widehat{\mathcal{G}}_{\text{C01}}$ derives from the estimation procedure proposed in Wille and Bühlmann [23] and is based on the 0-1 *conditional independence graph* G_{01} . This graph is defined as follows. For each pair of nodes (a, b) , we write $R_{a,b|\emptyset}$ for the correlation between the variable X_a and X_b and $R_{a,b|c}$ for the correlation of X_a and X_b conditionally on X_c . Then, there is an edge between a and b in G_{01} , if and only if $R_{a,b|\emptyset} \neq 0$ and $R_{a,b|c} \neq 0$ for all $c \in \Gamma \setminus \{a, b\}$, viz

$$a \stackrel{G_{01}}{\sim} b \iff \min\{|R_{a,b|c}|, c \in \{\emptyset\} \cup \Gamma \setminus \{a, b\}\} > 0. \quad (4)$$

Although the 0-1 conditional independence graph G_{01} does not usually coincide with the graph G_Σ , there is a close connection between both graphs in some cases (see [23]). Wille and Bühlmann [23] propose then to estimate the graph G_{01} in order to get an approximation of G_Σ . The following construction of the family $\widehat{\mathcal{G}}_{01}$ derives from their estimation procedure. We write $P(a, b|c)$ for the p -value of the likelihood ratio test of the hypothesis " $R_{a,b|c} = 0$ " and set

$$P_{\max}(a, b) = \max\{P(a, b|c), c \in \{\emptyset\} \cup \Gamma \setminus \{a, b\}\}.$$

For any $\alpha > 0$, the graph $\widehat{G}_{01,\alpha}$ is defined by

$$a \stackrel{\widehat{G}_{01,\alpha}}{\sim} b \iff P_{\max}(a, b) \leq \alpha$$

and the family $\widehat{\mathcal{G}}_{\text{C01}}$ is the family of nested graphs

$$\widehat{\mathcal{G}}_{\text{C01}} = \left\{ \widehat{G}_{01,\alpha}, \alpha > 0 \text{ and } \deg(\widehat{G}_{01,\alpha}) \leq D \right\}.$$

C01 Algorithm

1. Compute the $p(p-1)/2$ values $P_{\max}(a, b)$.
2. Order them.
3. Extract from these values the nested graphs $\{\widehat{G}_{01,\alpha} : \alpha > 0\}$.
4. Stop when the degree becomes larger than D .

Complexity. The algorithm goes very fast since its complexity is of order np^3 . The size of the family $\widehat{\mathcal{G}}_{C01}$ is smaller than pD .

Lasso-And family $\widehat{\mathcal{G}}_{LA}$

Description. The Lasso-And family $\widehat{\mathcal{G}}_{LA}$ derives from the estimation procedure proposed by Meinshausen and Bühlmann [17] and is based on the LARS-lasso algorithm of Hastie *et al.* [8]. For any $\lambda > 0$, we define the $p \times p$ matrix $\widehat{\theta}^\lambda$ by

$$\widehat{\theta}^\lambda = \operatorname{argmin} \{ \|\mathbf{X} - \mathbf{X}\theta'\|_{n \times p}^2 + \lambda \|\theta'\|_1 : \theta' \in \Theta \}, \quad (5)$$

where Θ is the set of $p \times p$ matrices with 0 on the diagonal and $\|\theta'\|_1 = \sum_{a \neq b} |\theta'_{a,b}|$. Then, we define the graph $\widehat{G}_{\text{and}}^\lambda$ by setting an edge between a and b if both $\widehat{\theta}_{a,b}^\lambda$ and $\widehat{\theta}_{b,a}^\lambda$ are non-zero, namely

$$a \stackrel{\widehat{G}_{\text{and}}^\lambda}{\sim} b \iff \widehat{\theta}_{a,b}^\lambda \neq 0 \text{ and } \widehat{\theta}_{b,a}^\lambda \neq 0.$$

This graph $\widehat{G}_{\text{and}}^\lambda$ is exactly the estimator (7) introduced by Meinshausen and Bühlmann [17]. We note that the size of $\widehat{G}_{\text{and}}^\lambda$ has a tendency to increase when the tuning parameter λ decreases. Hence, we define the family $\widehat{\mathcal{G}}_{LA}$ as the set of graphs $\widehat{G}_{\text{and}}^\lambda$ with λ large enough to ensure that $\deg(\widehat{G}_{\text{and}}^\lambda) \leq D$, viz

$$\widehat{\mathcal{G}}_{LA} = \left\{ \widehat{G}_{\text{and}}^\lambda, \lambda > \widehat{\lambda}_{\text{and},D} \right\}, \quad \text{where } \widehat{\lambda}_{\text{and},D} = \sup \left\{ \lambda, \deg(\widehat{G}_{\text{and}}^\lambda) > D \right\}.$$

From a computational point of view, the family $\widehat{\mathcal{G}}_{LA}$ can be efficiently computed with the LARS-lasso algorithm. The optimization problem (5) is broken into the p independent minimization problems

$$\widehat{\theta}_a^\lambda = \operatorname{argmin} \{ \|\mathbf{X}_a - \mathbf{X}v\|^2 + \lambda \|v\|_1 : v \in \mathbb{R}^p \text{ and } v_a = 0 \}, \quad \text{for any } a \in \Gamma, \quad (6)$$

with $\|v\|_1 = \sum_{b=1}^p |v_b|$. When λ decreases, the support of $\widehat{\theta}_a^\lambda$ is piecewise constant and the LARS-lasso algorithm provides the sequences $(\lambda_a^l)_{l \geq 1}$ of the values of λ where the support of $\widehat{\theta}_a^\lambda$ changes, as well as the sequence of the supports $(\operatorname{supp}(\widehat{\theta}_a^{\lambda_a^l}))_{l \geq 1}$. We work out the family $\widehat{\mathcal{G}}_{LA}$ by gathering these p sequences as described below.

LA Algorithm

1. Compute with LARS-lasso the $(\lambda_a^l, \operatorname{supp}(\widehat{\theta}_a^{\lambda_a^l}))_{l \geq 1}$ for all $a \in \Gamma$.
2. Order the sequence $\{\lambda_a^l : a \in \Gamma, l \geq 1\}$.
3. Compute $\widehat{G}_{\text{and}}^{\lambda_a^l}$ for all $\lambda_a^l > \widehat{\lambda}_{\text{and},D}$.

Complexity. The complexity of the LARS-lasso algorithm is unknown in general. Nevertheless, according to Hastie *et al.* [8] the algorithm requires $O(np(n \wedge p))$ operations in most cases. Hence, the whole complexity of the LA algorithm is generally of the order $p^2n(n \wedge p)$. Finally, the size of the family $\widehat{\mathcal{G}}_{LA}$ cannot be bounded uniformly, but it remains smaller than pD in practice.

Adaptive lasso family $\widehat{\mathcal{G}}_{EW}$

Description. The family $\widehat{\mathcal{G}}_{EW}$ is a modified version of $\widehat{\mathcal{G}}_{LA}$ inspired by the adaptive lasso of Zou [26]. The major difference between $\widehat{\mathcal{G}}_{EW}$ and $\widehat{\mathcal{G}}_{LA}$ lies in the replacement of the l^1 norm $\|\theta'\|_1$ in (5) by $\|\theta'/\widehat{\theta}^{\text{init}}\|_1$, where $\widehat{\theta}^{\text{init}}$ is a preliminary estimator of θ and $\theta'/\widehat{\theta}^{\text{init}}$ stands for the matrix with entries $(\theta'/\widehat{\theta}^{\text{init}})_{a,b} = \theta'_{a,b}/\widehat{\theta}_{a,b}^{\text{init}}$. Zou suggests to take for $\widehat{\theta}^{\text{init}}$ a ridge estimator. Here, we propose to use instead the Exponential Weights estimator $\widehat{\theta}^{EW}$ of Dalalyan and Tsybakov [5, 6]. The choice of this estimator appears more natural to us since it is designed for the sparse setting and enjoys nice theoretical properties. Moreover, we have observed on some simulated examples, that the adaptive lasso with the Exponential Weights initial estimator performs much better than the adaptive lasso with the ridge initial estimator.

To build the family $\widehat{\mathcal{G}}_{EW}$ we thus start by computing the Exponential Weight estimator $\widehat{\theta}^{EW}$. For each $a \in \Gamma$, we set $H_a = \{v \in \mathbb{R}^p : v_a = 0\}$ and

$$\widehat{\theta}_a^{EW} = \int_{H_a} v e^{-\beta\|\mathbf{X}_a - \mathbf{X}v\|_n^2} \prod_j (1 + (v_j/\tau)^2)^{-\alpha} \frac{dv}{\mathcal{Z}_a}, \quad (7)$$

with $\mathcal{Z}_a = \int_{H_a} e^{-\beta\|\mathbf{X}_a - \mathbf{X}v\|_n^2} \prod_j (1 + (v_j/\tau)^2)^{-\alpha} dv$ and $\alpha, \beta, \tau > 0$. We note that $\widehat{\theta}_a^{EW}$ with $\beta = n/(2\sigma_a^2)$ and $\sigma_a^2 = \text{var}(X_a | X_{-a})$ is simply the Bayesian estimator of θ_a with prior distribution $d\pi(v) \propto \prod_j (1 + (v_j/\tau)^2)^{-\alpha} dv$ on H_a . In the Gaussian setting, Dalalyan and Tsybakov [5] give a sharp and assumption-free sparse inequality for $\widehat{\theta}_a^{EW}$ with $\beta \leq n/(4\sigma_a^2)$, see Corollary 4 in [5].

The construction of $\widehat{\mathcal{G}}_{EW}$ is now similar to the construction of $\widehat{\mathcal{G}}_{LA}$. For any $\lambda > 0$ we set

$$\widehat{\theta}^{EW,\lambda} = \text{argmin} \left\{ \|\mathbf{X} - \mathbf{X}\theta'\|_{n \times p}^2 + \lambda \|\theta'/\widehat{\theta}^{EW}\|_1 : \theta' \in \Theta \right\}, \quad (8)$$

and we define the graph $\widehat{\mathcal{G}}_{or}^{EW,\lambda}$ by setting an edge between a and b if either $\widehat{\theta}_{b,a}^{EW,\lambda}$ or $\widehat{\theta}_{a,b}^{EW,\lambda}$ is non-zero:

$$a \stackrel{\widehat{\mathcal{G}}_{or}^{EW,\lambda}}{\sim} b \iff \widehat{\theta}_{a,b}^{EW,\lambda} \neq 0 \text{ or } \widehat{\theta}_{b,a}^{EW,\lambda} \neq 0.$$

Finally, the family $\widehat{\mathcal{G}}_{EW}$ is given by

$$\widehat{\mathcal{G}}_{EW} = \left\{ \widehat{\mathcal{G}}_{or}^{EW,\lambda}, \lambda > \widehat{\lambda}_{or,d}^{EW} \right\}, \quad \text{where} \quad \widehat{\lambda}_{or,d}^{EW} = \sup \left\{ \lambda, \text{deg}(\widehat{\mathcal{G}}_{or}^{EW,\lambda}) > d \right\}.$$

The Exponential Weight estimator $\widehat{\theta}^{EW}$ can be computed with a Langevin Monte-Carlo algorithm. We refer to Dalalyan and Tsybakov [6] for the details. Once $\widehat{\theta}^{EW}$ is computed, the family $\widehat{\mathcal{G}}_{EW}$ is obtained as before with the help of the LARS-lasso algorithm.

EW Algorithm

1. Compute $\widehat{\theta}^{EW}$ with a Langevin Monte-Carlo algorithm.
2. Compute with LARS-lasso the $(\lambda_a^l, \text{supp}(\widehat{\theta}^{\lambda_a^l}))_{l \geq 1}$ for all $a \in \Gamma$.
3. Order the sequence $\{\lambda_a^l : a \in \Gamma, l \geq 1\}$.
4. Compute $\widehat{G}_{\text{or}}^{EW, \lambda_a^l}$ for all $\lambda_a^l > \widehat{\lambda}_{\text{or}, D}$.

Complexity. The complexity of the first step depends on the choices of the tuning parameters. Some examples are given in Section 4.1. The complexity of steps 2, 3 and 4 is the same as for the LA-algorithm, of the order $p^2 n(n \wedge p)$ in practice. Finally, as for \widehat{G}_{LA} , we do not know a general bound for the size of \widehat{G}_{EW} , but it remains smaller than pD in practice.

3. Theoretical results

We present in this section some theoretical results which assess the performance of our selection procedure. We present two kind of results: a non-asymptotic oracle-like inequality concerning the estimation of θ and a consistency result for the estimation of G_{Σ} .

3.1. A non-asymptotic oracle-like inequality

Next theorem states an oracle-like inequality in the spirit of Theorem 1 in [11]. We associate to the graph \widehat{G} selected by the procedure of Section 2, the estimator $\tilde{\theta} = \widehat{\theta}_{\widehat{G}}$ of the matrix θ , where $\widehat{\theta}_G$ is given by (1) for any graph $G \in \widehat{\mathcal{G}}$. The quality of the estimation of θ is quantified by the MSEP of $\tilde{\theta}$ defined by

$$\text{MSEP}(\tilde{\theta}) = \mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \right].$$

We refer to the introduction of Giraud [11] for a discussion on the relevance of the use of the MSEP of $\tilde{\theta}$ to assess the quality of the estimator \widehat{G} . In the sequel, I stands for the identity matrix of size p .

Roughly speaking, next theorem ensures that the estimator $\tilde{\theta}$ performs almost as well as the best estimator in the family $\{\widehat{\theta}_G, G \in \widehat{\mathcal{G}}\}$.

Theorem 3.1. Assume that $n \geq 9$. Let $\widehat{\mathcal{G}}$ be any (data-driven) family of graphs with maximal degree $D_{\widehat{\mathcal{G}}} = \max\{\deg(G), G \in \widehat{\mathcal{G}}\}$ fulfilling

$$1 \leq D_{\widehat{\mathcal{G}}} \leq \gamma \frac{n}{2(1.1 + \sqrt{\log p})^2}, \quad \text{for some } \gamma < 1. \quad (9)$$

Then, the MSEP of the estimator $\tilde{\theta}$ is upper bounded by

$$\text{MSEP}(\tilde{\theta}) \leq L_{K,\gamma} \log(p) \left(\mathbb{E} \left[\inf_{G \in \widehat{\mathcal{G}}} \left(\text{MSEP}(\hat{\theta}_G) \right) \right] \vee \frac{\text{MSEP}(I)}{n} \right) + R_n. \quad (10)$$

where $L_{K,\gamma}$ is a positive constant depending on K and γ only and the residual term $R_n = R_n(\Sigma, \gamma)$ (made explicit in the proof) is of order $n^3 \text{tr}(\Sigma) e^{-n(\sqrt{\gamma}-\gamma)^2/4}$.

If we forget the term $n^{-1}\text{MSEP}(I)$ in (10), Theorem 3.1 states that under Condition (9) the MSEP of $\tilde{\theta}$ nearly achieves, up to a $\log(p)$ factor, the average minimal MSEP of the family of estimators $\{\hat{\theta}_G, G \in \widehat{\mathcal{G}}\}$. Hence, $\tilde{\theta}$ performs almost as well as the oracle up to a $\log p$ factor. This logarithmic factor is proved to be unavoidable from a minimax point of view (see [21] Sect. 4.2).

Let us now compare the additional term $n^{-1}\text{MSEP}(I)$ appearing in (10) with the risk $\text{MSEP}(\hat{\theta}_G)$. This additional term is equal to $n^{-1} \sum_a \sigma_a^2$, where σ_a^2 stands for the conditional variance of X_a given the remaining variables. Hence, this quantity is usually smaller than the risk $\text{MSEP}(\hat{\theta}_G)$ which is the sum of a bias term and a variance term of order $n^{-1} \sum_a d_a(G) \sigma_a^2$. Nevertheless, when the true graph G_Σ is empty and $\widehat{\mathcal{G}}$ contains the empty graph, the additional term $n^{-1}\text{MSEP}(I)$ is dominant and the estimator $\tilde{\theta}$ is not optimal. Such a drawback is actually unavoidable in model selection when the target is too close to zero (see Birgé and Massart [4] Sect.2.3.3 for a discussion).

Finally, we can compare the MSEP of $\tilde{\theta}$ to the MSEP of $\hat{\theta}_{G_\Sigma}$ when G_Σ belongs to $\widehat{\mathcal{G}}$ with large probability. Roughly speaking, the MSEP of $\tilde{\theta}$ is in this case smaller (up to a $\log p$ factor) than the MSEP of $\hat{\theta}_{G_\Sigma}$. This means that $\tilde{\theta}$ performs almost as well as if we knew the true graph G_Σ in advance.

Corollary 3.2. Under the assumption of the above theorem, if the minimal graph G_Σ belongs to the family $\widehat{\mathcal{G}}$ with large probability

$$\mathbb{P}(G_\Sigma \in \widehat{\mathcal{G}}) \geq 1 - L(\alpha) \exp(-\beta n^\delta), \quad \text{for some } \alpha, \beta, \delta > 0 \quad (11)$$

then, the MSEP of the estimator $\tilde{\theta}$ is upper bounded by

$$\text{MSEP}(\tilde{\theta}) \leq L_{K,\gamma} \log(p) \left(\text{MSEP}(\hat{\theta}_{G_\Sigma}) \vee \frac{\text{MSEP}(I)}{n} \right) + R_n. \quad (12)$$

where the residual term $R_n = R_n(\Sigma, \gamma, \alpha, \beta, \delta)$ is of order $n^3 \text{tr}(\Sigma) [e^{-n(\sqrt{\gamma}-\gamma)^2/4} + \sqrt{L(\alpha)} e^{-\frac{\beta}{2} n^\delta}]$.

3.2. Consistency of the selection procedure

The next theorem states, under mild assumptions, a consistency result for our selection procedure in a high dimensional setting. In the spirit of the results of Meinshausen and Bühlmann [17], we consider the case where the number of variables p increase with the sample size n .

We make the following assumptions:

$$\text{(H.1)} \quad p_n \geq n .$$

$$\text{(H.2)} \quad \deg(G_{\Sigma_n}) \leq \frac{n^s}{\log p_n} \wedge \frac{n}{\log^2 p_n} \text{ for some } s < 1 .$$

$$\text{(H.3)} \quad \min_{a \neq b, b \in \text{ne}_{G_{\Sigma_n}}(a)} \theta_{a,b}^2 \min_{a \neq b} \frac{\text{Var}(X_a | X_{-a})}{\text{Var}(X_b | X_{-b})} \geq n^{s'-1} \text{ for some } s' > s .$$

Theorem 3.3. *Assume that the family $\widehat{\mathcal{G}}$ of candidate graphs contains the true graph with probability going to 1 and (H.1), (H.2), (H.3) are fulfilled. Then, the estimation procedure GGMselect with $K > \left\lceil 3 \vee \frac{2.5}{(1-s)} \right\rceil$ and*

$$D_{\widehat{\mathcal{G}}} = \max\{\deg(G), G \in \widehat{\mathcal{G}}\} \leq \frac{n}{\log^2 p_n}$$

is consistent. More precisely, there exist some universal constant L and some integer $n_0 = n_0[K, s, s']$ not depending on the true graph G_{Σ_n} nor on the covariance Σ_n such that

$$\mathbb{P} \left[\widehat{\mathcal{G}} = G_{\Sigma_n} \right] \geq 1 - L p_n^{-1/2} - \mathbb{P} \left[G_{\Sigma_n} \notin \widehat{\mathcal{G}} \right], \quad \text{for any } n \geq n_0 .$$

Let us discuss the assumptions of the theorem and their similarity with some of the hypotheses made in Meinshausen and Bühlmann [17]. The Assumption (H.2) is met if p_n grows polynomially with respect to n and the degree of the true graph does not grow faster than n^κ with $\kappa < s$ (which corresponds to Assumptions 1 and 2 in [17]). We mention that (H.2) is not satisfied when p_n grows exponentially with n unless G_{Σ_n} is empty. It is actually impossible to consistently estimate a non-empty graph if p_n is of order $\exp(n)$, see Wainwright [22].

The Assumption (H.3) ensures that the conditional variances as well as the non-zero terms $\theta_{a,b}$ are large enough so that the edges can be detected. To compare with [17], Assumption (H.3) is met as soon as Assumption 2 and 5 in [17] are satisfied. In addition, we underline that we make no assumption on the l_1 -norm of the prediction coefficients or on the signs of $\theta_{a,b}$ (Assumption 4 and 6 in [17]).

Finally, we do not claim that the condition $K > \lceil 2.5/(1-s) \vee 3 \rceil$ is minimal to obtain consistency. It seems from simulation experiments that smaller choices of K also provide good estimations.

4. Numerical study

It is essential to investigate the performance of statistical procedures on data. Since we do not know the actual underlying graph of conditional dependences on real data sets, we opt for a numerical study with simulated data. Our aims in this study are to evaluate the feasibility of the GGMselect procedure and to compare its performances with those of recent graph-selection procedures.

Simulating the data The matrix \mathbf{X} is composed of n i.i.d. rows with Gaussian $\mathcal{N}_p(0, \Omega^{-1})$ distribution where the inverse covariance matrix Ω is constructed according to the following procedure. We set $\Omega = BB^T + D$, where B is a random sparse lower triangular matrix and D is a diagonal matrix with random entries of order 10^{-3} . The latter matrix D prevents Ω from having too small eigenvalues. To generate B we split $\{1, \dots, p\}$ into three consecutive sets I_1, I_2, I_3 of approximately equal size, and choose two real numbers η_{int} and η_{ext} between 0 and 1. For any a, b such that $1 \leq a < b \leq p$, we set $B_{a,b} = 0$ with probability $1 - \eta_{\text{int}}$ if a and b are in the same set, and we set $B_{a,b} = 0$ with probability $1 - \eta_{\text{ext}}$ if a and b belong to two different sets. Then, the lower diagonal values that have not been set to 0 are drawn according to a uniform law on $[-1, 1]$ and the diagonal values are drawn according to a uniform law on $[0, \varepsilon]$. Finally, we rescale Ω in order to have 1 on the diagonal of $\Sigma = \Omega^{-1}$. This matrix Σ defines a graph $G = G_\Sigma$ and a matrix θ defined as in Section 2.1. The sparsity of the graph is measured via a sparsity index noted I_s , defined as the average number of edges per nodes in the graph.

In our simulation study we set $\eta = \eta_{\text{int}} = 5\eta_{\text{ext}}$, and $\varepsilon = 0.1$. We evaluate the value of η corresponding to a desired value of the sparsity index I_s by simulation. Choosing I_s small, we get sparse graphs whose edges distribution is not uniform, see Figure 1.

GGMSelect: choice of graphs families Our procedure is applied for the families of graphs presented in Section 2.2. The methods are respectively denoted QE, C01, LA and EW.

As it was already mentioned in Section 2.2, the size of the family $\widehat{\mathcal{G}}_{\text{QE}}$ may be very large leading to memory size problems in the computational process. In that case, as soon as a memory size problem is encountered, the research between $\widehat{\mathcal{G}}_{K, \text{and}}$ and $\widehat{\mathcal{G}}_{K, \text{or}}$ is stopped and prolonged by a stepwise procedure.

The family $\widehat{\mathcal{G}}_{\text{EW}}$ is based on the calculation of exponential weight estimators $\widehat{\theta}^{\text{EW}}$. This calculation depends on parameters, denoted $\alpha, \beta, \sigma, \tau$ in [6], that defined the aggregation procedure, and on parameters, denoted h and T in [6], used in the Langevin Monte-Carlo algorithm. We chose these parameters as follows. The matrix \mathbf{X} being scaled such that the norm of each column equals 1, we took $\sigma = 1/\sqrt{n}$, and we set $\alpha = 0, \beta = 2/n, \tau = 1/\sqrt{n(p-1)}$ and $h = 10^{-3}, T = 200$. Using these parameters values we did not encountered convergence problems in our simulation study.

Our procedure depends on two tuning parameters: K occurring in the penalty function (see Equation 3) and D the maximum degree of the graph. We choose

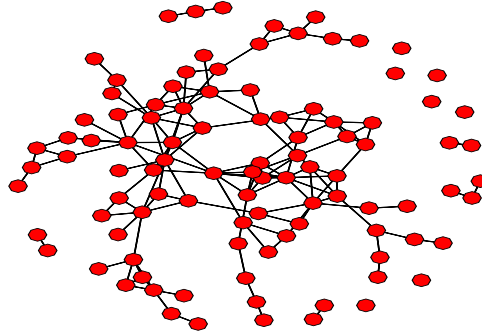


FIGURE 1. One simulated graph G with $p = 30$ and $I_s = 3$. The degree $\deg(G)$ of the graph equals 8.

$K = 2.5$ in all simulation experiments. In practice, we would like to choose D as large as possible, and this can be done for all methods except QE where the complexity of the algorithm increases exponentially with D .

All these methods are implemented in R-2.7.2 in the package `GGMselect` available on request.

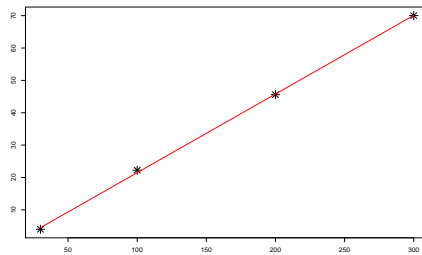
4.1. CPU times

We assess the practical feasibility of the methods we propose from the point of view of the memory size and computer time. To this aim, we simulate graphs with $p = 30, 100, 200, 300$ nodes, sparsity $I_s = 3$ and $n = 50$. The simulation were run on a Bi-Pro Xeon quad core 2.66 GHz with 24 Go RAM. The computer time being strongly dependent on the simulated graph we calculate the mean of computer times over $N_G = 100$ simulated graphs. For each of these graphs, one matrix \mathbf{X} is simulated. The results are given in Table 1. The maximum degree D of the estimated graph was set to 5, except for the QE method where $D = 3$ and 5. The maximum allowed memory size is exceeded for the QE method when $D = 5$ and $p = 100, 200$, and when $D = 3$ for $p = 300$. The LA and C01 methods are running very fast. The computing time for the EW method increases quickly with p : in this simulation study, it is roughly proportional to $\exp(\sqrt{p}/2)$, see Figure 2. This order of magnitude is obviously dependent on the choice of the parameters occurring in the Langevin Monte-Carlo algorithm for calculating $\hat{\theta}^{EW}$.

p	QE $D = 3$		QE $D = 5$		EW $D = 5$		LA $D = 5$	C01 $D = 5$
30	16	[1.9, 1366]	146	[125, 975]	7.4	[6, 9.3]	0.47	0.05
100	1956	[240, 5628]	>ams		112	[103, 121]	3.05	0.13
200	4240	[4008, 5178]	>ams		856	[813, 943]	8.0	0.65
300		>ams	>ams		4305	[4240, 4444]	14.7	2.12

TABLE 1

Means and ranges (in square brackets) of computing times in seconds calculated over $N_G = 100$ simulated graphs. For LA and C01 there is no variability in the computing times. >ams means that the maximum allowed memory size was exceeded.

FIGURE 2. Graphic of $\log^2(\text{CPU time})$ versus p for the EW method.

4.2. Methods comparison

We compare our methods with the following ones:

- the 0-1 conditional independence approach proposed by Wille and Bühlman [23], with the decision rule based on the adjusted p-values following the Benjamini-Hochberg procedure taking $\alpha = 5\%$.
- the lasso approach, with the two variants **and** and **or** proposed by Meinshausen and Bühlmann [17], taking $\alpha = 5\%$.
- the adaptive glasso method proposed by Fan *et al.* [9]. It works in two steps. First, the matrix Ω is estimated using the glasso method. Then the glasso procedure is run again using weights in the penalty that depend on the previous estimate of Ω , see Equation (2.5) in [9]. At each step the regularization parameter is calculated by K-fold cross-validation.

These methods will be denoted as **WB**, **MB.and**, **MB.or** and **Aglasso**. They were implemented in R-2.7.2 using the packages **lars** for the MB methods and the package **glasso** for the last one.

Assessing the performances of the methods We assess the performances of the investigated methods on the basis of $N_G \times N_X$ runs where N_G is the number of simulated graphs and N_X the number of matrices \mathbf{X} simulated for each of these graphs. We compare each simulated graph G with the estimated graphs \hat{G} by counting edges that are correctly identified as present or absent,

and those that are wrongly identified. We thus estimate the false discovery rate (or FDR) defined as the expected proportion of wrongly detected edges among edges detected as present, and the power defined as the expected proportion of rightly detected edges among edges present in the graph.

The statistical procedures designed to select graphs have one or several parameters that must be tuned. The quality of the final estimation is then affected as well by the intrinsic ability of the procedure to select an accurate graph, as by the parameter tuning. First, we investigate the first issue by varying the values of the tuning parameters and plotting *power versus FDR curves*. We choose $p = 100$, $n = 50$ and $I_s = 3$. Then, taking the point of view of a typical user, we compare the different procedures with the tuning parameter recommended in the literature. We investigate the effect of n by choosing $n = 30, 50, 100, 150$, keeping $p = 100$. We also evaluate the effect of graph sparsity taking $I_s = 1, 2, 3, 4, 5$, $p = 30$ to keep the computer time under reasonable values, and $n = 30$.

4.2.1. Power versus FDR curves when $p = 100$

The number of nodes p and the number of observations n being fixed to $p = 100$, $n = 50$, for each of the $N_G = 20$ simulated graphs, we estimated the FDR, the power and the MSEF on the basis of $N_X = 20$ simulations. These calculation are done for different values of the tuning parameter. The means over the N_G graphs are shown at Figure 3. The standard errors of the means over the N_G graphs are smaller than 0.0057 for the FDR, and 0.018 for the power.

Choice of the family of candidate graphs in our procedure The QE method presents good performances: the FDR stays small and the power is high. Though it was performed with $D = 3$, while EW, LA and C01 were performed with $D = 5$, it works the best. The EW method is more powerful than LA and C01 if one accepts a FDR greater than 2.5%.

Comparison with the other methods The procedures LA and C01 behave similarly to WB method. The MB.or method presents higher values of the power when the FDR is larger than 5%. The MB.and keeps down the FDR but lacks of power. The Aglasso method behaves completely in a different way: the curve stays under the others as long as the FDR is smaller than 20%. When the regularization parameter is chosen by 5-fold cross-validation, the power equals 59% at the price of a very large FDR equal to 90% (not shown). In the following we do not consider anymore the adaptive glasso method, and focus on methods that have a good control of the FDR.

4.2.2. Effect of the number of observations n

Keeping $p = 100$ and $I_s = 3$, the variations of the FDR and power values versus the number of observations, are shown at Figure 4. The QE method is applied with $D = 3$ while EW, LA and C01 are applied with $D = 5$. For all methods the

Power versus FDR

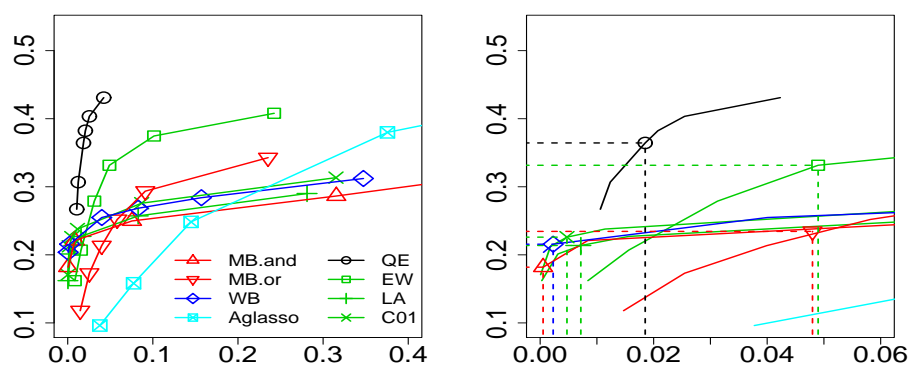


FIGURE 3. Graphics of power versus FDR for the case $p = 100$, $n = 50$ and $I_s = 3$. The marks on the first graphic correspond to different values of the tuning parameter. The curves for small FDR values are magnified on the second graphic. The FDR and power values corresponding to the tuning parameter recommended in the literature are superimposed on the curves (dashed lines) : $K = 2.5$ for GGMselect, $\alpha = 5\%$ for WB and MB methods. For Aglasso, with λ chosen by 5-fold cross-validation, the FDR equals 0.90 and the power equals 0.59 (not shown).

power increases with n while the FDR decreases for EW and increases for MB.or, LA and C01. QE and EW are the most powerful. When n is small, the QE method stays more powerful than EW in spite of a smaller D .

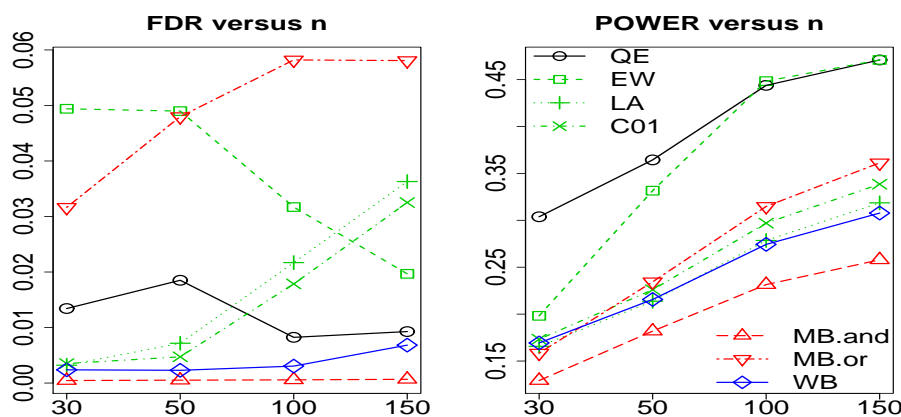


FIGURE 4. FDR and power estimated values as a function of n for $p = 100$ and $I_s = 3$. The results are calculated on the basis of $N_G = 20$ simulated graphs and $N_X = 20$ runs of matrices \mathbf{X} for each simulated graph. Our procedures were carried out with $K = 2.5$. The value of D was equal to 3 for the QE method and 5 for the others. For the procedures MB.or, MB.and and WB the tuning parameter α was taken equal to 5%.

4.2.3. Effect of graph sparsity

We have seen that when p is large, the GGMselect procedures using the graphs families QE and EW are powerful and have a good control of the FDR. Nevertheless, the simulated graphs were sparse, $I_s = 3$, and it may be worthwhile testing how the methods perform when the graph sparsity varies. Because the performances depend strongly on the simulated graph, the FDR and power are estimated on the basis of a large number of simulations: the number of simulated graphs N_G equals 50 and the number of simulated matrices \mathbf{X} for each graph, N_X equals 50. In order to keep reasonable computing times, we choose $p = 30$. The results are shown at Figure 5. The standard errors of the means over the N_G graphs are smaller than 0.0055 for the FDR, and 0.025 for the power.

For all methods the power decreases when I_s increases. The FDR values are slightly increasing with I_s for the EW and MB.or methods. The superiority of QE over the others is clear. EW is more powerful than LA, C01, MB and WB methods but its FDR is greater.

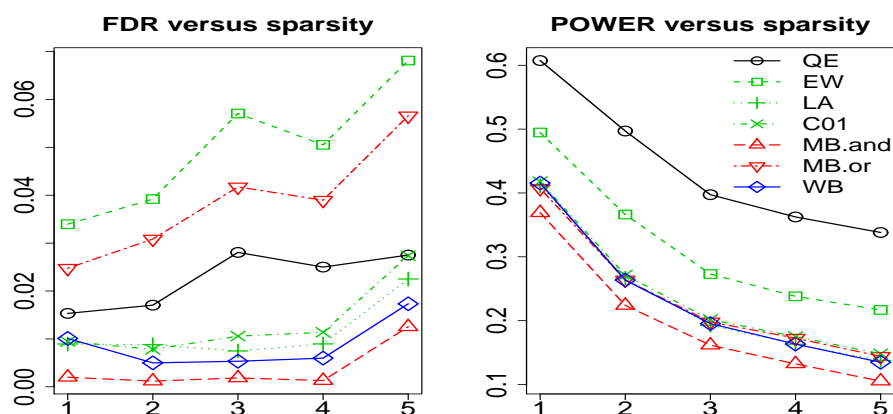


FIGURE 5. Graphs of FDR and power estimated values versus the graph sparsity I_s , for $p = 30$ and $n = 30$. The results are calculated on the basis of $N_G = 50$ simulated graphs and $N_X = 50$ runs of matrices \mathbf{X} for each simulated graph. Our procedures were carried out with $K = 2.5$ and $D = 5$. For the procedures MB.or, MB.and and WB the tuning parameter α was taken equal to 5%.

4.2.4. GGMselect : mixing the graphs families

Our procedure allows to mix several graphs families. It may happen that some graphs, or type of graphs, are known to be good candidates for modelling the observed data set. In that case, they can be considered in the procedure, and thus compete with $\hat{\mathcal{G}}_{EW}$ or $\hat{\mathcal{G}}_{QE}$. This can be done with the function `selectMyFam` of the package `GGMselect`.

Considering the results of our simulation study, we could ask if mixing $\hat{\mathcal{G}}_{LA}$ or $\hat{\mathcal{G}}_{C01}$ with $\hat{\mathcal{G}}_{EW}$ would not give a better control of the FDR than EW while keeping high values of the power. To answer this question we carried out simulation studies taking $\hat{\mathcal{G}}_{\text{mix}} = \hat{\mathcal{G}}_{C01} \cup \hat{\mathcal{G}}_{LA} \cup \hat{\mathcal{G}}_{EW}$ as the family of graphs. In all considered cases for p, n, I_s , the FDR and power values based on $\hat{\mathcal{G}}_{\text{mix}}$ are similar to those based on $\hat{\mathcal{G}}_{EW}$. This result can be explained by studying the behavior of the MSEP estimated by averaging the quantities $\|\Sigma^{1/2}(\hat{\theta}_{\hat{\mathcal{G}}} - \theta)\|^2$ over the $N_G \times N_X$ runs. The results are given at Figure 6. One can see that the smallest values of the MSEP are obtained for QE, then EW. Moreover, the MSEP decreases when the power increases, while it does not show any particular tendency when the FDR varies. Considering these tendencies together with the fact that our procedure aims at minimizing the MSEP, we can understand why we do not improve the performances of EW by considering $\hat{\mathcal{G}}_{\text{mix}}$.

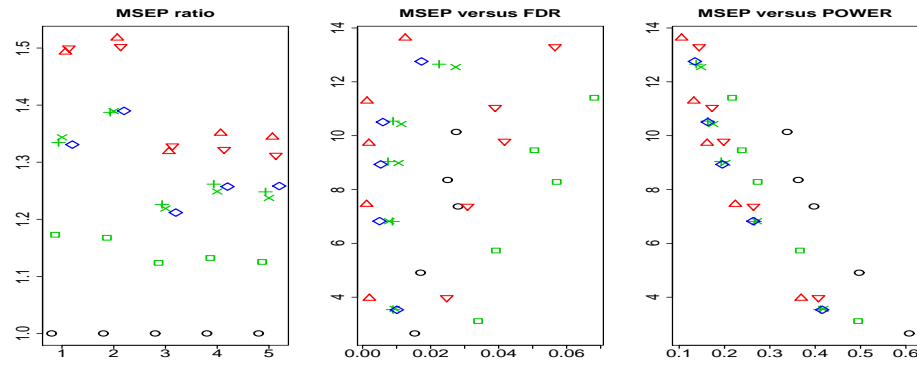


FIGURE 6. Values of the MSEP for the simulation results given at Figure 5. The first graphic on the left presents the ratio of the MSEP over the MSEP of the QE method. The two others present the MSEP versus the FDR and the power.

4.3. Summary

We recommend to use the QE method if the calculation of $\widehat{G}_{K,\text{and}}$ and $\widehat{G}_{K,\text{or}}$ is possible. Next, working out the family $\widehat{\mathcal{G}}_{\text{QE}}$ can always be done using some suitable algorithms if necessary (as a stepwise procedure for example). When p is large, QE can be used for small values of D ($D = 3$ or even $D = 2$). It may perform better than all the others when n is small. The procedure based on $\widehat{\mathcal{G}}_{\text{EW}}$ can be used for large p : the gain in power over LA, C01, MB and WB methods is significant, but the FDR is slightly greater. The LA and C01 methods are running very quickly, keep the FDR under control and are slightly more powerful than WB and MB. and.

5. Proofs

5.1. Proof of Theorem 3.1

We write \mathcal{G}_D for the family of all the graph with nodes in Γ and degree less than D . We remind the reader that for any graph $G \in \mathcal{G}_D$ we have noted Θ_G the space of $p \times p$ matrices θ such that $\theta_{a,b}$ is non zero if and only if there is an edge between a and b in G . We also set $\bar{\Theta}_{D_{\max}} = \cup_{G \in \mathcal{G}_{D_{\max}}} \Theta_G$. We set $\lambda = (1 - \sqrt{\gamma})^2$ and introduce the event

$$\mathbb{B} = \left\{ \lambda \|\Sigma^{1/2} A\|_{p \times p} \leq \frac{1}{\sqrt{n}} \|\mathbf{X} A\|_{n \times p} \leq \lambda^{-1} \|\Sigma^{1/2} A\|_{p \times p}, \text{ for all } A \in \theta + \bar{\Theta}_{D_{\max}} \right\}.$$

On this event we can control the L^2 -loss of $\tilde{\theta}$ by the empirical loss since

$$\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}} \leq \frac{\lambda^{-2}}{n} \|\mathbf{X}(\tilde{\theta} - \theta)\|_{n \times p}^2 \mathbf{1}_{\mathbb{B}}. \quad (13)$$

Furthermore, according to Lemma 1 in [11], we have $\mathbb{P}(\mathbb{B}^c) \leq 2e^{-n(\sqrt{\gamma}-\gamma)^2/2}$ when Condition (9) is met. To bound the risk of the procedure, we consider apart the events \mathbb{B} and \mathbb{B}^c .

5.1.1. Bound on $\mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}} \right]$

We have $\mathbf{X} = \mathbf{X}\theta + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ is a $n \times p$ matrix distributed as follows: for each $a \in \Gamma$, the column $\boldsymbol{\epsilon}_a$ is independent of \mathbf{X}_{-a} and is distributed according to the Gaussian law $\mathcal{N}(0, \sigma_a^2 I_n)$, with $\sigma_a^2 = 1/\Omega_{a,a}$. For any $G \in \mathcal{G}_D$, we write henceforth θ^G for the orthogonal projection of θ on Θ_G according to the Euclidean norm $\|\Sigma^{1/2} \cdot\|_{p \times p}$ on $\mathbb{R}^{p \times p}$. Similarly, we write $\bar{\theta}^G$ for the orthogonal projection of θ on Θ_G according to the (random) Euclidean norm $\|\mathbf{X} \cdot\|_{n \times p}$ on $\mathbb{R}^{p \times p}$. For any $G \in \mathcal{G}_D$, we write $d_a(G)$ for the degree of the node a in G and introduce the positive quantity

$$R(G) = \sum_{a=1}^p \left(1 + \frac{\text{pen}(d_a(G))}{n - d_a(G)} \right) (\|\mathbf{X}(\theta_a - \bar{\theta}_a^G)\|^2 + 2| \langle \mathbf{X}\theta_a - \mathbf{X}\bar{\theta}_a^G, \boldsymbol{\epsilon}_a \rangle |) + \sum_{a=1}^p \frac{\text{pen}(d_a(G))}{n - d_a(G)} \|\boldsymbol{\epsilon}_a\|^2,$$

where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the canonical norm and scalar product on \mathbb{R}^n . Following the same lines as in the beginning of the proof of Theorem 2 in Baraud *et al.* [2], we get for any G^* in $\hat{\mathcal{G}}$

$$\frac{K-1}{K} \|\mathbf{X}(\tilde{\theta} - \theta)\|_{n \times p}^2 \mathbf{1}_{\mathbb{B}} \leq R(G^*) \mathbf{1}_{\mathbb{B}} + \Delta(\hat{G}) \mathbf{1}_{\mathbb{B}} \quad (14)$$

with

$$\Delta(G) = \sum_{a=1}^p \sigma_a^2 \left(KU_{\text{ne}_G(a)} - \frac{\text{pen}(d_a(G))}{n - d_a(G)} V_{\text{ne}_G(a)} \right)_+$$

where $U_{\text{ne}_G(a)}$ and $V_{\text{ne}_G(a)}$ are two independent χ^2 random variables with $d_a(G) + 1$ and $n - d_a(G) - 1$ degrees of freedom.

We note that under Condition (9) there exists some constant $c(\gamma)$ depending on γ only, such that

$$\text{pen}(d) \leq c(\gamma)K(d+1) \log(p), \quad \text{for all } d \in \{0, \dots, D_{\max}\},$$

see Proposition 4 in Baraud *et al.* [2]. In particular, we have for any $G \in \mathcal{G}_D$

$$\frac{\text{pen}(d_a(G))}{n - d_a(G)} \leq \frac{c(\gamma)K(D_{\max} + 1) \log(p)}{n/2} \leq 4K\gamma c(\gamma) = L_{\gamma, K}.$$

Using this bound together with

$$|2 \langle \mathbf{X}\theta - \mathbf{X}\bar{\theta}_a^G, \boldsymbol{\epsilon}_a \rangle| \leq \|\mathbf{X}(\theta_a - \bar{\theta}_a^G)\|^2 + \sigma_a^2 \xi_{a,G}^2,$$

where for any $G \in \mathcal{G}$ and $a \in \{1, \dots, p\}$, the random variable

$$\xi_{a,G} = \langle \mathbf{X}(\theta_a - \bar{\theta}_a^G), \epsilon \rangle / (\sigma_a \|\mathbf{X}(\theta_a - \bar{\theta}_a^G)\|)$$

is standard Gaussian, we obtain

$$\begin{aligned} R(G) &\leq (1 + L_{\gamma,K}) \sum_{a=1}^p (2\|\mathbf{X}(\theta_a - \bar{\theta}_a^G)\|^2 + \sigma_a^2 \xi_{a,G}^2) + \frac{\text{pen}(d_a(G))}{n - d_a(G)} \|\epsilon_a\|^2 \\ &\leq 2(1 + L_{\gamma,K}) \|\mathbf{X}(\theta - \bar{\theta}^G)\|_{n \times p}^2 + (4 + L_{\gamma,K}) \sum_{a=1}^p \text{pen}(d_a(G)) \sigma_a^2 + r(\mathcal{G}_D) \end{aligned}$$

where

$$r(\mathcal{G}_D) = \sum_{a=1}^p \sigma_a^2 \left((1 + L_{\gamma,K}) \sum_{G \in \mathcal{G}} [\xi_{a,G}^2 - \text{pen}(d_a(G))]_+ + L_{\gamma,K} [\|\epsilon_a\|^2 / \sigma_a^2 - 3n/2]_+ \right).$$

Furthermore, we have $\|\mathbf{X}(\theta - \bar{\theta}^G)\|_{n \times p} \leq \|\mathbf{X}(\theta - \theta^G)\|_{n \times p}$ and on the event \mathbb{B} we also have $\|\mathbf{X}(\theta - \theta^G)\|_{n \times p}^2 \leq n\lambda^{-2} \|\Sigma^{1/2}(\theta - \theta^G)\|_{p \times p}^2$ so that on \mathbb{B}

$$R(G) \leq L'_{\gamma,K} \left(n\lambda^{-2} \|\Sigma^{1/2}(\theta - \theta^G)\|_{p \times p}^2 + \sum_{a=1}^p \text{pen}(d_a(G)) \sigma_a^2 \right) + r(\mathcal{G}_D),$$

with $L'_{\gamma,K} = \max(2 + 2L_{\gamma,K}, 4 + L_{\gamma,K})$. Putting this bound together with (13) and (14), we obtain

$$\begin{aligned} \|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}} &\leq \frac{K}{n\lambda^2(K-1)} \left(\inf_{G^* \in \hat{\mathcal{G}}} R(G^*) + \Delta(\hat{G}) \right) \mathbf{1}_{\mathbb{B}} \\ &\leq L''_{\gamma,K} \inf_{G^* \in \hat{\mathcal{G}}} \left(\|\Sigma^{1/2}(\theta - \theta^{G^*})\|_{p \times p}^2 + \sum_{a=1}^p \text{pen}(d_a(G^*)) \frac{\sigma_a^2}{n} \right) \\ &\quad + L''_{\gamma,K} n^{-1} \left(r(\mathcal{G}_D) + \Delta(\hat{G}) \right). \end{aligned}$$

We note that

$$n^{-1} \mathbb{E}(r(\mathcal{G}_D)) \leq \sum_{a=1}^p \frac{\sigma_a^2}{n} (1 + L_{\gamma,K}) (3 + \log(p))$$

and we get from the proof of Theorem 1 in [11] that

$$\begin{aligned} n^{-1} \mathbb{E}(\Delta(\hat{G})) &\leq n^{-1} \mathbb{E} \left(\sup_{G \in \hat{\mathcal{G}}_D} \Delta(G) \right) \\ &\leq K \sum_{a=1}^p \frac{\sigma_a^2}{n} (1 + \log(p)). \end{aligned}$$

Since $\text{pen}(d) \leq c(\gamma)K(d+1)\log(p)$, the latter bounds enforce the existence of constants $L_{\gamma,K}$ and $L'_{\gamma,K}$ depending on γ and K only, such that

$$\begin{aligned} & \mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}} \right] \\ & \leq L_{\gamma,K} \mathbb{E} \left[\inf_{G^* \in \widehat{\mathcal{G}}} \left(\|\Sigma^{1/2}(\theta - \theta^{G^*})\|_{p \times p}^2 + \sum_{a=1}^p (\log(p) \vee \text{pen}[d_a(G^*)]) \frac{\sigma_a^2}{n} \right) \right] \\ & \leq L'_{\gamma,K} \log(p) \left(\mathbb{E} \left[\inf_{G^* \in \widehat{\mathcal{G}}} \text{MSEP}(\widehat{\theta}_{G^*}) \right] \vee \sum_{a=1}^p \frac{\sigma_a^2}{n} \right). \end{aligned}$$

Finally, we note that $\sum_{a=1}^p \sigma_a^2/n = \text{MSEP}(I)$.

5.1.2. Bound on $\mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}^c} \right]$

We now prove the bound $\mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}^c} \right] \leq Ln^3 \text{tr}(\Sigma) \sqrt{\mathbb{P}(\mathbb{B}^c)}$. We have

$$\mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}^c} \right] = \sum_{a=1}^p \mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta}_a - \theta_a)\|^2 \mathbf{1}_{\mathbb{B}^c} \right]$$

and we will upper bound each of the p terms in this sum. Let a be any node in Γ . Given a graph G , the vector $[\widehat{\theta}_G]_a$ depends on G only through the neighborhood $\text{ne}_G(a)$ of a in G . Henceforth, we write $\widehat{\theta}_{\text{ne}_{\widehat{G}}(a)}$ for $\tilde{\theta}_a$ in order to emphasize this dependency. By definition $\widehat{\theta}_{\text{ne}_{\widehat{G}}(a)}$ is the least-squares estimator of θ_a with support included in $\text{ne}_{\widehat{G}}(a)$. Let us apply the same arguments as in the proof of Lemma 7.12 in [21]. By Cauchy-Schwarz inequality, we have

$$\mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta}_a - \theta_a)\|^2 \mathbf{1}_{\mathbb{B}^c} \right] \leq \sqrt{\mathbb{P}(\mathbb{B}^c)} \sqrt{\mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}_{\widehat{G}}(a)} - \theta_a)\|^4 \right]}. \quad (15)$$

Let $\mathcal{N}_D(a)$ be the set made of all the subsets of $\Gamma \setminus \{a\}$ whose size are smaller than $\gamma n / [2(1.1 + \sqrt{\log(p)})^2]$. By Condition (9), it holds that the estimated neighborhood $\text{ne}_{\widehat{G}}(a)$ belongs to $\mathcal{N}_D(a)$, so Hölder inequality gives

$$\begin{aligned} \mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}_{\widehat{G}}(a)} - \theta_a)\|^4 \right] &= \sum_{\text{ne}(a) \in \mathcal{N}_D(a)} \mathbb{E} \left[\mathbf{1}_{\text{ne}_{\widehat{G}}(a) = \text{ne}(a)} \|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^4 \right] \\ &\leq \sum_{\text{ne}(a) \in \mathcal{N}_D(a)} \mathbb{P} \left[\text{ne}_{\widehat{G}}(a) = \text{ne}(a) \right]^{1/u} \mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^{4v} \right]^{1/v} \\ &\leq \sum_{\text{ne}(a) \in \mathcal{N}_D(a)} \mathbb{P} \left[\text{ne}_{\widehat{G}}(a) = \text{ne}(a) \right]^{1/u} \sup_{\text{ne}(a) \in \mathcal{N}_D(a)} \mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^{4v} \right]^{1/v}, \end{aligned}$$

where $v = \lfloor \frac{n}{8} \rfloor$, and $u = \frac{v}{v-1}$ (we remind the reader that n is larger than 8). In particular, we have the crude bound

$$\begin{aligned} & \sqrt{\mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}_{\widehat{G}}(a)} - \theta_a)\|^4 \right]} \\ & \leq [\text{Card}(\mathcal{N}_D(a))]^{1/2v} \sup_{\text{ne}(a) \in \mathcal{N}_D(a)} \mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^{4v} \right]^{1/2v}, \end{aligned}$$

since the sum is maximum when every $\mathbb{P}[\text{ne}(a) = \text{ne}_{\widehat{G}}(a)]$ equals $[\text{Card}(\mathcal{N}_D(a))]^{-1}$. We first bound the term $[\text{Card}(\mathcal{N}_D(a))]^{1/2v}$. The size of the largest subset in $\mathcal{N}_D(a)$ is smaller than $n/(2 \log(p))$, so the cardinality of $\mathcal{N}_D(a)$ is smaller than $p^{\frac{D}{\varphi}}$. Since n is larger than 8, we get

$$[\text{Card}(\mathcal{N}_D(a))]^{1/2v} \leq \exp \left[\frac{n}{4 \lfloor n/8 \rfloor} \right] \leq L,$$

which ensures the bound

$$\sqrt{\mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}_{\widehat{G}}(a)} - \theta_a)\|^4 \right]} \leq L \sup_{\text{ne}(a) \in \mathcal{N}_D(a)} \mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^{4v} \right]^{1/2v}. \quad (16)$$

To conclude, we need to upper bound this supremum. Given a subset $\text{ne}(a)$ in $\mathcal{N}_D(a)$, we define $\theta_{\text{ne}(a)}$ as the vector in \mathbb{R}^p such that $\Sigma^{1/2}\theta_{\text{ne}(a)}$ is the orthogonal projection of $\Sigma^{1/2}\theta_a$ onto the linear span $\{\Sigma^{1/2}\beta : \text{supp}(\beta) \subset \text{ne}(a)\}$. Pythagorean inequality gives

$$\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^2 = \|\Sigma^{1/2}(\theta_{\text{ne}(a)} - \theta_a)\|^2 + \|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_{\text{ne}(a)})\|^2$$

and we obtain from Minkowski's inequality that

$$\begin{aligned} & \mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^{4v} \right]^{1/(2v)} \\ & \leq \|\Sigma^{1/2}(\theta_{\text{ne}(a)} - \theta_a)\|^2 + \mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_{\text{ne}(a)})\|^{4v} \right]^{1/(2v)}. \end{aligned}$$

The first term is smaller than $\text{Var}(X_a)$. In order to bound the second term, we use the following lemma which rephrases Proposition 7.8 in [21].

Lemma 5.1. *For any neighborhood $\text{ne}(a)$ and any $r > 2$ such that $n - |\text{ne}(a)| - 2r + 1 > 0$,*

$$\mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_{\text{ne}(a)})\|^{2r} \right]^{1/r} \leq Lr |\text{ne}(a)| n \text{Var}(X_a).$$

Since v is smaller than $n/8$ and since $|\text{ne}(a)|$ is smaller than $n/2$, it follows that for any model $\text{ne}(a) \in \mathcal{N}_D(a)$, $n - |\text{ne}(a)| - 4v + 1$ is positive and

$$\mathbb{E} \left[\|\Sigma^{1/2}(\widehat{\theta}_{\text{ne}(a)} - \theta_a)\|^{4v} \right]^{1/(2v)} \leq \text{Var}(X_a) [1 + Ln^2v] \leq Ln^3 \Sigma_{a,a}.$$

Gathering this last upper bound with (15) and (16), we get that

$$\mathbb{E} \left[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{p \times p}^2 \mathbf{1}_{\mathbb{B}^c} \right] \leq Ln^3 \text{tr}(\Sigma) \sqrt{\mathbb{P}(\mathbb{B}^c)}.$$

5.1.3. Conclusion

Finally, putting together the bound on $\mathbb{E}[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{\mathbb{B}}^2]$, the bound on $\mathbb{E}[\|\Sigma^{1/2}(\tilde{\theta} - \theta)\|_{\mathbb{B}^c}^2]$, and the bound $\mathbb{P}(\mathbb{B}^c) \leq 2pe^{-n(\sqrt{\gamma}-\gamma)^2/2}$, we obtain

$$\text{MSEP}(\tilde{\theta}) \leq L_{K,\gamma} \log(p) \left(\mathbb{E} \left[\inf_{G \in \hat{\mathcal{G}}} \left(\text{MSEP}(\hat{\theta}_G) \right) \right] \vee \frac{\text{MSEP}(I)}{n} \right) + R_n,$$

with $R_n \leq Ln^3 \text{tr}(\Sigma) e^{-n(\sqrt{\gamma}-\gamma)^2/4}$.

5.2. Proof of Corollary 3.2

The result is proved analogously except that we replace the event \mathbb{B} by

$$\mathbb{B}' = \mathbb{B} \cup \left\{ G_{\Sigma} \in \hat{\mathcal{G}} \right\}.$$

Hence, the residual term now satisfies

$$\begin{aligned} R_n &\leq Ln^3 \text{tr}(\Sigma) \sqrt{\mathbb{P}(\mathbb{B}^c)} \\ &\leq Ln^3 \text{tr}(\Sigma) \left[e^{-n(\sqrt{\gamma}-\gamma)^2/4} + \sqrt{L(\alpha)} e^{-\frac{\beta}{2} n^{\delta}} \right]. \end{aligned}$$

5.3. Proof of Theorem 3.3

In this proof, the notations $o(1)$, $O(1)$ respectively refer to sequences that converge to 0 or stay bounded when n goes to infinity. These sequences may depend on K , s , s' but *do not* depend on G_n , on the covariance Σ , or a particular subset $S \subset \Gamma$. The technical lemmas are postponed to Section 5.4. In the sequel, we omit the dependency of p and Σ on n for the sake of clarity. First, observe that the result is trivial if $n/\log(p)^2 < 1$, because the assumptions imply that G_{Σ} is the empty graph whereas the family $\hat{\mathcal{G}}$ contains at most the empty graph. In the sequel, we assume that $n/\log(p)^2 \geq 1$.

Let us set $D_{\max} = n/\log(p)^2$. We shall prove that for some $L > 0$,

$$\mathbb{P} \left(\text{Crit}(G_{\Sigma}) = \inf_{G', \text{deg}(G') \leq D_{\max}} \text{Crit}(G') \right) \geq 1 - Lp^{-1/2}, \quad (17)$$

for n larger than $n_0(K, s, s')$. Since \hat{G} minimizes the criterion $\text{Crit}(\cdot)$ on the family $\hat{\mathcal{G}}$, this will imply the result of the theorem.

In fact, we shall prove a slightly stronger result than (17). Let a be a node in Γ and let $\text{ne}(a)$ be a subset of $\Gamma \setminus \{a\}$. As defined in Section 5.1.2, $\widehat{\theta}_{\text{ne}(a)}$ is the least-squares estimator of θ_a whose support is included in $\text{ne}(a)$.

$$\widehat{\theta}_{\text{ne}(a)} = \arg \inf_{\theta'_a, \text{supp}(\theta'_a) \subset \text{ne}(a)} \|\mathbf{X}_a - \mathbf{X}\theta'_a\|_n^2 .$$

If G is a graph such that the neighborhood $\text{ne}_G(a)$ equals $\text{ne}(a)$, then $\widehat{\theta}_{\text{ne}(a)} = [\widehat{\theta}_G]_a$. We then define the partial criterion $\text{Crit}(a, \text{ne}(a))$ by

$$\text{Crit}(a, \text{ne}(a)) = \|\mathbf{X}_a - \mathbf{X}\widehat{\theta}_{\text{ne}(a)}\|_n^2 \left(1 + \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right) .$$

Observe that for any graph G , $\text{Crit}(G) = \sum_{a=1}^p \text{Crit}(a, \text{ne}_G(a))$. We note $\widehat{\text{ne}}(a)$ the set that minimizes the criterion $\text{Crit}(a, \cdot)$ among all subsets of size smaller than D_{\max} .

$$\widehat{\text{ne}}(a) = \arg \inf_{\text{ne}(a) \in \mathcal{N}_{D_{\max}}(a)} \text{Crit}(a, \text{ne}(a)) .$$

If for all nodes $a \in \Gamma$, the selected set $\widehat{\text{ne}}(a)$ equals $\text{ne}_{G_\Sigma}(a)$, then G_Σ minimizes the criterion $\text{Crit}(\cdot)$ over all graphs of degree smaller than D_{\max} . Consequently, the property (17) is satisfied if for any node $a \in \Gamma$, it holds that

$$\mathbb{P} [\widehat{\text{ne}}(a) = \text{ne}_{G_\Sigma}(a)] \geq 1 - 7p_n^{-3/2} , \quad (18)$$

for n larger than some $n_0[K, s, s']$.

Let us fix some node $a \in \Gamma$. We prove the lower bound (18) in two steps:

1. With high probability, the estimated neighborhood $\widehat{\text{ne}}(a)$ does not strictly contain the true one $\text{ne}_{G_\Sigma}(a)$.

$$\mathbb{P} [\widehat{\text{ne}}(a) \supsetneq \text{ne}_{G_\Sigma}(a)] \leq p_n^{-3/2} , \quad (19)$$

for n larger than some $n_0[K, s, s']$.

2. With high probability, the estimated neighborhood $\widehat{\text{ne}}(a)$ contains the true one $\text{ne}_{G_\Sigma}(a)$.

$$\mathbb{P} [\widehat{\text{ne}}(a) \not\supseteq \text{ne}_{G_\Sigma}(a)] \leq 6p_n^{-3/2} , \quad (20)$$

for n larger than some $n_0[K, s, s']$.

The remaining part of the proof is deserved to (19) and (20).

Let us recall some notations and let us introduce some other ones. The component X_a decomposes as

$$X_a = X\theta_a + \epsilon_a ,$$

where ϵ_a follows a centered normal distribution with variance $\Omega_{a,a}^{-1} = \text{Var}(X_a|X_{-a})$. The variables ϵ_a are independent of X_{-a} . Given a set $S \subset \Gamma$, Π_S stands for the projection of \mathbb{R}^n into the space generated by $(\mathbf{X}_a)_{a \in S}$, whereas Π_S^\perp denotes the projection along the space generated by $(\mathbf{X}_a)_{a \in S}$. The notation $\langle \cdot, \cdot \rangle_n$ refers to the empirical inner product associated with the norm $\|\cdot\|_n$. For any neighborhood $\text{ne}(a) \subset \Gamma \setminus \{a\}$ such that $|\text{ne}(a)| \leq D_{\max}$, let us define $\Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a))$ by

$$\Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a)) = \text{Crit}(a, \text{ne}(a)) - \text{Crit}(a, \text{ne}_{G_\Sigma}(a)) .$$

5.3.1. Bound on $\mathbb{P}(\widehat{\text{ne}}(a) \not\supseteq \text{ne}_{G_\Sigma}(a))$

We shall upper bound the probability that $\Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a))$ is negative for at least one of the neighborhoods $\text{ne}(a) \in \mathcal{N}_{D_{\max}}(a)$ such that $\text{ne}(a)$ strictly contains $\text{ne}_{G_\Sigma}(a)$. For such a set $\text{ne}(a)$, $\Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a))$ decomposes as (see e.g. Lemma 7.1 in [21])

$$\begin{aligned} \Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a)) &= \|\Pi_{\text{ne}(a)}^\perp \epsilon_a\|_n^2 \left[1 + \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right] - \|\Pi_{\text{ne}_{G_\Sigma}(a)}^\perp \epsilon_a\|_n^2 \left[1 + \frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n - |\text{ne}_{G_\Sigma}(a)|} \right] \\ &= -\|\Pi_{\text{ne}_{G_\Sigma}(a) \cap \text{ne}(a)}^\perp \epsilon_a\|_n^2 \left[1 + \frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n - |\text{ne}_{G_\Sigma}(a)|} \right] \\ &\quad + \|\Pi_{\text{ne}(a)}^\perp \epsilon_a\|_n^2 \left[\frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} - \frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n - |\text{ne}_{G_\Sigma}(a)|} \right] . \end{aligned}$$

Hence, $\Delta(m, \text{ne}_{G_\Sigma}(a)) > 0$ if

$$\begin{aligned} &\frac{\|\Pi_{\text{ne}_{G_\Sigma}(a) \cap \text{ne}(a)}^\perp \epsilon_a\|_n^2 / (|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|)}{\|\Pi_{\text{ne}(a)}^\perp \epsilon_a\|_n^2 / (n - |\text{ne}(a)|)} \\ &< \frac{\text{pen}(|\text{ne}(a)|) - \text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \left[1 + \frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n - |\text{ne}_{G_\Sigma}(a)|} \right]^{-1} . \quad (21) \end{aligned}$$

To conclude, it remains to prove that the bound (21) holds with high probability. Let us call A_1 the right expression of (21) and let us derive a lower bound of A_1 . Afterwards, we shall upper bound with high probability the left expression of (21).

Upper bound of A_1 . We first upper bound the penalty function.

Lemma 5.2. *Let $d_1 \geq d_2$ be two positive integers such that $d_1 \leq e^{-3/2}(p-1)$. We have*

$$\text{pen}(d_1) - \text{pen}(d_2) \geq 2K(d_1 - d_2) \log \left(\frac{p - d_1}{d_1} \right) . \quad (22)$$

A proof of this lemma is provided in Section 5.4. By Proposition 4 in [2], the penalty $\text{pen}(|\text{ne}_{G_\Sigma}(a)|)$ satisfies

$$\text{pen}(|\text{ne}_{G_\Sigma}(a)|) \leq LK \frac{|\text{ne}_{G_\Sigma}(a)|}{n} \log \left(\frac{p-1}{|\text{ne}_{G_\Sigma}(a)|} \right),$$

where L is some numerical constant. This last term converges towards 0 as n goes to infinity since $|\text{ne}_{G_\Sigma}(a)| \leq (n^s / \log(p)) \wedge (n / \log(p)^2)$ (Assumption 2). Gathering this upper bound with Lemma 5.2, we get

$$A_1 \geq 2K \frac{\log \left(\frac{p-|\text{ne}(a)|}{|\text{ne}(a)|} \right)}{1 + \frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n-|\text{ne}_{G_\Sigma}(a)|}} \geq 2K \log \left(\frac{p}{|\text{ne}(a)|} \right) (1 - o(1)). \quad (23)$$

Lower bound of the left part of (21). The random variables involved in this expression follow a Fisher distribution with $|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|$ and $n - |\text{ne}(a)|$ degrees of freedom. To conclude, we only need to compare the quantile of such a variable with the bound (23). Let $u \in (0, 1)$ and let $F_{D,N}^{-1}(u)$ denote the $1 - u$ quantile of a Fisher random variable with D and N degrees of freedom. By Lemma 1 in [3], it holds that

$$\begin{aligned} DF_{D,N}^{-1}(u) &\leq D + 2\sqrt{D \left(1 + 2\frac{D}{N}\right) \log \left(\frac{1}{u}\right)} \\ &\quad + \left(1 + 2\frac{D}{N}\right) \frac{N}{2} \left[\exp \left(\frac{4}{N} \log \left(\frac{1}{u} \right) \right) - 1 \right]. \end{aligned}$$

Let us set u to

$$u = \left\{ p^{3/2} e^{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \left(\frac{p - |\text{ne}_{G_\Sigma}(a)| - 1}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \right) \right\}^{-1}.$$

Since we consider the case $n / \log(p)^2 \geq 1$ and $p \geq n$, the term $4 / (n - |\text{ne}(a)|) \log(1/u)$ goes to 0 with n (uniformly w.r.t. $\text{ne}(a)$).

$$\begin{aligned} A_2 = F_{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|, n - |\text{ne}(a)|}^{-1}(u) &\leq 1 + 2\sqrt{\frac{1}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} (1 + o(1)) \log \left(\frac{1}{u} \right)} \\ &\quad + \frac{2}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} (1 + o(1)) \log \left(\frac{1}{u} \right). \end{aligned}$$

The term $\log(1/u) / |\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|$ goes to infinity with n (uniformly w.r.t. $\text{ne}(a)$). Hence, we get

$$A_2 \leq 1 + \frac{2}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \log \left(\frac{1}{u} \right) (1 + o(1)).$$

Applying the classical inequality $\log \binom{l}{k} \leq k \log(el/k)$, we obtain

$$\begin{aligned} A_2 &\leq \left[3 \frac{\log(p)}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} + 2 \log \left(\frac{p}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \right) \right] (1 + o(1)) \\ &\leq 5 \log \left(\frac{p}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \right) (1 + o(1)) . \end{aligned} \quad (24)$$

Conclusion. Let us compare the lower bound (23) of A_1 with the upper bound (24) of A_2 .

- Let us first assume that $|\text{ne}(a)| \leq 2|\text{ne}_{G_\Sigma}(a)|$. Then, we have

$$A_1 \geq 2K \log \left(\frac{p}{|\text{ne}_{G_\Sigma}(a)|} \right) (1 - o(1)) \geq 2K(1 - s) \log(p) (1 - o(1)) ,$$

since $|\text{ne}_{G_\Sigma}(a)| \leq n^s / \log(p) \leq p^s$. In particular,

$$A_2 \leq 5 \log \left(\frac{p}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \right) (1 + o(1)) < A_1 ,$$

for n large enough since we assume that $2K(1 - s) > 5$.

- If $|\text{ne}(a)| > 2|\text{ne}_{G_\Sigma}(a)|$, we also have

$$A_2 \leq 5 \log \left(\frac{p}{|\text{ne}(a)|} \right) (1 + o(1)) < A_1 ,$$

for n large enough since we assume that $2K > 5$.

It follows from Ineq. (21) and the definition of A_1 and A_2 that

$$\mathbb{P} [\Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a)) < 0] \leq \left\{ p^{3/2} e^{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \left(\frac{p - |\text{ne}_{G_\Sigma}(a)|}{|\text{ne}(a) \setminus \text{ne}_{G_\Sigma}(a)|} \right) \right\}^{-1} ,$$

for n larger than some positive constant that may depend on K, s , but does *not* depend on $\text{ne}(a)$. Applying this bound to any neighborhood $\text{ne}(a)$ that strictly contains $\text{ne}_{G_\Sigma}(a)$ yields Statement (19):

$$\mathbb{P} [\widehat{\text{ne}}(a) \not\supseteq \text{ne}_{G_\Sigma}(a)] \leq p^{-3/2} ,$$

for n large enough.

5.3.2. Bound on $\mathbb{P}(\widehat{\text{ne}}(a) \not\supseteq \text{ne}_{G_\Sigma}(a))$

Again, we shall prove that $\Delta[\text{ne}(a), \text{ne}_{G_\Sigma}(a)]$ is positive for $\text{ne}(a) \not\supseteq \text{ne}_{G_\Sigma}(a)$ with overwhelming probability. We recall that $\theta_{\text{ne}(a)}$ is the vector in \mathbb{R}^p such that $\Sigma^{1/2} \theta_{\text{ne}(a)}$ is the orthogonal projection of $\Sigma^{1/2} \theta_a$ onto the linear span $\{\Sigma^{1/2} \beta : \text{supp}(\beta) \subset \text{ne}(a)\}$. Moreover, $\|\Sigma^{1/2}(\theta_a - \theta_{\text{ne}(a)})\|^2 = \text{Var}(X_a | X_{\text{ne}(a)}) - \text{Var}(X_a | X_{-a})$ (see e.g. Lemma 7.1 in [21]).

Then, $\Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a))$ decomposes as

$$\begin{aligned} \Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a)) &= \left\| \Pi_{\text{ne}(a)}^\perp [\boldsymbol{\epsilon}_a + \mathbf{X}(\theta_a - \theta_{\text{ne}(a)})] \right\|_n^2 \left[1 + \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right] \\ &\quad - \left\| \Pi_{\text{ne}_{G_\Sigma}(a)}^\perp \boldsymbol{\epsilon}_a \right\|_n^2 \left[1 + \frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n - |\text{ne}_{G_\Sigma}(a)|} \right]. \end{aligned}$$

Let $\kappa = 6/7$ and let us define

$$E_{\text{ne}(a)} = \kappa^{-1} \left\langle \frac{\Pi_{\text{ne}(a)}^\perp \mathbf{X}(\theta - \theta_{\text{ne}(a)})}{\|\Pi_{\text{ne}(a)}^\perp \mathbf{X}(\theta - \theta_{\text{ne}(a)})\|_n}, \Pi_{\text{ne}(a)}^\perp \boldsymbol{\epsilon}_a \right\rangle_n^2 + \|\Pi_{\text{ne}(a)}^\perp \boldsymbol{\epsilon}_a\|_n^2.$$

We recall that $\langle \cdot, \cdot \rangle_n$ is the inner product associated to the norm $\|\cdot\|_n$. The quantity $\Delta(\text{ne}(a), \text{ne}_{G_\Sigma}(a))$ is positive if

$$\begin{aligned} (1 - \kappa) \|\Pi_{\text{ne}(a)}^\perp \mathbf{X}(\theta - \theta_{\text{ne}(a)})\|_n^2 &> E_{\text{ne}(a)} \left[1 + \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right] \\ &+ \|\boldsymbol{\epsilon}_a\|_n^2 \left[\frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n - |\text{ne}_{G_\Sigma}(a)|} - \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right]. \end{aligned} \quad (25)$$

We respectively call A_3 and A_4 the right and the left terms of the inequality. To conclude, we need to control the deviations of these terms in order to prove that (25) holds with high probability.

Upper Bound of A_3 . On an event \mathbb{A} of probability larger than $1 - 2p^{-3/2}$, the random variable $\|\boldsymbol{\epsilon}_a\|_n^2$ satisfies (see Lemma 1 in [16])

$$1 - 2\sqrt{\frac{3 \log(p)}{2n}} \leq \frac{\|\boldsymbol{\epsilon}_a\|_n^2}{\text{Var}(X_a|X_{-a})} \leq 1 + 2\sqrt{\frac{3 \log(p)}{2n}} + 3\frac{\log(p)}{n}.$$

Let us bound the other random variables involved in (25). As explained in the proof of Th.3.1 in [21], the random variables $\|\Pi_{\text{ne}(a)}^\perp \mathbf{X}(\theta - \theta_{\text{ne}(a)})\|_n^2$ and $E_{\text{ne}(a)}$ follow distributions of linear combinations of χ^2 random variables. We apply again Lemma 1 in [16]. On a event $\mathbb{A}_{\text{ne}(a)}$ of probability larger than $1 - 2p^{-3/2}e^{-|\text{ne}(a)|} \binom{p-1}{|\text{ne}(a)|}^{-1}$, it holds that

$$\begin{aligned} \frac{\|\Pi_{\text{ne}(a)}^\perp \mathbf{X}(\theta - \theta_{\text{ne}(a)})\|_n^2}{\text{Var}(X_a|X_{\text{ne}(a)}) - \text{Var}(X_a|X_{-a})} &\geq 1 - \frac{|\text{ne}(a)|}{n} \\ &\quad - 2\sqrt{\frac{\frac{3}{2} \log(p) + |\text{ne}(a)| [2 + \log(p-1)]}{n}} \end{aligned}$$

and

$$\begin{aligned} \frac{E_{\text{ne}(a)}}{\text{Var}(X_a|X_{-a})} &\leq \frac{|\text{ne}(a)| + \kappa^{-1}}{n} \\ &+ \frac{2}{n} \sqrt{(|\text{ne}(a)| + \kappa^{-2}) \left[|\text{ne}(a)| \left(2 + \log \left(\frac{p-1}{|\text{ne}(a)|} \right) \right) + \frac{3}{2} \log(p) \right]} \\ &+ \frac{2\kappa^{-1}}{n} \left[|\text{ne}(a)| \left(2 + \log \left(\frac{p-1}{|\text{ne}(a)|} \right) \right) + \frac{3}{2} \log(p) \right]. \end{aligned}$$

We derive that

$$\begin{aligned} \frac{E_{\text{ne}(a)}}{\text{Var}(X_a|X_{-a})} &\leq \frac{2\kappa^{-1}}{n} \left[|\text{ne}(a)| \log \left(\frac{p-1}{|\text{ne}(a)|} \right) + \frac{3}{2} \log(p) \right] (1 + o(1)) \\ &+ \frac{\sqrt{6|\text{ne}(a)| \log(p)}}{n} + \frac{\kappa^{-1}}{n}. \end{aligned}$$

- **CASE 1: $\text{ne}(a)$ is non empty.**

$$\frac{E_{\text{ne}(a)}}{\text{Var}(X_a|X_{-a})} \leq \kappa^{-1} \frac{2|\text{ne}(a)| \log \left(\frac{p-1}{|\text{ne}(a)|} \right) + 3 \log(p)}{n} (1 + o(1)).$$

Let us upper bound the terms involving $\text{pen}(|\text{ne}(a)|)$ in (25) on the event $\mathbb{A} \cap \mathbb{A}_{\text{ne}(a)}$.

$$\begin{aligned} &\left\{ E_{\text{ne}(a)} \left[1 + \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right] - \|\epsilon_a\|_n^2 \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right\} / \text{Var}(X_a|X_{-a}) \\ &\leq \frac{\kappa^{-1}}{n} \left(2|\text{ne}(a)| \log \left(\frac{p-1}{|\text{ne}(a)|} \right) + 3 \log(p) \right) (1 + o(1)) \\ &\quad - \frac{2K}{n} |\text{ne}(a)| \log \left(\frac{p-1}{|\text{ne}(a)|} \right) (1 + o(1)). \end{aligned}$$

This last quantity is negative for n large enough since $K \geq 3$.

- **CASE 2: $\text{ne}(a)$ is empty.** We get the upper bound

$$\begin{aligned} E_{\text{ne}(a)} \left[1 + \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \right] - \|\epsilon_a\|_n^2 \frac{\text{pen}(|\text{ne}(a)|)}{n - |\text{ne}(a)|} \\ &\leq \frac{\kappa^{-1} + 3 \log(p)}{n} \text{Var}(X_a|X_{-a}) \\ &\leq (3 + \kappa^{-1}) n^{s-1} \text{Var}(X_a|X_{-a}). \end{aligned}$$

Indeed, $\log(p)$ has to be smaller than n^s . If this is not the case, then $\text{ne}_{G_\Sigma}(a)$ should be empty and $\text{ne}(a)$ cannot satisfy $\text{ne}_{G_\Sigma}(a) \not\subseteq \text{ne}(a)$.

We conclude that on the event $\mathbb{A} \cap \mathbb{A}_{\text{ne}(a)}$,

$$A_3 \leq (3 + \kappa^{-1})n^{s-1}\text{Var}(X_a|X_{-a}) + \|\epsilon_a\|_n^2 \frac{\text{pen}(|\text{ne}_{G_\Sigma}(a)|)}{n - |\text{ne}_{G_\Sigma}(a)|},$$

for n large enough. Let us upper bound the penalty term as done in the upper bound of A_1 .

$$\text{pen}(\text{ne}_{G_\Sigma}(a)) \leq LK \frac{|\text{ne}_{G_\Sigma}(a)|}{n} \log \left(\frac{p-1}{|\text{ne}_{G_\Sigma}(a)|} \right).$$

Since $|\text{ne}_{G_\Sigma}(a)|$ is assumed to be smaller than $\frac{n^s}{\log(p)}$, the term A_3 is upper bounded as follows

$$A_3 \leq (K+1)n^{s-1}\text{Var}(X_a|X_{-a})O(1). \quad (26)$$

for n large enough.

Lower Bound of A_4 . Let us lower bound the left term A_4 in (25) on the event $\mathbb{A} \cap \mathbb{A}_{\text{ne}(a)}$.

$$\begin{aligned} A_4 &\geq (1 - o(1))(1 - \kappa) [\text{Var}(X_a|X_{\text{ne}(A)}) - \text{Var}(X_a|X_{-a})] \\ &\geq (1 - o(1))(1 - \kappa) \min_{b \in \Gamma \setminus \{a\}} (\theta_{a,b})^2 \min_{b,c \in \Gamma \setminus \{a\}} \frac{\text{Var}(X_b|X_{-b})}{\text{Var}(X_c|X_{-c})} \text{Var}(X_a|X_{-a}) \\ &\geq (1 - \kappa)(1 - o(1))n^{s'-1}\text{Var}(X_a|X_{-a}). \end{aligned}$$

Thanks to the last bound and (26) and since s' is larger than s , $A_3 < A_4$ on the event $\mathbb{A} \cap \mathbb{A}_{\text{ne}(a)}$ and for n large enough (not depending on $\text{ne}(a)$). Hence, for n large enough the inequality (25) holds simultaneously for all neighborhoods $\text{ne}(a)$ such that $\text{ne}_{G_\Sigma}(a) \not\subseteq \text{ne}(a)$ with probability larger than $1 - 2p^{-3/2} - 2(e/(e-1))p^{-3/2}$. We conclude that

$$\mathbb{P}(\widehat{\text{ne}}(a) \not\subseteq \text{ne}_{G_\Sigma}(a)) \leq 6p^{-3/2},$$

for n large enough.

5.4. Lemmas

Lemma 5.3. For any positive integer $d \leq e^{-3/2}(p-1)$,

$$\text{EDKhi} \left[d+1, n-d-1, \left[\binom{p-1}{d} (d+1)^2 \right]^{-1} \right] \geq d+1.$$

Lemma 5.4. For any positive number x and any positive integers d and N , $\text{EDKhi}(d, N, x)$ is an increasing function with respect to d and a decreasing function with respect to N .

Lemma 5.5. For any integer $d \geq 2$, the function

$$\frac{\mathbb{E}[(X_d - x \frac{X_N}{N})_+]}{\mathbb{E}[(X_2 - x)_+]}$$

is increasing with respect to x as soon as $x \geq d$.

Proof of Lemma 5.2. Let us write $L_1 = \log\left(\binom{p-1}{d_1}\right)$ and $L_2 = \log\left(\binom{p-1}{d_2}\right)$. Lemma 5.4 ensures that

$$\text{EDKhi}(d_1 + 1, n - d_1 - 1, e^{-L_1}) \geq \text{EDKhi}(d_2 + 1, n - d_2 - 1, e^{-L_1}) . \quad (27)$$

Let $x_1 \geq x_2$ be two positive numbers larger than some integer $d_2 + 1$. By Lemma 5.5, it holds that

$$\frac{\text{DKhi}(d_2 + 1, n - d_2 - 1, x_1)}{\text{DKhi}(d_2 + 1, n - d_2 - 1, x_2)} \geq \frac{\mathbb{E}[(X_2 - x_1)_+]}{\mathbb{E}[(X_2 - x_2)_+]} = e^{-(x_1 - x_2)/2} .$$

By Lemma 5.3, $\text{EDKhi}(d_2 + 1, n - d_2 - 1, e^{-L_2})$ is larger than $d_2 + 1$. Setting $x_1 = \text{EDKhi}(d_2 + 1, n - d_2 - 1, e^{-L_1})$ and $x_2 = \text{EDKhi}(d_2 + 1, n - d_2 - 1, e^{-L_2})$, we obtain

$$\text{EDKhi}(d_2 + 1, n - d_2 - 1, e^{-L_1}) - \text{EDKhi}(d_2 + 1, n - d_2 - 1, e^{-L_2}) \geq 2(L_1 - L_2) , \quad (28)$$

for $d_2 \geq 1$. Gathering the bounds (27), (28) with the definition (3) of the penalty enables to conclude

$$\text{pen}(d_1) - \text{pen}(d_2) \geq 2K(d_1 - d_2) \log\left(\frac{p - d_1}{d_1}\right) . \quad (29)$$

□

Proof of Lemma 5.3. We write henceforth X_d and X'_N for two independent χ^2 random variables with d and N degrees of freedom. Applying Jensen inequality we obtain that for any $x > 0$ and any $d \geq 2$

$$\begin{aligned} d \times \text{DKhi}(d, N, x) &= \mathbb{E} \left[\left(X_d - x \frac{X'_N}{N} \right)_+ \right] \\ &\geq \mathbb{E} [(X_d - x)_+] \\ &\geq \mathbb{E} [(X_2 - x)_+] = 2e^{-x/2} . \end{aligned}$$

Setting $x = \text{EDKhi}(d, N, e^{-L})$ with $L \geq 0$, we obtain

$$\text{EDKhi}(d, N, e^{-L}) \geq 2L - 2 \log(d), \quad \text{for } d \geq 2.$$

It follows that

$$\begin{aligned} \text{EDKhi} \left[d + 1, n - d - 1, \left[\binom{p-1}{d} (d+1)^2 \right]^{-1} \right] &\geq 2 \log \binom{p-1}{d} \\ &\geq 2d \log \left(\frac{p-1}{ed} \right) , \end{aligned}$$

which is larger than d if $d \leq e^{-3/2}(p-1)$. □

Proof of Lemma 5.4. By definition (3) of the function EDKhi, we only have to prove that $\text{DKhi}(d, N, x)$ is increasing with respect to d and decreasing with respect to n .

Conditioning on X_N (resp. X_d) it suffices to prove the two following facts:

FACT 1: Let d be a positive integer. For any positive number x ,

$$d\mathbb{E}[(X_{d+1} - x)_+] \geq (d+1)\mathbb{E}[(X_d - x)_+] .$$

FACT 2: Let N be a positive integer. For any positive numbers x and x' ,

$$\mathbb{E} \left[\left(x' - x \frac{X_N}{N} \right)_+ \right] \geq \mathbb{E} \left[\left(x' - x \frac{X_{N+1}}{N+1} \right)_+ \right] .$$

Proof of FACT 1. Let (Z_1, \dots, Z_{d+1}) be $d+1$ independent χ^2 random variables with 1 degree of freedom. Let $Y = \sum_{i=1}^{d+1} Z_i$ and for any $i \in \{1, \dots, d+1\}$, let $Y^{(i)}$ be the sum $Y^{(i)} = \sum_{j \neq i} Z_j$. The variable Y follows a χ^2 distribution with $d+1$ degrees of freedom, while the variables $Y^{(i)}$ follow χ^2 distribution with d degrees of freedom. It holds that

$$d(Y - x)_+ \geq \sum_{i=1}^{d+1} (Y^{(i)} - x)_+ . \quad (30)$$

Indeed, if all the variables $Y^{(i)}$ are larger than x , one observes that $d(Y - x)_+ = d(\sum_{i=1}^{d+1} Z_i - dx)$ while the second term equals $d\sum_{i=1}^{d+1} Z_i - d(d+1)x$. If some of the variables $Y^{(i)}$ are smaller than x , it is sufficient to note that the variables $Y^{(i)}$ are smaller than Y . We prove FACT 1 by integrating the inequality (30).

Proof of FACT 2. It is sufficient to prove that for any positive number x ,

$$\mathbb{E} \left[\left(x - \frac{X_N}{N} \right)_+ \right] \geq \mathbb{E} \left[\left(x - \frac{X_{N+1}}{N+1} \right)_+ \right] .$$

Observe that $\mathbb{E} \left[\left(x - \frac{X_N}{N} \right)_+ \right] = (x-1) + \mathbb{E} \left[\left(\frac{X_N}{N} - x \right)_+ \right]$. Hence, it remains to prove that

$$(N+1)\mathbb{E}[(X_N - Nx)_+] \geq N\mathbb{E}[(X_{N+1} - (N+1)x)_+] . \quad (31)$$

As in the proof of FACT 1, let (Z_1, \dots, Z_{d+1}) be $d+1$ independent χ^2 random variables with 1 degree of freedom. Let $Y = \sum_{i=1}^{d+1} Z_i$ and for any $i \in \{1, \dots, d+1\}$, let $Y^{(i)}$ be the sum $Y^{(i)} = \sum_{j \neq i} Z_j$. It holds that

$$\sum_{i=1}^{N+1} (Y^{(i)} - Nx)_+ \geq N(Y - (N+1)x)_+ . \quad (32)$$

This bound is trivial if $Y \leq (N+1)x$. If Y is larger than $(N+1)x$, then the second term equals $(N+1) \sum_{i=1}^{N+1} (Y^{(i)} - Nx)$, which is clearly smaller than the first term. Integrating the bound (32) enables to prove (31) and then FACT 2. \square

Proof of Lemma 5.5. We show that the derivate of the function $\mathbb{E} [(X_d - x \frac{X_N}{N})_+] / \mathbb{E} [(X_2 - x)_+]$ is non-negative for any $x \geq d$. Thus, we have to prove the following inequality:

$$\frac{\mathbb{E} \left[(X_d - x \frac{X_N}{N})_+ \right]}{\mathbb{E} \left[\frac{X_N}{N} \mathbf{1}_{X_d \geq x \frac{X_N}{N}} \right]} \geq \frac{\mathbb{E} [(X_2 - x)_+]}{\mathbb{P}(X_2 \geq x)} = 2.$$

Hence, we aim at proving that the function

$$\Psi(x) = \mathbb{E} \left[\left(X_d - x \frac{X_N}{N} \right)_+ \right] - 2 \mathbb{E} \left[\frac{X_N}{N} \mathbf{1}_{X_d \geq x \frac{X_N}{N}} \right]$$

is positive. Observe that $\Psi(x)$ converges to 0 when x goes to infinity. Let us respectively note $f_{X_d}(t)$ and $f_{\frac{X_N}{N}}(t)$ the densities of X_d and X_N/N .

$$\Psi'(x) = \int_{t=0}^{\infty} t \left[2t f_{X_d}(xt) - \int_{u=xt}^{\infty} f_{X_d}(u) du \right] f_{\frac{X_N}{N}}(t) dt.$$

Integrating by part the density of a χ^2 distribution, we get the lower bound

$$\int_{u=xt}^{\infty} f_{X_d}(u) du \geq \frac{(1/2)^{d/2}}{\Gamma(d/2)} 2(xt)^{d/2-1} e^{-xt/2}.$$

$$\begin{aligned} \Psi'(x) &\leq \frac{(1/2)^{d/2-1}}{\Gamma(d/2)} \int_{t=0}^{\infty} t(xt)^{d/2-1} e^{-xt/2} (t-1) f_{\frac{X_N}{N}}(t) dt \\ &\leq \frac{(1/2)^{(N+d)/2-1}}{\Gamma(d/2)\Gamma(N/2)} N^{N/2} x^{d/2-1} \int_{t=0}^{\infty} t^{d/2} (t-1) t^{N/2-1} e^{-(x+N)t/2} dt \\ &\leq \frac{2N^{N/2} x^{d/2-1}}{\Gamma(d/2)\Gamma(N/2)(x+N)^{(d+N)/2}} \int_{t=0}^{\infty} t^{(d+N)/2-1} \left(\frac{2t}{x+N} - 1 \right) e^{-t} dt \\ &\leq \frac{2N^{N/2} x^{d/2-1}}{\Gamma(d/2)\Gamma(N/2)(x+N)^{(d+N)/2}} \left[\frac{2\Gamma\left(\frac{d+N}{2} + 1\right)}{x+N} - \Gamma\left(\frac{d+N}{2}\right) \right] \\ &\leq \frac{2N^{N/2} x^{d/2-1} \Gamma\left(\frac{d+N}{2}\right)}{\Gamma(d/2)\Gamma(N/2)(x+N)^{(d+N)/2}} \left[\frac{d+N}{x+N} - 1 \right] \leq 0, \end{aligned}$$

since $x \geq d$. Hence, Ψ is decreasing to 0 for x larger than d and it is therefore non-negative. \square

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