INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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**N° 0001**

October 2008

Thème COM

ISSN 0249-6399 ISRN INRIA/RR--0001--FR+ENG

INSTITUT NATIONAL
DE RECHERCHE
EN INFORMATIQUE
ET EN AUTOMATIQUE

INRIA

centre de recherche
PARIS - ROCQUENCOURT

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Thème COM — Systèmes communicants
Équipe-Projet Hipercom

Rapport de recherche n° 0001 — October 2008 — 21 pages

**Abstract:** Motivated by the recent refutation of information loss paradox in black hole by Hawking, we investigate the new concept of *non unitary random walks*. In a non unitary random walk, we consider that the state 0, called the *black hole*, has a probability weight that decays exponentially in $e^{-\lambda t}$ for some $\lambda > 0$. This decaying probabilities affect the probability weight of the other states, so that the the apparent transition probabilities are affected by a repulsion factor that depends on the factors $\lambda$ and black hole lifetime $t$. If $\lambda$ is large enough, then the resulting transition probabilities correspond to a neutral random walk. We generalize to *non unitary gravitational walks* where the transition probabilities are function of the distance to the black hole. We show the surprising result that the black hole remains attractive below a certain distance and becomes repulsive with an exactly reversed random walk beyond this distance. This effect has interesting analogy with so-called dark energy effect in astrophysics.

**Key-words:** random walk, black hole, unitarity, dark energy, continued fractions, asymptotic analysis

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# Marches aléatoires non unitaires

**Résumé :** Motivés par la récente réfutation par Hawking du paradoxe de a perte d'information dans les trous noirs, nous introduisons le nouveau concept de *marches aléatoires non unitaires*. Dans une marche aléatoire non unitaire, nous considérons que l'état 0, aussi appelé le *trou noir*, a un poids de probabilité qui décroît exponentiellement en $\exp^{-\lambda t}$ pour un certain $\lambda > 0$. Cette décroissance exponentielle affecte les poids des autres états, de telle manière que les probabilités apparentes de transition entre les états sont affectées par un facteur de répulsion qui dépend des quantités $\lambda$ et de la durée de vie $t$ du trou noir. Si $\lambda$ est suffisamment large, alors les probabilités apparentes correspondent à une marche aléatoire neutre. Nous généralisons aux *marches gravitationnelles non unitaires* où les probabilités de transition dépendent de la distance au trou noir. Nous montrons le résultat surprenant où le trou noir reste attractif en dessous d'une certaine distance et devient strictement répulsif en deça en inversant strictement la marche aléatoire. Cette effet présente une intéressante analogie avec l'effet de l'énergie sombre en astrophysique.

**Mots-clés :** marche aléatoire, trou noir, unitarité, énergie sombre, fractions continues, analyse asymptotique

# 1 Introduction and motivation

A black hole is a massive stellar object that absorb everything trapped by its gravitational field, including light. In 1976, Stephen Hawking [2], the famous astrophysicist, discovered that the black holes are not absolutely black and eternal. Indeed black holes evaporate via Hawking radiation and eventually disappear. The problem is in the fact that Hawking radiation does not carry any information and therefore the information contained in the objects absorbed by the black hole during its lifetime simply disappear. This information loss paradox leads to fundamental question in theoretical physics since it apparently violates the principle of unitarity in quantum physics.

In 2006 Hawking [5] refuted its argument about the information loss paradox. His refutation is still the object of controversy but established the foundation of non unitary Markov system. Basically Hawking stated that if black holes would have a non unitary effect, then the impact of this non unitary effect on any black hole history would vanish exponentially with time and therefore only the effect in unitary metric would remain at infinity. In other words a black hole tends almost surely to be unitary and to release its information without loss during its lifetime.

The motivation of this paper is to extend the analysis of non unitary effect on a system involving a black hole. We describe a simple model of non unitary effect as a probability weight that decays exponentially with time. What is interesting is the effect of the unitary loss on a halo of matter that surround a black hole. The matter acts like a gas where particles collide at random times like in a *random walk*. In physics such random walk problems are investigated via finite element simulation with particles traveling between finite boxes of space. Each box is a state in the random walk. In a non unitary random walk, we consider that the state 0, called the *black hole* state has a probability weight that decays exponentially in $e^{-\lambda t}$ for some $\lambda > 0$.

We will show that such non unitary random walk shows a behavior which is very sensitive to its tuning parameters, and thus the simulation must be done in the most realistic situation. The number of states needed to simulate a random walk in a galactic halo may exceed $10^{20}$, assuming spherical symmetry in order to consider only distinct distances to the central black hole. This makes the simulation very hard to perform. Furthermore the impact of the unitarity effect stands in the *future* of the particle, namely in the quantity of probability weight it will lose after eventually ingested by the black hole. Therefore the simulation must compute the probability weight of all potential trajectories. In a (classic) unitary universe, all these trajectories sums to one. But in a non unitary universe this is not the case, and computation must involve all possible trajectory within the whole life time of the black hole. A super-massive black hole, like the black hole in the center of our galaxy, has a lifetime of order $10^{90}$ years. Such simulation is rather impossible to make: either we have to iterate on vector of $10^{20}$ coefficients over $10^{90}$ steps, or we iterate during $\log_2 10^{90}$ steps, on the successive squares of the non unitary matrix $10^{20} \times 10^{20}$. The object of this paper is to analytically guess the results of such simulations by investigating the asymptotics of *non unitary random walks*.

We show the surprising result that the black hole remains attractive below a certain distance and becomes repulsive or at least neutral beyond this distance.

This effect shows an interesting analogy with so-called dark energy effect in astrophysics.

## 2 Non Unitary effect and the Schrödinger's rabbit

In a (honest) unitary universe all probabilities sum to one. If you take a rabbit at time $t = 0$ then one month later the rabbit will be either alive with probability $\alpha > 0$ or dead with probability $\beta > 0$ and $\alpha + \beta = 1$. But in quantum physics the sum of probabilities is equal to the integral $\int |\Psi|^2$ of the wave function $\Psi$ of the universe. Therefore the sum of probabilities being equal to one is not only a mathematical statement but also a physical statement. Unitarity principle states that for all time $t$: $\int |\Psi|^2 = 1$.

But in a non unitary universe we may have $\int |\Psi|^2 \neq 1$. The loss of information implies a non unitary universe. Hawking mentions metrics pertinent for black holes (so-called anti-deSitter metrics) where $\int |\Psi|^2 = e^{-\lambda t}$ for some $\lambda > 0$. In other word let assume a rabbit which at time $t = 0$ is either in state $s_0$ inside a black hole with a non unitary metric with probability $\alpha$ or in state $s_1$ outside a black hole in an unitary metric. If blackhole lifetime is $t$ then at time $t$ the probability weight of state $s_0$ will be $r_0(t) = e^{-\lambda t}$ while the probability weight of state $s_1$ will remain at $r_1(t) = 1$. Consequently the sum of probabilitities of the rabbit equals $\alpha r_0(t) + \beta r_1(t) \neq 1$ and when $t$ increases the *apparent* probability of state $s_0$ defined as $\frac{\alpha r_0(t)}{\alpha r_0(t)+\beta r_1(t)} \to 0$. In other words the rabbit never falls in non unitary metric and stays in unitary metric.

This is equivalent as if an investor has one euro at time $t = 0$, and put $\alpha$ cents in a black hole market, and keep $\beta$ cents in his pocket. If the black hole market looses value at rate $-\lambda$, then at the end of day the investor will have most of his remaining money in his pocket.

In fact Hawking point is not as trivial as exposed above: he does not consider a rabbit, but the whole black hole and considers that its quantum state is a superposition of unitary and non unitary metrics. He shows that on the path integral over the black hole lifetime, the contributions of non unitary metric vanish exponentially, so that only the unitary metric contributions remain.

In the following we are not considering state superposition, we assume that either the rabbit is inside the black hole (state $s_0$) or far away outside the black hole (state $s_1$) so that the state functions don't overlap. In this hypotheses we treat the problem in a classical way. We can see the rabbit as a classical equivalent of Schrödinger cat in an non unitary universe. The Schrödinger cat can be in the superposition of two states $s_0$ and $s_1$ (for instance dead or alive). We introduce the Schrödinger rabbit which is either in state $s_0$ or in state $s_1$, but probability weight of state $s_0$ decays exponentially with time.

We shall notice that black hole typical lifetime $t$ can be extremely large. In particular massive black holes at center of galaxies with mass of about $10^4$ solar masses, would evaporate in a time that exceeds $10^{80}$ years [4] greatly exceeding the current age of our universe ($10^{10}$ years)!

## 3 Modeling non unitary systems

### 3.1 Non unitary Markov processes

We still consider a two state process with state $s_0$ and a state $s_1$. State $s_0$ is an absorbing state that mimics a black hole and during the black hole lifetime the probability weight decays with rate $-\lambda$. We consider that the time is discretized and during the black hole lifetime, at each time unit the rabbit has a probability $q$ to fall in the black hole and probability $p$. When the rabbit is in the black hole it stays inside until the end of the black hole lifetime.

Or in other words, an investor has one euro in its pocket at time $t = 0$ and he has a contract with its black hole bank to move every month a fraction $q$ of his pocket money in the black hole market.

This system mimics the effect of a black hole which is permanently attractive on any object in its vicinity. We will see that if the black hole exerts an attraction which is more effective than its unitary decay, then the previous result is highly modified.

We still have $r_0(t) = \rho^t$, but about $r_1(t)$, the transition from lifetime $t$ and lifetime $t-1$ follows the identity $r_1(t) = qr_0(t-1) + pr_1(t-1)$ which makes $r_1(t) < 1$. Notice that the previous identity is time forward, since the black hole lifetime decreases when the time goes forward. We call such system a non unitary Markov process with transition matrix:

$$\mathbf{R} = \begin{bmatrix} \rho & q \\ 0 & p \end{bmatrix} .$$

We call $\rho$ the one-step unitary decay. The matrix $\mathbf{R}$ is not unitary in the Markovian sense, since vector $[1,1]$ is not an eigenvector and 1 is not an eigenvalue:

$$[1,1]\mathbf{R} \neq [1,1] .$$

The eigenvalues of matrix $\mathbf{R}$ are $\rho$ and $p$. We expect to have different behaviour when $p > \rho$ or when $\rho > p$. The objective is to estimate the apparent attraction of the black hole, or more precisely, the apparent repulsion $\tilde{p}(t) = p\frac{r_1(t-1)}{r_1(t)}$. We have

$$[r_0(t), r_1(t)] = [1,1]\mathbf{R}^t .$$

We prove the following very simple theorem whose purpose is to introduce our methodology

**Theorem 1.** *In the two-state model, when $\rho > p$ the black hole is apparently attractive, when $\rho \leq p$, the black hole is apparently repulsive when $t \to \infty$.*

*Proof.* From the fact that $r_1(t) = q\frac{\rho^t - p^t}{\rho - p} + p^t$ we get $\tilde{p}(t) = \frac{p}{\rho} + O((\frac{p}{\rho})^t)$ when $\rho > p$. In this case the attraction is stronger than the non unitary effect and the rabbit eventually falls in the black hole which exert an apparent attraction $1 - \tilde{p}(t)$ still stronger than the apparent repulsion $\tilde{p}(t)$. In the opposite case, when $\rho < p$, we have $\tilde{p}(t) = 1 + O((\frac{\rho}{p})^t)$, the non unitary effect is stronger than the attraction and the black hole is apparently fully repulsive. When $p = \rho$, we get $r_1(t) = tqp^{t-1} + p^t$ and $\tilde{p}(t) = 1 + O(\frac{1}{t})$. □

## 3.2 Non unitary random walks

In this section we investigate the system where there are several states: an absorbing black hole state $s_0$ and a infinite sequence of states $s_1, s_2, \ldots, s_n, \ldots$. If the rabbit is in state $s_0$ its stay there for ever, or at least for the remaining black hole lifetime, if the rabbit is in state $s_n$, for $n > 0$, then at the time unit it has probability $q$ to shift down to state $s_{n-1}$, and probability $p$ to shift up to state $s_{n+1}$.

We denote $r_n(t)$ the sum of probabilities when the rabbit is on state $s_n$ when the black hole has a remaining lifetime $t$. We denote $K_n(u) = \sum_{t \geq 0} r_n(t)u^t$.

Let $T_n$ the time at which the rabbit hits the black hole starting at state $s_n$ at time 0. We have $r_n(t) = \sum_{\theta \leq t} P(T_n = \theta)\rho^{t-\theta} + P(T_n > t)$, and thus

$$K_n(u) = F_n(u)\frac{1}{1-\rho u} + (1 - F_n(u))\frac{1}{1-u}$$

with $F_n(u)$ the probability generating function of $T_n$. $F_n(u)$ satisfies $F_n(u) = quF_{n-1}(u) + puF_{n+1}(u)$, and therefore $F_n(u) = (F(u))^n$, with $1 = \frac{qu}{F(u)} + puF(u)$, which leads to

$$F(u) = \frac{1 - \sqrt{1 - 4pqu^2}}{2up} \ .$$

Our aim is to evaluate the apparent repulsion $\tilde{p}_n(t) = p\frac{r_{n+1}(t-1)}{r_n(t)}$. By Cauchy we have

$$r_n(t) = \frac{1}{2i\pi} \oint K_n(u) \frac{du}{u^{n+1}} \ .$$

The generating function $K_n(u)$ has two main set of singularities, one at $u = \frac{1}{\rho}$ and another one at $\pm u(p)$ with $u(p) = \frac{1}{2\sqrt{pq}}$. We expect a change in behavior when one has smaller modulus than the other one. Notice that $u = 1$ is not a singularity since $1 - F_n(1) = 0$ and by Lhospital rule, $K_n(u)$ can be analytically extended beyond $u = 1$.

#### 3.2.1 Case $\rho > 2\sqrt{pq}$

**Theorem 2.** *When $\rho > 2\sqrt{pq}$, the black hole is apparently attractive for all states $s_n$ with $n > 1$, when $t \to \infty$.*

*Proof.* In this case $u = \frac{1}{\rho}$ is the dominant singularity in $K_n(u)$ and we get the estimate

$$r_n(t) = (F(\frac{1}{\rho}))^n \rho^t + O((u(p)^{-t}) \ ,$$

and

$$\tilde{p}_n(t) = \frac{p}{\rho} F(\frac{1}{\rho}) + O((\rho u(p))^{-t}) \ .$$

We notice that $\frac{\tilde{p}_n(t)}{p} > 1$ since $1 = \frac{q}{\rho F(\frac{1}{\rho})} + \frac{p}{\rho} F(\frac{1}{\rho}) > \frac{p}{\rho} F(\frac{1}{\rho})$. □

#### 3.2.2 Case $\rho < 2\sqrt{pq}$

**Theorem 3.** *When $\rho < 2\sqrt{pq}$, the black hole is apparently repulsive for all state $n$ with $n > 1$, when $t \to \infty$ and $n = o(\sqrt{t})$.*

*Proof.* In this case $u = \pm\frac{1}{2\sqrt{pq}}$ are the dominant singularities in $K_n(u)$. Let $u(p) = \frac{1}{2\sqrt{pq}}$. When $u = \pm(u(p) + \delta u)$ we have

$$F^n(u) = \frac{1}{(2up)^n} \left(1 - 2n(pq)^{\frac{1}{4}} \sqrt{\delta u} + O(n^2 \delta u)\right)$$

**Bounded values of $n$** Applying Flajolet Odlyzko [3] asymptotic result, namely

$$\begin{aligned} r_n(t) \quad &= \quad \frac{\rho^{n+t}}{(2p)^n} + \sqrt{\pi} \frac{n}{t^{\frac{3}{2}}} \frac{(pq)^{\frac{1}{4}}}{(2p)^n (u(p))^{n+t}} \\ &\left(\frac{1}{1-\rho u(p)} - \frac{1}{1-u(p)} + \frac{(-1)^{n+t}}{1+\rho u(p)}) - \frac{(-1)^{n+t}}{1+u(p)} + O(\frac{n}{t})\right) \end{aligned}$$

Notice the oscillation on the odd and even values of $n + t$ which are due to the model artefact that the random walk only affect the closest neighbor states of the current state. These oscillation does not affect the values of the apparent repulsion $\tilde{p}_n(t)$ for $n > 1$ which satisfies:

$$\tilde{p}_n(t) = \frac{1}{2} \frac{n+1}{n} \left(1 + O(\frac{1}{t}) + O(\frac{n}{t})\right)$$

which converge to $\frac{1}{2}$ as soon as $t \to \infty$ with $\frac{n}{t} \to 0$. In other words the apparent random random walk tends to be neutral with attraction balancing the repulsion.

For $n = 1$, since $r_0(t) = \rho^t \ll r_1(t) = O(\frac{1}{u(p)^t})$ we have $\tilde{p}_1(t) \to 1$: the black hole is apparently 100% repulsive on the last state before, and the random walk on the remaining states is apparently neutral. In other word the random walk is apparently neutral and reflective on the last state before the black hole.

**Unbounded values of** $n$ If we consider unbounded value of $n$, but with $n = o(\sqrt{t})$, we need a more involved analysis shows that since

$$r_n(t) = \frac{1}{2i\pi(2p)^n} \oint \left(1 - \sqrt{1 - 4pqu^2}\right)^n g_\rho(u) \frac{du}{u^{n+t+1}}$$

with $g_\rho(u) = \left(\frac{1}{1-\rho u} - \frac{1}{1-u}\right)$. by deformation of the integral loop we get for any arbitrary $z < \frac{1}{\rho}$

$$r_n(t) = I_n(t, z) + J_n(t, z) + O(\frac{1}{(2p)^n z^{n+t}})$$

with

$$I_n(t, z) = \frac{1}{2i\pi(2p)^n} \int_{u(p)}^{z} \left(\left(1 + i\sqrt{4pqu^2 - 1}\right)^n - \left(1 - i\sqrt{4pqu^2 - 1}\right)^n\right) g_\rho(u) \frac{du}{u^{n+t+1}} \quad (1)$$

$$J_n(t, z) = \frac{-1}{2i\pi(2p)^n} \int_{-z}^{-u(p)} \left(\left(1 + i\sqrt{4pqu^2 - 1}\right)^n - \left(1 - i\sqrt{4pqu^2 - 1}\right)^n\right) g_\rho(u) \frac{du}{u^{n+t+1}} \quad (2)$$

With change of variable $u = (1 + \frac{v}{n+t})u(p)$, we get

$$u^{n+t+1} = \left(1 + O(\frac{v^2}{n+t+1})\right) \exp(v) \quad (3)$$

$$\left(1 + i\sqrt{4pqu^2 - 1}\right)^n = \left(1 + O(\frac{vn}{n+t})\right) \exp(i\frac{n}{\sqrt{n+t+1}}\sqrt{2v}) \quad (4)$$

$$\left(1 - i\sqrt{4pqu^2 - 1}\right)^n = \left(1 + O(\frac{vn}{n+t})\right) \exp(-i\frac{n}{\sqrt{n+t+1}}\sqrt{2v}) \quad (5)$$

$$g_\rho(u) = \left(1 + O(\frac{v}{n+t})\right) g_\rho(u(p)) \quad (6)$$

and consequently

$$I_n(t, z) = \frac{g_\rho(u(p))}{(n+t)\pi(2p)^n u(p)^{n+t+1}} \left(\int_0^{(z-u(p))\frac{n+t+1}{u(p)}} \sin(\frac{n}{\sqrt{n+t}}\sqrt{2v}) e^{-v} dv + O(\frac{n}{n+t})\right)$$
$$\frac{g_\rho(u(p))}{(n+t)\pi(2p)^n u(p)^{n+t+1}} \left(\sqrt{\frac{\pi}{2}} \frac{n}{\sqrt{n+t}} \exp(-\frac{1}{2}\frac{n^2}{n+t}) + O(\frac{n}{n+t})\right)$$

Similarly with the change of variable $u = -(1 + \frac{v}{n+t})u(p)$ we get

$$J_n(t, z) = \frac{g_\rho(-u(p))(-1)^{n+t+1}}{(n+t)\pi(2p)^n u(p)^{n+t+1}} \left(\sqrt{\frac{\pi}{2}} \frac{n}{\sqrt{n+t}} \exp(-\frac{1}{2}\frac{n^2}{n+t}) + O(\frac{n}{n+t})\right)$$

which validate the asymptotics obtained via Flajolet-Odlyzko method to unbounded values of $n$, provided that $n = o(\sqrt{t})$. □

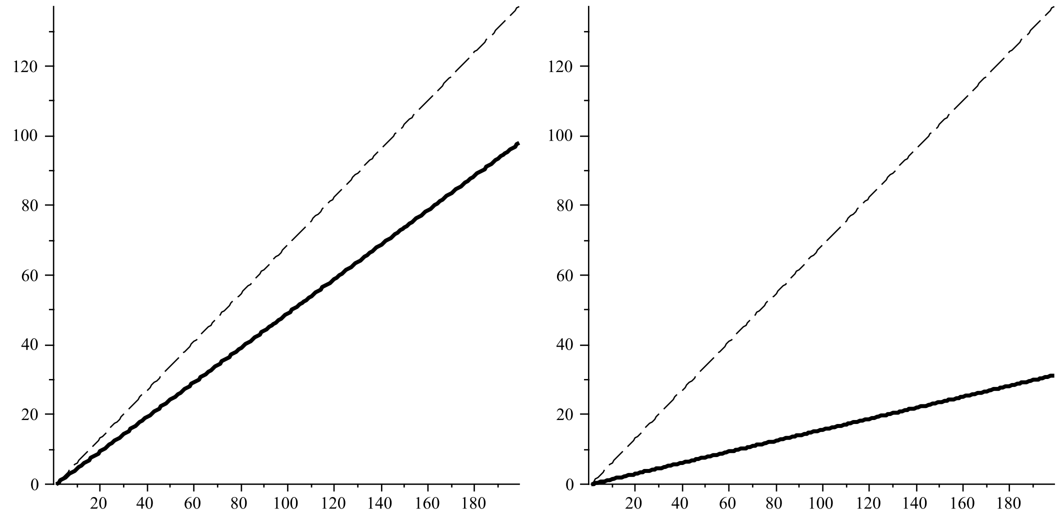


Figure 1: actual potential (dashed) and apparent asymptotic potential (plain) for $p = 0.2$ for $\rho = 0.9$ (left) and $\rho = 0.81$ both greater than $2\sqrt{pq} = 0.8$.

### 3.3 Random walk potential

Let $V_n = n\frac{1}{2}\log\frac{q}{p}$ that we call the potential of state $s_n$. Clearly the state $s_0$ is the state with the lowest potential. Notice that the analysis does not change if we add to the potential an arbitrary constant.

Noticing that $\frac{1-\tilde{p}_n(t)}{\tilde{p}_n(t)} = \frac{q}{p}\frac{r_{n+1}(t-1)}{r_{n-1}(t-1)}$. The apparent potential $\tilde{V}_n(t) = V_n + \frac{1}{2}\log r_{n+1}(t-1) + \frac{1}{2}\log r_n(t-1) = \sum_{i\leq n}\frac{1}{2}\log\frac{1-\tilde{p}_i(t)}{\tilde{p}_i(t)}$.

It comes from the previous analysis that when $\rho > u(p)$ we have an apparent potential $\tilde{V}_n(t)$ tends to be equal to $n\frac{1}{2}\log\frac{1-\tilde{p}}{\tilde{p}}$, modulo an arbitrary constant term, with $\tilde{p} = \frac{p}{\rho}F(\frac{1}{\rho})$, and in this case state $s_0$ is the state with the lowest potential. The black hole is still attractive.

When $\rho < u(p)$, the apparent potential tends to be flat, excepted for a peak on the black hole state.

Figure 1and 2 show the asymptotic apparent potentials when $\rho$ is above or below $2\sqrt{pq}$. Notice that as expected the apparent repulsion does not change very much as soon as $\rho < 2\sqrt{pq}$.

## 4 Non unitary concave walk

We call a concave walk a random walk where the transition pair of probabilities $p_n, q_n$ depends on the state and $p_n$ increases with $n$ with $\lim p_n \leq \frac{1}{2}$ when $n \to \infty$. For example $p_n = \frac{1}{2} - \frac{\beta}{n^2}$ if one wants to simulate the random walk of a particle around a gravitationally heavy object such as a stellar body. Or $p_n = \frac{1}{2} - \frac{\beta}{n}$ if one wants to simulate the random walk of a particle in a gravitationally heavy medium of density decreasing in the inverse of distance to the center (state $s_0$). A gravitational walk can be used to simulate the behavior of a particle in a gas where each collision with another particle give a random momentum.

Similarly as with uniform walk we define the actualpotential of the random walk as

$$V_n = \sum_{j=1}^{j=n} \frac{1}{2}\log(\frac{q_j}{p_j}) \ .$$

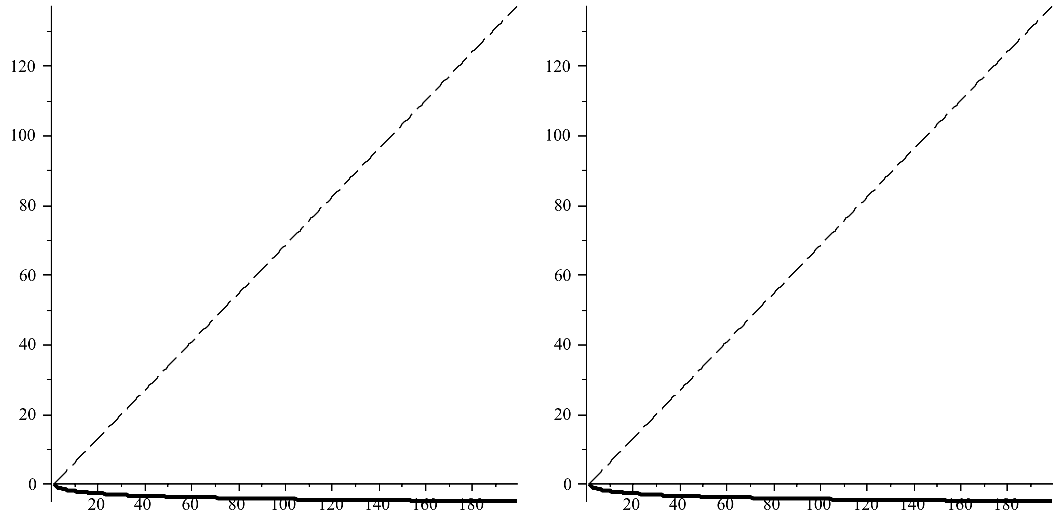


Figure 2: actual potential (dashed) and apparent asymptotic potential (plain) for $p = 0.2$ for $\rho = 0.79$ (left) and $\rho = 0.3$ both smaller than $2\sqrt{pq} = 0.8$.

Let $G_i(u)$ be the probability generating function of the time to get down from state $s_i$ to state $s_{i-1}$. We have

$$F_n(u) = \prod_{i=1}^{i=n} G_i(u)$$

We have the recursion

$$G_n(u) = q_n u F_{n-1}(u) + p_n u F_{n+1}(u) \ ,$$

or

$$G_n(u) = q_n u + p_n u G_{n+1}(u) G_n(u) \ .$$

We therefore get the recursion

$$G_n(u) = \frac{q_n u}{1 - p_n u G_{n+1}(u)} \ , \qquad (7)$$

which resolves into the continuous fraction as noticed by [ref]

$$G_n(u) = \frac{q_n u}{1 - \frac{p_n q_{n+1} u^2}{1 - \frac{p_{n+1} q_{n+2} u^2}{1 - \cdots}}} \ .$$

We notice that for all $n$, function $G_n(u)$ is odd: $G_n(-u) = -G_n(u)$. Let us denote $F(u,p) = \frac{1-\sqrt{1-4pqu^2}}{2pu}$. We have for all $u > 1$ real:

$$F(u, p_n) \leq G_n(u) \leq F(u, p_\infty)$$

## 4.1 Quasi-continuous walk

We define that we are in quasi-continuity condition when the values $p_n$ does not vary quickly. For example $p_n = p(\alpha n)$ where $p()$ is a continuous function and $\alpha$ is a small number. In quasi-continuity condition function $G_n()$ is close to the fixed point of the functionnal equation $G_n(u) = \frac{q_n u}{1 - p_n u G_n(u)}$, or in other words and uniformly in $n$:

$$\lim_{\alpha \to 0} G_n(u) = F(u, p_n) = \frac{1 - \sqrt{1 - 4 p_n q_n u^2}}{2 p_n u} \ .$$

### 4.2 Stable walks

We will assume that the random walk probabilities takes a fixed value $p_\infty$ beyond state $N$: $\forall n \geq N$ $p_n = p_\infty$. Therefore for all $n \geq N$ $G_n = \frac{1-\sqrt{1-4p_\infty q_\infty u^2}}{2p_\infty u}$. We will investigate the bi-valued random walk where $\forall n < N$ $p_n = p_1$.

Let $u_n = 2\sqrt{p_n q_n}$ and $\bar{u} = u_N$. The main singularity of $G_N(u)$ is $\bar{u}$ since $\forall n \geq N$: $G_n(u) = G_N(u) = G_N(\bar{u})\left(1 - D_N\sqrt{1-\frac{u^2}{\bar{u}^2}}\right)$ for some $D_N$, for instance $D_N = 1$. Using recursion 7, we see that $u(p_\infty)$ is also the main singularity of all $G_n(u)$ for $n < N$

**Properties of** $G_n(u)$ Let $s(u) = 1 - \frac{u^2}{\bar{u}^2}$. Let $R_n(u)$ denotes $2p_n u G_n(u)$. Notice that $R_n(u)$ is even. We have the recursion

$$R_n(u) = \frac{2p_n q_n u^2}{1 - \frac{p_n}{2}p_{n+1}R_{n+1}(u)} \tag{8}$$

We split

$$R_n(u) = H_n(u) - \sqrt{s(u)}Q_n(u)$$

with $H_n(u)$ and $Q_n(u)$ analytical defined and bounded for $|u| < |\bar{u}| + \varepsilon$ for some $\varepsilon > 0$. Notice that there is only one possible decomposition of $R_n(u)$. When $n \geq N$, we have $R_n = 1 - \sqrt{s(u)}$: $H_n(u) = Q_n(u) = 1$.

From recursion (8) we get

$$\begin{aligned} H_n(u) &= 2p_n q_n u^2 \frac{1}{1-\frac{p_n}{2p_{n+1}}H_{n+1}(u) - \frac{p_n}{2p_{n+1}}\sqrt{s(u)}Q_{n+1}(u)} \\ &= 2p_n q_n u^2 \frac{1-\frac{p_n}{2p_{n+1}}H_{n+1}(u)+\frac{p_n}{2p_{n+1}}\sqrt{s(u)}Q_{n+1}(u)}{(1-\frac{p_n}{2p_{n+1}}H_{n+1}(u))^2+(\frac{u^2}{\bar{u}^2}-1)(\frac{p_n}{2p_{n+1}}Q_{n+1}(u))^2} \end{aligned} \tag{9}$$

**Theorem 4.** *There exists $\varepsilon > 0$ such that $H_n(u)$ and $Q_n(u)$ are uniformly bounded for $|u| < \bar{u} + \varepsilon$.*

To prove the theorem we formally identify $R_N(u) = w$ and define the bivariate generating function $R_n(u, w)$ via the recursion

$$R_n(u, w) = \frac{2p_n q_n u^2}{1 - \frac{p_n}{2q_{n+1}}R_{n+1}(u, w)} \tag{10}$$

The function $R_n(u, w)$ is analytical and has positive Taylor coefficients. We notice the identity

$$R_n(u) = R_n(u, 1 - \sqrt{s(u)}) \ .$$

**Lemma 1.** *There exists $\varepsilon > 0$ such that for all $u$ such that $|u| \leq \bar{u} + \varepsilon$ and for all $w$ such that $|w| \leq 1 + \varepsilon$ we have $|R_n(u, w)| \leq 1 + \varepsilon$.*

*Proof.* We notice that $2p_n q_n = \frac{u_n^2}{2}$ is an increasing function of $n$. We also have $\frac{p_n}{p_{n+1}} \leq 1$. Let $n < N$, there exist $\varepsilon > 0$ such that if $|u| \leq \bar{u} + \varepsilon$ and $|R_{n+1}(u, w)| \leq 1 + \varepsilon$, then $|R_n(u, w)| \leq 1 + \varepsilon$.

Indeed from recursion (10):

$$|R_n(u, w)| \leq |4p_n q_n u^2| \frac{1}{2 - |R_{n+1}(u, w)|} \leq |4p_n q_n u^2| \frac{1}{1 - \varepsilon} \ ,$$

and this we set $|u| < u_{N-1}$ thus, $|4p_n q_n u^2| \leq \frac{|u|^2}{u_{N-1}^2} \leq 1 - \varepsilon^2$, for some $\varepsilon$ such that $|u| \leq \bar{u} + \varepsilon$. □

**Proof of Theorem 4** Now we have to express $H_n(u)$ and $Q_n(u)$ with $R_n(u,w)$. Let define $R_n(u,1-x)$ we denote $H_n(u,y)$ the $x$-even part of $R_n(u,1-x)$, namely $H_n(u,x^2)=\frac{1}{2}(R_n(u,1-x)+R_n(u,1+x))$ and $Q_n(u,y)$ the$x$-odd part of $R_n(u,1-x)$, namely $\frac{-1}{2x}(R_n(u,1-x)-R_n(u,1+x))$. Both are analytical in $u$ and $y$ and we have the relation

$$\left\{\begin{array}{lcl} H_n(u) & = & H_n(u,1-\frac{u^2}{\bar{u}^2}) \\ Q_n(u) & = & Q_n(u,1-\frac{u^2}{\bar{u}^2}) \end{array}\right.$$

It remains to prove that $H_n(u,x^2)$ and $Q_n(u,x^2)$ are bounded, which is given by the integral representation

$$\left\{\begin{array}{r} H_n(u,y)=\frac{1}{2i\pi}\oint zH_n(u,1-z)\frac{dz}{z^2-y} \\ Q_n(u,y)=\frac{1}{2i\pi}\oint H_n(u,1-z)\frac{dz}{z^2-y} \end{array}\right.$$

with appropriate integral loops. This last identities give the formal proof that the $H_n(u)$ and $Q_n(u)$ are analytical and uniformly bounded for $|u|\leq\bar{u}+\varepsilon$ for some $\varepsilon>0$. This terminates the end of the proof of theorem 4.

**Theorem 5.** *There exists a neighborhood of $\bar{u}$ such that the following holds for all $n\geq N$: (i) $\log(G_n(u))$ exists, (ii) let the decomposition $\log G_n(u)=h_n(u)-\sqrt{s(u)}q_n(u)$, functions $h_n(u)$ and $q_n(u)$ are uniformly bounded, (iii) $q_n(u)$ decays exponentially when $n$ decreases.*

**Lemma 2.** *For all $u$ such that $|u|\leq\bar{u}+\varepsilon$ and for all $|w|\leq 1+\varepsilon$, and for all $n<N$ we have $|R_n(u,w)|\leq\min\{4p_n\bar{u}^2,1\}(1+\varepsilon)$*

*Proof.* We just reuse recursion (10) and notice that when $n<N$

$$\begin{array}{rcl} |R_n(u,w)| & \leq & \frac{p_nq_n}{p_{N-1}q_{N-1}}4p_{N-1}q_{N-1}|u|\frac{1}{2-|R_{n+1}(u,w)|} \\ & \leq & \frac{p_nq_n}{p_{N-1}q_{N-1}}(1+\varepsilon) \end{array}$$

We end the proof of the lemma with the fact that $\frac{p_nq_n}{p_{N-1}q_{N-1}}\leq\frac{p_nq_n}{p_\infty q_\infty}=4p_nq_n\bar{u}^2\leq 4p_n\bar{u}^2$. □

**Corollary 1.** *For all $u$ such that $|u|\leq\bar{u}+\varepsilon$ and for all $n\leq N$ the functions $\frac{1}{p_n}H_n(u)$ and $\frac{1}{p_n}Q_n(u)$ are uniformly bounded.*

**Lemma 3.** *For all $u$ such that $|u|\leq\bar{u}$: $|R_n(u)|\leq\frac{p_n}{p_\infty}\leq 1$*

*Proof.* When $|u|\leq\bar{u}$ we have $|G_n(u)|\leq G_n(\bar{u})\leq\frac{1}{2p_\infty\bar{u}}$. □

**Lemma 4.** *For all $\varepsilon>0$, there exists a neighborhood of $\bar{u}$ such that $Q_n(u)$ exponentially decays when $n$ decreases and $|Q_n(u)|\leq Q_n^{1-\varepsilon}(\bar{u})$.*

*Proof.* We rewrite recursion (8) with

$$\begin{array}{rcl} H_n(u) & = & \frac{1}{2p_nq_nu^2}H_n(u)\bar{H}_n(u)\left(1-\frac{p_n}{2p_{n+1}}H_{n+1}(u)\right) \\ Q_n(u) & = & \frac{1}{4p_{n+1}q_nu^2}H_n(u)\bar{H}_n(u)Q_{n+1}(u) \end{array}$$

with $\bar{H}_n(u)=H_n(u)+\sqrt{s(u)}Q_n(u)$.

Furthermore, since $\frac{1}{p_n}R_n(u)=\frac{1}{p_n}H_n(u)-\sqrt{s(u)}\frac{1}{p_n}Q_n(u)$, $\frac{1}{p_n}H_n(u)$ and $\frac{1}{p_n}Q_n(u)$, are uniformly bounded for all $n\leq N$, thanks to Cauchy theorem, they are also uniformly continuous. Since $s(\bar{u})=0$, For any $\varepsilon>0$ there exists a neighborhood of $\bar{u}$ such that $|R_n(u)|<\frac{p_n}{p_\infty}(1+\varepsilon)$, uniformly for all $n\leq N$.

Similarly, since $\bar{R_n}(u) = R_n(u) + 2\sqrt{s(u)}Q_n(u)$ we also have a neighborhood of $\bar{u}$ where $|\bar{R_n}(u)| \leq \frac{p_n}{p_\infty}(1+\varepsilon)$.

Consequently for $u$ in this neighborhood (assuming $|u-\bar{u}| < \varepsilon\bar{u}$), we have

$$\begin{aligned}
|Q_n(u)| &= \tfrac{1}{4p_{n+1}p_n|u|^2}\left|R_n(u)\bar{R_n}(u)\right||Q_{n+1}(u)| \\
&\leq \tfrac{1}{4p_{n+1}p_n|u|^2}\tfrac{p_n^2}{p_\infty^2}(1+\varepsilon)^2|Q_{n+1}(u)| \\
&\leq \tfrac{1}{4p_{n+1}p_n\bar{u}^2}\tfrac{p_n^2}{p_\infty^2}(1+\varepsilon)^2\tfrac{\bar{u}^2}{|u|^2}|Q_{n+1}(u)| \\
&\leq \tfrac{4p_\infty q_\infty}{4p_{n+1}q_n}\tfrac{p_n^2}{p_\infty^2}(1+\varepsilon)^4|Q_{n+1}(u)| \\
&= \tfrac{p_n}{p_{n+1}}\tfrac{q_\infty}{q_n}\tfrac{p_n}{p_\infty}(1+\varepsilon)^4|Q_{n+1}(u)| \leq \tfrac{q_\infty}{q_n}\tfrac{p_n}{p_\infty}(1+\varepsilon)^4|Q_{n+1}(u)|
\end{aligned}$$

Therefore $Q_n(u)$ decays exponentially like $\prod_{j=n}^{j=N-1}\frac{q_\infty}{q_j}\frac{p_j}{p_\infty}(1+\varepsilon)^4$ as soon as $\frac{q_\infty}{q_n}\frac{p_n}{p_\infty}(1+\varepsilon)^4 < 1$. □

**Corollary 2.** *There exists $\varepsilon > 0$ and a real number $\beta$ such that for all $n \leq N$ when $|u-\bar{u}| \leq \varepsilon$ such that $|Q_n(u)-Q_n(\bar{u})| \leq Q_n(\bar{u})\left(\exp((N-n)|u-\bar{u}|)\beta\right)-1)$.*

**Lemma 5.** *We have $H_n(u) = R_n(\bar{u}) > 2p_n\bar{u}$.*

*Proof.* we have $R_n(u) = 2p_n\bar{u}G_n(\bar{u})$ and also the fact that $G_n(\bar{u}) \geq 1$. □

**End of proof of theorem 5** Since $\frac{1}{p_n}R_n(\bar{u})$ is uniformly bounded from below and $\frac{1}{p_n}R_n(u)$ is uniformly bounded and continuous around $\bar{u}$, then there exists a neighborhood of $\bar{u}$ where $\frac{1}{p_n}R_n(u)$ and $\frac{1}{p_n}H_n(u)$ are non zero and uniformly bounded from above and from below. If this neighborhood is a simple disk, then $\log\frac{1}{p_n}R_n(u)$ and $\log\frac{1}{p_n}H_n(u)$ exist. We also have the identity

$$\log\frac{1}{p_n}R_n(u) = h_n(u) - \sqrt{s(u)}q_n(u)$$

with

$$\begin{aligned}
h_n(u) &= \log\tfrac{1}{p_n}H_n(u) + \tfrac{1}{2}\log(1-s(u)\tfrac{Q_n^2(u)}{H_n^2(u)}) \\
q_n(u) &= \tfrac{1}{2\sqrt{s(u)}}\log\tfrac{H_n(u)-\sqrt{s(u)}Q_n(u)}{H_n(u)+\sqrt{s(u)}Q_n(u)}
\end{aligned}$$

Notice that $q_n(u)$ is of order $\frac{Q_n(u)}{H_n(u)}$ and decays exponentially as $Q_n(u)$ when $n$ decreases. This terminates the proof of theorem 5

We denote $D_n = q_n(\bar{u})$ we have

$$D_n = \frac{p_n}{q_n}G_n(\bar{u})G_{n+1}(\bar{u})D_{n+1}$$

**Quasi-continuous concave walk** We have the assymptotic estimate.

$$\log F_n(u) = \sum_{j=1}^{n}\log G_j(u) = \frac{1}{\alpha}\int_0^{\alpha n}\log F(u,p(x))dx + O(1)$$

and

$$\log D_n = \sum_{j=1}^{n}\log\frac{p_j}{q_j}G_j(\bar{u})G_{j+1}(\bar{u}) = \frac{1}{\alpha}\int_{\alpha n}^{\infty}\log\left(\frac{p(x)}{q(x)}F^2(\bar{u},p(x))\right)dx + O(1)\ .$$

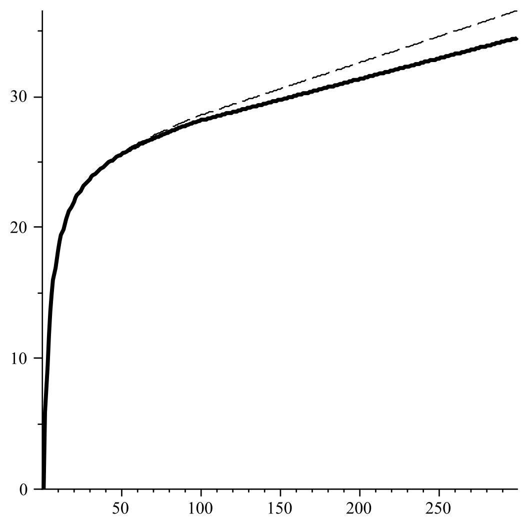


Figure 3: actual potential (dashed) and apparent asymptotic potential (plain) for a concave walk with $p_n = 0.48\exp(-\frac{100}{n^2})$ $N = 100$, $t = 10,000$, $\rho = \exp(0.0005)\bar{u}^{-1}$.

### 4.3 Case where $\rho\bar{u} > 1$

This is the simplest of the three cases. The quantity $\frac{1}{\rho}$ is the main singularity and lead to a simple pole. Quantity $\bar{u}$ is the second singularity whose contribution will be detailed in the next section. If we assume that $\frac{1}{\rho} < z < \bar{u}$ we get

$$r_n(t) = F_n(\frac{1}{\rho})\rho^{-t} + O(F_n(z)\frac{z^{-t}}{1-\rho z})$$

With $F_n(u) = \prod_{j=1}^{j=n} G_j(u)$ we get

$$\tilde{p}_n(t) = p_n\frac{1}{\rho}G_{n+1}(\frac{1}{\rho}) + O((\rho z)^{-t})$$

In quasi-continuity condition we have or all $u < u_n$: $G_n(u) = F(u, p(\alpha n)) + O(\alpha)$ and therefore we get the result we had for uniform random walk that is

$$\tilde{p}_n(t) = p(\alpha n)\frac{1}{\rho}F(\frac{1}{\rho}, p(\alpha n)) + O(\alpha) + O((\rho z)^{-t}) \ .$$

Figure 3 shows the apparent potential when $\rho\bar{u} > 1$.

### 4.4 Case where $\rho\bar{u} < 1$

**Theorem 6.** *When $\rho\bar{u} < 1$, with $n = o(\sqrt{t})$we have the asymptotic estimate*

$$r_n(t) = \sqrt{\pi}\frac{F_n(\bar{u})}{(n+t)^{\frac{3}{2}}}\frac{(p_\infty q_\infty)^{\frac{1}{4}}}{(\bar{u})^t}\left(\sum_{j=1}^{j=n} D_j\right) (g_\rho(\bar{u}) + (-1)^{n+t}g_\rho(-\bar{u}))\left(1 + O(\frac{N}{n+t})\right) + O((1+\varepsilon)^n z^{-t})$$

*With $F_n(u) = \prod_{j=1}^{j=n} G_j(u)$,.*

*Proof.* We define $z = \bar{u}(1+\varepsilon)$ which is the upper-limit of the radius where $|u| \leq z$ implies $|R_n(u)| < 1 + \varepsilon$. In this section we assume that $\frac{1}{\rho} > z$, *i.e.* the disk $|u| \leq z$ does not contain any other singularoties than $\pm\bar{u}$ for $F_n(u)g_\rho(u)$.

We have

$$F_n(u) = \prod_{j=1}^{j=n} G_j(u) = \frac{1}{u^n} \exp\left(\sum_{j=1}^{j=n} h_j(u) - \sqrt{s(u)} q_j(u)\right)$$

We have $h_n(\bar{u}) = \log(\bar{u} G_n(\bar{u}))$. Let $D_n = q_n(\bar{u}) = \frac{Q_n(\bar{u})}{H_n(\bar{u})}$. We have the expression

$$D_n = \frac{p_n}{q_n} G_n(\bar{u}) G_{n+1}(\bar{u}) D_{n+1}$$

Which gives $D_n = \prod_{j=n}^{j=N} \frac{p_j}{q_j} G_j(\bar{u}) G_{j+1}(\bar{u})$. Notice that $\frac{p_n}{q_n} G_n(\bar{u}) G_{n+1}(\bar{u}) \leq 1$ since $\frac{p_n}{q_n} \leq \frac{p_\infty}{q_\infty}$

$$\frac{p_n}{q_n} G_n(\bar{u}) G_{n+1}(\bar{u}) \leq \frac{p_\infty}{q_\infty} F^2(\bar{u}, p_\infty) = \frac{p_\infty}{q_\infty} \frac{1}{4(p\infty)^2 \bar{u}^2} = 1$$

Our aim is to find an accurate estimate of $r_n(t)$ via the Cauchy formula

$$r_n(t) = \frac{1}{2i\pi} \oint \prod \frac{R_j(u)}{p_j} \frac{du}{u^{n+t}} g_\rho(u)$$

As in the previous section we have the estimate

$$r_n(t) = I_n(t,z) + J_n(t,z) + O((1+\varepsilon)^n z^{-t})$$

The key of our analysis is the estimate of $I_n(z,t)$ and $J_n(z,t)$. This is an integration with the factor $u^n F_n(u) = \exp(\sum_{j=1}^{j=n} h_j(u) - \sqrt{s(u)} q_j(u))$. By doing the change of variable $u = (1 + \frac{v}{n+t})\bar{u}$ we get

$$\begin{aligned}
\frac{1}{u^t} F_n(u) &= \frac{1}{\bar{u}^{n+t}} \exp(\textstyle\sum_{j=1}^{j=n} h_j(\bar{u}) + \frac{1}{n+t} O_1(v) - \sqrt{-2v} \frac{D_j}{\sqrt{n+t}} \\
&\quad + \sqrt{-2v} \frac{1}{(n+t)^{\frac{1}{2}}} O_2(D_j(e^{(N-j)|u-\bar{u}|\beta} - 1))) e^{-v} \\
&= \frac{1}{\bar{u}^t} \prod_{j=1}^{j=n} G_j(\bar{u}) \left(1 - \sqrt{-2v} \frac{\sum_{j=1}^{j=n} D_j}{\sqrt{n+t}} + \frac{n}{n+t} O_1'(v)\right. \\
&\quad \left. + \sqrt{-2v} \frac{1}{(n+t)^{\frac{3}{2}}} \sum_{j=1}^{j=n} O_2'(v D_j (N-j))\right) e^{-v}
\end{aligned}$$

Notice that $O_1(v)$ and $O_1'(v)$ are both analytic in $v$ with the domain of definition including the integration paths.

Therefore expression $I_n(t,z)$ we have

$$\begin{aligned}
I_n(t,z) &= \frac{1}{2i\pi} \int_{\bar{u}}^{z} \left(\exp\left(\sum_{j=1}^{j=n} h_j(u) + i\sqrt{\frac{u^2}{\bar{u}}^2 - 1}\, q_j(u)\right)\right. \\
&\quad \left. - \exp\left(\sum_{j=1}^{j=n} h_j(u) - i\sqrt{\frac{u^2}{\bar{u}}^2 - 1}\, q_j(u)\right)\right) g_\rho(u) \frac{du}{u^{n+t+1}} \\
&= \frac{\bar{u}^{-t}}{\pi} \prod_{j=1}^{j=n} G_j(\bar{u}) \left(\sum_{j=1}^{j=n} D_j \sqrt{2v} e^{-v} + \sum_{j=1}^{j=n} O_2'(D_j) \frac{N}{(n+t)^{\frac{3}{2}}}\right) + O((1+\varepsilon)^n z^{-t}).
\end{aligned}$$

Notice that the term in $O_1'(v)\frac{n}{n+t}$ disappears when we substract the term in $-i\sqrt{\frac{u^2}{\bar{u}^2} - 1}$:

$$\begin{aligned}
r_n(t) &= \sqrt{\pi} \frac{F_n(\bar{u})}{(n+t)^{\frac{3}{2}}} \frac{(p_\infty q_\infty)^{\frac{1}{4}}}{(\bar{u})^t} \left(\sum_{j=1}^{j=n} D_j\right) \\
&\quad (g_\rho(\bar{u}) + (-1)^{n+t} g_\rho(-\bar{u}))\,(1 + O(\tfrac{N}{n+t})) + O((1+\varepsilon)^n z^{-t})
\end{aligned}$$

□

With $F_n(u) = \prod_{j=1}^{j=n} G_j(u)$, we get the following evaluation for the apparent repulsion:

$$\tilde{p}_n(t) = p_n G_{n+1}(\bar{u}) u_N \frac{\sum_{j=1}^{j=n+1} D_j}{\sum_{j=1}^{j=n} D_j}(1 + O(\frac{N}{n+t}))$$

In quasi-continuity condition we can identify $G_n(u)$ with $F(u, p_n)$ and $D_{n+1-k} = (\frac{p_n}{q_n} F^2(u, p_n))^k D_{n+1}$. This leads to $\sum_{j=1}^{j=n+1} D_j = \frac{1}{1-\frac{p_n}{q_n}F^2(u,p_n)} D_{n+1}$ and $\sum_{j=1}^{j=n} D_j = \frac{\frac{p_n}{q_n}F^2(u,p_n)}{1-\frac{p_n}{q_n}F^2(u,p_n)} D_{n+1}$

$$\lim_{\alpha\to 0, t\to\infty} \tilde{p}_n(t) = \frac{\bar{u}}{F(\bar{u}, p_n)} q_n \ .$$

It turns out that the black hole is simply repulsive since $\tilde{p}_n(t) \geq (u_N)^2 2 p_n q_n = \frac{1}{2}\left(1 + \sqrt{1 - (\frac{\bar{u}}{u_n})^2}\right) \geq \frac{1}{2}$, and the closer we are to the black hole, the stronger is the repulsion. When $n > N$ we get naturally $\tilde{p}_n(t) \approx \frac{1}{2}$: the random walk is apparently neutral beyond the state $s_N$ where the coefficients are stable.

In the limit case where $u_N \approx 1$, we would have $\tilde{p}_n(t) \approx q_n$: the random walk is apparently strictly *reversed*.

**Corollary 3.** *We have* $r_n(t) = P(T_n = t)(\frac{1}{1-\rho\bar{u}} + \frac{(-1)^{n+t}}{1+\rho\bar{u}})(1+O(\frac{N}{n+t})) + P(T_n > t)$

## 4.5 Case where $\bar{u}\rho \approx 1$

We consider the case where $\rho \approx \bar{u}^{-1}$ but with $\bar{u} > 1$.

**Case** $\frac{1}{\rho} < \bar{u}$

**Theorem 7.** *For* $n = o(\sqrt{t})$ *we have*

$$\begin{aligned} r_n(t) &= \rho^t F_n(\tfrac{1}{\rho}) + \bar{u}^{-t}\sqrt{\pi} F_n(\bar{u}) \frac{\sum_{j=1}^{j=n} D_j}{(n+t)^{\frac{3}{2}}} (p_\infty q_\infty)^{\frac{1}{4}} \left(g_\rho(\bar{u}) + (-1)^{n+t} g_\rho(-\bar{u})\right)\left(1 + O(\tfrac{N}{n+t})\right) \\ &\quad + O((1+\varepsilon)^n z^{-t}) \ . \end{aligned}$$

*Proof.* We assume that $\frac{1}{\rho} - \bar{u} = \frac{1}{o(t)} = o(\frac{1}{\sqrt{t}})$. It suffice to add the contribution of both $\frac{1}{\rho}$ and $\bar{u}$:

$$\begin{aligned} r_n(t) &= \rho^t F_n(\tfrac{1}{\rho}) + \bar{u}^{-t}\sqrt{\pi} F_n(\bar{u}) \frac{\sum_{j=1}^{j=n} D_j}{(n+t)^{\frac{3}{2}}} (p_\infty q_\infty)^{\frac{1}{4}} \left(g_\rho(\bar{u}) + (-1)^{n+t} g_\rho(-\bar{u})\right)\left(1 + O(\tfrac{N}{n+t})\right) \\ &\quad + O((1+\varepsilon)^n z^{-t}) \end{aligned}$$

Notice that the term in $\bar{u}^{-t}$ is smaller than the term in $\rho^t$. □

**Case** $\frac{1}{\rho} > \bar{u}$

**Theorem 8.** *Assuming* $\rho\bar{u} = 1 - \frac{1}{o(t)}$, *when* $n = o(\sqrt{t})$ *we have the estimate:*

$$\begin{aligned} \frac{r_n(t)}{F_n(\bar{u})} &= \rho^t + \bar{u}^{-t}\sqrt{\pi} \frac{\sum_{j=1}^{j=n} D_j}{(n+t)^{\frac{3}{2}}} (p_\infty q_\infty)^{\frac{1}{4}} \left(g_\rho(\bar{u}) + (-1)^{n+t} g_\rho(-\bar{u})\right) \\ &\quad + O(\frac{1}{\sqrt{\frac{1}{\rho} - \bar{u}}}) \rho^t \sum_{j=1}^{j=n} D_j \end{aligned}$$

*Proof.* This case is interesting because the second main singularity $\frac{1}{\rho}$ stands right in the integration path of $I_n(t, z)$, and $g_\rho(u)$ becomes singular at $u = \frac{1}{\rho}$. To remove this annoying singularity we bend the integration path of $I_n(t, z)$

so that it avoids the point $\frac{1}{\rho}$ and therefore the singularity at $u = \frac{1}{\rho}$ becomes a simple pole. Anyhow the detour will introduce a correction term of order $F_n(\bar{u})g_\rho(\bar{u})\sum_{j=1}^{n} D_j\sqrt{1-\frac{1}{(\rho\bar{u})^2}}\rho^t$. That is an error term in $F_n(\bar{u})\rho^t O(\frac{1}{\sqrt{\frac{1}{\rho}-\bar{u}}})\sum_{j=1}^{n} D_j$.

We assume that $\frac{1}{\rho}-\bar{u}$ is both $o(\frac{1}{\sqrt{t}})$ and $\frac{1}{o(t)}$. We have $F_n(\frac{1}{\rho}) = F_n(\bar{u})\left(1+O(\frac{n}{\sqrt{t}})\sum_{j=1}^{j=n} D_j\right)$. Therefore we get

$$\begin{aligned}\frac{r_n(t)}{F_n(\bar{u})} &= \rho^t + \bar{u}^{-t}\sqrt{\pi}\frac{\sum_{j=1}^{j=n} D_j}{(n+t)^{\frac{3}{2}}}(p_\infty q_\infty)^{\frac{1}{4}}\left(g_\rho(\bar{u})+(-1)^{n+t}g_\rho(-\bar{u})\right)\\ &\quad + O(\frac{1}{\sqrt{\frac{1}{\rho}-\bar{u}}})\rho^t\sum_{j=1}^{j=n} D_j\end{aligned}$$

□

Notice that $g_\rho(\bar{u}) = O(\frac{1}{\frac{1}{\rho}-\bar{u}}) = o(t)$ and the error term is negligible in front of the term in $\bar{u}^{-t}$. Since $F_n(\frac{1}{\rho}) = F_n(\bar{u})(1+O(\frac{n}{\sqrt{t}}))$ and $\sum_{j=1}^{j=n+1} D_j = \frac{1}{1-\frac{p_n}{q_n}F^2(\bar{u},p_n)}D_{n+1}$ in quasi continuous condition, we get (removing the error term)

$$\tilde{p}_n(t) \approx p_n\frac{1}{\rho}F(\frac{1}{\rho},p_{n+1})\frac{\rho^t+\bar{u}^{-t}\sqrt{\pi}(p_\infty q_\infty)^{\frac{1}{4}}(n+t)^{-\frac{3}{2}}\frac{1}{1-\frac{p_n}{q_n}F^2(\bar{u},p_n)}D_{n+1}(g_\rho(\bar{u})+(-1)^{n+t}g_\rho(-\bar{u}))}{\rho^t+\bar{u}^{-t}\sqrt{\pi}(p_\infty q_\infty)^{\frac{1}{4}}(n+t)^{-\frac{3}{2}}\frac{\frac{p_n}{q_n}F^2(\bar{u},p_n)}{1-\frac{p_n}{q_n}F^2(\bar{u},p_n)}D_{n+1}(g_\rho(\bar{u})+(-1)^{n+t}g_\rho(-\bar{u}))} .$$

We cannot say that any of the two terms in $\rho^t$ and in $\bar{u}^{-t}$ is negligible in front of the other one.

**Corollary 4.** *In quasi continuous condition there is a state $B$ such that the black hole is attractive before this state and attractive beyond and $B = \frac{z}{\alpha}$ such that*

$$\int_z^\infty \log\left(\frac{p(x)}{q(x)}F^2(\bar{u},p(x))\right)dx = \alpha\log\left(\rho^t\bar{u}^t t^{\frac{3}{2}}(1-\rho\bar{u})\right) .$$

*Proof.* We observe that if we define state $B$ such that

$$\rho^t \approx \bar{u}^{-t}\sqrt{\pi}(p_\infty q_\infty)^{\frac{1}{4}}\frac{1}{1-\frac{p_B}{q_B}F^2(\bar{u},p_B)}D_{B+1}(g_\rho(\bar{u})+(-1)^{n+t}g_\rho(-\bar{u})) .$$

When $n < B$ then the term in $\rho^t$ will be preponderant and in this case

$$\tilde{p}_n(t) \approx p_n\frac{1}{\rho}F(\frac{1}{\rho},p_n)$$

in other word the black hole is attractive.

When $n > B$, then

$$\tilde{p}_n(t) \approx q_n\frac{1}{\rho F(\frac{1}{\rho},p_n)}$$

in other words, the black hole is repulsive beyond state $B$. Notice that the change of mode is sharp: there is a brisky change of the value of $\tilde{p}_n(t)$ in a small set of contiguous states, leading to an edge in the apparent random walk potential (see figure 4 and following).

Notice that $\tilde{p}_n(t) \to \frac{1}{2}$ when $n$ increases: the random walk is asymptotically neutral.

Ignoring $O(\alpha)$ terms, the change of mode occurs on state $B = \frac{z}{\alpha}$ such that

$$\int_z^\infty \log\left(\frac{p(x)}{q(x)}F^2(\bar{u},p(x))\right)dx = \alpha\log\left(\rho^t\bar{u}^t t^{\frac{3}{2}}(1-\rho\bar{u})\right)$$

□

Figures 4 and 5 shows the apparent potential of a concave stable walk with different $\rho < \bar{u}^{-1}$.

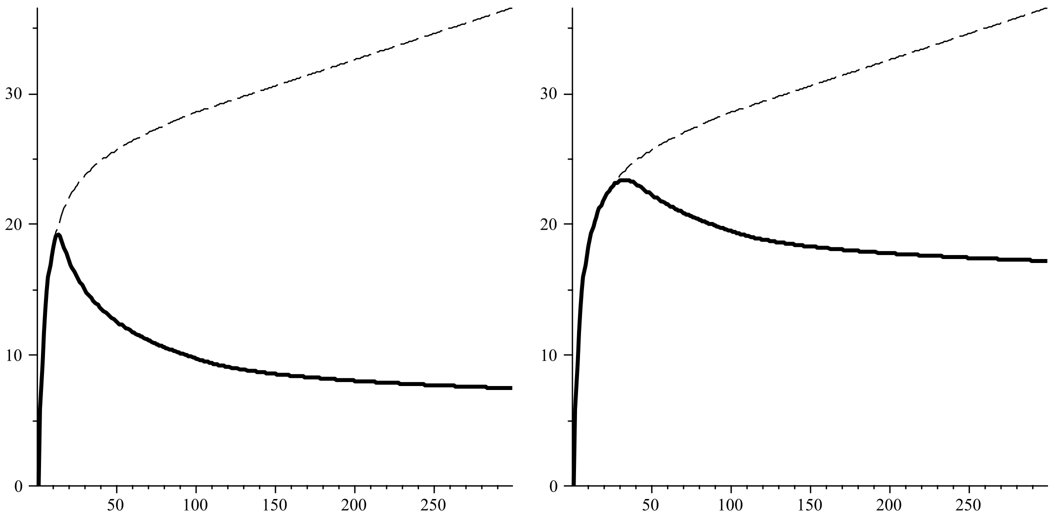


Figure 4: actual potential (dashed) and apparent asymptotic potential (plain) for a concave walk with $p_n = 0.48\exp(-\frac{100}{n^2})$ $N = 100$, $t = 10,000$, $\rho = \exp(-0.002)\bar{u}^{-1}$ (left), $\rho = \exp(-0.001)\bar{u}^{-1}$ (right).

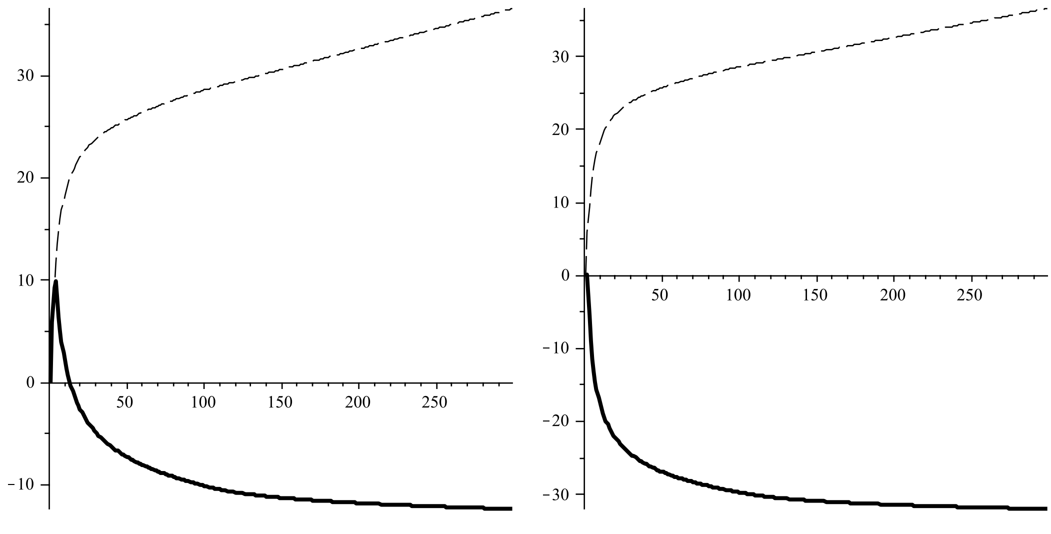


Figure 5: actual potential (dashed) and apparent asymptotic potential (plain) for a concave walk with $p_n = 0.48\exp(-\frac{100}{n^2})$ $N = 100$, $t = 10,000$, $\rho = \exp(-0.004)\bar{u}^{-1}$ (left), $\rho = \exp(-0.006)\bar{u}^{-1}$ (right).

### 4.6 Generalized concave walks

In this subsection we don't consider anymore that the random walk coefficient are stable beyond a fixed step $N$. In this case we simply assume that $\lim p_n = p_\infty$ and we denote $\bar{u} = u(p_\infty)$.

The main difficulty is in the convergence of the decomposition functions $R_n(u) = H_n(u) - \sqrt{s(u)}Q_n(u)$. Therefore we assume that the series $\sqrt{p_\infty - p_n}$ converge. We can prove that there exists $\varepsilon > 0$ and $N$ such for all $u$ with $|u| < \bar{u}(1+\varepsilon)$ and for all $n > N$:

$$R_n(u,v) = v + O(\sum_{j\geq n} p_\infty - p_j)$$

A more thorough analysis shows that lemma 1 can be extended to the statement that when $|u| < u_N$ the following bounding condition holds: $|R_N(u,v)| \leq 1 + \sqrt{1 - \frac{|u|^2}{u_N^2}}$. To this end we assume that $1 - \frac{|u|^2}{u_N^2} \leq (1-\varepsilon)\sqrt{1 - \frac{\bar{u}^2}{u_N^2}} \leq (1-\varepsilon)\sqrt{2(p_\infty - p_N)})$. But $\sum_{j\geq n} p_\infty - p_j = o(\sqrt{p_\infty - p_n}$. Therefore by letting $N$ large enough such that $|R_N(u,v)| \leq 1 + \sqrt{1 - \frac{|u|^2}{u_N^2}}$, we can export the results of stable random concave walks to general concave walk.

## 5 Non unitary Gravitational walks

We call gravitational walk a concave walk where $p_\infty = \frac{1}{2}$. Or, in other words $\bar{u} = 1$. For example $p_n = \frac{1}{2} - \frac{\alpha^2}{n^2}$. In this case we have $4p_n q_n$ which converge to one as $(p_\infty - p_n)^2$.

**Theorem 9.** *We have the estimate* $r_n(t) = \rho^t + P(T_n > t) + P(T_n = t)(\frac{1}{1-\rho} + \frac{(-1)^{n+t}}{1+\rho})(1 + O(\frac{N}{\sqrt{n+t}}))$ *with*

$$\begin{aligned} P(T_n = t) &= \sqrt{\pi/2}\left(\sum_{j=1}^{j=n} D_j\right) t^{-\frac{3}{2}} \left(1 + O(\tfrac{N}{n+t})\right) \\ P(T_n > t) &= \sqrt{2\pi}\left(\sum_{j=1}^{j=n} D_j\right) t^{-\frac{1}{2}} \left(1 + O(\tfrac{N}{n+t})\right) . \end{aligned}$$

*Proof.* The analysis of the stable gravitational walk and the decomposition $F_n(u) = \prod_{j=1}^{j=n} G_j(u)$ and $R_n(u) = 2p_n u G_n(u) = H_n(u) - \sqrt{s(u)}Q_n(u)$ remains with the asymptotic estimates. Nevertheless the contribution of $g_\rho(\bar{u})$ becomes singular when $\bar{u} = 1$. In this case we have to develop the asymptotics of $P(T_n > t)$ separately and this become the preponderant term. It comes that $r_n(t) = \rho^t + P(T_n > t) + P(T_n = t)(\frac{1}{1-\rho} + \frac{(-1)^{n+t}}{1+\rho})(1 + O(\frac{N}{\sqrt{n+t}}))$ with

$$\begin{aligned} P(T_n = t) &= \sqrt{\pi/2}\left(\sum_{j=1}^{j=n} D_j\right) t^{-\frac{3}{2}} \left(1 + O(\tfrac{N}{n+t})\right) \\ P(T_n > t) &= \sqrt{2\pi}\left(\sum_{j=1}^{j=n} D_j\right) t^{-\frac{1}{2}} \left(1 + O(\tfrac{N}{n+t})\right) \end{aligned}$$

And we have

$$D_n = \frac{p_n}{q_n} D_{n+1} = \prod_{j=n}^{j=\infty} \frac{p_j}{q_j} .$$

□

Notice that:

$$D_n = \exp\left(2(V_n - V_\infty)\right) .$$

The transition to generalized gravitational walks is somewhat different because $4p_n q_n$ converges to one like $(p_\infty - p_n)^2$ since $p_\infty = \frac{1}{2}$ instead of converging

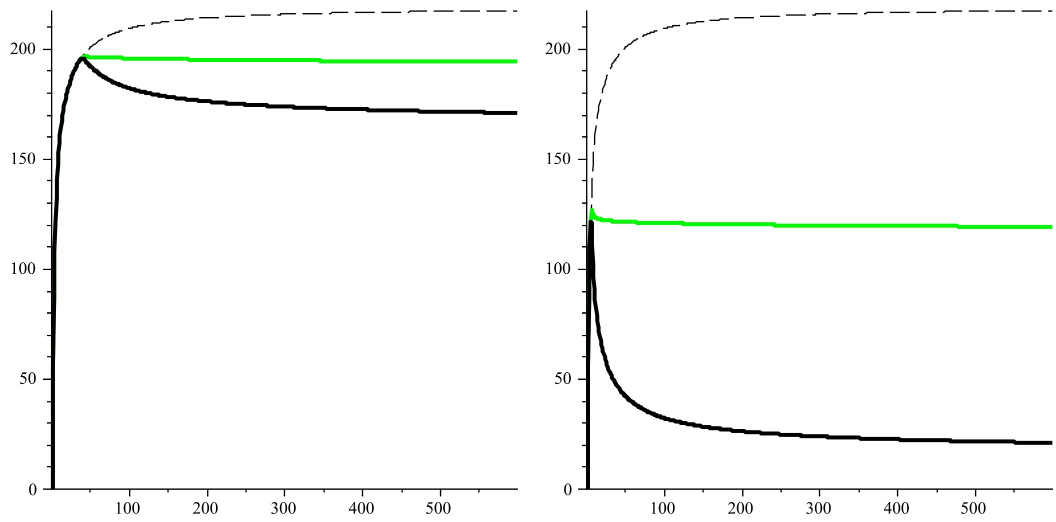


Figure 6: actual potential (dashed) and apparent asymptotic potential (plain) for a gravitational walk with $p_n = \frac{1}{2}\exp(-\frac{1000}{n^2})$, $t = 10^8$, $\rho = \exp(-50.10^{-8})$ (left), $\rho = \exp(-200.10^{-8})$ (right), in green we display the average value between actual and apparent potential.

to $4p_\infty q_\infty$ like $p_\infty - p_n$ when $p_\infty < \frac{1}{2}$. Therefore the condition of the transition to the asymptotics of the previous section is now that the series in $p_\infty - p_n$ converges (instead of the series in $\sqrt{p_\infty - p_n}$).

We consider the case where $\rho$ is very close to one

**Corollary 5.** *When* $\frac{1}{o(t)} < 1-\rho < o(\frac{1}{\sqrt{t}})$ *we have*

$$r_n(t) = \rho^t + P(T_n > t)(1 + o(1))$$

*Proof.* For this model we get $\frac{1}{1-\rho}P(T_n = t) = o(P(T_n > t))$ and since $F_n(1) = 1$:

$$r_n(t) = \rho^t + P(T_n > t)(1 + o(1))$$

□

In other words the non unitary effect is equivalent to end pay back model: if the rabbit reaches the black hole before time $t$, the unitary effect is $\rho^t$, otherwise it is 1 where we should have $r_n(t) = \rho^t + P(T_n > t)(1-\rho^t)$.

The quasi continuous condition brings a similar conclusion as in the previous section. Defining

$$\int_z^\infty \log\left(\frac{p(x)}{q(x)}\right) dx = \alpha \log\left(\rho^t t^{\frac{1}{2}}\right)$$

The black hole is attractive until state $B = \frac{z}{\alpha}$ with $\tilde{p}_n(t) \approx p_n$: the attraction is unchanged. Beyond state $B$ the random walk is repulsive with $\tilde{p}_n \approx q_n$:. the attraction is completely reversed beyond that state. See figure 6

In quasi continuous situation, we denote $V(y) = \int_0^y \frac{1}{2}\log(\frac{q(x)}{p(x)})dx$. For $x = \alpha n$ it turns out that $V_n = \frac{1}{\alpha}V(x) + O(1)$ and $D_n = \exp(-\frac{2}{\alpha}(V(x) - V(\infty))$. Similarly $\sum_{j=1}^{j=n} D_j = \frac{\exp(\frac{2}{\alpha}(V(x)-V(\infty))}{1-\frac{p(x)}{q(x)}}$. Denoting $\rho = \exp(\frac{\beta}{t})$, we have

$$r_n = e^{-\beta} + \frac{\sqrt{2\pi}}{\sqrt{t}} \frac{\exp(\frac{2}{\alpha}(V(x) - V(\infty))}{1 - \frac{p(x)}{q(x)}} \left(1 + \frac{1}{2\beta} + O(\frac{1}{t})\right) .$$

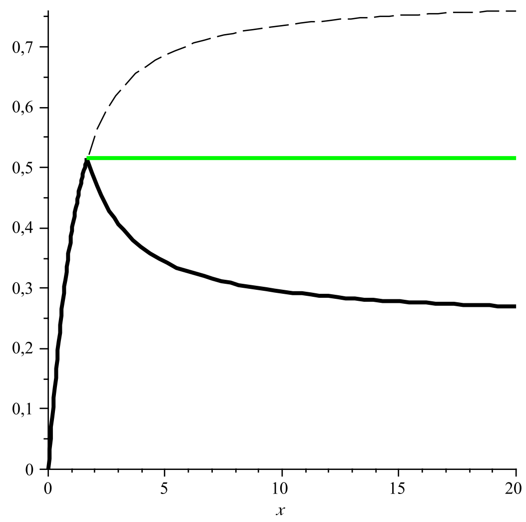


Figure 7: actual potential $V(x)$ (dashed) and apparent asymptotic potential $\tilde{V}(x)$ (plain) for a gravitational walk with $p_n = 1 - \frac{1}{1+\exp(-\frac{1}{1+x^2})}$, $t = 10^{90}$, $\beta = 10^{20}$, $\alpha = \frac{1}{2}.10^{-20}$, in green we display the average value between actual and apparent potential.

Assuming $t$ and $\frac{1}{\alpha}$ large the change of mode occurs for $z$ such that

$$\beta = \frac{1}{2}\log t + \frac{2}{\alpha}(V(\infty) - V(z)) + \log(1 - \frac{p(z)}{q(z)})$$

We can investigate a special case where the computations are tractable. For example when $\log(\frac{q(x)}{p(x)}) = \frac{1}{1+x^2}$, namely $p_n = 1 - \frac{1}{1+\exp(-\frac{1}{1+x^2})}$. In this case we have $V(x) = \frac{1}{2}\arctan(x)$. Denoting $\tilde{V}(x) = \alpha\tilde{V}_n$, figure 7 displays the actual and apparent potential for this case with parameters tuned to galactic orders of magnitudes: $t = 10^{90}$, $\alpha = \frac{1}{2}.10^{-20}$, $\beta = 10^{20}$. We get $z \approx 1.7$.

The bimodal aspect of the apparent gravitationnal, and in particular the exact reversion of the random walk beyond critical distance $z$ give some reminiscence about the mysterious *dark energy* effect. Since 1998, it has been noted that the expansion of the universe is in acceleration, fact that contradicts the usual decelerated expansion scheme predicted by general relativity. In order to explain this discrepancy, astrophysicist have imagined a so-called dark energy that contributes to the acceleration of the expansion. In classic newtonnian physics, the dark energy is equivalent to a medium that excerts a repulsion that is twice is gravitational attraction, a kind of force which does not exist in the standard models at least at low energy. In the gravitational walk model the dark energy effect would be exactly the effect of random walk reversion of low energy usual matter beyond the critical distance, assuming that gallactic black holes are non unitary.

It should be noted that in the model presented in figure 7 we would have a non unitary effect of less than $10^{-70}$ per time unit, inside a black hole. It means that the impact of the non unitary effect within the critical distance is not measurable, and that only the extremely large black hole lifetimes would make it apparent beyond the critical distance. It should also be noted that the repulsion is not due to a force in the physical sense, but would be due to a weird effect in future trajectory weight summation in the random walk.

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ISSN 0249-6399