Universal convex coverings

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Abstract

In every dimension $d \geq 1$, we establish the existence of a constant $v_d > 0$ and of a discrete subset $\mathcal{U}_d$ of $\mathbb{R}^d$ such that the following holds: $\mathcal{C} + \mathcal{U}_d = \mathbb{R}^d$ for every convex set $\mathcal{C} \subset \mathbb{R}^d$ of volume at least v_d and $\mathcal{U}_d$ contains at most $\log(r)^{d-1} r^d$ points at distance at most r from the origin, for every large r.[1]

1 Introduction

Fix a dimension $d \geq 1$ and consider the volume associated to the standard Lebesgue measure on $\mathbb{R}^d$. Given $v > 0$, a subset $\mathcal{U}$ of $\mathbb{R}^d$ is a *$v-$universal convex covering* of $\mathbb{R}^d$ if we have $\mathcal{C} + \mathcal{U} = \mathbb{R}^d$ for every convex subset $\mathcal{C}$ of $\mathbb{R}^d$ of volume strictly greater than v. Here, $\mathcal{C} + \mathcal{U}$ denotes the set of all points of the form $P + Q$ with P in $\mathcal{C}$ and Q in $\mathcal{U}$.

For every positive t, a subset $\mathcal{U}$ of $\mathbb{R}^d$ is a $v-$universal convex covering if and only if $t\mathcal{U}$ is a $(t^d v)-$universal convex covering. The properties in which we are interested are thus independent of the particular value of v. We call $\mathcal{U}$ a *universal convex covering of* $\mathbb{R}^d$ if $\mathcal{U}$ is a $v-$universal convex covering of $\mathbb{R}^d$ for some $v > 0$.

Our main result is the following.

Theorem 1.1. *Let $d \geq 1$. Up to translation and rescaling, any universal convex covering of the Euclidean vector space $\mathbb{R}^d$ has at least $\ell_d(r) = r^d$ points at distance at most r from the origin. There exists a universal convex covering $\mathcal{U}_d$ of $\mathbb{R}^d$ with at most $u_d(r) = \log(r)^{d-1} r^d$ points at distance at most r from the origin, for every large r.*

The first part of theorem 1.1 is obvious since, in any dimension d, one can consider the unit cube as the convex set $\mathcal{C}$. In dimension $d = 1$, the second part is obvious too since $\mathcal{I} + \mathbb{Z} = \mathbb{R}$ for every interval $\mathcal{I}$ of length strictly

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greater than 1, hence one can choose $\mathcal{U}_1 = \mathbb{Z}$. However, in dimension $d \geq 2$, there is a factor of $\log^{d-1}$ between the easy lower bound ℓ_d and the upper bound u_d.

Whilst it is surely possible to improve these results, I would be very surprised by the existence of universal coverings in dimension d achieving the lower bound ℓ_d for $d \geq 2$.

Proposition 2.1 below suggests thus the following question.

Question 1.2. *Let $\mathcal{S}$ be a discrete subset of the Euclidean plane $\mathbb{R}^2$ such that*

$$\sharp\{x \in S \mid \ \| x \| \leq R\} \leq R^2 + 1$$

for all $R \geq 0$. Does the complement $\mathbb{R}^2 \setminus \mathcal{S}$ of $\mathcal{S}$ necessarily contain triangles of arbitrarily large area?

By Proposition 2.1, the anwser to question 1.2 (which has an obvious generalization to the case of dimension $d > 2$) is YES if and only if there is no universal covering of the plane achieving the lower bound ℓ_2 for $d = 2$.

Call a subset $\mathcal{S}$ of $\mathbb{R}^n$ *uniformly discrete* if there exists a neighbourhood $\mathcal{O}$ of the origin such that $x - y \notin \mathcal{O}$ for every pair (x, y) of distinct elements in $\mathcal{S}$.

As long as the answer to question 1.2 is unknown or if the answer is NO, the following question and its higher-dimensional generalisations is also interesting.

Question 1.3. *Does the complement of a uniformly discrete subset $\mathcal{S}$ of the plane necessarily contain triangles of arbitrarily large area?*

Universal convex coverings are related to sphere coverings, see [1] for an overview, or more generally to coverings of $\mathbb{R}^d$ by translates of a fixed convex body. Rogers proved in [4] that every convex body of $\mathbb{R}^d$ covers $\mathbb{R}^d$ with density at most $d(5 + \log d + \log\log d)$ for a suitable covering. Erdös and Rogers in [2] showed the existence of such a covering which furthermore covers no point with multiplicity exceeding $ed(5+\log d+\log\log d)$. Chapter 31 of [3] contains an account of subsequent developpements.

There appears to be no result in the literature closely related to universal convex coverings and featuring results similar to theorem 1.1.

This paper is organized as follows. In section 2 we collect some preliminary facts. In section 3 we construct recursively a sequence of sets $(\mathcal{U}_d)_{d\geq 1}$ such that $\mathcal{U}_1 = \mathbb{Z}$ and $\mathcal{U}_d \subset \mathbb{R}^d$ for every $d \geq 1$, and we show, by induction on $d \geq 1$, that $\mathcal{U}_d$ is a universal convex covering of $\mathbb{R}^d$. In section 4, we define growth classes of functions and we introduce a natural equivalence relation on them, which is compatible with the natural partial order on increasing positive functions. Finally, we show in section 5 that the growth class of the universal convex covering $\mathcal{U}_d$ constructed in section 3 is represented by u_d. This implies theorem 1.1 by rescaling $\mathcal{U}_d$ suitably.

2 Preliminaries

For any subset $\mathcal{S}$ of $\mathbb{R}^d$, let

$$-\mathcal{S} = \{-Q \in \mathbb{R}^d \mid Q \in \mathcal{S}\}$$

denote the set of all opposite vectors.

Proposition 2.1. *Choose $v > 0$. A subset $\mathcal{U}$ of $\mathbb{R}^d$ is a $v-$universal convex covering if and only if every convex subset of $\mathbb{R}^d$ with volume at least v intersects $\mathcal{U}$ non-trivially.*

Proof. Consider a convex subset $\mathcal{C}$ of $\mathbb{R}^d$ with volume at least v. Then $-\mathcal{C}$ is a convex set of the same volume. For any point Q in $\mathbb{R}^d$, Q belongs to $\mathcal{C} + \mathcal{U}$ if and only if the convex set $-\mathcal{C} + Q$ intersects $\mathcal{U}$. □

The growth function $f_{\mathcal{S}}$ of a subset $\mathcal{S} \subset \mathbb{R}^d$ without accumulation points is defined as follows: for r an arbitrary positive real number, $f_{\mathcal{S}}(r)$ denotes the number of points of $\mathcal{S}$ at distance at most r from the origin.

Universal convex coverings are stable under affine bijections and $v-$universal convex coverings are stable under affine bijections which preserve the volume. Thus we consider the growth class with respect to the equivalence relation $\sim$ defined as follows: for any increasing non-negative functions f and g, we have $f \sim g$ if there exists a real number $t \geq 1$ such that $f(r) \leq g(tr) \leq f(t^2 r)$ for every $r \geq t$.

Growth functions of sets without accumulation points related by affine bijections are equivalent under this equivalence relation.

For any nonzero integer n, let $v_2(n)$ denote the 2-valuation of n: this is the unique integer k such that n is 2^k times an odd integer. Write any point x of $\mathbb{R}^d$ as $x = (x_i)_{1 \leq i \leq d}$, use the coordinate functions π_i defined by $\pi_i(x) = x_i$ and let $\rho^{(i)}$ denote the projection of $\mathbb{R}^d$ onto $\mathbb{R}^{d-1}$ obtained by erasing the $i-$th coordinate x_i and defined by

$$\rho^{(i)}(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, \hat{x_i}, x_{i+1}, \ldots, x_d) \ .$$

3 From dimension d to dimension $d + 1$

Let $\mathcal{U}$ denote a subset of $\mathbb{R}^d$. For every $1 \leq i \leq d+1$, let $\varphi_i^d(\mathcal{U})$ denote the set of points $x = (x_j)_{1 \leq j \leq d+1}$ in $\mathbb{R}^{d+1}$ such that $x_i \in \mathbb{Z} \setminus \{0\}$ and $2^{v_2(x_i)/d} \rho^{(i)}(x)$

belongs to $\mathcal{U}$. Finally, let

$$\varphi_d(U) = \bigcup_{i=1}^{d+1} \varphi_i^d(U).$$

For example, $\varphi_1(\mathbb{Z}) \subset \mathbb{R}^2$ is the set of all points $(x, y) \in (\mathbb{Z}[\frac{1}{2}])^2$ such that $xy \in \mathbb{Z}\backslash\{0\}$ or $xy = 0$ and $x+y \in \mathbb{Z}\backslash\{0\}$. Otherwise stated, a point (x, y) of $\varphi_1(\mathbb{Z})$ is either a non-zero element of $\mathbb{Z}^2$ or it has two non-zero coordinates and belongs to the set

$$\bigcup_{n=0}^{\infty} \left((2^n\mathbb{Z}) \times (2^{-n}\mathbb{Z})\right) \cup \left((2^{-n}\mathbb{Z}) \times (2^n\mathbb{Z})\right) .$$

Proposition 3.1. *Let $\mathcal{U}$ be a $v-$universal convex covering of $\mathbb{R}^d$. Then $\varphi_d(\mathcal{U})$ is a $v'-$universal covering of $\mathbb{R}^{d+1}$ with v' given by*

$$v' = 4^{d+1} \max(1, 4v).$$

The value of v' in proposition 3.1 is not optimal and can easily be improved.

Let $(\mathcal{U}_d)_{d\geq 1}$ denote the sequence of sets defined recursively by $\mathcal{U}_1 = \mathbb{Z}$ and, for every $d \geq 1$,

$$\mathcal{U}_{d+1} = \varphi_d(\mathcal{U}_d).$$

Proposition 3.1 implies the following result.

Corollary 3.2. *For every $d \geq 1$, the set $\mathcal{U}_d$ is a universal convex covering of $\mathbb{R}^d$.*

Proof of proposition 3.1. By proposition 2.1, it is enough to show that the volume of any convex set $\mathcal{C}$ which does not intersect $\varphi_d(\mathcal{U})$ is bounded by v'. Without loss of generality, we may assume that $\mathcal{C}$ is open. Let L denote the diameter of $\mathcal{C}$ with respect to the L^∞ norm on $\mathbb{R}^{d+1}$ defined by $\| x \|_\infty = \max_{1\leq i\leq d+1}(|x_i|)$. Two cases arise.

First, if $L \leq 4$, the volume of $\mathcal{C}$ satisfies $vol(\mathcal{C}) \leq 4^{d+1} \leq v'$.

The remaining case is when $L > 4$. Hence we assume that $L > 4$ and we must show that the volume $vol(\mathcal{C})$ of $\mathcal{C}$ is at most $4^{d+2}v$.

Note that there exists an index $1 \leq i \leq d + 1$ such that $\pi_i(\mathcal{C}) =]a, b[\subset \mathbb{R}$ is an open interval of length L. Thus one can pick two real numbers α and β such that

$$a < a + \frac{L}{4} \leq \alpha < \alpha + \frac{L}{4} \leq \beta < \beta + \frac{L}{4} \leq b$$

and $\alpha\beta \geq 0$ (or, equivalently, α and β are of the same sign).

Then the interval $]\alpha, \beta[$ contains an integer k such that $|k| = 2^m \geq L/8$. This implies that $\mathcal{C}' = \pi_i^{-1}(\{k\})$ is a convex set of $\mathbb{R}^d$ which does not intersect

$2^{-m/d}\mathcal{U}$. By proposition 2.1, the volume $vol(\mathcal{C}')$ of $\mathcal{C}'$ is at most $v/2^m \leq 8v/L$.

Let $\mathcal{C}_-$ denote the set of points x in $\mathcal{C}$ such that $x_i \leq k$, and let $\mathcal{C}_+$ denote the set of points x in $\mathcal{C}$ such that $x_i \geq k$. Then,

$$vol(\mathcal{C}_-) \leq (k-a)\ vol(\mathcal{C}') \left(\frac{b-a}{b-k}\right)^d \leq L\,\frac{8v}{L}\left(\frac{L}{L/4}\right)^d \leq 2\cdot 4^{d+1}v.$$

The same inequality holds for $vol(\mathcal{C}_+)$. Since $vol(\mathcal{C}) = vol(\mathcal{C}_+) + vol(\mathcal{C}_-)$, we have $vol(\mathcal{C}) \leq 4^{d+2}v \leq v'$. □

4 Growth classes

Let $\mathcal{G}_0$ denote the set of positive and increasing functions f defined on an interval $[M(f), +\infty[$, where $M(f)$ is a finite real number which may depend on f. Then $\mathcal{G}_0$ is equipped with a preorder relation $\preceq$ defined by $f \preceq g$ if there exists $t \geq 1$ such that $f(x) \leq g(tx)$ for every $x \geq t$.

The set $\mathcal{G}$ of *(affine) growth classes* is the quotient set of $\mathcal{G}_0$ by the equivalence relation $\sim$ defined by $f \sim g$ if there exists $t \geq 1$ such that, for every $x \geq t$,

$$g(x) \leq f(tx) \leq g(t^2x).$$

The preorder relation $\preceq$ on $\mathcal{G}_0$ induces a partial order on $\mathcal{G}$.

Recall that for every $a > 0$ and $x > 0$, $\ell_a(x) = x^a$, hence $\ell_a \in G_0$. A function $f \in \mathcal{G}$ is *polynomially bounded* if there exists $a > 0$ such that $f \preceq \ell_a$. If f is polynomially bounded, f has *critical exponent* $a > 0$ if $\ell_b \preceq f \preceq \ell_c$ for every positive b and c such that $b < a < c$. Additionally, a non-zero function f has *critical exponent* 0 if $f \preceq \ell_b$ for every $b > 0$.

Equivalently, a function $f \in \mathcal{G}$ is polynomially bounded if $\limsup_{x\to\infty} \frac{\log(f(x))}{\log(x)} < \infty$ and we have $a = \lim_{x\to\infty} \frac{\log(f(x))}{\log(x)}$ if $f \in \mathcal{G}$ is polynomially bounded with critical exponent a.

It can happen that a polynomially bounded function has no critical exponent. This is the case if $\sup\{a \mid \ell_a \preceq f\} < \inf\{a \mid f \preceq \ell_a\}$.

Any function f with critical exponent a can be written as $f = \ell_a\, h$, where the (not necessarily eventually increasing) function h is such that, for every $b > 0$, there exists a finite x_b such that $x^{-b} \leq h(x) \leq x^b$ for every $x \geq x_b$.

The notions of polynomial boundedness and critical exponent of functions in $\mathcal{G}_0$ are well behaved with respect to the preorder relation $\preceq$ on $\mathcal{G}_0$, hence these can also be defined on suitable growth classes in $\mathcal{G}$.

Given $\mathcal{S} \subset \mathbb{R}^d$ without accumulation points, the choice of a (not necessarily Euclidean) norm on $\mathbb{R}^d$ yields an increasing non-negative function $f_\mathcal{S}$ such that $f_{\mathcal{S}}(r)$ is the number of elements of $\mathcal{S}$ whose norm is at most r. The growth class of $f_{\mathcal{S}}$ is independent of the norm, hence one can call it the growth class of $\mathcal{S}$. Two subsets of $\mathbb{R}^d$ related by a translation belong to the same growth class. Growth classes are invariant under the action of the group of affine bijections of $\mathbb{R}^d$.

A set $\mathcal{S} \subset \mathbb{R}^d$ is *sparse* if its growth class is strictly smaller than ℓ_d. We say that $\mathcal{S} \subset \mathbb{R}^d$ is *nearly uniform* if it has a polynomially bounded growth class of critical exponent d. The growth class of a nearly uniform set can be represented by a function $h\ell_d$, where h encodes the "asymptotic density" of $\mathcal{S}$ up to affine bijections.

For example, $\mathbb{Z}^d$ and $\mathbb{N}^d$ are both nearly uniform sets and in the same growth class ℓ_d.

A more concise and less precise reformulation of theorem 1.1 is as follows.

Theorem 4.1. *In every dimension, there exist nearly uniform universal convex coverings.*

We conclude this section with a remark.

One can also define growth classes for measurable subsets $\mathcal{A} \subset \mathbb{R}^d$ and for any given measure μ and norm on $\mathbb{R}^d$, by replacing $f_{\mathcal{A}}$ by the function $f^{\mu}_{\mathcal{A}}$ such that $f^{\mu}_{\mathcal{A}}(r)$ denotes the μ measure of the intersection of $\mathcal{A}$ with the ball of radius r centered at the origin.

5 Growth class of $\mathcal{U}_d$

Recall that $u_d(r) = \log(r)^{d-1} r^d$ for every $r \geq 1$.

Proposition 5.1. *The universal convex covering $\mathcal{U}_d$ defined in corollary 3.2 belongs to the growth class of u_d.*

Proof of theorem 1.1. By proposition 5.1, there exists a constant c_d such that the set $\mathcal{U}_d$ constructed in corollary 3.2 has at most $c_d u_d(r)$ elements at distance at most r from the origin. Considering the rescaled set $t\mathcal{U}_d$ for $t > c_d^{1/d}$ ends the proof. □

Proof of proposition 5.1. We proceed by induction on the dimension d. For $d = 1$, $u_1(r) = r$ hence $\mathcal{U}_1 = \mathbb{Z}$ belongs to the growth class of u_1.

Before starting the proof of the induction step, let us remark that the growth class of the function u_d contains all functions in $\mathcal{G}$ which can be written as $\lambda(r)u_d(r)$ where $r \longmapsto \lambda(r)$ is a bounded function. This fact allows to neglect bounded factors involved in u_d or u_{d+1}.

We assume now that $\mathcal{U}_d$ is of growth class u_d for some $d \geq 1$. Up to a bounded factor, the growth class of $\mathcal{U}_{d+1}$ is described by the set $\mathcal{B} \subset \mathbb{N} \times \mathbb{R}^d$ defined as

$$\mathcal{B} = \bigcup_{m \geq 1} (m, 2^{-v_2(m)/d} \mathcal{U}_d) = \bigcup_{n \geq 0} (2^n(1 + 2\mathbb{N}), 2^{-n/d} \mathcal{U}_d).$$

We work with the L^∞ norm $\| x \|_\infty$ already encountered in section 3. Using the fact that growth classes are increasing and that bounded factors in u_{d+1} can be neglected, it is enough to compute the growth function

$$\beta(r) = \sharp \left(\mathcal{B} \cap \{ x \in \mathbb{R}^{d+1} \mid \| x \|_\infty < r \} \right)$$

counting elements of $\mathcal{B}$ in open balls of radius a power of 2.

Neglecting boundary effects and using the fact that the set $\{1, \ldots, 2^m - 1\}$ contains exactly 2^{m-n-1} integers of the form $2^n(1 + 2\mathbb{N})$, we have

$$\beta(2^m) \sim \sum_{n=0}^{m-1} 2^{m-n-1} u_d(2^{m+n/d})$$

$$\sim \sum_{n=0}^{m-1} 2^{m-n-1} \left(m + \frac{n}{d} \right)^{d-1} 2^{dm+n} \sim m^d 2^{(d+1)m}$$

which shows that β is in the growth class of u_{d+1}. $\square$

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References

[1] Conway, J. H.; Sloane, N. J. A. Sphere packings, lattices and groups. Third edition. Springer-Verlag, New York, 1999.

[2] Erdös, P.; Rogers, C. A. *Covering space with convex bodies*, Acta Arith. 7 1961/1962 281–285.

[3] Gruber, Peter M. Convex and discrete geometry. Grundlehren der Mathematischen Wissenschaften, 336. Springer, Berlin, 2007.

[4] Rogers, C. A. *A note on coverings*, Mathematika 4 (1957), 1–6.

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