

SPECTRAL AND SCATTERING THEORY OF SPACE-CUTOFF CHARGED $P(\varphi)_2$ MODELS

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ABSTRACT. We consider in this paper space-cutoff charged $P(\varphi)_2$ models arising from the quantization of the non-linear charged Klein-Gordon equation:

$$(\partial_t + iV(x))^2 \phi(t, x) + (-\Delta_x + m^2)\phi(t, x) + g(x)\partial_{\bar{z}}P(\phi(t, x), \bar{\phi}(t, x)) = 0,$$

where $V(x)$ is an electrostatic potential, $g(x) \geq 0$ a space-cutoff and $P(\lambda, \bar{\lambda})$ a real bounded below polynomial. We discuss various ways to quantize this equation, starting from different CCR representations. After describing the construction of the interacting Hamiltonian H we study its spectral and scattering theory. We describe the essential spectrum of H , prove the existence of asymptotic fields and of wave operators, and finally prove the *asymptotic completeness* of wave operators. These results are similar to the case when $V = 0$.

1. INTRODUCTION

1.1. Charged Klein-Gordon equations. Let us consider the charged Klein-Gordon equation:

$$(1.1) \quad (\partial_t + iV(x))^2 \phi(t, x) + (-\Delta_x + m^2)\phi(t, x) = 0,$$

where $\phi : \mathbb{R}_t \rightarrow L^2(\mathbb{R}^d; \mathbb{C})$, $m > 0$ is the mass. The equation (1.1) describes a charged field minimally coupled to an external electrostatic field given by the potential V . As is well known, after introducing the φ and π fields by

$$\varphi(t) = \phi(t), \quad \pi(t) = \partial_t \phi(t) + iV\phi(t),$$

one can interpret (1.1) as a Hamiltonian system on the symplectic space

$$\mathcal{Y} = \{y = (\pi, \varphi) : \pi, \varphi \in L^2(\mathbb{R}^d, \mathbb{C})\}$$

equipped with the (complex) symplectic form

$$(\pi, \varphi)\omega(\pi', \varphi') = \int_{\mathbb{R}^d} \bar{\pi}(x)\varphi'(x) - \bar{\varphi}(x)\pi'(x)dx.$$

for the classical Hamiltonian

$$\begin{aligned} & h_V(\pi, \varphi) \\ &= \int_{\mathbb{R}^d} \bar{\pi}(x)\pi(x)dx + \int_{\mathbb{R}^d} \overline{\nabla_x \varphi(x)} \cdot \nabla_x \varphi(x) + m^2 \bar{\varphi}(x)\varphi(x)dx \\ & \quad + i \int_{\mathbb{R}^d} \bar{\varphi}(x)V(x)\pi(x) - \bar{\pi}(x)V(x)\varphi(x)dx \end{aligned}$$

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In order to obtain a *stable quantization* of (1.1), ie a CCR representation of (\mathcal{Y}, ω) in a Hilbert space \mathcal{H} with the property that the time evolution is implemented by a positive Hamiltonian, it is necessary that the classical Hamiltonian $h_V(\pi, \varphi)$ is *positive*. If this is the case, one can equip \mathcal{Y} with a *Kähler structure*, ie a complex structure j such that

$$(y|y')_{\text{dyn}} := y\omega jy' + iy\omega y'$$

is a scalar product on \mathcal{Y} . The completion of the pre-Hilbert space $(\mathcal{Y}, j, (\cdot|\cdot)_{\text{dyn}})$, denoted by \mathcal{Z} is called the *one-particle space*. The stable quantization is then the Fock representation on the bosonic Fock space $\Gamma_s(\mathcal{Z})$, and the time evolution is unitarily implemented by the group e^{itH_V} , where $H_V = d\Gamma(h_V)$ is a second quantized Hamiltonian.

An alternative quantization is obtained by considering first the Klein-Gordon equation (1.1) for $V(x) \equiv 0$. Let us denote by j_0 (resp. \mathcal{Z}_0) the associated complex structure (resp. one-particle space). As is well known, \mathcal{Z}_0 can be unitarily identified with $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$.

The dynamics for $V = 0$ is unitarily implemented by e^{itH_0} on the Fock space $\Gamma_b(\mathcal{Z}_0)$, for $H_0 = d\Gamma(\omega)$, where $\omega = \epsilon \oplus \epsilon$ is the one-particle energy and $\epsilon = (-\Delta_x + m^2)^{\frac{1}{2}}$.

One can then try to implement the dynamics for $V \neq 0$ by considering the Fock representation on $\Gamma_s(\mathcal{Z}_0)$ and by giving a meaning to the formal expression:

$$H = d\Gamma(\omega) + i \int_{\mathbb{R}^d} \overline{\varphi}(x)V(x)\pi(x) - \overline{\pi}(x)V(x)\varphi(x)dx,$$

where $\varphi(x)$, $\pi(x)$ are the *quantized* φ and π fields. Note that the two CCR representations above are in general not unitarily equivalent.

It turns out that it is possible to give a meaning to H , in *one space dimension* ($d = 1$), provided the potential V is small enough as we will see in Sect. 4.

1.2. Non-linear perturbations. We assume now that $d = 1$. Let us fix a positive space cutoff function $g : \mathbb{R} \rightarrow \mathbb{R}^+$, decreasing fast enough at infinity and a bounded below real potential $P(\lambda, \overline{\lambda})$. We consider now the *non-linear* charged Klein-Gordon equation:

$$(1.2) \quad (\partial_t + iV(x))^2 \phi(t, x) + (-\Delta_x + m^2)\phi(t, x) + g(x)\partial_{\overline{z}}P(\phi(t, x), \overline{\phi}(t, x)) = 0.$$

The usual procedure to quantize (1.2) is to start from a quantization of (1.1) (ie (1.2) for $g(x) \equiv 0$), leading to the Hamiltonians H_V or H_0 (depending on the choice of the CCR representation), and to implement the interacting dynamics by giving a meaning to either:

$$(1.3) \quad H_V + \int_{\mathbb{R}} g(x)P(\varphi(x), \overline{\varphi}(x))dx,$$

or:

$$(1.4) \quad H_0 + i \int_{\mathbb{R}} \overline{\varphi}(x)V(x)\pi(x) - \overline{\pi}(x)V(x)\varphi(x)dx + \int_{\mathbb{R}} g(x)P(\varphi(x), \overline{\varphi}(x))dx.$$

The choice (1.3) seems difficult, because both the one-particle energy h_V and the φ , π fields are not very explicit in the Fock representation for the complex structure j .

In this paper we will adopt the choice (1.4).

The associated Hamiltonian will be constructed in Sect. 4. We will show that if $|\lambda| < \lambda_{\text{quant}}$, where the constant λ_{quant} is defined in (3.8), the formal expression

$$H := H_0 + i\lambda \int_{\mathbb{R}} \overline{\varphi}(x)V(x)\pi(x) - \overline{\pi}(x)V(x)\varphi(x)dx + \int_{\mathbb{R}} g(x) :P(\varphi(x), \overline{\varphi}(x)) : dx,$$

is well defined as a bounded below selfadjoint operator.

The rest of the paper is devoted to the spectral and scattering theory of H , which is studied in Sect. 5. We will use the results of [GP], where an abstract class of QFT Hamiltonians are considered, so most of our task is to prove that our Hamiltonian H satisfies the abstract hypotheses of [GP]. This will be done in Subsect. 5.1.

The first result is the *HVZ theorem*, describing the essential spectrum of H . We obtain that

$$\sigma_{\text{ess}}(H) = [\inf\sigma(H) + m, +\infty[,$$

which implies that H has a ground state.

The second results deal with the *scattering theory* of H , which is formulated in terms of *asymptotic fields*. These are (formally) defined as the limits:

$$\lim_{t \rightarrow \pm\infty} e^{itH} \phi(e^{it\omega} F) e^{-itH} =: \phi^\pm(F), \quad F \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}).$$

It follows then from the stability condition $|\lambda| < \lambda_{\text{quant}}$ and abstract arguments that the two asymptotic CCR representations

$$F \mapsto \phi^\pm(F)$$

are of *Fock type*, ie unitarily equivalent to a sum of Fock representations.

The main problem of scattering theory is now to identify the spaces of *asymptotic vacua*, ie the spaces of vectors annihilated by all *asymptotic annihilation operators* $a^\pm(F)$. Applying the abstract results of [GP], we show that the asymptotic vacua coincide with the *bound states* of H . This result, called the *asymptotic completeness* of wave operators, is the main result of this paper.

1.3. Notation. In this subsection we collect some useful notation and results.

Scales of Hilbert spaces

If \mathfrak{h} is a Hilbert space and ϵ a linear operator on \mathfrak{h} , its domain will be denoted by Dome . The closure of a closeable operator a will be denoted by a^{cl} .

If ϵ is selfadjoint, we write $\epsilon > 0$ if $\epsilon \geq 0$ and $\text{Ker}\epsilon = \{0\}$. If $\epsilon > 0$ and $s \in \mathbb{R}$, ϵ^s is well defined as a selfadjoint operator and we denote by $\epsilon^s \mathfrak{h}$ the completion of $\text{Dome}\epsilon^{-s}$ for the norm $\|\epsilon^{-s} h\|$. Clearly $\epsilon^s \mathfrak{h}$ are Hilbert spaces and ϵ^t is isometric from $\epsilon^s \mathfrak{h}$ to $\epsilon^{s+t} \mathfrak{h}$.

Fourier transform

Let $\mathfrak{h} = L^2(\mathbb{R})$. We denote by \mathcal{F} the unitary Fourier transform:

$$\mathcal{F}u(k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ik \cdot x} u(x) dx.$$

We denote also by \widehat{f} the usual Fourier transform of f :

$$\widehat{f}(k) = \int_{\mathbb{R}} e^{-ik \cdot x} u(x) dx,$$

so that if V is the operator of multiplication by the function V one has

$$(1.5) \quad \mathcal{F}V\mathcal{F}^{-1}u(k_1) = (2\pi)^{-1} \int_{\mathbb{R}} \widehat{V}(k_1 - k_2) u(k_2) dk_2.$$

If $\epsilon = (-\Delta_x + m^2)^{\frac{1}{2}}$ for $m > 0$ then $\epsilon^s L^2(\mathbb{R})$ is equal to the Sobolev space $H^{-s}(\mathbb{R})$ with the norm

$$\|f\|_{H^{-s}(\mathbb{R})}^2 = \int_{\mathbb{R}} (k^2 + m^2)^{-s} |\mathcal{F}u(k)|^2 dk.$$

Pseudodifferential calculus

Set $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$. For $m \in \mathbb{R}$ we will denote by $S^m(\mathbb{R})$ the space

$$S^m(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : |f^{(\alpha)}(x)| \leq C_n \langle x \rangle^{m-\alpha}, \alpha \in \mathbb{N}\}.$$

For $m, p \in \mathbb{R}$ we denote by $S^{m,p}(\mathbb{R}^2)$ the space

$$S^{m,p}(\mathbb{R}^2) = \{f \in C^\infty(\mathbb{R}^2) : |\partial_x^\alpha \partial_k^\beta f(x, k)| \leq C_{\alpha,\beta} \langle x \rangle^{m-\alpha} \langle k \rangle^{p-\beta}, \alpha, \beta \in \mathbb{N}\}.$$

If $a \in S^{m,p}(\mathbb{R}^2)$, we denote by $\text{Op}^w(a) = a^w(x, D_x)$ the *Weyl quantization* of a , defined as:

$$\text{Op}^w(a)u(x) = (2\pi)^{-1} \int e^{i(x-y) \cdot k} a\left(\frac{x+y}{2}, k\right) u(y) dy dk,$$

as an operator on $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R}) = \bigcap_{m \in \mathbb{R}} S^m(\mathbb{R})$ is the Schwartz class.

The operator $\text{Op}^w(a)$ is bounded on $L^2(\mathbb{R})$ if $a \in S^{m,p}(\mathbb{R}^2)$ for $m, p \leq 0$, and belongs to the Hilbert-Schmidt class iff $a \in L^2(\mathbb{R}^2)$. One has then

$$(1.6) \quad \|\text{Op}^w a\|_{\text{HS}}^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} |a(x, k)|^2 dx dk.$$

2. CHARGED KLEIN-GORDON EQUATION

In this section we detail the arguments given in Subsect. 1.1. The results of this section are standard, they can be found for example in Palmer [Pa]. For simplicity we consider the one dimensional case, although the results of Subsect. 2.1 hold in any space dimension.

2.1. Charged Klein-Gordon equation as a Hamilton equation. Let $m > 0$ and $V : \mathbb{R} \rightarrow \mathbb{R}$ a real measurable potential such that

$$(2.1) \quad V, \nabla_x V \in L^\infty(\mathbb{R}).$$

We consider the Cauchy problem for the charged Klein-Gordon equation:

$$(2.2) \quad \begin{cases} (\partial_t + iV(x))^2 \phi(t, x) + (-\Delta_x + m^2) \phi(t, x) = 0, \\ \phi(0, x) = \varphi(x), \\ \partial_t \phi(0, x) + iV(x) \phi(0, x) = \pi(x), \end{cases}$$

where $\phi : \mathbb{R} \rightarrow L^2(\mathbb{R}; \mathbb{C})$, describing a charged scalar field of mass m minimally coupled to the electrostatic potential V .

Note that (2.2) is invariant under *time-reversal*, ie if $\phi(t, x)$ is a solution, so is $\bar{\phi}(-t, x)$. In terms of Cauchy data, time-reversal becomes the involution:

$$(2.3) \quad \kappa : (\pi, \varphi) \mapsto (-\bar{\pi}, \bar{\varphi}).$$

Let us set

$$\varphi(t) := \phi(t), \quad \pi(t) = \partial_t \phi(t) + iV \phi(t),$$

and

$$\mathcal{Y} = \{y = (\pi, \varphi) : \pi, \varphi \in L^2(\mathbb{R})\}.$$

We transform (2.2) into the first order evolution equation on $\mathcal{Y} := L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$:

$$\begin{bmatrix} \pi(t) \\ \varphi(t) \end{bmatrix} = r_t \begin{bmatrix} \pi(0) \\ \varphi(0) \end{bmatrix}$$

Formally we have,

$$(2.4) \quad \partial_t \begin{bmatrix} \pi(t) \\ \varphi(t) \end{bmatrix} = \begin{bmatrix} -iV & -\epsilon^2 \\ \mathbb{1} & -iV \end{bmatrix} \begin{bmatrix} \pi(t) \\ \varphi(t) \end{bmatrix},$$

for $\epsilon := (-\Delta_x + m^2)^{\frac{1}{2}}$.

If we equip \mathcal{Y} with the (sesquilinear) anti-symmetric form:

$$(\pi, \varphi)\omega(\pi', \varphi') = (\pi|\varphi') - (\varphi|\pi'),$$

and the Hamiltonian:

$$(2.5) \quad h_V(\pi, \varphi) = \|\pi\|^2 + \|\epsilon\varphi\|^2 + i(\varphi|V\pi) - i(\pi|V\varphi),$$

we see that (2.4) are the associated Hamilton equations. If we prefer to forget the complex structure of \mathcal{Y} , we write

$$(2.6) \quad \varphi =: \varphi_1 + i\varphi_2, \quad \pi =: \pi_1 + i\pi_2,$$

and equip \mathcal{Y} (as a real vector space) with the real symplectic form $\text{Re}\omega$ and the Hamiltonian

$$(2.7) \quad \begin{aligned} h_{V,\mathbb{R}}(\pi, \varphi) &= \frac{1}{2}h(\pi, \varphi) \\ &= \frac{1}{2}\|\pi_1\|^2 + \frac{1}{2}\|\pi_2\|^2 + \frac{1}{2}\|\epsilon\varphi_1\|^2 + \frac{1}{2}\|\epsilon\varphi_2\|^2 \\ &\quad + (\pi_1|V\varphi_2) - (\pi_2|V\varphi_1). \end{aligned}$$

2.2. Stable quantization. A *stable* quantization of the symplectic dynamics r_t is a CCR representation of the symplectic space (\mathcal{Y}, ω) :

$$\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$$

in some Hilbert space \mathcal{H} such that there exists a *positive* selfadjoint operator H on \mathcal{H} implementing r_t , ie:

$$e^{itH}W(y)e^{-itH} = W(r_t y), \quad y \in \mathcal{Y}, \quad t \in \mathbb{R}.$$

As is well known (see eg [BSZ]), in order for a stable quantization to exist, it is necessary that the classical Hamiltonian $h_V(\pi, \varphi)$ is positive. The violation of the positivity of $h_V(\pi, \varphi)$ is connected with the so called *Klein paradox*.

Let us assume the following stronger positivity:

$$(2.8) \quad \pm i((\varphi|V\pi) - (\pi|V\varphi)) \leq \delta (\|\pi\|^2 + \|\epsilon\varphi\|^2) \quad \pi \in L^2(\mathbb{R}), \quad \varphi \in \text{Dome}, \quad \text{for } 0 \leq \delta < 1.$$

Note that (2.8) implies that the energy norms $h_0(\cdot)^{\frac{1}{2}}$ and $h_V(\cdot)^{\frac{1}{2}}$ are equivalent.

The construction of the stable quantization is then as follows:

- (1) one considers the *energy space* \mathcal{Y}_{en} which is the completion of $L^2(\mathbb{R}) \oplus H^1(\mathbb{R})$ for the norm $h_V(\pi, \varphi)^{\frac{1}{2}}$;

- (2) clearly $t \rightarrow r_t$ is a strongly continuous group of isometries of \mathcal{Y}_{en} , and we denote by a its generator ie $r_t =: e^{t a}$. From (2.4) we see that

$$a = \begin{bmatrix} -iV & \Delta_x - m^2 \\ \mathbb{1} & -iV \end{bmatrix},$$

is anti-selfadjoint on $\text{Doma} = H^1(\mathbb{R}) \oplus H^2(\mathbb{R})$. Moreover from (2.8) we see that $\text{Ker} a = \{0\}$.

- (3) we consider now the polar decomposition of a :

$$h_V := (-a^2)^{\frac{1}{2}}, \quad a =: j h_V = j h_V,$$

and we see that j is an anti-involution (a complex structure) on \mathcal{Y}_{en} , such that ωj is a symmetric positive definite form.

- (4) we equip \mathcal{Y}_{en} with the complex structure j and the scalar product

$$(y_1 | y_2)_{\text{dyn}} := y_1 \omega j y_2 + i y_1 \omega y_2.$$

- (5) denoting by \mathcal{Z} the completion of \mathcal{Y}_{en} for $(\cdot | \cdot)$, we obtain a complex Hilbert space, such that h_V extends to \mathcal{Z} as a positive selfadjoint operator. The stable quantization of the charged Klein-Gordon equation is obtained by taking the Hilbert space:

$$\mathcal{H} = \Gamma_s(\mathcal{Z}),$$

where $\Gamma_s(\mathcal{Z})$ is the bosonic Fock space over \mathcal{Z} , the CCR representation

$$\mathcal{Z} \supset \mathcal{Y}_{\text{en}} \ni y \mapsto W(y) \in U(\mathcal{H})$$

where $W(y)$ are the Fock Weyl operators, and the physical Hamiltonian

$$H = d\Gamma(h_V),$$

where $d\Gamma(h_V)$ is the second quantization of h_V .

2.3. Alternative choice of the complex structure. Let us consider the charged Klein-Gordon equation (2.2) for $V = 0$ and denote with the subscript 0 the associated objects.

By the same procedure as above we can equip \mathcal{Y} with the *free* complex structure j_0 . A very convenient feature of j_0 is that if \mathcal{Z}_0 is the associated Hilbert space, then the map:

$$U : \mathcal{Z}_0 \ni (\pi, \varphi) \mapsto (\epsilon^{-\frac{1}{2}} \pi + i \epsilon^{\frac{1}{2}} \varphi, \epsilon^{-\frac{1}{2}} \bar{\pi} + i \epsilon^{\frac{1}{2}} \bar{\varphi}) \in L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$$

is unitary. This allows to identify \mathcal{Z}_0 with an explicit Hilbert space. In terms of neutral fields π_i, φ_i the map W becomes:

$$(2.9) \quad (\pi, \varphi) \mapsto (\epsilon^{-\frac{1}{2}} \pi_1 + i \epsilon^{\frac{1}{2}} \varphi_1, \epsilon^{-\frac{1}{2}} \pi_2 + i \epsilon^{\frac{1}{2}} \varphi_2) \in L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d).$$

As is well known (see eg [Pa]), there exists an invertible symplectic transformation u on \mathcal{Y} such that

$$j_0 = u^{-1} j u.$$

(This actually holds for any pair of Kähler complex structures on a symplectic space).

Therefore $u : \mathcal{Z} \rightarrow \mathcal{Z}_0$ and its second quantization $\Gamma(u) : \Gamma_s(\mathcal{Z}) \rightarrow \Gamma_s(\mathcal{Z}_0)$ are unitary. The Fock representation of CCR on $\Gamma_s(\mathcal{Z})$ is unitarily equivalent to the following Bogoliubov representation on $\Gamma_s(\mathcal{Z}_0)$:

$$(2.10) \quad W_V(f) := W_0(uf), \quad f \in \mathcal{Y},$$

where $W_0(\cdot)$ is the Fock representation on $\Gamma_s(\mathcal{Z}_0)$. This allows to work on the more convenient Fock space $\Gamma_s(\mathcal{Z}_0)$. The positive Hamiltonian on $\Gamma_s(\mathcal{Z}_0)$ implementing the dynamics e^{ta} in the Bogoliubov representation $W_V(\cdot)$ is then

$$d\Gamma(h_V),$$

where we still denote by h_V acting on \mathcal{Z}_0 the operator uh_Vu^{-1} .

2.4. Quantization of the non-linear charged Klein-Gordon equation. Let $P(z_1, z_2)$ be a polynomial on \mathbb{C}^2 such that $\mathbb{C} \ni z \mapsto P(z, \bar{z})$ is real and bounded below. Let also g a positive function (typically $g \in C_0^\infty(\mathbb{R})$). We consider now the non-linear Klein-Gordon equation:

$$(2.11) \quad (\partial_t + iV(x))^2 \phi(t, x) + (-\Delta_x + m^2)\phi(t, x) + g(x)\partial_{\bar{z}}P(\phi(t, x), \bar{\phi}(t, x)) = 0.$$

The quantization of (2.11) for $g(x) \equiv 0$, outlined in Subsect. 2.3 leads to the free Hamiltonian

$$d\Gamma(h_V), \text{ acting on } \Gamma_s(\mathcal{Z}_0),$$

and to the Bogoliubov representation of CCR $W_V(\cdot)$ defined in (2.10).

Denoting by $\phi_V(f)$ for $f \in \mathcal{Y}$ the Segal field operators associated to the CCR representation (2.10), one sets:

$$\varphi_V(x) = \phi_V(\delta_x, 0), \quad x \in \mathbb{R}$$

which are the corresponding φ fields. The natural way to quantize (2.11) is now to try to make sense of the Hamiltonian

$$(2.12) \quad H_V = d\Gamma(h_V) + \int_{\mathbb{R}} g(x)P(\varphi_V(x), \bar{\varphi}_V(x))dx.$$

If (possibly after some Wick ordering of the interaction term), the above Hamiltonian is well defined, one can set

$$\phi_V(t, f) = e^{itH_V} \phi_V(f) e^{-itH},$$

which leads to the quantization of (2.11) in the Bogoliubov representation (2.10).

The difficulty with this method is of course to make sense of H_V , since neither the one-particle Hamiltonian h_V nor the φ fields $\varphi_V(x)$ are explicitly known.

Actually if V decays fast enough at infinity, it is possible to find a symplectic transformation u such that uh_Vu^{-1} equals the *free* one-particle energy and additionally u is *real*, ie commutes with the time-reversal operator κ in (2.3). This opens the possibility to rigorously construct the Hamiltonian (2.12). We plan to come back to this problem in a subsequent paper.

An alternative way, which we will follow in this paper, is as follows:

- (1) one considers the stable quantization of (2.2) for $V = 0$, leading to the usual complex structure j_0 . It is convenient to use the neutral fields π_i, φ_i $i = 1, 2$ as in (2.6), and to identify the one-particle space \mathcal{Z}_0 with $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ as in (2.9).
- (2) the free Hamiltonian is now

$$H_0 = d\Gamma(\epsilon \oplus \epsilon), \text{ acting on } \mathcal{H} = \Gamma_s(L^2(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

which implements the dynamics e^{ta_0} in the Fock representation for the complex structure j_0 .

(3) one sets for $x \in \mathbb{R}$:

$$(2.13) \quad \begin{aligned} \varphi_1(x) &:= \phi\left(\epsilon^{-\frac{1}{2}}\delta_x \oplus 0\right), & \varphi_2(x) &:= \phi\left(0 \oplus \epsilon^{-\frac{1}{2}}\delta_x\right) \\ \pi_1(x) &:= \phi\left(i\epsilon^{\frac{1}{2}}\delta_x \oplus 0\right), & \pi_2(x) &:= \phi\left(0 \oplus i\epsilon^{\frac{1}{2}}\delta_x\right), \end{aligned}$$

where $\phi(f)$ are the Segal field operators. These operators are well defined as selfadjoint operators after integration against test functions.

(4) setting with a slight abuse of notation

$$P(\lambda_1, \lambda_2) := P(\lambda_1 + i\lambda_2, \lambda_1 - i\lambda_2),$$

one tries to rigorously define as a selfadjoint operator the formal expression:

$$(2.14) \quad H = d\Gamma(\epsilon \oplus \epsilon) + \int_{\mathbb{R}} g(x)P(\varphi_1(x), \varphi_2(x))dx + \int_{\mathbb{R}} V(x) (\pi_1(x)\varphi_2(x) - \pi_2(x)\varphi_1(x)) dx,$$

corresponding to the hamiltonian $h_{\mathbb{R}}(\pi, \varphi)$ defined in (2.7). This will be done in Sect. 4.

3. LOCAL CHARGE OPERATOR

In the rest of the paper we set

$$\mathfrak{h} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \quad \mathcal{H} = \Gamma_s(\mathfrak{h}).$$

The elements of \mathfrak{h} will be denoted by $F = (f_1, f_2)$. The one-particle energy is

$$\omega := \epsilon \oplus \epsilon \text{ acting on } \mathfrak{h},$$

and

$$H_0 := d\Gamma(\omega)$$

The (total) *number operator* N is

$$N := d\Gamma(\mathbb{1} \oplus \mathbb{1}),$$

equal to $N_1 + N_2$, where

$$N_1 := d\Gamma(\mathbb{1} \oplus 0), \quad N_2 := d\Gamma(0 \oplus \mathbb{1}).$$

We will also use the partial creation/annihilation operators

$$a_1^\sharp(f) = a^\sharp(f \oplus 0) \quad a_2^\sharp(f) = a^\sharp(0 \oplus f), \quad f \in L^2(\mathbb{R}).$$

3.1. Local charge operator. Set

$$Q(V) := \int_{\mathbb{R}} V(x) (\pi_1(x)\varphi_2(x) - \pi_2(x)\varphi_1(x)) dx,$$

where $\varphi_i(x), \pi_i(x)$ are defined in (2.13). For the moment it is only a formal expression.

We will call $Q(V)$ a *local charge operator*.

To work with well defined objects, we introduce the UV cutoff fields, $\varphi_i^\kappa(x), \pi_i^\kappa(x)$, for $\kappa \gg 1$, obtained by replacing δ_x by $F(\kappa^{-1}D_x)\delta_x$ where $F \in C_0^\infty(\mathbb{R})$ is a cutoff function with $F(0) = 1$. We denote by $Q^\kappa(V)$ the cutoff charge operator, which is for example well defined on $\text{Dom}N$.

Lemma 3.1. *Assume that $V \in L^\infty(\mathbb{R})$ and $V' \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Then there exists a constant C such that:*

$$(3.1) \quad \|Q(V)(N+1)^{-1}\| \leq C(\|V\|_\infty + \|V\|_2 + \|V'\|_\infty).$$

Proof. For ease of notation we will remove the UV cutoff. To get a rigorous proof, it suffices to put back the UV cutoff, letting $\kappa \rightarrow +\infty$ in the various estimates.

Introducing the creation/annihilation operators $a_i^\sharp(f)$ for $i = 1, 2$ we have:

$$\begin{aligned} & \pi_1(x)\varphi_2(x) - \pi_2(x)\varphi_1(x) \\ &= \frac{i}{2} \left(a_1^*(\epsilon^{\frac{1}{2}}\delta_x) - a_1(\epsilon^{\frac{1}{2}}\delta_x) \right) \left(a_2^*(\epsilon^{-\frac{1}{2}}\delta_x) + a_2(\epsilon^{-\frac{1}{2}}\delta_x) \right) \\ & - \frac{i}{2} \left(a_2^*(\epsilon^{\frac{1}{2}}\delta_x) - a_2(\epsilon^{\frac{1}{2}}\delta_x) \right) \left(a_1^*(\epsilon^{-\frac{1}{2}}\delta_x) + a_1(\epsilon^{-\frac{1}{2}}\delta_x) \right) \\ &= \frac{i}{2} \left(a_1^*(\epsilon^{\frac{1}{2}}\delta_x)a_2(\epsilon^{-\frac{1}{2}}\delta_x) + a_1^*(\epsilon^{-\frac{1}{2}}\delta_x)a_2(\epsilon^{\frac{1}{2}}\delta_x) \right. \\ & \quad \left. - a_2^*(\epsilon^{-\frac{1}{2}}\delta_x)a_1(\epsilon^{\frac{1}{2}}\delta_x) - a_2^*(\epsilon^{\frac{1}{2}}\delta_x)a_1(\epsilon^{-\frac{1}{2}}\delta_x) \right) \\ & + \frac{i}{2} \left(a_1^*(\epsilon^{\frac{1}{2}}\delta_x)a_2^*(\epsilon^{-\frac{1}{2}}\delta_x) - a_1^*(\epsilon^{-\frac{1}{2}}\delta_x)a_2^*(\epsilon^{\frac{1}{2}}\delta_x) \right) \\ & + \frac{i}{2} \left(a_2(\epsilon^{\frac{1}{2}}\delta_x)a_1(\epsilon^{-\frac{1}{2}}\delta_x) - a_1(\epsilon^{\frac{1}{2}}\delta_x)a_2(\epsilon^{-\frac{1}{2}}\delta_x) \right) \\ &=: R^{a^*,a}(x) + R^{a^*,a^*}(x) + R^{a,a}(x). \end{aligned}$$

It is convenient to pass to the momentum representation using the unitary Fourier transform \mathcal{F} . It follows that

$$(3.2) \quad \begin{aligned} a_i^*(\epsilon^s\delta_x) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \epsilon^s(k) e^{-ikx} a_i^*(k) dk, \\ a_i(\epsilon^s\delta_x) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \epsilon^s(k) e^{ikx} a_i(k) dk. \end{aligned}$$

Let us first consider the term

$$Q^{a^*,a}(V) = \int_{\mathbb{R}} V(x) R^{a^*,a}(x) dx.$$

Using the above transformation and (1.5) we see that

$$(3.3) \quad Q^{a^*,a}(V) = d\Gamma \left(\begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \right).$$

for

$$(3.4) \quad b = \frac{i}{2} (\epsilon^{\frac{1}{2}} V \epsilon^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} V \epsilon^{\frac{1}{2}}).$$

Since $V, V' \in L^\infty(\mathbb{R})$, V is a bounded operator on $H^1(\mathbb{R})$ hence by interpolation and duality also on $H^{\frac{1}{2}}(\mathbb{R})$ and $H^{-\frac{1}{2}}(\mathbb{R})$. This implies that b is bounded. Clearly this implies that (3.1) holds for $Q^{a^*,a}(V)$. Let us now consider the term

$$Q^{a^*,a^*}(V) = \int_{\mathbb{R}} V(x) R^{a^*,a^*}(x) dx.$$

Using (3.2), we obtain that:

$$\begin{aligned} & Q^{a^* a^*}(V) \\ &= \int_{\mathbb{R}^2} R(k_1, k_2) a_1^*(k_1) a_2^*(k_2) dk_1 dk_2, \end{aligned}$$

where:

$$(3.5) \quad R(k_1, k_2) = \frac{i}{4\pi} \widehat{V}(k_1 + k_2) \left(\epsilon(k_1)^{\frac{1}{2}} \epsilon(k_2)^{-\frac{1}{2}} - \epsilon(k_1)^{-\frac{1}{2}} \epsilon(k_2)^{\frac{1}{2}} \right).$$

We note that:

$$\begin{aligned} & |\epsilon(k_1)^{\frac{1}{2}} \epsilon(k_2)^{-\frac{1}{2}} - \epsilon(k_1)^{-\frac{1}{2}} \epsilon(k_2)^{\frac{1}{2}}| \\ &= \epsilon(k_1)^{-\frac{1}{2}} \epsilon(k_2)^{-\frac{1}{2}} |\epsilon(k_1) - \epsilon(k_2)| \\ &= \epsilon(k_1)^{-\frac{1}{2}} \epsilon(k_2)^{-\frac{1}{2}} \left| \frac{k_1^2 - k_2^2}{\epsilon(k_1) + \epsilon(k_2)} \right| \\ &= |k_1 + k_2| \left| \frac{k_1 - k_2}{\epsilon(k_1) + \epsilon(k_2)} \right| \epsilon(k_1)^{-\frac{1}{2}} \epsilon(k_2)^{-\frac{1}{2}} \\ &\leq |k_1 + k_2| \epsilon(k_1)^{-\frac{1}{2}} \epsilon(k_2)^{-\frac{1}{2}}. \end{aligned}$$

Hence

$$(3.6) \quad |R(k_1, k_2)| \leq C |\widehat{V}'(k_1 + k_2)| \epsilon(k_1)^{-\frac{1}{2}} \epsilon(k_2)^{-\frac{1}{2}}.$$

Arguing for example as in [DG], we obtain that

$$(3.7) \quad \|R\|_{L^2(\mathbb{R}^2)} \leq C \|V'\|_{L^2(\mathbb{R})}.$$

Using now the N_τ estimates (see [GJ]), we obtain (3.1) for $Q^{a^* a^*}(V)$. The same estimate holds also for $Q^{a^* a}(V)$. \square

3.2. Coupling constant. Let us fix a potential $V \in L^\infty(\mathbb{R})$ with $V' \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. We set:

$$(3.8) \quad (\lambda_{\text{quant}})^{-1} := \frac{1}{2} \|\epsilon^{-1}V + V\epsilon^{-1}\|_{B(L^2(\mathbb{R}))} + \frac{1}{m} \|\epsilon^{-\frac{1}{2}}[V, \epsilon]\epsilon^{-\frac{1}{2}}\|_{\text{HS}}.$$

Lemma 3.2. *Assume that $|\lambda| < \lambda_{\text{quant}}$. Then:*

(1) *there exists $0 \leq \delta < 1$ and $C \geq 0$ such that*

$$\pm \lambda Q(V) \leq \delta d\Gamma(\omega) + C.$$

(2) *there exists $c > 0$ such that*

$$\omega_{\lambda V} := \begin{bmatrix} \epsilon & \lambda b \\ \lambda b^* & \epsilon \end{bmatrix} \geq c \mathbb{1}.$$

Proof. Set $c_0 = \|\epsilon^{-\frac{1}{2}}b\epsilon^{-\frac{1}{2}}\| = \frac{1}{2} \|V\epsilon^{-1} + \epsilon^{-1}V\|$. Clearly

$$\pm \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \leq c_0 \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix},$$

hence

$$\pm Q^{a^* a}(V) \leq c_0 d\Gamma(\omega).$$

From the N_τ estimates (see eg [GJ]) we get that

$$\pm(Q^{a^* a^*}(V) + Q^{a^* a}(V)) \leq c_1(N + 1) \leq c_1 m^{-1}(d\Gamma(\omega) + m),$$

for

$$c_1 = 2\|R(\cdot, \cdot)\|_{L^2(\mathbb{R}^2)},$$

for $R(k_1, k_2)$ defined in (3.5). Changing k_2 to $-k_2$ and using (1.5), we see that

$$c_1 = \|\epsilon^{\frac{1}{2}}V\epsilon^{-\frac{1}{2}} - \epsilon^{-\frac{1}{2}}V\epsilon^{\frac{1}{2}}\|_{\text{HS}} = \|\epsilon^{-\frac{1}{2}}[V, \epsilon]\epsilon^{-\frac{1}{2}}\|_{\text{HS}}.$$

These estimates clearly imply the lemma. \square

4. CHARGED $P(\varphi)_2$ HAMILTONIANS

In this section we construct the charged $P(\varphi)_2$ Hamiltonians formally defined in (2.14). We also prove some resolvent estimates known as *higher order estimates*.

4.1. Charged $P(\varphi)_2$ Hamiltonians. Let

$$P(\lambda_1, \lambda_2) = \sum_{|\alpha|=0}^{\deg P} a_\alpha \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}$$

be a real bounded below polynomial on \mathbb{R}^2 . Clearly P is bounded below iff $\deg P = 2m$ is even and

$$\inf_{\theta \in [0, 2\pi[} \sum_{|\alpha|=2n} a_\alpha \cos \theta^{\alpha_1} \sin \theta^{\alpha_2} > 0.$$

Let also $g \in L^2(\mathbb{R})$ be a real function. We consider the interaction term

$$H_I = \int_{\mathbb{R}} g(x) :P(\varphi_1(x), \varphi_2(x)) : dx,$$

where $\varphi_i(x)$ are defined in Subsect. 3.1 and $:$ denotes the Wick ordering.

By the usual arguments (see eg [GJ], [DG]) one can show that H_I is a *Wick polynomial*, ie a finite sum of terms of the form:

$$\text{Wick}(w_{p,q}) = \int_{\mathbb{R}^{p+q}} w_{p,q}(k_1, \dots, k_p, k'_1, \dots, k'_q) \prod_1^p a_{s_i}^*(k_i) \prod_1^q a_{r_j}(k'_j) dK dK',$$

where $s_i, r_j \in \{1, 2\}$ and the kernels $w_{p,q}$ are in $L^2(\mathbb{R}^{p+q})$.

Using the N_τ estimates (see eg [GJ, DG]) one can prove that H_I is a symmetric operator on $\text{Dom} N^m$.

Proposition 4.1. *Assume that $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \geq 0$. Then*

- (1) $H_0 + H_I$ is essentially selfadjoint on $\text{Dom} H_0 \cap \text{Dom} H_I$.
- (2) the operator $H_1 = \overline{H_0 + H_I}$ is bounded below.
- (3) for any $0 < \epsilon$ there exists C_ϵ such that

$$H_0 \leq (1 + \epsilon)H_1 + C_\epsilon.$$

Proof. The proof is an immediate modification of arguments in the standard $P(\varphi)_2$ model. One introduces the Q -space representation associated to the canonical conjugation $F \mapsto \overline{F}$ on $L^2(\mathbb{R}; \mathbb{C}^2)$, which allows to identify $\Gamma_s(\mathfrak{h})$ with $L^2(Q, d\mu)$ for a probability measure μ . The operator H_I can be seen as a multiplication operator on $L^2(Q, d\mu)$ such that $H_I \in L^p(Q)$ for some $p > 2$ and $e^{-tV} \in L^1(Q)$ for some $t > 0$. To obtain the second estimate one uses the fact that $g \geq 0$ and P is bounded below. Using then that e^{-tH_0} is hypercontractive,

one obtains (1) and (2). The same argument show that for any $\epsilon > 0$ $\epsilon H_0 + H_I$ is bounded below, which implies (3). \square

The following *higher order estimates* are easily seen to hold for H_1 , with the same proof as in usual $P(\varphi)_2$ Hamiltonians.

Proposition 4.2. *Assume that $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \geq 0$. Then there exists $b > 0$ such that for all $\alpha \in \mathbb{N}$, the following higher order estimates hold:*

$$(4.1) \quad \begin{aligned} \|N^\alpha(H_1 + b)^{-\alpha}\| &< \infty, \\ \|H_0 N^\alpha(H_1 + b)^{-m-\alpha}\| &< \infty, \\ \|N^\alpha(H_1 + b)^{-1}(N + 1)^{1-\alpha}\| &< \infty. \end{aligned}$$

Theorem 4.3. *Assume that $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $g \geq 0$, $V \in L^\infty(\mathbb{R})$, $V' \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$.*

- (1) *for any λ with $|\lambda| < \lambda_{\text{quant}}$, the quadratic form $H_0 + \lambda Q(V) + H_I$ with domain $\text{Dom}H_0 \cap \text{Dom}N^m$ is closeable and bounded below,*
- (2) *the domain of the closure of the above quadratic form equals $\text{Dom}|H_1|^{\frac{1}{2}}$,*
- (3) *The associated bounded below selfadjoint operator will be denoted by H and called a charged $P(\varphi)_2$ Hamiltonian.*

Proof. From Lemma 3.2 we know that if $|\lambda| < \lambda_{\text{quant}}$ then $\pm\lambda Q(V) \leq \delta H_0 + C$, for some $0 < \delta < 1$. By (3) of Prop. 4.1, this implies that as quadratic form $Q(V)$ is H_1 -bounded with relative bound strictly less than 1. The theorem follows then from the KLMN theorem. \square

4.2. Higher order estimates and essential selfadjointness. In this subsection we check that the higher order estimates of Prop. 4.2 extend to the full Hamiltonian H . As a consequence we will find a suitable core for H .

Proposition 4.4. *Assume that $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $g \geq 0$, $V \in L^\infty(\mathbb{R})$, $V' \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Let $|\lambda| < \lambda_{\text{quant}}$ and H the charged $P(\varphi)_2$ Hamiltonian constructed in Thm. 4.3. Then there exists $b > 0$ such that for all $\alpha \in \mathbb{N}$, the following higher order estimates hold:*

$$(4.2) \quad \begin{aligned} \|N^\alpha(H + b)^{-\alpha}\| &< \infty, \\ \|H_0 N^\alpha(H + b)^{-m-\alpha}\| &< \infty, \\ \|N^\alpha(H + b)^{-1}(N + 1)^{1-\alpha}\| &< \infty. \end{aligned}$$

Corollary 4.5. *The Hamiltonian H is essentially selfadjoint on $\text{Dom}H_0 \cap \text{Dom}N^m$. Consequently:*

$$H = (H_0 + \lambda Q(V) + H_I)^{\text{cl}} = (d\Gamma(\omega_{\lambda V}) + \lambda Q^{a^* a^*}(V) + \lambda Q^{a a}(V) + H_I)^{\text{cl}},$$

where we recall that:

$$(4.3) \quad \omega_{\lambda V} := \begin{bmatrix} \epsilon & \lambda b \\ \lambda b^* & \epsilon \end{bmatrix}.$$

Proof. It follows from Prop. 4.4 that for p large enough $\text{Dom}H^p \subset \text{Dom}(H_0) \cap \text{Dom}N^m$. This implies the corollary since $\text{Dom}H^p$ is a core for H . \square

In the rest of this subsection we will explain the proof of Prop. 4.4, which is a rather easy adaptation of the standard proof by Rosen [Ro]. We will only give the main steps, referring the reader for example to [DG, Sect. 7] for details.

Lattices

The proof in [Ro] relies on the introduction of a family H_n of (volume and ultra-violet) cutoff Hamiltonians. These Hamiltonians are obtained by considering an increasing sequence $\mathfrak{h}_n \subset \mathfrak{h}$ of finite dimensional subspaces of \mathfrak{h} such that $\bigcup_{n \in \mathbb{N}} \mathfrak{h}_n$ is dense in \mathfrak{h} . Moreover one assumes that the isometric projections $\pi_n : \mathfrak{h} \rightarrow \mathfrak{h}_n$ commute with the conjugation $F \mapsto \overline{F}$ on $\mathfrak{h} = L^2(\mathbb{R}; \mathbb{C}^2)$.

The subspaces \mathfrak{h}_n are defined as follows: for $v \gg 1$, consider the lattice $v^{-1}\mathbb{Z}$ and let

$$\mathbb{R} \ni k \mapsto [k]_v \in v^{-1}\mathbb{Z}$$

be the integer part mod $v^{-1}\mathbb{Z}$. For $\gamma \in v^{-1}\mathbb{Z}$, let $e_\gamma(k) = v^{\frac{1}{2}} \mathbb{1}_{[-(2v)^{-1}, (2v)^{-1}]}(k - \gamma)$. Set also for $\kappa \gg 1$ $\Gamma_{\kappa, v} = \{\gamma \in v^{-1}\mathbb{Z} : |\gamma| \leq \kappa\}$, and let

$$\mathfrak{h}_{\kappa, v} := \text{Span}\{e_\gamma \oplus 0, 0 \oplus e_\gamma : \gamma \in \Gamma_{\kappa, v}\}.$$

We choose then a sequence (κ_n, v_n) tending to (∞, ∞) in such a way that $\Gamma_{\kappa_n, v_n} \subset \Gamma_{\kappa_{n+1}, v_{n+1}}$ and set $\mathfrak{h}_n := \mathfrak{h}_{\kappa_n, v_n}$.

Cutoff Hamiltonians

Let us explain how to define the associated cutoff Hamiltonians. Since $\mathfrak{h} = \mathfrak{h}_n \oplus \mathfrak{h}_n^\perp$, there exists by the exponential law a unitary map $U_n : \Gamma_s(\mathfrak{h}_n) \otimes \Gamma_s(\mathfrak{h}_n^\perp) \rightarrow \Gamma_s(\mathfrak{h})$. If W is a bounded operator on $\Gamma_s(\mathfrak{h})$, one can define its projection to $\Gamma_s(\mathfrak{h}_n)$:

$$(4.4) \quad \Pi_n W := U_n (\Gamma(\pi_n) W \Gamma(\pi_n)^* \otimes \mathbb{1}) U_n^{-1}.$$

This definition extends to Wick polynomials, for example if $W = \prod_1^p a^*(F_i) \prod_1^q a(G_i)$, then:

$$\Pi_n W = \prod_1^p a^*(\pi_n^* \pi_n F_i) \prod_1^q a(\pi_n^* \pi_n G_i).$$

We set now:

$$H_{0, n} := d\Gamma(\epsilon_n \oplus \epsilon_n), \quad H_{I, n} := \Pi_n H_I, \quad Q_n(V) := \Pi_n Q(V),$$

where $\Pi_n W$ is defined in (4.4) and

$$\epsilon_n(k) = \epsilon([k]_{v_n}),$$

in the momentum space representation. Note that $\epsilon_n \oplus \epsilon_n$ commutes with $\pi_n^* \pi_n$. The construction of the cutoff Hamiltonians H_n is done in the next proposition.

Proposition 4.6. (1) *Let $|\lambda| < \lambda_{\text{quant}}$. Then there exists $0 < \delta < 1$ and $C > 0$ such that uniformly for n large enough:*

$$\pm \lambda Q_n(V) \leq \delta H_{0, n} + C.$$

- (2) the Hamiltonian $H_{1,n} = H_{0,n} + H_{I,n}$ is essentially selfadjoint on $\text{Dom}d\Gamma(\omega) \cap \text{Dom}N^m$ and there exists $b > 0$ such that

$$0 \leq H_{1,n} + b, \quad \forall n \in \mathbb{N}.$$

- (3) there exists $0 < \delta < 1$ and $b > 0$ such that

$$\pm \lambda Q_n(V) \leq \delta(H_{1,n} + b), \quad \forall n \in \mathbb{N}.$$

- (4) Let H_n the bounded below selfadjoint operator associated to the quadratic form $H_{1,n} + \lambda Q_n(V)$ with domain $\text{Dom}|H_{1,n}|^{\frac{1}{2}}$. Then

$$s\text{-}\lim_{n \rightarrow \infty} (H_n + b)^{-1} = (H + b)^{-1},$$

where H is the charged $P(\varphi)_2$ Hamiltonian defined in Thm. 4.3.

To prove (1) we note that (modulo the trivial factors U_n):

$$(4.5) \quad \Pi_n W = \Gamma(\pi_n^* \pi_n) W \Gamma(\pi_n^* \pi_n).$$

Since $|\lambda| < \lambda_{\text{quant}}$ there exists $0 < \delta < 1$ and $C > 0$ such that $\lambda Q(V) \leq \delta d\Gamma(\epsilon \oplus \epsilon) + C$. Using (4.5) we get that

$$\lambda Q_n(V) \leq \delta \Gamma(\pi_n^* \pi_n) d\Gamma(\epsilon \oplus \epsilon) \Gamma(\pi_n^* \pi_n) + C \leq \delta d\Gamma(\epsilon \oplus \epsilon) + C,$$

since $\pi_n^* \pi_n$ commutes with $\epsilon \oplus \epsilon$. Clearly for any $\alpha > 0$ one has

$$(1 + \alpha)^{-1} \epsilon_n \leq \epsilon \leq (1 + \alpha) \epsilon_n, \quad \text{if } n \text{ is large enough.}$$

This implies (1). Statement (2) is standard (see eg [DG, Sect. 7]). It follows also that for any $\epsilon > 0$ there exists C_ϵ such that uniformly in n :

$$H_{0,n} \leq (1 + \epsilon) H_{1,n} + C_\epsilon,$$

which implies (3). It remains to prove (4). Since $Q_n(V)$ are uniformly $H_{1,n}$ -form bounded with relative bound strictly less than 1, there exists $b \gg 1$ such that $(H_{1,n} + b)^{-\frac{1}{2}} \lambda Q_n(V) (H_{1,n} + b)^{-\frac{1}{2}}$ has norm less than some $\delta < 1$ uniformly in n , and:

$$(H_n + b)^{-1} = (H_{1,n} + b)^{-\frac{1}{2}} (\mathbb{1} + (H_{1,n} + b)^{-\frac{1}{2}} \lambda Q_n(V) (H_{1,n} + b)^{-\frac{1}{2}})^{-1} (H_{1,n} + b)^{-\frac{1}{2}},$$

It follows that

$$(H_n + b)^{-1} = \sum_{k=0}^{+\infty} (H_{1,n} + b)^{-1} (-\lambda Q_n(V) (H_{1,n} + b)^{-1})^k$$

as a norm convergent series. The same formula holds for $(H + b)^{-1}$. Therefore it suffices to prove that for all $k \in \mathbb{N}$:

$$(4.6) \quad s\text{-}\lim_{n \rightarrow \infty} (H_{1,n} + b)^{-1} (Q_n(V) (H_{1,n} + b)^{-1})^k = (H_1 + b)^{-1} (Q(V) (H_1 + b)^{-1})^k.$$

The arguments in [SHK, Prop. 4.8] easily extend to yield that

$$(4.7) \quad (H_{1,n} + b)^{-1} \rightarrow (H_1 + b)^{-1} \text{ in norm.}$$

(Note that H_1 is essentially a standard $P(\varphi)_2$ Hamiltonian). Moreover

$$(4.8) \quad \sup_{n \in \mathbb{N}} \|N(H_{1,n} + b)^{-1}\| < \infty.$$

This implies using Lemma 3.1 that $Q_n(V)(H_{1,n} + b)^{-1}$ is uniformly bounded. Hence (4.6) will follow from

$$(4.9) \quad s\text{-}\lim_{n \rightarrow \infty} (H_1 + b)^{-1} (Q_n(V)(H_1 + b)^{-1})^k = (H_1 + b)^{-1} (Q(V)(H_1 + b)^{-1})^k.$$

Now $Q_n(V)(H_1 + b)^{-1}$ is uniformly bounded and converges strongly to $Q(V)(H_1 + b)^{-1}$, which implies (4.9). This completes the proof of the proposition. \square

Proof of Prop. 4.4. The key point of the proof of the higher order estimates is to consider the multicommutators:

$$R_{i,n}(k_1, \dots, k_p) := \text{ad}_{a_i(k_1)} \cdots \text{ad}_{a_i(k_p)}(H_{I,n} + Q_n(V)), \quad k_1, \dots, k_p \in \mathbb{R},$$

for $i = 1, 2$ where $\text{ad}_A B = [A, B]$. The key step is then to prove that there exists $b > 0$ such that for all $\lambda_1, \lambda_2 \geq b$ one has:

$$(4.10) \quad \|(H_n + \lambda_1)^{-\frac{1}{2}} R_{i,n}(k_1, \dots, k_p) (H_n + \lambda_2)^{-\frac{1}{2}}\| \leq C_p \prod_1^p F_n(k_i),$$

where

$$(4.11) \quad \sup_{n \in \mathbb{N}} \int_{\mathbb{R}} |F_n(k)|^2 \epsilon(k)^{-\delta} dk < \infty, \quad \forall \delta > 0.$$

Note that it is only necessary to bound multicommutators with $a_i(k)$ for a fixed $i = 1, 2$. Indeed this follows from the fact that it suffices to prove the higher order estimates with N, H_0 replaced by $N_i, H_{0,i}$ for:

$$N_i = \int_{\mathbb{R}} a_i^*(k) a_i(k) dk, \quad H_{0,i} = \int_{\mathbb{R}} \epsilon(k) a_i^*(k) a_i(k) dk.$$

We first note that since $H_{0,n} \leq C(H_n + b)$ uniformly in n , it suffices to prove (4.10) with $(H_{0,n} + \lambda)^{-\frac{1}{2}}$ instead of $(H_n + \lambda)^{-\frac{1}{2}}$.

Clearly the multicommutator $R_i(\cdots)$ is the sum of the two multicommutators with $H_{I,n}$ and $Q_n(V)$. The multicommutators with $H_{I,n}$ are estimated as in [Ro], [DG], yielding:

$$(4.12) \quad \|(H_0 + \lambda_1)^{-\frac{1}{2}} \text{ad}_{a_i(k_1)} \cdots \text{ad}_{a_i(k_p)} H_{I,n} (H_0 + \lambda_2)^{-\frac{1}{2}}\| \leq C_p \prod_1^p \epsilon(k_i)^{-\frac{1}{2}},$$

so that (4.11) is satisfied.

Let us now estimate the multicommutators with $Q_n^{a^* a}(V)$. Abusing notation, we will still denote by π_n the projection from $L^2(\mathbb{R})$ onto $\text{Span}\{e_\gamma : \gamma \in \Gamma_{\kappa_n, v_n}\}$. Let $b_n = \pi_n^* \pi_n b \pi_n^* \pi_n$, where b is the operator defined in (3.4) and denote also by $b_n(k_1, k_2)$ its kernel in the momentum representation. Then

$$\text{ad}_{a_1(k)} Q_n^{a^* a}(V) = a_2(b_n(k, \cdot)),$$

and the similar formula with the indices 1 and 2 exchanged. Using the well known estimate

$$(4.13) \quad \|a(f)(d\Gamma(b) + 1)^{-\frac{1}{2}}\| \leq \|b^{-\frac{1}{2}} f\|,$$

for $b \geq 0$, we get

$$\|\text{ad}_{a_1(k)} Q_n^{a^* a}(V)(H_{0,n} + b)^{-\frac{1}{2}}\| \leq \|\epsilon_n(\cdot)^{-\frac{1}{2}} b_n(k, \cdot)\|_{L^2(\mathbb{R})} =: F_n(k),$$

hence to prove (4.11) it suffices to show that $\epsilon(k_1)^{-\delta/2}\epsilon(k_2)^{-\frac{1}{2}}b_n(k_1, k_2) \in L^2(\mathbb{R}^2)$ uniformly in n . This is equivalent to the fact that $\epsilon^{-\delta/2}b_n\epsilon^{-\frac{1}{2}}$ is Hilbert-Schmidt uniformly in n . Clearly this is true if $\epsilon^{-\delta/2}b\epsilon^{-\frac{1}{2}}$ is Hilbert-Schmidt. Working in the momentum representation we need to consider the integrals:

$$I_1 = \int \epsilon(k)^{-1-\delta}|\widehat{V}|^2(k' - k)dkdk',$$

$$I_2 = \int \epsilon(k)^{1-\delta}\epsilon(k')^{-2}|\widehat{V}|^2(k' - k)dkdk'.$$

I_1 is clearly convergent since $V \in L^2(\mathbb{R})$. To estimate I_2 , we use the Peetre inequality:

$$(4.14) \quad 1 + |x| \leq 2(1 + |x - y|)(1 + |y|), \quad x, y \in \mathbb{R},$$

and obtain that I_2 is convergent. In fact $\epsilon(k)^{(1-\delta)/2}\widehat{V} \in L^2(\mathbb{R})$ since $V' \in L^2(\mathbb{R})$.

Let us now consider the multicommutators with $Q_n^{a^* a^*}(V)$. Recall that the kernel $R(k_1, k_2)$ of $Q_n^{a^* a^*}(V)$ was defined in (3.5) and set $R_n = \Gamma(\pi_n^* \pi_n)R$. Then:

$$\text{ad}_{a_1(k)}Q_n^{a^* a^*}(V) = a_2^*(R_n(k, \cdot)).$$

Using again (4.13), we get that

$$\|(H_{0,n} + b)^{-\frac{1}{2}}\text{ad}_{a_1(k)}Q_n^{a^* a^*}(V)\| \leq \|\epsilon_n(\cdot)^{-\frac{1}{2}}R_n(k, \cdot)\|_{L^2(\mathbb{R})} =: F_n(k).$$

Now(4.11) follows from the fact that $R(k_1, k_2) \in L^2(\mathbb{R}^2)$, shown in (3.7).

The proof of the higher order estimates can now be completed as in [Ro], [DG]. In particular the strong resolvent convergence in Prop. 4.6 (4) is needed to apply the principle of cutoff independence in [Ro]. \square

5. SPECTRAL AND SCATTERING THEORY FOR CHARGED $P(\varphi)_2$ HAMILTONIANS

In this section we study the spectral and scattering theory of charged $P(\varphi)_2$ Hamiltonians. We will use the results of [GP]. In [GP], we introduced an abstract class of QFT Hamiltonians formally given by

$$H = d\Gamma(\omega) + \text{Wick}(w),$$

on a bosonic Fock space $\Gamma_s(\mathfrak{h})$, where $\omega \geq 0$ is a selfadjoint operator on the one-particle space \mathfrak{h} and $\text{Wick}(w)$ is a Wick polynomial associated to a kernel w .

Our main task in this section will be to explain how to fit charged $P(\varphi)_2$ Hamiltonians into the abstract framework of [GP] and to check the various abstract hypotheses there. The results on spectral and scattering theory are then obtained as simple applications of the general results of [GP].

5.1. Charged $P(\varphi)_2$ Hamiltonians as abstract QFT Hamiltonians. The class of abstract QFT Hamiltonians in [GP] is described in terms of three types of hypotheses, which will be briefly explained below.

Hypotheses on the Hamiltonian

One first requires (see [GP, Subsect. 3.1]) that the Hamiltonian H is the closure of $d\Gamma(\omega) + \text{Wick}(w)$ where $\text{Wick}(w)$ is a Wick polynomial with L^2 kernels and is bounded

below. This follows from Corollary 4.5. In our case we take for ω the operator $\omega_{\lambda V}$ defined in (4.3).

One also requires that $\omega \geq m_1 > 0$, which follows from Lemma 3.2.

Moreover one asks that any power of the number operator should be controlled by a sufficiently high power of the resolvent of H (see [GP, Subsect. 3.1]). This follows from the higher order estimates, which were proved in Prop. 4.4.

Hypotheses on the one-particle Hamiltonian

One requires that the one-particle energy ω has a sufficiently nice spectral and scattering theory. The precise statements can be found in [GP, Subsect. 3.2]. They are formulated in terms of two additional selfadjoint operators on the one-particle Hilbert space \mathfrak{h} .

The first one, denoted by $\langle X \rangle$ is called a *weight operator*, used to measure the propagation of one-particle states to infinity. The second, denoted by a is a *conjugate operator*, used in the Mourre commutator method. Moreover one introduces a dense subspace \mathcal{S} of \mathfrak{h} , preserved by the operators $\omega, a, \langle X \rangle$ on which (multi)-commutators between these three operators can be unambiguously defined.

To verify them in our case it is convenient to assume that the electrostatic potential is smooth. More precisely we will assume that $V \in S^{-\mu}(\mathbb{R})$, for some $\mu > 0$, where the classes $S^m(\mathbb{R})$ are defined in Subsect. 1.3.

The one-particle Hamiltonian in our case is $\omega_{\lambda V}$ defined in (4.3). For the weight operator, we choose:

$$\langle X \rangle := \begin{bmatrix} \langle x \rangle & 0 \\ 0 & \langle x \rangle \end{bmatrix},$$

and for the conjugate operator

$$a = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \quad c = \frac{1}{2} \left(x \cdot \frac{D_x}{\epsilon(D_x)} + \frac{D_x}{\epsilon(D_x)} \cdot x \right).$$

For the subspace \mathcal{S} we choose $\mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R})$ where $\mathcal{S}(\mathbb{R})$ is the Schwartz class.

Assuming that $V \in S^{-\mu}(\mathbb{R})$ for some $\mu > 0$, it is a tedious but straightforward exercise in pseudodifferential calculus to check that the hypotheses in [GP, Subsect. 3.2] are satisfied.

Hypotheses on the interaction

The final set of hypotheses concerns the kernel w of the interaction $\text{Wick}(w)$ (see [GP, Subsect. 3.3]). In our case they correspond to the fact that each kernel $w_{p,q}$, considered as an element of $\otimes^{p+q} L^2(\mathbb{R}; \mathbb{C}^2)$ should be in the domain of $d\Gamma(\langle x \rangle^s)$ for some $s > 1$.

The interaction term $\text{Wick}(w)$ is the sum of the $P(\varphi)_2$ interaction $\int_{\mathbb{R}} :P(\varphi_1(x), \varphi_2(x)) : dx$ and $\lambda Q^{a^* a^*}(V) + \lambda Q^{a a}(V)$.

For the first term, the hypotheses above are satisfied if $\langle x \rangle^s g \in L^2(\mathbb{R})$ (see [DG, Subsect. 6.3]).

Lemma 5.1. *Assume that $V \in S^{-\mu}(\mathbb{R})$ for some $\mu > \frac{1}{2}$. Let $R(k_1, k_2)$ be the kernel of $Q^{a^* a^*}(V)$ and $Q^{a a}(V)$ defined in (3.5). Then $|D_{k_i}|^s R \in L^2(\mathbb{R}^2)$ for $i = 1, 2$.*

Proof. Using (1.5), we see that $R(k_1, -k_2)$ is the distribution kernel of

$$\begin{aligned} & \frac{i}{2} \mathcal{F}(\epsilon(D_x)^{\frac{1}{2}} V \epsilon(D_x)^{-\frac{1}{2}} - \epsilon(D_x)^{-\frac{1}{2}} V \epsilon(D_x)^{\frac{1}{2}}) \mathcal{F}^{-1} \\ &= \frac{1}{2} \mathcal{F}(\epsilon(D_x)^{-\frac{1}{2}} [\epsilon(D_x), iV] \epsilon(D_x)^{-\frac{1}{2}}) \mathcal{F}^{-1}. \end{aligned}$$

We need to prove that $|D_{k_1}|^s R(k_1, -k_2) \in L^2(\mathbb{R}^2)$ or equivalently that the operator $C = \langle x \rangle^s \epsilon(D_x)^{-\frac{1}{2}} [\epsilon(D_x), V] \epsilon(D_x)^{-\frac{1}{2}}$ is Hilbert Schmidt on $L^2(\mathbb{R})$.

From the pseudodifferential calculus, we obtain that $C = \text{Op}^w(c)$, where $c(x, k)$ is a symbol satisfying:

$$|\partial_x^\alpha \partial_k^\beta c(x, k)| \leq C_{\alpha, \beta} \langle x \rangle^{-\mu-1+s} \langle k \rangle^{-1-\beta}, \quad \alpha, \beta \in \mathbb{N},$$

and $\text{Op}^w(a)$ denotes the Weyl quantization of a . Using (1.6) we see the conclusion of the lemma holds if $\mu > \frac{1}{2}$. \square

5.2. Spectrum of charged $P(\varphi)_2$ Hamiltonians. In the rest of this section we assume:

$$\langle x^s \rangle g \in L^2(\mathbb{R}), \quad g \in L^1(\mathbb{R}), \quad g \geq 0, \quad V \in S^{-s}(\mathbb{R}), \quad \text{for some } s > 1.$$

Moreover we assume as before that:

$$|\lambda| < \lambda_{\text{quant}}.$$

Theorem 5.2 (HVZ Theorem). *One has*

$$\sigma_{\text{ess}}(H) = [\inf \sigma(H) + m, +\infty[.$$

Consequently H has a ground state.

The theorem follows from [GP, Thm. 7.1] and the fact that $\sigma_{\text{ess}}(\omega_{\lambda V}) = [m, +\infty[$.

5.3. Asymptotic fields. For $F \in \mathfrak{h}$ we set $F_t := e^{-it\omega} h$. The results of this subsection follow from [GP, Thm. 4.1], taking into account [GP, Remark 4.2]. The fact that $\omega_{\lambda V}$ restricted to its continuous spectral subspace is unitarily equivalent to ω follow easily from standard two-body scattering theory, using that $V \in S^{-s}(\mathbb{R})$ for $s > 1$.

Theorem 5.3. (1) *for all $F \in \mathfrak{h}$ the strong limits*

$$(5.1) \quad W^\pm(F) := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} W(F_t) e^{-itH}$$

exist. They are called the asymptotic Weyl operators.

(2) *the map*

$$(5.2) \quad \mathfrak{h} \ni F \mapsto W^\pm(F)$$

is strongly continuous.

(3) *the operators $W^\pm(F)$ satisfy the Weyl commutation relations:*

$$W^\pm(F) W^\pm(G) = e^{-i\frac{1}{2}\text{Im}(F|G)} W^\pm(F + G).$$

(4) *the Hamiltonian preserves the asymptotic Weyl operators:*

$$(5.3) \quad e^{itH} W^\pm(F) e^{-itH} = W^\pm(F_{-t}).$$

5.4. Wave operators and asymptotic completeness. For $F \in \mathfrak{h}$, let $a^{\pm\sharp}(F)$ the asymptotic creation/annihilation operators associated to the asymptotic Weyl operators (see eg [GP, Subsect. 8.1]). The following theorem describes the construction of *wave operators* and their main property, the *asymptotic completeness*.

Theorem 5.4. *Set:*

$$\begin{aligned}\Omega^\pm &: \mathcal{H}_{\text{pp}}(H) \otimes \Gamma_s(\mathfrak{h}) \rightarrow \Gamma_s(\mathfrak{h}) \\ \Psi \otimes \prod_1^n a^*(F_i)\Omega &\mapsto \prod_1^n a^{\pm*}(F_i)\Psi.\end{aligned}$$

The operators Ω^\pm are called the wave operators. Set also

$$H^\pm = H|_{\mathcal{H}_{\text{pp}}(H)} \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega), \text{ acting on } \mathcal{H}_{\text{pp}}(H) \otimes \Gamma_s(\mathfrak{h}).$$

The operators H^\pm are called the asymptotic Hamiltonians. Then:

- (1) Ω^\pm are unitary operators;
- (2) Ω^\pm intertwine the asymptotic Weyl operators with the Fock Weyl operators:

$$\Omega^\pm \mathbb{1} \otimes W(F) = W^\pm(F)\Omega^\pm, \quad \forall F \in \mathfrak{h},$$

- (3) Ω^\pm intertwine the asymptotic Hamiltonians with the Hamiltonian H :

$$\Omega^\pm H^\pm = H\Omega^\pm.$$

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