THE SPACES $\mathrm{H}^n(\mathfrak{osp}(1|2), M)$ FOR SOME WEIGHT MODULES M

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Abstract. We entirely compute the cohomology for a natural and large class of $\mathfrak{osp}(1|2)$ modules M. We study the restriction to the $\mathfrak{sl}(2)$ cohomology of M and apply our results to the module $M = \mathfrak{D}_{\lambda,\mu}$ of differential operators on the super circle, acting on densities.

1. Introduction

The simplest Lie superalgebra is the algebra $\mathfrak{osp}(1|2)$. For such an algebra, the notion of Cartan subalgebra and weight module is well known (see section 2 for definitions and notations). In this paper, we consider such a weight module M, with moreover the assumption that one of the odd element (noted here A) acts through a surjective map.

This generalizes the notion of $\ell\downarrow$ modules for $\mathfrak{sl}(2)$ [8], a class of modules admitting a finite dimensional and nontrivial extension, but our main motivation is the study of deformations of some actions of vector fields on the supercircle or the superspace $\mathbb{R}^{1|1}$, this theory was developped by Ovsienko and many other authors and some conjectures about the cohomology of natural modules coming from the action of $\mathfrak{osp}(1|2)$ on differential operators on densities were presented (see [4, 3, 5]). The first cohomology group for this module was computed by Basdouri and Ben Ammar [2], it was conjectured that the second cohomology group would be generated by cup-product of nontrivial 1 cocycles, that the 2 cocycles whose $\mathfrak{sl}(2)$ restriction is trivial are trivial, and so one.

In this paper, we first entirely determine the cohomology for our $\mathfrak{osp}(1|2)$ module M and prove that the restriction map is one to one from $\mathrm{H}^n(\mathfrak{osp}(1|2), M)$ to $\mathrm{H}^n(\mathfrak{sl}(2), M)$. Then we apply this to the module of the differential operators on densities, computing completely their cohomologies and explicitely describing the cocycles.

2. Definitions and notations

First, we define the Lie superalgebra $\mathfrak{osp}(1|2)$ and the module M. We define the superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2)$ as the real algebra whose basis is (H, X, Y, A, B). The elements H, X and Y are even (with parity 0, or in $\mathfrak{g}_0$) and the elements A, B are odd (with parity 1, or in $\mathfrak{g}_1$), the bracket is graded antisymmetric, we denote this property by

$$[U, V] = -(-1)^{UV}[V, U].$$

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The commutation relations are :

$$\begin{array}{lll} [H,X]=X, & [H,Y]=-Y, & [X,Y]=2H, \\ [H,A]=\frac{1}{2}A, & [X,A]=0, & [Y,A]=-B, \\ [H,B]=-\frac{1}{2}B, & [X,B]=A, & [Y,B]=0, \\ [A,A]=2X, & [A,B]=2H, & [B,B]=-2Y. \end{array}$$

The bracket satisfies the graded Jacobi identity

$$(-1)^{UW}[[U,V],W]+(-1)^{VU}[[V,W],U]+(-1)^{WU}[[W,V],U]=0.$$

We consider the subalgebra $\mathbb{R}H$ as the Cartan subalgebra of $\mathfrak{osp}(1|2)$, its adjoint action is trivially split, with roots 0, $\pm\frac{1}{2}$, ± 1.

The even subalgebra $\mathfrak{g}_0$ of $\mathfrak{osp}(1|2)$ is of course the simple Lie algebra $\mathfrak{sl}(2)$. From the relations, it is clear that, as a graded Lie algebra, $\mathfrak{osp}(1|2)$ is generated by its odd part $\mathfrak{g}_1 = \mathrm{Span}(A,B)$.

We consider here a special class of $\mathfrak{osp}(1|2)$ modules M. We first suppose M is a complex $\mathbb{Z}_2$ graded vector space $M_0 \oplus M_1$ (the elements of M_i are said homogenous with parity i) and the H action is diagonalized on M, that is we decompose M (and thus M_0 and M_1) into weight spaces M^α (resp. M_i^α) :

$$M=\bigoplus_{\alpha\in\Sigma} M^\alpha, \quad Hv_\alpha=\alpha v_\alpha,\ \forall v_\alpha\in M^\alpha.$$

($\Sigma\subset\mathbb{C}$ is the set of weights).

If V is a H-invariant vector subspace, then V itself can be decomposed in $V=\bigoplus_{\alpha\in\Sigma}V^\alpha$ with $V^\alpha=M^\alpha\cap V$. For instance, each M_i can be decomposed.

The commutation relations imply directly

$$\begin{array}{ll} AM_i^\alpha & \subset M_{i+1}^{\alpha+\frac{1}{2}},\ XM_i^\alpha\subset M_i^{\alpha+1}, \\ BM_i^\alpha & \subset M_{i+1}^{\alpha-\frac{1}{2}},\ YM_i^\alpha\subset M_i^{\alpha-1}. \end{array}$$

Then we add the condition that the action of A is onto (or equivalently X is onto). This conditions implies that M does not have any minimal weight vector v, with weight α_0. Indeed, if such a vector exists, the relation $v=Aw=A\sum_{\beta\in\Sigma}w_\beta$ ($w_\beta\in M^\beta$) implies

$$Hv=\alpha_0 v=\textstyle\sum_\beta HAw_\beta=\sum_\beta(\beta+\frac{1}{2})Aw_\beta=\sum_\beta\alpha_0 Aw_\beta,$$

or $Aw_\beta=0$ if $\beta\neq\alpha_0-\frac{1}{2}$, and $0\neq v=Aw_{\alpha_0-\frac{1}{2}}$, therefore $\alpha_0-\frac{1}{2}\in\Sigma$, which is impossible. Then our modules M are infinite dimensional.

For $\mathfrak{sl}(2)$, the simple modules for which X are onto are the modules $\ell\downarrow$. It is well known that these modules are the only (with the 'symmetric' case $\ell\uparrow$) $\mathfrak{sl}(2)$-modules admitting finite dimensional nontrivial extensions for some values of ℓ (see [8]).

We now consider the cohomology groups $\mathrm{H}^n(\mathfrak{osp}(1|2),M)$ of these modules. A n cochain is a mapping f from $\mathfrak{osp}(1|2)^n$ to M which is n linear and graded antisymmetric:

$$f(U_1,\dots,U_i,\dots,U_j,\dots,U_n)=-(-1)^{U_iU_j}f(U_1,\dots,U_j,\dots,U_i,\dots,U_n).$$

Defining the graded sign $\varepsilon_U(\sigma)$ for a permutation $\sigma\in\mathfrak{S}_n$ acting on the elements U_i as the product $\varepsilon(\sigma)\varepsilon(\tau)$ of the usual sign $\varepsilon(\sigma)$ of σ by the sign of the induced permutation τ on the set of indices i for odd elements U_i, we have :

$$f(U_{\sigma(1)},\dots,U_{\sigma(n)})=\varepsilon_U(\sigma)f(U_1,\dots,U_n).$$

Due to this property, we use the following notation:

$$U_1\cdots U_n=\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon_U(\sigma)(U_{\sigma^{-1}(1)}\otimes\cdots\otimes U_{\sigma^{-1}(n)})$$

and for any $\sigma\in\mathfrak{S}_n$,

$$f(U_1\dots U_n)=\varepsilon_U(\sigma)f(U_{\sigma^{-1}(1)}\otimes\cdots\otimes U_{\sigma^{-1}(n)})=\varepsilon_U(\sigma)f(U_{\sigma^{-1}(1)},\dots,U_{\sigma^{-1}(n)}).$$

The cochain f is homogeneous with parity f if $f(\mathfrak{g}_{i_1}\otimes\cdots\otimes\mathfrak{g}_{i_n})\subset M_{f+\sum i_j}$. The space of n cochain is denoted $C^n(\mathfrak{osp}(1|2),M)$, or C^n if no confusion is possibe.

On such a cochain f, the coboundary operator is defined, using the Koszul rule for signs, by the relation (see [7] for instance):

$$(\partial f)(U_0,\dots,U_n)=\sum_{i=0}^{n}(-1)^i(-1)^{U_i(f+U_0+\cdots+U_{i-1})}U_if(U_0,\dots,\hat{\imath},\dots,U_n)+$$
$$+\sum_{0\le i<j\le n}(-1)^{i+j}(-1)^{U_i(U_0+\cdots+U_{i-1})}(-1)^{U_j(U_0+\cdots+\hat{\imath}+\cdots+U_{j-1})}f([U_i,U_j],U_0,\dots,\hat{\imath},\dots,\hat{\jmath},\dots,U_n).$$

If f is a n cochain, ∂f is a $n+1$ cochain with the same parity f, we can verify directly that $\partial\circ\partial=0$ (or we can use a shift on degree and usual cohomology computations). The n cocycles are the n cochains such that $\partial f=0$, the n coboundaries are the cochains in the image of ∂, we put as usual

$$Z^n=\ker(\partial:C^n\longrightarrow C^{n+1}),\ B^n=\partial(C^{n-1}),\ \mathrm{H}^n(\mathfrak{osp}(1|2),M)=Z^n/B^n.$$

$\mathrm{H}^n(\mathfrak{osp}(1|2),M)$ is the n^{th} cohomology group for the module M.

3. The cohomology

3.1. $\mathfrak{osp}(1|2)$ cohomology.

The cohomology is described by the following

Theorem 3.1. (The groups $\mathrm{H}^n(\mathfrak{osp}(1|2),M)$)

Let us denote by $\ker A$ *(respectively* $\ker B$*) the subspaces of* M*, kernel of the morphism* $v\mapsto Av$ *(respectively* $v\mapsto Bv$*) in* M*. Then we have the following linear isomorphisms:*

(i) $\mathrm{H}^0(\mathfrak{osp}(1|2),M)=\ker A\cap\ker B$.

(ii) $\mathrm{H}^1(\mathfrak{osp}(1|2),M)\simeq(\ker A\cap\ker B)\oplus((\ker A)^{-\frac12}/B((\ker A)^0))$.

(iii) $\mathrm{H}^2(\mathfrak{osp}(1|2),M)\simeq(\ker A)^{-\frac12}/B((\ker A)^0)$.

(iv) $\mathrm{H}^n(\mathfrak{osp}(1|2),M)=0$ *if* $n>2$.

The realization of these isomorphisms will be explicitly detailed in the proof. Before to prove this theorem, we shall give some preliminary results.

First we say that a n cochain f is reduced if $f(AU_2\cdots U_n)=0$ for any $U_2,\dots,U_n$ in $\{A,H,B,Y\}$. Observe that if f is reduced then we have also $f(XU_2\cdots U_n)=0$ for any $U_2,\dots,U_n$ in $\{A,H,B,Y\}$, since

$$0=(\partial f)(A^2U_2\cdots U_n)=-f([A,A]U_2\cdots U_n)=-2f(XU_2\cdots U_n).$$

Proposition 3.2. (Each cochain is cohomologous to a reduced one)
Let f be a n cochain. Then there exists a $n-1$ cochain g such that $f-\partial g$ is reduced.

Proof. If $n=0$, any cochain is reduced and there is nothing to do. Suppose now $n>0$, Recall that the vectors X, A, H, B, Y are root vectors with respective weight $1, \frac{1}{2}, 0, -\frac{1}{2}, -1$. Define the weight of $U_1\otimes\cdots\otimes U_n$ as the sum of the weights of the vectors U_i.

First, we kill $f(A^n)$. Indeed, if g_0 is the $n-1$ cochain such that $g_0(U_1\ldots U_{n-1})=0$ except if $U_1=\cdots=U_{n-1}=A$ and $g_0(A^{n-1})=v$ where v is such that $nAv=(-1)^f f(A^n)$ then $(\partial g_0)(A^n)=f(A^n)$ and $f_0=f-\partial g_0$ vanishes on A^n. If $n=1$, the proposition is proved.

Now, by induction, we suppose there is g_k such that $f_k=f-\partial g_k$ vanishes on any product of the form $A^{n-k}U_{n-k+1}\cdots U_n$ with $U_j\in\{A,H,B,Y\}$.

Suppose $n-k>1$ and consider a $k+1$ product of the form $U_{n-k}\cdots U_n$. If one of the U_i is A, f_k vanishes on $A^{n-k-1}U_{n-k}\cdots U_n$, if there is no such U_i, but if $U_i=U_j=H$, then f_k vanishes on $A^{n-k-1}U_{n-k}\cdots U_n$. The monomial T with maximal weight for which $f_k(A^{n-k-1}T)$ could be not zero is thus HB^k and its weight is $w(T)=-\frac{k}{2}$.

By induction, we can suppose f'_k vanishes for any monomial of the form $A^{n-k}S$ and any monomial of the form $A^{n-k-1}T$ with $w(T)>\ell$ ($\ell\leq-\frac{k}{2}+\frac{1}{2}$). We choose now $g^\ell_{k+1}(U_1\cdots U_{n-1})=0$ except if $U_1\ldots U_{n-1}=A^{n-k-2}U_{n-k}\cdots U_n$ and $w(U_{n-k}\cdots U_n)=\ell$ and $U_j\in\{H,B,Y\}$. Then for such a monomial,

$$\begin{aligned}0&=g^\ell_{k+1}([A,A]A^{n-k-3}U_{n-k}\cdots U_n)=g^\ell_{k+1}([A,U_j]A^{n-k-2}U_{n-k}\cdots\hat{\jmath}\cdots U_n)\\&=g^\ell_{k+1}([U_i,U_j]A^{n-k-1}U_{n-k}\cdots\hat{\imath}\cdots\hat{\jmath}\cdots U_n)\end{aligned}$$

and

$$(\partial g^\ell_{k+1})(A^{n-k-1}U_{n-k}\cdots U_n)=(-1)^{g^\ell_{k+1}}(n-k-1)Ag^\ell_{k+1}(A^{n-k-2}U_{n-k}\cdots U_n).$$

We can then choose the value of g^ℓ_{k+1} such that $f'_k-\partial g^\ell_{k+1}$ vanishes on any monomial of the form $A^{n-k-1}T$ with $w(T)\geq\ell$.

By induction, we prove there is g' such that $f'=f-\partial g'$ is reduced.

Proposition 3.3. (Localization for cocycles)
Suppose than f is a n reduced cocycle. Then

- *(i) If $n>0$, $f=0$ if and only if $f(B^n)=0$.*
- *(ii) If $n>1$, any reduced cocycle vanishing on HB^{n-1} is a coboundary.*

Proof. *(i)* With the antisymmetry condition on f, the only possibly non vanishing terms for f are monomials containing B (as odd vector) and H and Y as even vector, but each of them at most one time. $f(U_1\cdots U_n)=0$ except if $U_1\cdots U_n$ is HB^{n-1} or B^n or YB^{n-1} and, if $n>1$, HYB^{n-2}.

Now, the cocycle relation allows us to compute these vectors with the only knowledge of $f(B^n)$:

$$\begin{aligned}(\partial f)(AB^n)&=(-1)^f Af(B^n)+\textstyle\sum_{i=1}^n(-1)^i(-1)^{(i-1)}f([A,B]B^{n-1})\\&=(-1)^f Af(B^n)-2nf(HB^{n-1})=0,\end{aligned}$$

and

$$\begin{aligned}(\partial f)(B^{n+1}) &= \textstyle\sum_{i=0}^{n}(-1)^i(-1)^{f+i}Bf(B^n) + \sum_{0\leq i<j\leq n}(-1)^{i+j}(-1)^{i+j-1}f([B,B]B^{n-1})\\ &= (n+1)\left[(-1)^f Bf(B^n) + nf(YB^{n-1})\right] = 0,\end{aligned}$$

$$\begin{aligned}(\partial f)(HB^n) &= Hf(B^n) + \textstyle\sum_{i=1}^{n}(-1)^i(-1)^{f+i-1}Bf(HB^{n-1}) +\\ &\qquad\quad + \textstyle\sum_{j=1}^{n}(-1)^j(-1)^{j-1}f([H,B]B^{n-1}) +\\ &\qquad\quad + \textstyle\sum_{1\leq i<j\leq n}(-1)^{i+j}(-1)^{i+j}f([B,B]HB^{n-2})\\ &= (H+\tfrac{n}{2}id)f(B^n) - n\left[(-1)^f Bf(HB^{n-1}) + (n-1)f(HYB^{n-2})\right] = 0.\end{aligned}$$

Thus a reduced cocycle f is completely determined by the vector $f(B^n)$, especially $f=0$ if and only if $f(B^n)=0$.

(ii) Suppose now $f(HB^{n-1})=0$ and $n>1$. Then our computation proves that $f(B^n)$ is in the kernel of A. We define a $n-1$ cochain g by putting $g(U_1\cdots U_{n-1})=0$ except for

$$g(YB^{n-2}) = \frac{1}{n(n-1)}f(B^n).$$

Then

$$(\partial g)(B^n) = n(n-1)g(YB^{n-2}) = f(B^n).$$

and

$$\begin{aligned}(\partial g)(AU_1\cdots U_{n-1}) &= (-1)^g Ag(U_1\cdots U_{n-1}) +\\ &\quad + \textstyle\sum_{j=1}^{n-1}(-1)^j(-1)^{U_j(g+1+U_2+\cdots+U_{j-1})}U_j g(AU_1\cdots\hat{\jmath}\cdots U_{n-1}) +\\ &\quad + \textstyle\sum_{j=1}^{n-1}(-1)^j(-1)^{U_j(U_1+\cdots+U_{j-1})}g([A,U_j]U_1\cdots\hat{\jmath}\cdots U_{n-1}) +\\ &\quad + \textstyle\sum_{1\leq i<j\leq n-1}(-1)^{i+j}(-1)^{U_i(1+U_1+\cdots+U_{i-1})+U_j(1+U_1+\cdots\hat{\imath}\cdots+U_{j-1})}\\ &\qquad\quad g([U_i,U_j]AU_1\cdots\hat{\imath}\cdots\hat{\jmath}\cdots U_{n-1})\\ &= \textstyle\sum_{j=1}^{n-1}(-1)^j(-1)^{U_j(U_1+\cdots+U_{j-1})}g([A,U_j]U_1\cdots\hat{\jmath}\cdots U_{n-1}).\end{aligned}$$

This is non vanishing only if $[A,U_j]=B$ and $U_1\cdots\hat{\jmath}\cdots U_{n-1}=YB^{n-3}$ or $[A,U_j]=Y$ and $U_1\cdots\hat{\jmath}\cdots U_{n-1}=B^{n-2}$. In the first case, we are computing $\partial g(AY^2B^{n-3})$, but, the antisymmetry condition on ∂g gives $\partial g(AY^2B^{n-3})=0$. In the second case there is no such U_j. Thus ∂g vanishes on any monomial $AU_1\dots U_{n-1}$.

Now $f-\partial g$ is a cocycle vanishing on any $AU_2\cdots U_n$ and on B^n, thus $f=\partial g$.

Proof of Theorem 3.1.

(i) If $n=0$, there is no coboundaries, the cocycles are the vector $f\in M$ such that $(\partial f)(U)=(-1)^{fU}Uf=0$ for any U in $\mathfrak{osp}(1|2)$ these vectors are in $\ker A\cap\ker B$. Conversely, since A and B generate $\mathfrak{osp}(1|2)$ as an algebra, each vector in $\ker A\cap\ker B$ is 0 cocycle.

(ii) Suppose $n=1$. We saw that up to a coboundary, f is vanishing on A and X. Thus $f(H)$ belongs to $\ker A$ since

$$(\partial f)(AH) = (-1)^f Af(H) = 0.$$

Let us now decompose $f(H)$ on weight vectors :

$$f(H) = \textstyle\sum_{\alpha\in\Sigma}v_\alpha, \qquad Hv_\alpha = \alpha v_\alpha,\ Av_\alpha = 0.$$

Put $g=\sum_{\alpha\neq 0}\frac{1}{\alpha}v_\alpha$. Then $(\partial g)(A)=(-1)^g Ag=0$ and $(\partial g)(H)=Hg=\sum_{\alpha\neq 0}v_\alpha$. The 1 cocycle $f'=f-\partial g$ is reduced and satisfies $f'(H)\in\ker A\cap\ker H$. Now

$$0=(\partial f')(HB)=Hf'(B)-(-1)^f Bf'(H)+\frac{1}{2}f'(B)\quad=(H+\frac{1}{2}id)f'(B)-(-1)^f Bf'(H).$$

The first term is in $\bigoplus_{\alpha\neq-\frac{1}{2}}M^\alpha$, the second one in $B(\ker H)\subset M^{-\frac{1}{2}}$. Thus these two terms vanish. Therefore $f'(H)$ is in $\ker A\cap\ker B$. We now suppose $f(H)\in\ker A\cap\ker B$. Then $(H+\frac{1}{2}id)f(B)=0$.

On the other hand, we have $Af(B)=2(-1)^f f(H)$. Thus $f(B)$ is in the affine space of solutions for these two last equations. The corresponding linear space is $(\ker A)^{-\frac{1}{2}}$. But we can still add a coboundary ∂g to f with $Ag=Hg=0$, then $f(B)$ becomes $f(B)+(-1)^g Bg$. That means, we can impose to look for solution in an affine space parallel to $(\ker A)^{-\frac{1}{2}}/B(\ker A\cap\ker H)$.

To be more precise, let us choose a supplementary space V to $(\ker A)^{-\frac{1}{2}}$ in $M^{-\frac{1}{2}}$ and a supplementary space W to $B((\ker A)^0)$ in $(\ker A)^{-\frac{1}{2}}$:

$$M^{-\frac{1}{2}}=(\ker A)^{-\frac{1}{2}}\oplus V=B((\ker A)^0)\oplus W\oplus V.$$

Up to a coboundary, $f(H)$ belongs to $\ker A\cap\ker B$ and $f(B)$ to $W\oplus V$. Write $f(B)=w+v$, we get $Af(B)=Av=2(-1)^f f(H)$. This relation characterizes v since $A|_V$ is one-to-one. We associate to f the vector $(f(H),w)$ in $(\ker A\cap\ker B)\oplus W$.

Conversely, let u be in $\ker A\cap\ker B$, homogeneous with parity u and v the unique vector in V_{u+1} such that $Av=2(-1)^u u$. Choose any w in W and define a map $f:\mathfrak{osp}(1|2)\longrightarrow M$ by putting $f(A)=f(X)=0$, $f(H)=u$, $f(B)=v+w$, and $f(Y)=-(-1)^f B(v+w)$. Then we verify directly that

$$\begin{aligned}(\partial f)(AX)&=(\partial f)(AA)=(\partial f)(AH)=0\\(\partial f)(AB)&=(-1)^f A(v+w)-2u=(-1)^f Av-2u=0,\\(\partial f)(AY)&=-AB(v+w)-(v+w)=-(AB+BA)(v+w)-(v+w)\\&=-(2H+id)(v+w)=0.\end{aligned}$$

The map ∂f is then a reduced 2 cocycle and moreover, we have

$$\partial f(B^2)=(-1)^f 2Bw+2(-(-1)^f Bw)=0.$$

Thus $\partial f=0$, that is, f is a 1 cocycle.

Now, suppose f is a coboundary, then there is g such that

$$Ag=0\ \text{ and }\ f(H)=u=Hg.$$

This implies $H^2g=Hu=0$, thus $g\in(\ker A)^0$ and $u=0$, thus $v=0$ and $f(B)=w=(-1)^g Bg\in B((\ker A)^0)\cap W$, thus $w=0$. Conversely, if $v=0$ and $w=(-1)^g Bg$ with $g\in(\ker A)^0$, then $f'=f-\partial g$ is a reduced 1 cocycle such that $f'(B)=0$, thus $f'=0$ and f is a coboundary. Thus, the map $f\mapsto(u,w)$ realizes an isomorphism between $\mathrm{H}^1(\mathfrak{osp}(1|2),M)$ and $(\ker A\cap\ker B)\oplus W$.

We proved *(ii)* since $W\simeq(\ker A)^{-\frac{1}{2}}/B((\ker A)^0)$.

(iii) Suppose $n \geq 2$ and f is a reduced n cocyle. Since

$$0 = (\partial f)(AHB^{n-1}) = (-1)^f Af(HB^{n-1}),$$

we get as above: $f(HB^{n-1})$ is in $\ker A$.

We decompose $f(HB^{n-1}) = \sum v_\alpha$ with $(H - \alpha id)v_\alpha = Av_\alpha = 0$. Define the $n-1$ cochain g by $g(U_1 \dots U_{n-1}) = 0$ except for $g(B^{n-1})$ and $g(YB^{n-2})$ and

$$g(B^{n-1}) = \textstyle\sum_{\alpha \neq -\frac{n-1}{2}} \frac{1}{\alpha + \frac{n-1}{2}} v_\alpha, \quad (-1)^g Ag(YB^{n-2}) = g(B^{n-1}).$$

Then ∂g is a n cocycle, the only non vanishing terms in $\partial g(AU_2 \cdots U_n)$ are $Ag(YB^{n-2})$ and $g([A,Y]B^{n-2})$. Both happen only if $U_2 \dots U_n = YB^{n-2}$ and

$$(\partial g)(AYB^{n-2}) = (-1)^g Ag(YB^{n-2}) - g(B^{n-1}) = 0.$$

Thus $f' = f - \partial g$ is a reduced n cocycle and $f'(HB^{n-1}) = v_{-\frac{n-1}{2}} \in (\ker A)^{-\frac{n-1}{2}}$. From now on, we suppose f is a reduced n cocycle such that $f(HB^{n-1}) \in (\ker A)^{-\frac{n-1}{2}}$.

Suppose now $n = 2$.

If $f(HB)$ is in $B(\ker A \cap \ker H)$, we put $g(X) = g(A) = 0$ and $(-1)^g Bg(H) = f(HB)$ with $Ag(H) = Hg(H) = 0$, then we choose $g(B)$ such that $Ag(B) = (-1)^g 2g(H)$ and $g(Y)$ such that $Ag(Y) = (-1)^g g(B)$. Then $f - \partial g$ is a 2 cocycle vanishing on AX, AA, AB and AY and on HB. We saw that $f - \partial g$ is then a coboundary. Thus, f is a coboundary.

Conversely, let w be a vector in $(\ker A)^{-\frac{1}{2}}/B((\ker A)^0)$ (or in the supplementary space W for $B((\ker A)^0)$ in $(\ker A)^{-\frac{1}{2}}$). Then

$$\begin{aligned} ABw &= -w = (AB + BA)w, \\ -2Bw &= 2HBw = (AB + BA)Bw, \\ AB^2w &= -2Bw - BABw = -Bw. \end{aligned}$$

We put $f(XU) = f(AU) = 0$, for any U, $f(HB) = w$, put $f(B^2) = -4(-1)^f Bw$, $f(HY) = -(-1)^f Bw$, and $f(YB) = 2B^2w$. The 3 cocycle ∂f vanishes on A^2U for any U, we consider it on AHB, AB^2, AYB and AYH.

$$\begin{aligned} (\partial f)(AHB) &= (-1)^f Aw = 0, \\ (\partial f)(AB^2) &= -4ABw - 2f([A,B]B) = 4w - 4w = 0, \\ (\partial f)(AHY) &= -ABw + f([A,Y]H) = w - w = 0, \\ (\partial f)(AYB) &= (-1)^f Af(YB) - f([A,Y]B) + f([A,B]Y) \\ &= (-1)^f \left[2AB^2w + 4Bw - 2Bw\right] = 0. \end{aligned}$$

∂f is a reduced 3 cocycle, we moreover have

$$(\partial f)(B^3) = (-1)^f 3Bf(B^2) - 3f([B,B]B) = -12B^2w + 6f(YB) = 0.$$

Thus $\partial f = 0$, f is then a reduced 2 cocycle. Now if $f = \partial g$, then

$$w = f(HB) = \partial g(HB) = \left(H + \tfrac{1}{2}id\right) g(B) - Bg(H).$$

Let $g(B) = \sum_\alpha u_\alpha$, $g(H) = \sum_\alpha x_\alpha$ and $g(A) = \sum_\alpha y_\alpha$ where u_α, x_α and y_α are in M^α, then we get

$$w = \textstyle\sum_{\alpha \neq -\frac{1}{2}} \left((\alpha + \frac{1}{2})u_\alpha - Bx_{\alpha + \frac{1}{2}}\right) - Bx_0.$$

But w is in W, thus, $(\alpha+\frac{1}{2})u_\alpha - Bx_{\alpha+\frac{1}{2}} = 0$ if $\alpha \neq -\frac{1}{2}$ and then $w = -Bx_0$. Moreover, we have

$$0 = f(HA) = (H - \tfrac{1}{2}id)g(A) - Ag(H) = \textstyle\sum_{\alpha\neq\frac{1}{2}} \Big((\alpha-\tfrac{1}{2})y_\alpha - Ax_{\alpha-\frac{1}{2}}\Big) - Ax_0.$$

Thus, $Ax_0 = 0$, therefore $x_0 \in (\ker A)^0$ and $w = -Bx_0 \in W \cap B((\ker A)^0) = \{0\}$, this implies $f = 0$.

We proved the point *(iii)*.

(iv) Suppose $n > 2$. We saw that any n cocycle f can be choosen such that f is reduced and $f(HB^{n-1}) \in (\ker A)^{-\frac{n-1}{2}}$. We define g by $g(U_1 \dots U_{n-1}) = 0$ except

$$g(HYB^{n-3}) = -\frac{1}{(n-1)(n-2)} f(HB^{n-1})$$

and $g(YB^{n-2})$, choosen such that

$$Ag(YB^{n-2}) - 2(n-2)g(HYB^{n-3}) = 0.$$

Then $(\partial g)(AHB^{n-2}) = 0$ and if $U_2 \dots U_n \neq HB^{n-2}$, then the only non vanishing terms in $(\partial g)(AU_2 \dots U_n)$ have the form $\pm g([A,U_j]U_2 \dots \hat{\jmath} \dots U_n)$ with $[A,U_j] = Y$, which is impossible, or $[A,U_j] = B$, this means $U_j = Y$, but there is another index $i \neq j$ with $U_i = Y$ and this is still impossible or $[A,U_j] = H$, this means $U_j = B$ and $U_2 \dots U_n = HB^{n-2}$, which is impossible. Thus $f - \partial g$ is reduced and vanishes on HB^{n-1}, it is a coboundary, f is a coboundary, $\mathrm{H}^n(\mathfrak{osp}(1|2), M) = 0$.

3.2. **Restriction to $\mathfrak{sl}(2)$.** We keep our notations.

Lemma 3.4. (Characterization for $B((\ker A)^0)$)

Let w be a vector in M such that $w \in (\ker A)^{-\frac{1}{2}}$ and $Bw \in Y((\ker X)^0)$. Then w is in $B((\ker A)^0)$.

Proof. We suppose $Bw = B^2v$, with $Hv = A^2v = 0$. Thus $ABv + BAv = 0$ and

$$2HBv = -Bv = AB^2v + BABv, \qquad ABw = AB^2v = -Bv - BABv.$$

But

$$2Hw = (AB + BA)w = ABw = -w.$$

Or $w = B(v + ABv)$. But

$$2HAv = (AB + BA)Av = ABAv = Av.$$

Finally:

$$A(v + ABv) = Av + A^2Bv = Av - ABAv = 0.$$

This proves our lemma.

Let f be a n cochain for $\mathfrak{osp}(1|2)$. Its restriction $f|_{\mathfrak{sl}(2)}$ to $\mathfrak{sl}(2)^n$ is a n cochain for the $\mathfrak{sl}(2)$ module M. If f is a cocycle (resp. a coboundary), $f|_{\mathfrak{sl}(2)}$ is a cocycle (resp. a coboundary). The map $f \mapsto f|_{\mathfrak{sl}(2)}$ defines a map φ from $\mathrm{H}^n(\mathfrak{osp}(1|2), M)$ to $\mathrm{H}^n(\mathfrak{sl}(2), M)$.

Proposition 3.5. (Restriction of $\mathfrak{osp}(1|2)$ cocycle and triviality)

A n cocycle f for $\mathfrak{osp}(1|2)$ is a coboundary (a trivial cocycle) if and only if its restriction $f|_{\mathfrak{sl}(2)}$ is a $\mathfrak{sl}(2)$ coboundary. Or: φ is one to one.

Proof. We just consider $n \leq 2$ and f choosen as in Theorem 3.1.

A 0 cocycle is a vector f in $\ker A \cap \ker B$, it is trivial if and only if $f = 0$.

A 1 cocycle is cohomologous to a cocycle f such that:

$$\begin{aligned} &f(A) = f(X) = 0, \quad f(H) = u \in (\ker A)^0, \\ &f(B) = v + w \in V \oplus W, \quad f(Y) = -(-1)^f B(v+w). \end{aligned}$$

Here V is a supplementary space for $(\ker A)^{-\frac{1}{2}}$ in $M^{-\frac{1}{2}}$, W a supplementary space for $B((\ker A)^0)$ in $(\ker A)^{-\frac{1}{2}}$, and v is the only vector in V such that $Av = 2(-1)^f u$. We saw that f is characterized by u and w.

Suppose there is g in M such that $(f - \partial g)|_{\mathfrak{sl}(2)}$ vanishes, thus $Hg = f(H) = u$, since u is in M^0, this relation forces $u = 0$, therefore $v = 0$. Now $Xg = f(X) = 0$, $Hg = f(H) = 0$ and $Yg = f(Y) = -(-1)^f Bw$. Our lemma says that w is in $B((\ker A)^0)$, thus $w = 0$, $f = 0$.

A 2 cocycle is cohomologous to a cocycle f such that:

$$\begin{array}{llllll} f(AU) & = f(XU) = 0, & f(HB) & = w \in W, & f(BB) & = -4(-1)^f Bw, \\ f(HY) & = -(-1)^f Bw, & f(YB) & = 2B^2 w. & & \end{array}$$

And f is characterized by w.

Suppose there is g in $C^1(\mathfrak{sl}(2), M)$ such that $(f - \partial g)|_{\mathfrak{sl}(2)}$ vanishes, put:

$$g(X) = \textstyle\sum_\alpha x_\alpha, \quad g(H) = \sum_\alpha h_\alpha, \quad g(Y) = \sum_\alpha y_\alpha, \quad (x_\alpha, h_\alpha, y_\alpha \in M^\alpha).$$

We get

$$\begin{aligned} f(XH) &= Xg(H) - Hg(X) - g([X,H]) = \textstyle\sum_{\alpha \neq 1}((-\alpha+1)x_\alpha + Xh_{\alpha-1}) + Xh_0 = 0, \\ f(HY) &= Hg(Y) - Yg(H) - g([H,Y]) = \textstyle\sum_{\gamma \neq -1}((\gamma+1)y_\gamma - Yh_{\gamma+1}) - Yh_0 = -(-1)^f Bw. \end{aligned}$$

Since Bw is in M^{-1}, this implies $Hh_0 = Xh_0 = 0$ and $Yh_0 = (-1)^f Bw$. Our lemma says that w is in $B((\ker A)^0)$, therefore $w = 0$ and $f = 0$.

Remark 3.6. *In the same way as for Theorem 3.1, it is easy to compute the cohomology for the $\mathfrak{sl}(2)$ module M. Here it is:*

$$\begin{aligned} &\mathrm{H}^0(\mathfrak{sl}(2), M) = \ker X \cap \ker Y, \qquad \mathrm{H}^1(\mathfrak{sl}(2), M) \simeq (\ker X \cap \ker Y) \oplus (\ker X)^{-1}/Y((\ker X)^0), \\ &\mathrm{H}^2(\mathfrak{sl}(2), M) \simeq (\ker X)^{-1}/Y((\ker X)^0), \qquad \mathrm{H}^{>2}(\mathfrak{sl}(2), M) = 0. \end{aligned}$$

4. Application to $\mathfrak{D}_{\lambda,\mu}$

4.1. **Differential operators on weighted densities.**

We define the superspace $\mathbb{R}^{1|1}$ in terms of its superalgebra of functions, denoted by $C^\infty(\mathbb{R}^{1|1})$ and consisting of elements of the form:

$$F(x, \theta) = f_0(x) + f_1(x)\theta,$$

where x is the even variable, θ is the odd variable ($\theta^2 = 0$) and $f_0(x)$, $f_1(x) \in C^\infty(\mathbb{R})$. We consider the contact bracket on $C^\infty(\mathbb{R}^{1|1})$ defined on $C^\infty(\mathbb{R}^{1|1})$ by:

$$\{F, G\} = FG' - F'G + \tfrac{1}{2}\eta(F)\overline{\eta}(G),$$

where $\eta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ and $\overline{\eta} = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x}$. Let $\mathrm{Vect}(\mathbb{R}^{1|1})$ be the superspace of vector fields on $\mathbb{R}^{1|1}$:

$$\mathrm{Vect}(\mathbb{R}^{1|1}) = \left\{ F_0\partial_x + F_1\partial_\theta \mid \ F_i \in C^\infty(\mathbb{R}^{1|1}) \right\},$$

where ∂_θ stands for $\frac{\partial}{\partial\theta}$ and ∂_x stands for $\frac{\partial}{\partial x}$. We can realize the algebra $\mathfrak{osp}(1|2)$ as a subalgebra of $\mathrm{Vect}(\mathbb{R}^{1|1})$:

$$\mathfrak{osp}(1|2) = \mathrm{Span}(X_1,\, X_x,\, X_{x^2},\, X_{x\theta},\, X_\theta).$$

where, the vector field X_G is defined for any $G \in C^\infty(\mathbb{R}^{1|1})$ by

$$X_G = G\partial_x + \tfrac{1}{2}\eta(G)\overline{\eta}.$$

Here, we have $(-X_x, X_1, -X_{x^2}, 2X_\theta, X_{x\theta}) = (H, X, Y, A, B)$. The bracket on $\mathfrak{osp}(1|2)$ is then given by $[X_F,\, X_G] = X_{\{F,G\}}$.

We denote by $\mathfrak{F}_\lambda$ the space of all weighted densities on $\mathbb{R}^{1|1}$ of weight λ:

$$\mathfrak{F}_\lambda = \left\{ F(x,\theta)\alpha^\lambda \mid F(x,\theta) \in C^\infty(\mathbb{R}^{1|1}) \right\} \qquad (\alpha = dx + \theta d_\theta).$$

The action of $\mathfrak{osp}(1|2)$ on $\mathfrak{F}_\lambda$ is given by

$$X_G(F\alpha^\lambda) = ((G\partial_x + \frac{1}{2}\eta(G)\overline{\eta})(F) + \lambda G'F)\alpha^\lambda.$$

Any differential operator A on $\mathbb{R}^{1|1}$ defines a linear mapping from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any λ by: $A : F\alpha^\lambda \mapsto A(F)\alpha^\mu$, $\mu \in \mathbb{R}$, thus, the space of differential operators becomes a family of $\mathfrak{osp}(1|2)$ modules denoted $\mathfrak{D}_{\lambda,\mu}$, for the natural action:

$$X_G \cdot A = X_G \circ A - (-1)^{AG} A \circ X_G.$$

For more details see, for instance [1, 2, 3, 5]

4.2. **Cohomology.**

Let us consider the $\mathfrak{osp}(1|2)$-module $\mathfrak{D}_{\lambda,\mu}$ of differential operators on densities on $\mathbb{R}^{1|1}$.

We put here $p = \mu - \lambda$ and choose the following basis for $\mathfrak{D}_{\lambda,\mu}$:

$$a_{m,k} = x^m\partial_x^k,\ b_{m,k} = x^m\theta\partial_\theta\partial_x^k,\ c_{m,k} = x^m\theta\partial_x^k,\ d_{m,k} = x^m\partial_\theta\partial_x^k - x^m\theta\partial_x^{k+1}.$$

(Here, m and k are natural integral numbers), we say that $a_{m,k}$ and $b_{m,k}$ are even vectors (see below) and $c_{m,k}$ and $d_{m,k}$ are odd vectors.

In fact they are weight vectors for the action of H:

$$\begin{array}{llll} Ha_{m,k} & = (k-m-p)a_{m,k} & Hb_{m,k} & = (k-m-p)b_{m,k} \\ Hc_{m,k} & = (k-m-p-\frac{1}{2})c_{m,k} & Hd_{m,k} & = (k-m-p+\frac{1}{2})d_{m,k}. \end{array}$$

Similarly, a direct computation give the following relations for the A and B actions on these vectors :

$$\begin{array}{llll} A\ a_{m,k} & = mc_{m-1,k}, & A\ b_{m,k} & = d_{m,k}, \\ A\ c_{m,k} & = a_{m,k}, & A\ d_{m,k} & = mb_{m-1,k} \end{array}$$

and

$$\begin{array}{llll} Ba_{m,k} & = (m-2k+2p)c_{m,k} - kd_{m,k-1}, & Bb_{m,k} & = d_{m+1,k} - (2\lambda+k)c_{m,k}, \\ Bc_{m,k} & = a_{m+1,k} + kb_{m,k-1}, & Bd_{m,k} & = (m-2k+2p-1)b_{m,k} + (2\lambda+k)a_{m,k}. \end{array}$$

From these formulas (or directly), we can compute the X and Y actions, getting:

$$\begin{aligned} Xa_{m,k} &= ma_{m-1,k}, & Xb_{m,k} &= mb_{m-1,k}, \\ Xc_{m,k} &= mc_{m-1,k}, & Xd_{m,k} &= md_{m-1,k}, \end{aligned}$$

and

$$\begin{aligned} Ya_{m,k} &= (2k-2p-m)a_{m+1,k} + k(2\lambda+k-1)a_{m,k-1} + kb_{m,k-1}, \\ Yb_{m,k} &= (2k-2p-m)b_{m+1,k} + k(2\lambda+k)b_{m,k-1}, \\ Yc_{m,k} &= (2k-2p-m-1)c_{m+1,k} + k(2\lambda+k-1)c_{m,k-1}, \\ Yd_{m,k} &= (2k-2p-m+1)d_{m+1,k} + k(2\lambda+k)d_{m,k-1} - (2\lambda+2k+1)c_{m,k}. \end{aligned}$$

From these formulas, we immediately get

$$\ker A \cap \ker B = \begin{cases} \mathrm{Span}(a_{0,0}) & \text{if } \ p = 0, \\ \mathrm{Span}(d_{0,k}) & \text{if } \ p = k + \frac{1}{2}, \ k \in \{0,1,2,\dots\} \ \text{ and } \ 2\lambda + k = 0, \\ 0 & \text{elsewhere.} \end{cases}$$

and

$$(\ker A)^{-\frac{1}{2}} = \begin{cases} \mathrm{Span}(a_{0,k}) & \text{if } \ p = k + \frac{1}{2}, \ k \in \{0,1,2,\dots\}, \\ \mathrm{Span}(d_{0,k}) & \text{if } \ p = k + 1, \ k \in \{0,1,2,\dots\}, \\ 0 & \text{elsewhere.} \end{cases}$$

Moreover, if $p = k + \frac{1}{2}$, $(\ker A)^0 = \mathrm{Span}(d_{0,k})$, $B((\ker A)^0) = \mathrm{Span}(a_{0,k})$ if $2\lambda + k \neq 0$, 0 if it is not the case. Similarly, if $p = k+1$, then $B((\ker A)^0) = B(\mathrm{Span}(a_{0,k+1})) = \mathrm{Span}(d_{0,k})$.

Now we deduce :

Proposition 4.1. (The cohomology for $\mathfrak{D}_{\lambda,\mu}$)

The dimensionalities for the cohomology groups $\mathrm{H}^n(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$ *are:*

(i) $$\dim(\mathrm{H}^0(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) = \begin{cases} 1 & \textit{if } \ \lambda = \mu, \\ 1 & \textit{if } \ \lambda = -\frac{k}{2} \ \textit{and} \ \mu = \frac{k+1}{2}, \ k \in \{0,1,2,\dots\}, \\ 0 & \textit{in the other cases.} \end{cases}$$

(ii) $$\dim(\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) = \begin{cases} 1 & \textit{if } \ \lambda = \mu, \\ 2 & \textit{if } \ \lambda = -\frac{k}{2} \ \textit{and} \ \mu = \frac{k+1}{2}, \ k \in \{0,1,2,\dots\}, \\ 0 & \textit{in the other cases.} \end{cases}$$

(iii) $$\dim(\mathrm{H}^2(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) = \begin{cases} 1 & \textit{if } \ \lambda = -\frac{k}{2} \ \textit{and} \ \mu = \frac{k+1}{2}, \ k \in \{0,1,2,\dots\}, \\ 0 & \textit{in the other cases.} \end{cases}$$

(iv) $\dim(\mathrm{H}^n(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) = 0.$

We refind here the results of [2] for the H^1.

To be more preciseу, in the following, we give explicit basis for these cohomology groups

(i) $\mathrm{H}^0(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\lambda}) = \mathrm{Span}(id)$ and $\mathrm{H}^0\left(\mathfrak{osp}(1|2), \mathfrak{D}_{-\frac{k}{2},\frac{k+1}{2}}\right) = \mathrm{Span}(\partial_\theta \partial_x^k - \theta \partial_x^{k+1})$.

(ii) The space $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\lambda})$ is spanned by the cohomology class of the reduced 1 cocycle h_λ defined by:

$$h_\lambda(X) = h_\lambda(A) = 0, \quad h_\lambda(H) = -id, \quad h_\lambda(B) = \theta\,\cdot \quad \text{and} \quad h_\lambda(Y) = -2x\,\cdot\,.$$

While the space $\mathrm{H}^1\left(\mathfrak{osp}(1|2),\mathfrak{D}_{-\frac{k}{2},\frac{k+1}{2}}\right)$ is spanned by the cohomology classes of the reduced 1 cocycles f_k and $\widetilde{f}_k$ defined respectively by:

$$f_k(X)=f_k(A)=0,\quad f_k(H)=\partial_\theta\partial_x^k-\theta\partial_x^{k+1},\quad f_k(B)=\theta\partial_\theta\partial_x^k \text{ and } f_k(Y)=2xf_k(H),$$

$$\widetilde{f}_k(X)=\widetilde{f}_k(A)=\widetilde{f}_k(H)=0,\quad \widetilde{f}_k(B)=\partial_x^k \text{ and } \widetilde{f}_k(Y)=-2k\partial_\theta\partial_x^{k-1}+2\theta(k+1)\partial_x^k.$$

(iii) A similar realization of $\mathrm{H}^2\left(\mathfrak{osp}(1|2),\mathfrak{D}_{-\frac{k}{2},\frac{k+1}{2}}\right)$ is easy, we prefer to give an explicit, nontrivial, reduced 2 cocycle as a cup product. Let

$$\Omega_k(U,V)=(f_k\vee h_{-\frac{k}{2}})(U,V):=f_k(U)\circ h_{-\frac{k}{2}}(V)-(-1)^{UV}f_k(V)\circ h_{-\frac{k}{2}}(U).$$

Since f_k and $h_{-\frac{k}{2}}$ are cocyles, a direct computation shows that Ω_k is a 2 cocycle, it is nontrivial since its restriction to $\mathfrak{sl}(2)\times\mathfrak{sl}(2)$ is nontrivial:

$$\Omega_k(X_f,X_g)=-(-1)^k\omega(f,g)(k\partial_\theta\partial_x^{k-1}-(k+1)\theta\partial_x^k)$$

where ω is the Gelfand-Fuchs cocycle defined by $\omega(f,g)=f'g''-g'f''$.

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