On the Dirichlet boundary control of the heat equation with a final observation

Part I: A space-time mixed formulation and penalization

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Abstract: We are interested in the optimal control problem of the heat equation where the quadratic cost functional involves a final observation and the control variable is a Dirichlet boundary condition. We first prove that this problem is well-posed. Next, we check its equivalence with a fixed point problem for a space-time mixed system of parabolic equations. Finally, we introduce a Robin penalization on the Dirichlet boundary control for the mixed problem and analyze the convergence when the penalty parameter tends to zero.

Résumé: Nous considérons le problème de contrôle suivant: minimiser une fonctionnelle quadratique qui fait intervenir la solution de l'équation de la chaleur au temps final en agissant sur la frontière du domaine. Nous prouvons d'abord que ce problème est bien posé. Puis nous vérifions son équivalence avec un problème de point fixe pour un système d'équations paraboliques admettant une formulation mixte. Nous introduisons finalement une pénalisation du problème mixte et nous analysons la convergence lorsque le paramètre de pénalisation tend vers zéro.

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1. Introduction.

Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d = 1$, 2 or 3, with a Lipschitz-continuous boundary $\partial\Omega$, and let T be a positive real number. We set:

$$Q = \Omega \times]0, T[, \qquad \Sigma = \partial\Omega \times]0, T[. \tag{1.1}$$

We are interested in the optimal control problem that consists in minimizing the quadratic functional $\mathscr{J}$ defined by

$$\mathscr{J}(v) = \frac{1}{2} \int_\Omega |y_v(\boldsymbol{x}, T) - y_T(\boldsymbol{x})|^2 \, d\boldsymbol{x} + \frac{\beta}{2} \int_\Sigma |v(\boldsymbol{\tau}, t)|^2 \, d\boldsymbol{\tau} dt, \tag{1.2}$$

where, for each v in $L^2(\Sigma)$, y_v denotes the solution of the heat equation with a Dirichlet boundary condition

$$\begin{cases} \partial_t y_v - \Delta y_v = 0 & \text{in } Q, \\ y_v = v & \text{on } \Sigma, \\ y_v(\cdot, 0) = 0 & \text{on } \Omega. \end{cases} \tag{1.3}$$

The datum of the problem is the function y_T in $L^2(\Omega)$ and the parameter β is a positive constant. The domain of the cost function $\mathscr{J}$ is not $L^2(\Sigma)$, since the final observation $y_v(\cdot, T)$ may not belong to $L^2(\Omega)$ for some v in $L^2(\Sigma)$ (see [11, Chap. 9]).

The main idea of this paper is to introduce a penalization term on the Dirichlet condition, namely to replace the second line in (1.3) by a Robin type boundary condition,

$$\varepsilon \partial_\nu y_v^\varepsilon + y_v^\varepsilon = v \quad \text{on } \Sigma, \tag{1.4}$$

where ε is a small positive parameter, aimed to tend to zero.

An immediate advantage of this approach is the possibility to define the final observation $y_v^\varepsilon(\cdot, T)$ in $L^2(\Omega)$ for all $v \in L^2(\Sigma)$; we are therefore allowed to consider the optimal control problem on the whole Lebesgue space $L^2(\Sigma)$. On the other hand, from a numerical point of view, taking into account the Dirichlet condition is scarcely made exactly; it is rather approximated by adding a penalization term. The Robin penalization issue has been considered for elliptic problems for the linear case in [1] and for the non-linear case in [4]; we also refer to [8] and [9] for the extension to the steady Navier-Stokes system.

The specific difficulty for the optimal control is that a final observation is involved in the cost function while the convergence of the penalized observation $y_v^\varepsilon(\cdot, T)$ towards the observation $y_v(\cdot, T)$ is not guaranteed. Nevertheless, we establish the convergence of the penalized optimal control towards to the non-penalized one, without any restrictive assumptions on its solution. The key idea of the proof relies on a mixed space-time variational formulation of two parabolic equations, one on the state y_v and the other on the its adjoint state (denoted by p_v). Thanks to this coupling, the standard saddle-point theory of [3, Chap. II] may be applied (see also [6, §I.4]) and leads to well-posedness and convergence results. It can also be noted that this penalized mixed problem is the first-order optimal condition of the penalized control problem.

An outline of the paper is as follows:

• In Section 2, we prove the well-posedness of problem (1.3), and thus of the minimization problem.

• In Section 3, we consider a mixed problem consisting of two parabolic equations, the state equation and its adjoint problem. We prove first its equivalence with the initial problem, second its well-posedness.

• Section 4 is devoted to the treatment of the mixed problem by penalization, in view of the discretization. We state a convergence result in the general case, then we exhibit the convergence rate of the penalization under some additional smoothness assumptions.

2. The optimal control problem.

Before tackling the optimal control problem we have to address the first difficulty that consists in giving a mathematical sense to the system

$$\begin{cases} \partial_t y_u - \Delta y_u = 0 & \text{in } Q, \\ y_u = u & \text{on } \Sigma, \\ y_u(\cdot, 0) = 0 & \text{on } \Omega, \end{cases} \tag{2.1}$$

for all functions u in $L^2(\Sigma)$. We follow here the variational approach dealt with in [11, Th. 9.1].

We introduce some notation: For any separable Banach space X provided with the norm $\|\cdot\|_X$, we denote by $L^2(0,t;X)$ the space of measurable functions v from $(0,t)$ in X such that

$$\|v\|_{L^2(0,t;X)} = \Big(\int_0^t \|v(\cdot,s)\|_X^2\, ds\Big)^{\frac{1}{2}} < +\infty. \tag{2.2}$$

For any positive integer m, we introduce the space $H^m(0,t;X)$ of functions in $L^2(0,t;X)$ such that all their time derivatives up to the order m belong to $L^2(0,t;X)$. We also use the space $\mathscr{C}(0,t;X)$ of continuous functions v from $[0,t]$ in X. Finally, on Ω we consider the full scale of Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, and also the analogous spaces $H^s(\partial\Omega)$ on its boundary. We denote by $H_0^1(\Omega)$ the closure in $H^1(\Omega)$ of the space $\mathscr{D}(\Omega)$ of infinitely differentiable functions with a compact support in Ω, and by $H^{-1}(\Omega)$ its dual space.

We associate with any function f in $L^2(Q)$ the solution $r(f)$ of the heat equation

$$\begin{cases} -\partial_t r(f) - \Delta r(f) = f & \text{in } Q, \\ r(f) = 0 & \text{on } \Sigma, \\ r(f)(\cdot, T) = 0 & \text{on } \Omega. \end{cases} \tag{2.3}$$

It is readily checked, see [12, Chap. 4, Th. 1.1], that this problem admits a unique solution in $L^2(0,T;H_0^1(\Omega)) \cap \mathscr{C}^0(0,T;L^2(\Omega))$. Moreover, by multiplying the previous equation by $-\Delta r(f)$, we easily derive that this solution is such that $\Delta r(f)$ belongs to $L^2(Q)$ and satisfies

$$\|\Delta r(f)\|_{L^2(Q)} \le \|f\|_{L^2(Q)}. \tag{2.4}$$

We also have the following result. Let $\boldsymbol{\nu}$ denote the unit normal vector to $\partial\Omega$ which is outward to Ω.

Lemma 2.1. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$ or is a polygon ($d = 2$) or a polyhedron ($d = 3$). Then, for any data f in $L^2(Q)$, the normal derivative $\partial_\nu r(f)$ belongs to $L^2(\Sigma)$ and there exists a constant c only depending on the geometry of Ω such that*

$$\|\partial_\nu r(f)\|_{L^2(\Sigma)} \le c\, \|f\|_{L^2(Q)}. \tag{2.5}$$

Proof: This result is a consequence of the regularity properties of the solution of the Laplace equation (i.e., the fact that $\Delta r(f)$ belongs to $L^2(Q)$) with homogeneous Dirichlet boundary conditions, see [7, §2.2.2] or [5]: For a.e. t in $[0,T]$,

1) if Ω is of class $\mathscr{C}^{1,1}$, $r(f)(\cdot,t)$ belongs to $H^2(\Omega)$; thus, $\partial_\nu r(f)(\cdot,t)$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$;
2) if Ω is a polygon or a polyhedron, there exists an $s>0$ such that $r(f)(\cdot,t)$ belongs to $H^{\frac{3}{2}+s}(\Omega)$; thus, $\partial_\nu r(f)(\cdot,t)$ belongs to $H^s(\gamma)$ for each edge ($d=2$) or face ($d=3$) γ of Ω, hence to $L^2(\partial\Omega)$.
In both cases, we have

$$\|\partial_\nu r(f)(\cdot,t)\|_{L^2(\partial\Omega)} \le c\,\|\Delta r(f)(\cdot,t)\|_{L^2(\Omega)}.$$

By integrating the square of the previous estimate on $]0,T[$ and using (2.4), we obtain the desired estimate.

The results of the previous lemma are still valid for much more general domains (for instance, curved polygons or polyhedra). From now on, we work with a domain Ω which satisfies one of the assumptions of Lemma 2.1. We now consider the following variational problem, for any datum u in $L^2(\Sigma)$ (here, $\boldsymbol{\tau}$ stands for the coordinate on $\partial\Omega$):

Find y_u in $L^2(Q)$ such that

$$\forall f \in L^2(Q), \quad \int_Q y_u(\boldsymbol{x},t) f(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt = -\int_\Sigma u(\boldsymbol{\tau},t)\big(\partial_\nu r(f)\big)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt. \tag{2.6}$$

The reason for this is stated in the next proposition.

Proposition 2.2. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$ or is a polygon ($d=2$) or a polyhedron ($d=3$). Problems* (2.1) *and* (2.6) *are equivalent, in the sense that a function y_u in $L^2(Q)$ is a solution of* (2.1) *in the distribution sense if and only if it is a solution of* (2.6).

Proof: Let y_u be a solution of problem (2.6), and let z be any function in $\mathscr{D}(\Omega\times]0,T[)$. Setting $f=-\partial_t z - \Delta z$, we observe from (2.6) that

$$0 = \int_Q (-\partial_t z - \Delta z)(\boldsymbol{x},t) y_u(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt,$$

which means that y_u satisfies the first line of (2.1) in the distribution sense. Using this property yields that, for any z in $\mathscr{D}(\Omega\times[0,T[)$ (i.e. satisfying $z(\cdot,T)=0$),

$$0 = \int_Q (-\partial_t z - \Delta z)(\boldsymbol{x},t) y_u(\boldsymbol{x},t)\, \mathrm{d}\boldsymbol{x}\, dt = \int_\Omega z(x,0) y_u(x,0)\, d\boldsymbol{x},$$

whence the initial condition in (2.1). Finally, we have, for any z in $\mathscr{D}(\overline{\Omega}\times]0,T[)$ vanishing on $\partial\Omega$,

$$\begin{aligned} -\int_\Sigma u(\boldsymbol{\tau},t)(\partial_\nu z)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt &= \int_Q (-\partial_t z - \Delta z)(\boldsymbol{x},t) y_u(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt \\ &= -\int_\Sigma y_u(\boldsymbol{\tau},t)(\partial_\nu z)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt. \end{aligned}$$

By density, this formula holds for any z in $\mathscr{D}(]0,T[; H^2(\Omega)\cap H^1_0(\Omega))$. When Ω has a boundary of class $\mathscr{C}^{1,1}$, the trace operator: $z \mapsto \partial_\nu z$ maps $H^2(\Omega)\cap H^1_0(\Omega)$ onto $H^{\frac{1}{2}}(\partial\Omega)$,

which yields the boundary condition in (2.1). Similarly, when Ω is a polygon ($d=2$) or a polyhedron ($d=3$), the range of the trace operator: $z \mapsto \partial_\nu z$ is dense in $L^2(\gamma)$ for each edge ($d=2$) or face ($d=3$) γ of Ω, whence the boundary condition in (2.1).
Conversely, the fact that any solution of (2.1) is a solution of (2.6) follows from the same arguments, together with the density of $\mathscr{D}(\overline{\Omega}\times[0,T])$ in $L^2(0,T;H^s(\Omega))\cap H^1(0,T;L^2(\Omega))$, $\frac{3}{2}<s\le 2$ (see [12, §4.2] for instance).

Owing to the Lax–Milgram lemma, the next corollary is a direct consequence of Lemma 2.1.

Corollary 2.3. *For any datum u in $L^2(\Sigma)$, problem* (2.6) *has a unique solution y_u in $L^2(Q)$.*

In view of the next sections, we prove a technical result. Let B be the operator defined from a subspace $\mathbb{D}(B)$ of $L^2(\Sigma)$ into $L^2(\Omega)$ by: $Bu = y_u(\cdot,T)$, where y_u is the solution of problem (2.6). Obviously, the domain $\mathbb{D}(B)$ of B is the space of functions u in $L^2(\Sigma)$ such that $y_u(\cdot,T)$ belongs to $L^2(\Omega)$, and it is readily checked that the operator B is closed. Similarly, we introduce the operator B^* defined from a subspace $\mathbb{D}(B^*)$ of $L^2(\Omega)$ into $L^2(\Sigma)$ by: $B^*\varphi = -\partial_\nu q_\varphi$, where q_φ is the solution of

$$\begin{cases} -\partial_t q_\varphi - \Delta q_\varphi = 0 & \text{in } Q, \\ q_\varphi = 0 & \text{on } \Sigma, \\ q_\varphi(\cdot,T) = \varphi & \text{on } \Omega. \end{cases} \tag{2.7}$$

The domain $\mathbb{D}(B^*)$ of B^* is the space of functions φ in $L^2(\Omega)$ such that $\partial_\nu q_\varphi$ belongs to $L^2(\Sigma)$.

Lemma 2.4. *The following Green's formula holds for any u in $\mathbb{D}(B)$ and φ in $\mathbb{D}(B^*)$*

$$\int_\Omega \varphi(\boldsymbol{x})y_u(\boldsymbol{x},T)\,d\boldsymbol{x} + \int_\Sigma u(\boldsymbol{\tau},t)(\partial_\nu q_\varphi)(\boldsymbol{\tau},t)d\boldsymbol{\tau}dt = 0. \tag{2.8}$$

Proof: First, for any smooth enough function u and any φ in $\mathbb{D}(B^*)$, multiplying the first line of problem (2.7) by y_u and integrating by parts yield the formula. Next, we observe that, for any function u in $\mathbb{D}(B)$, there exists a sequence $(u_n)_n$ in $\mathscr{D}(\Sigma)$ for instance which converges to u in $L^2(\Sigma)$. It is readily checked that the corresponding sequence $(y_{u_n})_n$ converges to y_u in $L^2(Q)$ and that the sequence $\big(y_{u_n}(\cdot,T)\big)_n$ converges to $y_u(\cdot,T)$ in $L^2(\Omega)$ (indeed, we recall from [10, Chap. 3] that y_u belongs to $\mathscr{C}^0(0,T;H^{-1}(\Omega))$, see also [13]). Thus, applying formula (2.8) for each u_n and φ and passing to the limit leads to this formula for u and φ.

Formula (2.8) can be written equivalently as

$$\int_\Omega \varphi(\boldsymbol{x})Bu(\boldsymbol{x})\,d\boldsymbol{x} = \int_\Sigma u(\boldsymbol{\tau},t)B^*\varphi(\boldsymbol{\tau},t)d\boldsymbol{\tau}dt,$$

so that B^* is the adjoint operator of B. For the sake of completeness, we now state the full Green's formula, associated with the problem

$$\begin{cases} -\partial_t q_\varphi(f) - \Delta q_\varphi(f) = f & \text{in } Q, \\ q_\varphi(f) = 0 & \text{on } \Sigma, \\ q_\varphi(f)(\cdot,T) = \varphi & \text{on } \Omega. \end{cases} \tag{2.9}$$

We skip its proof since it is a direct consequence of Lemma 2.4 and problem (2.6).

Corollary 2.5. *The following Green's formula holds for any f in $L^2(Q)$, for any u in $\mathbb{D}(B)$ and φ in $\mathbb{D}(B^*)$*

$$\int_Q f(\boldsymbol{x},t) y_u(\boldsymbol{x},t)\, d\boldsymbol{x}dt + \int_\Omega \varphi(\boldsymbol{x}) y_u(\boldsymbol{x},T)\, d\boldsymbol{x} + \int_\Sigma u(\boldsymbol{\tau},t)\big(\partial_\nu q_\varphi(f)\big)(\boldsymbol{\tau},t) d\boldsymbol{\tau} dt = 0. \quad (2.10)$$

To go further, we observe from Corollary 2.3 that, for any v in $L^2(\Sigma)$, problem (1.3) has a unique solution y_v in $L^2(Q)$. Moreover, it is checked in [12, Chap. 4, §15] that, when Ω has a boundary of class $\mathscr{C}^{1,1}$, this solution belongs $L^2(0,T;H^{\frac{1}{2}}(\Omega))$ and in [10, Chap. 3] (see also [13]) that it belongs to $\mathscr{C}^0(0,T;H^{-1}(\Omega))$. However, these regularity properties are not sufficient for handling the minimization problem. So we introduce the space

$$\mathbb{V} = \big\{v \in L^2(\Sigma);\ y_v(\cdot,T) \in L^2(\Omega)\big\}, \quad (2.11)$$

equipped with the norm

$$\|v\|_{\mathbb{V}} = \Big(\|v\|^2_{L^2(\Sigma)} + \|y_v(\cdot,T)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}. \quad (2.12)$$

It is readily checked that $\mathbb{V}$ is a Hilbert space for the scalar product associated with this norm. Note also that it coincides with the domain $\mathbb{D}(B)$ introduced above.

We now consider the optimal control problem:

Find u in $\mathbb{V}$ such that

$$\mathscr{J}(u) = \min_{v\in\mathbb{V}} \mathscr{J}(v), \quad (2.13)$$

where the functional $\mathscr{J}$ is defined in (1.2). Writing the first-order optimality condition yields that this problem admits the equivalent variational formulation:

Find u in $\mathbb{V}$ such that

$$\forall v \in \mathbb{V}, \qquad \int_\Omega y_u(\boldsymbol{x},T) y_v(\boldsymbol{x},T)\, d\boldsymbol{x} + \beta \int_\Sigma u(\boldsymbol{\tau},t) v(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt = \int_\Omega y_T(\boldsymbol{x}) y_v(\boldsymbol{x},T)\, d\boldsymbol{x}. \quad (2.14)$$

It follows from the definition (2.12) of the norm $\|\cdot\|_{\mathbb{V}}$ that the bilinear form in the left-hand side of this equation is continuous on $\mathbb{V}\times\mathbb{V}$ and elliptic on $\mathbb{V}$ (with ellipticity constant equal to $\min\{1,\beta\}$). We have thus the following result.

Theorem 2.6. *For any datum y_T in $L^2(\Omega)$, problem (2.14) has a unique solution u in $\mathbb{V}$. Moreover, this solution satisfies*

$$\min\{1,\sqrt{2\beta}\}\, \|u\|_{\mathbb{V}} \le \|y_T\|_{L^2(\Omega)}. \quad (2.15)$$

3. A space-time mixed formulation.

Currently, the control u is expressed by means of the adjoint state. We are led to study the adjoint problem: For any u in $\mathbb{V}$, we consider the solution p_u of

$$\begin{cases} -\partial_t p_u - \Delta p_u = 0 & \text{in } Q, \\ p_u = 0 & \text{on } \Sigma, \\ p_u(\cdot, T) = y_u(\cdot, T) - y_T & \text{on } \Omega. \end{cases} \tag{3.1}$$

This problem has a unique solution p_u in $L^2(0,T;H_0^1(\Omega)) \cap \mathscr{C}^0(0,T;L^2(\Omega))$, see [12, Chap. 4, Th. 1.1]. Let us introduce the space

$$\mathbb{X} = \Big\{ q \in L^2(0,T;H_0^1(\Omega)) \cap \mathscr{C}^0(0,T;L^2(\Omega)); \\ \partial_\nu q \in L^2(\Sigma) \text{ and } \partial_t q + \Delta q \in L^2(Q) \Big\}. \tag{3.2}$$

The following statement holds.

Proposition 3.1. *A function u in $\mathbb{V}$ is a solution of problem* (2.14) *if and only if the solution p_u of the adjoint problem* (3.1) *belongs to $\mathbb{X}$ and satisfies*

$$u = \beta^{-1}\, \partial_\nu p_u \quad \text{on } \Sigma. \tag{3.3}$$

Proof: It is performed in two steps.
1) Let u be any solution of (2.14) and v belong to $\mathbb{V}$. When multiplying y_v by $-\partial_t p_u - \Delta p_u$, we easily derive that

$$0 = -\int_Q (\partial_t p_u)(\boldsymbol{x},t) y_v(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt - \int_Q (\Delta p_u)(\boldsymbol{x},t) y_v(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt \\ = -\int_\Omega \big(y_u(\boldsymbol{x},T) - y_T(\boldsymbol{x})\big) y_v(\boldsymbol{x},T)\, d\boldsymbol{x} - \int_\Sigma (\partial_\nu p_u)(\boldsymbol{\tau},t) v(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt.$$

When comparing with (2.14), we obtain

$$\int_\Sigma (\partial_\nu p_u)(\boldsymbol{\tau},t) v(\boldsymbol{\tau},t)\, d\boldsymbol{\tau} dt = \beta \int_\Sigma u(\boldsymbol{\tau},t) v(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}.$$

Thus, $\partial_\nu p_u$ is equal to $\beta\, u$ in the distribution sense, hence belongs to $L^2(\Sigma)$ (so that p_u belongs to $\mathbb{X}$) and satisfies (3.3).
2) Conversely, let p_u be any solution of (3.3) in $\mathbb{X}$. Problem (3.1) yields that u belongs to $\mathbb{V}$. Then formula (2.8) combined with (3.3) gives, for all v in $\mathbb{V}$,

$$0 = \int_\Omega \big(y_u(\boldsymbol{x},T) - y_T(\boldsymbol{x})\big) y_v(\boldsymbol{x},T)\, d\boldsymbol{x} + \int_\Sigma (\partial_\nu p_u)(\boldsymbol{\tau},t) v(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt \\ = \int_\Omega \big(y_u(\boldsymbol{x},T) - y_T(\boldsymbol{x})\big) y_v(\boldsymbol{x},T)\, d\boldsymbol{x} + \beta \int_\Sigma u(\boldsymbol{\tau},t) v(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt,$$

whence (2.14).

Thanks to the equivalence property proved in Proposition 3.1, from now on we work with problem (3.3). In what follows, we consider the coupled system where the control u is eliminated

$$\begin{cases} -\partial_t p - \Delta p = 0 & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(\cdot, T) = y(\cdot, T) - y_T & \text{on } \Omega, \end{cases} \qquad \begin{cases} \partial_t y - \Delta y = 0 & \text{in } Q, \\ y = \beta^{-1}\, \partial_\nu p & \text{on } \Sigma, \\ y(\cdot, 0) = 0 & \text{on } \Omega. \end{cases} \tag{3.4}$$

Indeed, we have the next property.

Proposition 3.2. *Problem* (3.1) *has a solution* p_u *in* $\mathbb{X}$ *which satisfies* (3.3) *if and only if problem* (3.4) *has a solution* (p, y) *in* $\mathbb{X} \times L^2(Q)$ *such that* p *coincides with* p_u*.*

Proof: As previously, it is performed in two steps.
1) Let p_u in $\mathbb{X}$ satisfy (3.3). Then, if y_u denotes the function associated with u by (2.1), equation (3.3) implies that (p_u, y_u) belongs to $\mathbb{X} \times L^2(Q)$ and satisfies (3.4).
2) Conversely, let (p, y) be a solution of (3.4) in $\mathbb{X} \times L^2(Q)$. Setting $\tilde{u} = \beta^{-1}\, \partial_\nu p$, we derive from Corollary 2.3 that y coincides with the solution $y_{\tilde{u}}$. Thus, p is equal to the solution $p_{\tilde{u}}$ in $\mathbb{X}$ of problem (3.1). All this yields that (3.3) is satisfied.

We are now interested in proving the well-posedness of problem (3.4). We first state a preliminary lemma.

Lemma 3.3. *The quantity* $\|\cdot\|_{\mathbb{X}}$ *defined by*

$$\|q\|_{\mathbb{X}} = \Big(\|\partial_\nu q\|^2_{L^2(\Sigma)} + \|q(\cdot, T)\|^2_{L^2(\Omega)} + \|\partial_t q + \Delta q\|^2_{L^2(Q)} \Big)^{\frac{1}{2}}, \tag{3.5}$$

is a norm on $\mathbb{X}$*. Moreover the space* $\mathbb{X}$ *equipped with the scalar product associated with this norm is a Hilbert space.*

Proof: Let q be any function in $\mathbb{X}$ such that $\|q\|_{\mathbb{X}}$ is zero. Thus, q is a solution of the system

$$\begin{cases} -\partial_t q - \Delta q = 0 & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(\cdot, T) = 0 & \text{on } \Omega, \end{cases}$$

hence it is zero on Q. Thus, $\|\cdot\|_{\mathbb{X}}$ is a norm. Let now $(q_n)_n$ be a Cauchy sequence in $\mathbb{X}$. Since $L^2(\Sigma)$, $L^2(\Omega)$ and $L^2(Q)$ are Banach spaces, there exists a triple (k, q_T, f) in $L^2(\Sigma) \times L^2(\Omega) \times L^2(Q)$ such that $(\partial_\nu q_n)_n$ converges to k in $L^2(\Sigma)$, $q_n(\cdot, T)$ converges to q_T in $L^2(\Omega)$ and $-\partial_t q_n - \Delta q_n$ converges to f in $L^2(Q)$. Thus, by noting that the system

$$\begin{cases} -\partial_t q - \Delta q = f & \text{in } Q, \\ \partial_\nu q = k & \text{on } \Sigma, \\ q(\cdot, T) = q_T & \text{on } \Omega, \end{cases}$$

admits a unique solution q in $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$, we observe that the sequence $(q_n)_n$ converges to q in $\mathbb{X}$. Then, $\mathbb{X}$ is a Banach space. Since $L^2(\Sigma)$, $L^2(\Omega)$ and $L^2(Q)$ are Hilbert spaces, it is also a Hilbert space.

Problem (3.4) which couples two parabolic equations can be put under a mixed variational form

Find (p, y) *in* $\mathbb{X} \times L^2(Q)$ *such that*

$$\begin{aligned} &\forall q \in \mathbb{X}, \qquad a(p,q) + b(q,y) = -\int_\Omega y_T(\boldsymbol{x}) q(\boldsymbol{x},T)\, d\boldsymbol{x}. \\ &\forall z \in L^2(Q), \qquad b(p,z) = 0, \end{aligned} \tag{3.6}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$\begin{aligned} a(p,q) &= \beta^{-1} \int_\Sigma (\partial_\nu p)(\boldsymbol{\tau},t)(\partial_\nu q)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}\, dt + \int_\Omega p(\boldsymbol{x},T) q(\boldsymbol{x},T)\, d\boldsymbol{x}, \\ b(q,z) &= -\int_Q (\partial_t q + \Delta q)(\boldsymbol{x},t)\, z(\boldsymbol{x},t)\, d\boldsymbol{x}. \end{aligned} \tag{3.7}$$

We have the following result.

Proposition 3.4. *Assume that the domain* Ω *has a boundary of class* $\mathscr{C}^{1,1}$ *or is a polygon* ($d = 2$) *or a polyhedron* ($d = 3$)*. Problems* (3.4) *and* (3.6) *are equivalent, in the sense that a pair* (p, y) *in* $\mathbb{X} \times L^2(Q)$ *is a solution of* (3.4) *in the distribution sense if and only it is a solution of* (3.6)*.*

Proof: Let (p, y) be a solution of problem (3.4) in $\mathbb{X} \times L^2(Q)$. We easily derive from this problem that it satisfies, for all (q, z) in $\mathbb{X} \times L^2(Q)$,

$$a(p,q) = \beta^{-1} \int_\Sigma (\partial_\nu p)(\boldsymbol{\tau},t)(\partial_\nu q)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau} dt + \int_\Omega \big(y(\boldsymbol{x},T) - y_T(\boldsymbol{x})\big) q(\boldsymbol{x},T)\, d\boldsymbol{x}, \quad b(p,z) = 0,$$

and also, by integrating by parts (this relies on the same arguments as for Proposition 2.2)

$$b(q,y) = -\int_\Omega y(\boldsymbol{x},T) q(\boldsymbol{x},T)\, d\boldsymbol{x} - \beta^{-1} \int_\Sigma (\partial_\nu p)(\boldsymbol{\tau},t)(\partial_\nu q)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau} dt.$$

These two equations yield that (p, y) is a solution of problem (3.6). Conversely, if (p, y) is a solution of problem (3.6), the second equation in (3.6) yields the first line on the system for p in (3.4) while the first equation applied with z in $\mathscr{D}(\Omega \times]0, T[)$ leads to the first line on the system for y. Next, appropriate choices of z give the boundary and initial or final conditions, owing to the same arguments as for Proposition 2.2.

The study of problem (3.6) relies on the properties of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ that we now establish.

Lemma 3.5. *The form* $b(\cdot,\cdot)$ *is continuous on* $\mathbb{X} \times L^2(Q)$*, with norm equal to* 1*. Moreover, it satisfies the inf-sup condition, for a constant* $\alpha > 0$ *only depending on the geometry of* Ω*,*

$$\forall z \in L^2(Q), \quad \sup_{q \in \mathbb{X}} \frac{b(q,z)}{\|q\|_{\mathbb{X}}} \geq \alpha \, \|z\|_{L^2(Q)}. \tag{3.8}$$

Proof: The continuity of $b(\cdot,\cdot)$ follows from the definition (3.5) of $\|\cdot\|_{\mathbb{X}}$. On the other hand, for any z in $L^2(Q)$, we observe from Lemma 2.1 that the problem

$$\begin{cases} -\partial_t q - \Delta q = z & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(\cdot, T) = 0 & \text{on } \Omega, \end{cases}$$

has a unique solution q in $\mathbb{X}$, which satisfies

$$\|q\|_{\mathbb{X}} \le c\,\|z\|_{L^2(Q)}.$$

We also have

$$b(q,z) = \|z\|^2_{L^2(Q)},$$

which leads to the desired inf-sup condition.

It remains to check the ellipticity of the form $a(\cdot,\cdot)$ on the kernel of the form $b(\cdot,\cdot)$, namely on the space

$$\mathbb{K} = \big\{q \in \mathbb{X};\ \forall z \in L^2(Q),\ b(q,z) = 0\big\}. \tag{3.9}$$

The following characterization of $\mathbb{K}$ is readily checked

$$\mathbb{K} = \big\{q \in \mathbb{X};\ -\partial_t q - \Delta q = 0 \text{ a.e. in } Q\big\}. \tag{3.10}$$

Furthermore, it follows from Lemma 3.5 that it is a closed subspace of $\mathbb{X}$, hence a Hilbert space.

Lemma 3.6. *The form $a(\cdot,\cdot)$ is continuous on $\mathbb{X}\times\mathbb{X}$, with norm equal to $\max\{1,\beta^{-1}\}$. Moreover, it satisfies the ellipticity property*

$$\forall q \in \mathbb{K}, \quad a(q,q) \ge \min\{1,\beta^{-1}\}\,\|q\|^2_{\mathbb{X}}. \tag{3.11}$$

Proof: There also, the continuity of $a(\cdot,\cdot)$ follows from the definition (3.5) of $\|\cdot\|_{\mathbb{X}}$. Similarly, the ellipticity property is a direct consequence of this definition, combined with (3.10).

We are now in a position to prove the main result of this section.

Theorem 3.7. *For any datum y_T in $L^2(\Omega)$, problem (3.6) has a unique solution (p,y) in $\mathbb{X}\times L^2(Q)$. Moreover, this solution satisfies*

$$\|p\|_{\mathbb{X}} + \|y\|_{L^2(Q)} \le c\,\|y_T\|_{L^2(\Omega)}, \tag{3.12}$$

and a function u in $\mathbb{V}$ is a solution of problem (3.3) if and only if it coincides with $\beta^{-1}\partial_\nu p$ on Σ.

Proof: Lemmas 3.5 and 3.6 yield the well-posedness of problem (3.6) together with estimate (3.12), see [6, Chap. I, Thm 4.1]. The last part of the statement thus follows from Propositions 3.2 and 3.4.

4. The penalized problem.

The part y of the solution of problem (3.4) is not smooth and the variational formulation (2.6) does not seem appropriate for any type of discretization. So, in view of the finite element discretization, the Dirichlet conditions in both systems of (3.4) can be penalized by Robin type conditions.

Let ε be a real parameter, $0 < \varepsilon < 1$. We consider the penalized optimal control problem: *Find u^ε in $L^2(\Sigma)$ such that*

$$\mathscr{J}^\varepsilon(u^\varepsilon) = \min_{v \in L^2(\Sigma)} \mathscr{J}^\varepsilon(v), \tag{4.1}$$

where the quadratic functional $\mathscr{J}^\varepsilon$ is now defined by

$$\mathscr{J}^\varepsilon(v) = \frac{1}{2} \int_\Omega |y_v^\varepsilon(\boldsymbol{x}, T) - y_T(\boldsymbol{x})|^2 \, d\boldsymbol{x} + \frac{\beta}{2} \int_\Sigma |v(\boldsymbol{\tau}, t)|^2 \, d\boldsymbol{\tau} dt, \tag{4.2}$$

and, for each v in $L^2(\Sigma)$, y_v^ε denotes the solution of the penalized heat equation

$$\begin{cases} \partial_t y_v^\varepsilon - \Delta y_v^\varepsilon = 0 & \text{in } Q, \\ \varepsilon \, \partial_\nu y_v^\varepsilon + y_v^\varepsilon = v & \text{on } \Sigma, \\ y_v \varepsilon(\cdot, 0) = 0 & \text{on } \Omega. \end{cases} \tag{4.3}$$

It is readily checked that, for any datum y_T in $L^2(\Omega)$, problem (4.1) has a unique solution.

In a first step and in view of the equivalence properties stated in Section 3, we introduce the penalized coupled system

$$\begin{cases} -\partial_t p^\varepsilon - \Delta p^\varepsilon = 0 & \text{in } Q, \\ \varepsilon \, \partial_\nu p^\varepsilon + p^\varepsilon = 0 & \text{on } \Sigma, \\ p^\varepsilon(\cdot, T) = y^\varepsilon(\cdot, T) - y_T & \text{on } \Omega, \end{cases} \qquad \begin{cases} \partial_t y^\varepsilon - \Delta y^\varepsilon = 0 & \text{in } Q, \\ \varepsilon \, \partial_\nu y^\varepsilon + y^\varepsilon = \beta^{-1} \, \partial_\nu p^\varepsilon & \text{on } \Sigma, \\ y^\varepsilon(\cdot, 0) = 0 & \text{on } \Omega. \end{cases} \tag{4.4}$$

Studying this new system requires the introduction of the space

$$\mathbb{X}_\varepsilon = \Big\{ q \in L^2(0, T; H^1(\Omega)) \cap \mathscr{C}^0(0, T; L^2(\Omega)); \\ \varepsilon \, \partial_\nu q + q = 0 \text{ on } \Sigma \text{ and } \partial_t q + \Delta q \in L^2(Q) \Big\}. \tag{4.5}$$

Standard arguments yield that, for any $y^\varepsilon(\cdot, T)$ in $L^2(\Omega)$, the equation for p^ε has a unique solution in $\mathbb{X}_\varepsilon$ and that, for any p^ε in $\mathbb{X}_\varepsilon$, the equation for y^ε has a unique solution in $L^2(0, T; H^1(\Omega)) \cap \mathscr{C}^0(0, T; L^2(\Omega))$. We must now prove the existence of a solution for the full system (4.4).

The variational problem corresponding to system (4.4) reads in the same terms as (3.6) and consists therefore in

Finding $(p^\varepsilon, y^\varepsilon)$ in $\mathbb{X}_\varepsilon \times L^2(Q)$ such that

$$\begin{aligned} &\forall q \in \mathbb{X}_\varepsilon, \qquad a(p^\varepsilon, q) + b(q, y^\varepsilon) = -\int_\Omega y_T(\boldsymbol{x}) q(\boldsymbol{x}, T) \, d\boldsymbol{x}. \\ &\forall z \in L^2(Q), \qquad b(p^\varepsilon, z) = 0. \end{aligned} \tag{4.6}$$

The following result holds.

Proposition 4.1. *Problems* (4.4) *and* (4.6) *are equivalent, in the sense that a pair* $(p^\varepsilon, y^\varepsilon)$ *in* $\mathbb{X}_\varepsilon \times L^2(Q)$ *is a solution of* (4.4) *in the distribution sense if and only it is a solution of* (4.6).

Proof: We prove successively the two parts of the equivalence.
1) Let $(p^\varepsilon, y^\varepsilon)$ in $\mathbb{X}_\varepsilon \times L^2(Q)$ be a solution of problem (4.4). It is readily checked that $b(p^\varepsilon, z)$ is zero for all z in $L^2(Q)$. On the other hand, as already noted, the function y^ε belongs to $L^2(0,T;H^1(\Omega)) \cap \mathscr{C}^0(0,T;L^2(\Omega))$. Using this regularity, we see by integration by parts that

$$\begin{aligned} b(q, y^\varepsilon) = &- \int_\Omega q(\boldsymbol{x},T) y^\varepsilon(\boldsymbol{x},T)\, d\boldsymbol{x} - \int_\Sigma (\partial_\nu q)(\boldsymbol{\tau},t) y^\varepsilon(\boldsymbol{\tau},t)\, d\boldsymbol{\tau} dt \\ &\qquad + \int_\Sigma q(\boldsymbol{\tau},t)(\partial_\nu y^\varepsilon)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau} dt \\ = &- \int_\Omega q(\boldsymbol{x},T) y^\varepsilon(\boldsymbol{x},T)\, d\boldsymbol{x} - \beta^{-1} \int_\Sigma (\partial_\nu q)(\boldsymbol{\tau},t)(\partial_\nu p^\varepsilon)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau} dt \\ &\qquad + \int_\Sigma (\varepsilon \partial_\nu q + q)(\boldsymbol{\tau},t)(\partial_\nu y^\varepsilon)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau} dt. \end{aligned}$$

By using the fact that, since q belongs to $\mathbb{X}_\varepsilon$, $\varepsilon\partial_\nu q + q$ vanishes on Σ and adding $a(p^\varepsilon, q)$, we obtain the first line of (4.6).
2) Conversely, let $(p^\varepsilon, y^\varepsilon)$ be a solution of problem (4.6). Letting q and z run through $\mathscr{D}(Q)$ yields the first lines of the two systems (4.4) (in the distribution sense) . The boundary condition on p^ε follows from the fact that it belongs to $\mathbb{X}_\varepsilon$ and the boundary condition on y^ε are obtained by letting z run though $\mathscr{D}(\overline{\Omega} \times]0,T[)$. Similarly, the initial and final conditions are obtained by letting z run though $\mathscr{D}(\Omega \times [0,T])$. This concludes the proof.

When equipped with the norm $\|\cdot\|_{\mathbb{X}}$, the space $\mathbb{X}_\varepsilon$ is clearly a Hilbert space. Moreover, owing to very similar arguments as in the proof of Lemmas 3.5 and 3.6, we check that:
1) The form $b(\cdot,\cdot)$ is continuous on $\mathbb{X}_\varepsilon \times L^2(Q)$, with norm equal to 1 and satisfies the inf-sup condition (3.8) on these new spaces;
2) The form $a(\cdot,\cdot)$ is continuous on $\mathbb{X}_\varepsilon \times \mathbb{X}_\varepsilon$, with norm equal to $\max\{1, \beta^{-1}\}$ and satisfies the ellipticity property (3.11) on the kernal

$$\mathbb{K}_\varepsilon = \big\{ q \in \mathbb{X}_\varepsilon;\ \forall z \in L^2(Q),\ b(q,z) = 0 \big\}. \tag{4.7}$$

As a consequence, we have the following result.

Theorem 4.2. *For any datum* y_T *in* $L^2(\Omega)$, *problem* (4.6) *has a unique solution* $(p^\varepsilon, y^\varepsilon)$ *in* $\mathbb{X}_\varepsilon \times L^2(Q)$. *Moreover, this solution satisfies, for a constant* c *independent of* ε,

$$\|p^\varepsilon\|_{\mathbb{X}} + \|y^\varepsilon\|_{L^2(Q)} \le c\, \|y_T\|_{L^2(\Omega)}. \tag{4.8}$$

The aim is to prove the convergence of the penalized solution $(p^\varepsilon, y^\varepsilon)$ toward (p, y) without any further assumption on the regularity of this solution. Let us observe first that the definition (3.5) of the norm $\|\cdot\|_{\mathbb{X}}$ yields that it is still a norm on the more general space

$$\overline{\mathbb{X}} = \Big\{ q \in L^2(0,T;H^1(\Omega)) \cap \mathscr{C}^0(0,T;L^2(\Omega)); \\ \partial_\nu q \in L^2(\Sigma) \text{ and } \partial_t q + \Delta q \in L^2(Q) \Big\}. \tag{4.9}$$

Proposition 4.3. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$ or is a polygon ($d=2$) or a polyhedron ($d=3$). Let (p,y) be a solution of problem* (3.6)*. The family of solutions $(p^\varepsilon, y^\varepsilon)$ of problem* (4.6) *for $0<\varepsilon<1$ tends to (p,y) weakly in $\overline{\mathbb{X}} \times L^2(Q)$ when ε tends to zero.*

Proof: It follows from estimate (4.8) that there exists a sequence $(\varepsilon_n)_n$ tending to zero such that the sequence $(p^{\varepsilon_n}, y^{\varepsilon_n})_n$ converges weakly in $\overline{\mathbb{X}} \times L^2(Q)$. So we must now check that any weak limit (φ, η) of a sequence $(p^{\varepsilon_n}, y^{\varepsilon_n})_n$ is the solution of problem (3.6) since the desired result is a consequence of Theorem 3.7. This is performed in two steps.
1) We recall that y is the solution of problem (2.6), where $r(f)$ is the solution of problem (2.3). It is readily checked that each y^ε is the solution in $L^2(Q)$ of

$$\forall f \in L^2(Q), \quad \int_Q y^\varepsilon(\boldsymbol{x},t) f(\boldsymbol{x},t)\, d\boldsymbol{x}dt = -\beta^{-1} \int_\Sigma (\partial_\nu p^\varepsilon)(\boldsymbol{\tau},t) \big(\partial_\nu r^\varepsilon(f)\big)(\boldsymbol{\tau},t)\, d\boldsymbol{\tau}dt, \tag{4.10}$$

where $r^\varepsilon(f)$ is now the solution of the problem

$$\begin{cases} -\partial_t r^\varepsilon(f) - \Delta r^\varepsilon(f) = f & \text{in } Q, \\ \varepsilon\, \partial_\nu r^\varepsilon(f) + r^\varepsilon(f) = 0 & \text{on } \Sigma, \\ r^\varepsilon(f)(\cdot,T) = 0 & \text{on } \Omega. \end{cases} \tag{4.11}$$

By subtracting this equation from (2.3) and combining [2, Thm 5.1] with a more precise version of Lemma 2.1 (using the further regularity property that $\partial_\nu r(f)$ belongs to $H^s(\partial\Omega)$ for some s, $0 < s \le \frac{1}{2}$), we obtain

$$\|\partial_\nu\big(r(f) - r^\varepsilon(f)\big)\|_{L^2(\Sigma)} \le c\, \varepsilon^s\, \|f\|_{L^2(Q)}. \tag{4.12}$$

Next, passing to the limit in (4.10) yields that η is a solution of

$$\begin{cases} \partial_t \eta - \Delta \eta = 0 & \text{in } Q, \\ \eta = \beta^{-1}\, \partial_\nu \varphi & \text{on } \Sigma, \\ \eta(\cdot,0) = 0 & \text{on } \Omega, \end{cases}$$

which is the second system in (3.4).
2) We derive from (4.8) that

$$\|p^\varepsilon\|_{L^2(\Sigma)} = \varepsilon\, \|\partial_\nu p^\varepsilon\|_{L^2(\Sigma)} \le \varepsilon\, \|p^\varepsilon\|_{\mathbb{X}} \le c\, \varepsilon\, \|y_T\|_{L^2(\Omega)},$$

so that φ vanishes on Σ, hence belongs to $\mathbb{X}$. Passing to the limit in the first line of (4.4) is easy. Finally, it can be checked that all y^ε belong to $\mathscr{C}^0(0,T;L^2(\Omega))$ and that the norms

$\|y^\varepsilon(\cdot,T)\|_{L^2(\Omega)}$ are thus bounded independently of ε. As a consequence, there exists a subsequence $(y^{\varepsilon_{n'}}(\cdot,T))_{n'}$ which converges to $\eta(\cdot,T)$ weakly in $L^2(\Omega)$. We also have

$$\|p^\varepsilon(\cdot,T)\|_{L^2(\Omega)} \le c\,\|p^\varepsilon\|_{\mathbb{X}},$$

which yields the weak convergence of a subsequence $(p^{\varepsilon_{n''}}(\cdot,T))_{n''}$ in $L^2(\Omega)$. By combining these two weak convergences, we pass to the limit in the third line of (4.4). We thus derive that φ is the solution of the first system in (3.4).

This last result can be strengthened according to the following argument.

Lemma 4.4. *The following property holds*

$$\lim_{\varepsilon\to 0} \|p^\varepsilon\|_{\mathbb{X}} = \|p\|_{\mathbb{X}}, \qquad \lim_{\varepsilon\to 0} \|y^\varepsilon\|_{L^2(Q)} = \|y\|_{L^2(Q)}. \tag{4.13}$$

Proof: By taking q equal to p in (3.6) and q equal to p^ε in (4,3) we obtain

$$\|p\|_{\mathbb{X}}^2 = a(p,p) = -\int_\Omega y_T(\boldsymbol{x})p(\boldsymbol{x},T)\,d\boldsymbol{x}, \qquad \|p^\varepsilon\|_{\mathbb{X}}^2 = a(p^\varepsilon,p^\varepsilon) = -\int_\Omega y_T(\boldsymbol{x})p^\varepsilon(\boldsymbol{x},T)\,d\boldsymbol{x}..$$

Thus, passing to the limit and using Proposition 4.3 yields the first convergence property. On the other hand, taking f equal to y in (2.6) and f equal to y^ε in (4.10) gives

$$\|y\|_{L^2(Q)}^2 = -\beta^{-1}\int_\Sigma (\partial_\nu p)(\boldsymbol{\tau},t)\big(\partial_\nu r(y)\big)(\boldsymbol{\tau},t)\,d\boldsymbol{\tau}dt,$$
$$\|y^\varepsilon\|_{L^2(Q)}^2 = -\beta^{-1}\int_\Sigma (\partial_\nu p^\varepsilon)(\boldsymbol{\tau},t)\big(\partial_\nu r^\varepsilon(y^\varepsilon)\big)(\boldsymbol{\tau},t)\,d\boldsymbol{\tau}dt,$$

where $r(y)$ and $r^\varepsilon(y^\varepsilon)$ are defined in (2.3) and (4.11), respectively. The strong convergence of $\partial_\nu r(y^\varepsilon)$ to $\partial_\nu r(y)$ is easily derived from Lemma 2.1 together with a compactness argument, while the strong convergence of $\partial_\nu r^\varepsilon(y^\varepsilon)$ to $\partial_\nu r(y^\varepsilon)$ follows from (4.12) and (4.8). All this gives the second convergence property.

By combining Lemma 4.4 with Proposition 4.3, we obtain the following result.

Corollary 4.5. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$ or is a polygon ($d=2$) or a polyhedron ($d=3$). Let (p,y) be a solution of problem (3.6). The family of solutions $(p^\varepsilon,y^\varepsilon)$ of problem (4.6) for $0<\varepsilon<1$ tends to (p,y) strongly in $\overline{\mathbb{X}}\times L^2(Q)$ when ε tends to zero.*

Combining this last property with Theorem 3.7, we also obtain the following result.

Corollary 4.6. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$ or is a polygon ($d=2$) or a polyhedron ($d=3$). The family of solutions u^ε of problem (4.1) for $0<\varepsilon<1$ tends to the solution u of problem (2.1) strongly in $L^2(\Sigma)$ when ε tends to zero.*

We now evaluate the error due to the penalty method. Indeed, it is readily checked from the same arguments as in the proof of Proposition 4.3 that the error between the solutions (p, y) of problem (3.6) and $(p^\varepsilon, y^\varepsilon)$ of problem (4.6) satisfy, at least when y is smooth enough,

$$\begin{aligned} &\forall q \in \mathbb{X}_\varepsilon, \qquad a(p - p^\varepsilon, q) + b(q, y - y^\varepsilon) = -\varepsilon \int_\Sigma (\partial_\nu q)(\boldsymbol{\tau}, T)(\partial_\nu y)(\boldsymbol{\tau}, T)\, d\boldsymbol{x}. \\ &\forall z \in L^2(Q), \qquad b(p - p^\varepsilon, z) = 0. \end{aligned} \tag{4.14}$$

By inserting a q^ε in this equation and using the ellipticity property of the form $a(\cdot, \cdot)$ on $\mathbb{K}_\varepsilon$, we easily derive from a triangle inequality that, when y is smooth enough,

$$\|p - p^\varepsilon\|_{\mathbb{X}} \leq 2 \inf_{q^\varepsilon \in \mathbb{K}_\varepsilon} \|p - q^\varepsilon\|_{\mathbb{X}} + c\, \varepsilon\, \|\partial_\nu y\|_{L^2(\Sigma)}.$$

By applying [6, Chap. II, Thm 1.1] (see also [3, Chap. II]), it follows from the inf-sup condition satisfied by the form $b(\cdot, \cdot)$ on $\mathbb{X}_\varepsilon \times L^2(Q)$ that

$$\|p - p^\varepsilon\|_{\mathbb{X}} + \|y - y^\varepsilon\|_{L^2(Q)} \leq c \big(\inf_{q^\varepsilon \in \mathbb{X}_\varepsilon} \|p - q^\varepsilon\|_{\mathbb{X}} + \varepsilon\, \|\partial_\nu y\|_{L^2(\Sigma)} \big). \tag{4.15}$$

So it remains to evaluate the distance of p to $\mathbb{X}_\varepsilon$.

Lemma 4.7. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$. Let p be any function in $\mathbb{X}$ such that $\partial_\nu p$ belongs to $H^1(\Sigma)$. Then, the following estimate holds*

$$\inf_{q^\varepsilon \in \mathbb{X}_\varepsilon} \|p - q^\varepsilon\|_{\mathbb{X}} \leq c\, \varepsilon\, \|\partial_\nu p\|_{H^1(\Sigma)}. \tag{4.16}$$

Proof: With the function p, we associate the element $q^\varepsilon = p + \varepsilon\, e^\varepsilon$, where e^ε is the solution in $\overline{\mathbb{X}}$ of the equation

$$\begin{cases} -\partial_t e^\varepsilon - \Delta e^\varepsilon = 0 & \text{in } Q, \\ \varepsilon\, \partial_\nu e^\varepsilon + e^\varepsilon = -\partial_\nu p & \text{on } \Sigma, \\ e^\varepsilon(\cdot, T) = 0 & \text{on } \Omega. \end{cases}$$

It is readily checked that q^ε belongs to $\mathbb{X}_\varepsilon$. Moreover, we have

$$\|p - q^\varepsilon\|_{\mathbb{X}} = \varepsilon\, \|\partial_\nu e^\varepsilon\|_{L^2(\Sigma)}.$$

Since $\partial\Omega$ is of class of class $\mathscr{C}^{1,1}$, it follows from [2, Lemma 4.6] (see also the proof of this lemma) that

$$\|\partial_\nu e^\varepsilon\|_{L^2(\Sigma)} \leq c\, \|\partial_\nu p\|_{H^1(\Sigma)},$$

which concludes the proof.

Inserting (4.16) into (4.15) leads to the following statement.

Proposition 4.8. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$ and that the solution (p, y) of problem (3.6) is such that $(\partial_\nu p, \partial_\nu y)$ belongs to $H^1(\Sigma) \times L^2(\Sigma)$. The following a priori error estimate holds between this solution and the solution $(p^\varepsilon, y^\varepsilon)$ of problem (4.6)*

$$\|p - p^\varepsilon\|_{\mathbb{X}} + \|y - y^\varepsilon\|_{L^2(Q)} \leq c\, \varepsilon\, \big(\|\partial_\nu p\|_{H^1(\Sigma)} + \|\partial_\nu y\|_{L^2(\Sigma)} \big). \tag{4.17}$$

Remark 4.9. A closer look at [2, §4] yields that the result of Lemma 4.7, hence of Proposition 4.8, still holds with a weaker assumption on $\partial_\nu p$ (this assumption involves spaces of different regularity with respect to space and time, so that we have rather skip their definitions). Moreover at least part of this regularity can be deduced from the regularity assumption on $\partial_\nu y$.

Owing to Proposition 3.1 and its analogue for the penalized control problem, we also derive from Proposition 4.8 the following result (indeed, it is readily checked that, when u is in $H^1(\Sigma)$, $(\partial_\nu p, \partial_\nu y)$ belongs to $H^1(\Sigma) \times L^2(\Sigma)$).

Corollary 4.10. *Assume that the domain Ω has a boundary of class $\mathscr{C}^{1,1}$ and that the solution u of problem* (2.1) *belongs to $H^1(\Sigma)$. The following a priori error estimate holds between this solution and the solution u^ε of problem* (4.1)

$$\|u - u^\varepsilon\|_{L^2(\Sigma)} \leq c\,\varepsilon\,\|u\|_{H^1(\Sigma)}. \tag{4.18}$$

In view of the discretization, we must write another formulation of system (4.4). We have already observed that the part y^ε of the solution of problem (4.4) is more regular than expected. Thus, setting

$$\mathbb{Z} = L^2(0,T;H^1(\Omega)) \cap \mathscr{C}^0(0,T;L^2(\Omega)), \tag{4.19}$$

we observe that system (4.4) admits the equivalent variational formulation

Find $(p^\varepsilon, y^\varepsilon)$ in $\mathbb{Z} \times \mathbb{Z}$ such that

$$p^\varepsilon(\cdot,T) = y^\varepsilon(\cdot,T) - y_T \qquad \text{and} \qquad y^\varepsilon(\cdot,0) = 0, \tag{4.20}$$

and that, for a.e. t in [0,T],

$$\begin{aligned}
\forall q \in H^1(\Omega), \quad & -\int_\Omega (\partial_t p^\varepsilon)(\boldsymbol{x})q(\boldsymbol{x})\,d\boldsymbol{x} + \int_\Omega (\mathbf{grad}\,p^\varepsilon)(\boldsymbol{x})\cdot(\mathbf{grad}\,q)(\boldsymbol{x})\,d\boldsymbol{x} \\
&\qquad + \varepsilon^{-1}\int_{\partial\Omega} p^\varepsilon(\boldsymbol{\tau})q(\boldsymbol{\tau})\,d\boldsymbol{\tau} = 0, \\
\forall z \in H^1(\Omega), \quad & \int_\Omega (\partial_t y^\varepsilon)(\boldsymbol{x})z(\boldsymbol{x})\,d\boldsymbol{x} + \int_\Omega (\mathbf{grad}\,y^\varepsilon)(\boldsymbol{x})\cdot(\mathbf{grad}\,z)(\boldsymbol{x})\,d\boldsymbol{x} \\
&\qquad + \varepsilon^{-1}\int_{\partial\Omega} y^\varepsilon(\boldsymbol{\tau})z(\boldsymbol{\tau})\,d\boldsymbol{\tau} = -\varepsilon^{-2}\beta^{-1}\int_{\partial\Omega} p^\varepsilon(\boldsymbol{\tau})z(\boldsymbol{\tau})\,d\boldsymbol{\tau}.
\end{aligned} \tag{4.21}$$

The modification of the right-hand side of the last equation is due to the fact that now we wish to work on the space $\mathbb{Z}$ instead of $\mathbb{X}$ and thus not to make any assumption on the regularity of $\partial_\nu p^\varepsilon$.

As a conclusion, we successfully studied the convergence of the Robin penalization of the optimal control problem with a final observation when acting on the Dirichlet boundary condition The forthcoming work is twofold: First, to conduct a numerical analysis of a time scheme/finite element discretization of the mixed system (4.4) by using formulation (4.20)–(4.21); second, to validate our convergence results by some computational experiments. These are the subject of Part II of the current work.

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