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Wavelet estimation of the derivatives of an unknown function from a convolution model

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Abstract We observe a stochastic process where a convolution product of an unknown function f and a known function g is corrupted by Gaussian noise. We wish to estimate the d-th derivatives of f from the observations. To reach this goal, we develop an adaptive estimator based on wavelet block thresholding. We prove that it achieves near optimal rates of convergence under the mean integrated squared error (MISE) over a wide range of smoothness classes.

Keywords Deconvolution · Derivatives function estimation · Wavelets · Block thresholding

1 Motivation

We observe the stochastic process $\{Y(t);\ t \in [0,1]\}$ where

$$dY(t) = (f \star g)(t)dt + \epsilon n^{-1/2} dW(t), \qquad t \in [0,1], \qquad n \in \mathbb{N}^*, \qquad (1)$$

$\epsilon > 0$ is a fixed constant, $(f \star g)$ is the convolution product:

$$(f \star g)(t) = \int_0^1 f(t-u)g(u)du,$$

$\{W(t);\ t \in [0,1]\}$ is a non-observed standard Brownian motion, f is an unknown function and g is a known function. We assume that f and g belong to $\mathbb{L}^2_{per}([0,1]) = \{h;\ \ h \text{ is 1-periodic on } [0,1] \text{ and } \int_0^1 h^2(t)dt < \infty\}$. The general goal is to estimate an unknown quantity depending on f from $\{Y(t);\ t \in [0,1]\}$. The convolution model (1) illustrates the action of a linear time-invariant system on an input signal f when the data are corrupted with

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additional noise. See, for instance, Bertero and Boccacci (1998) and Neelamani, Choi and Baraniuk (2004). This is a standard deconvolution problem in the field of function estimation. For related results on (1), we refer to Cavalier and Tsybakov (2002), Cavalier et al. (2004), Johnstone et al. (2004) and Cavalier (2008). Extensions of (1) can be found in Willer (2005), Cavalier and Raimondo (2007) and Pensky and Sapatinas (2009).

The estimation of f has received a lot of attention (see e.g. Cavalier and Tsybakov (2002), Johnstone et al. (2004) and Chesneau (2008)). In this paper, we focus on a more general problem: estimate the d-th derivative of f: $f^{(d)}$ with $d \in \mathbb{N}$ (we set $f^{(0)} = f$). This is of interest to detect possible bumps, concavity or convexity properties of f. For the standard nonparametric models (density, regression, ...), the estimation of $f^{(d)}$ has been investigated in several papers starting with Bhattacharya (1967). For references using wavelet methods, let us cite Prakasa Rao (1996), Chaubey and Doosti (2005) and Chaubey et al. (2006). However, to the best of our knowledge, the estimation of $f^{(d)}$ from (1) is a new challenge.

Considering the *ordinary smooth case* where the Fourier coefficients of g decrease in a polynomial fashion (to be described in (7)), we develop an adaptive wavelet estimator $\widehat{f}_{n,d}$ of $f^{(d)}$. It is constructed from a periodised Meyer wavelet basis and a block thresholding rule known under the named of BlockJS. This construction has been initially elaborated by Cai (1999) for the standard Gaussian noise model. Further details and recent developments on BlockJS can be found in Cavalier and Tsybakov (2001), Tsybakov (2004) and Chesneau et al. (2008).

To measure the performance of $\widehat{f}_{n,d}$, we consider the asymptotic minimax approach under the mean integrated squared error (MISE) over a wide range of smoothness spaces: the Besov balls. More precisely, we aim to evaluate the smallest bound w_n such that

$$\sup_{f \in B^s_{\pi,r}(M)} \mathbb{E}\left(\int_{-\infty}^{\infty} \left(\widehat{f}_{n,d}(x) - f^{(d)}(x) \right)^2 dx \right) \leq w_n,$$

where $B^s_{\pi,r}(M)$ is the Besov ball (to be defined in subsection 2.2). In this study, we obtain

$$w_n = \begin{cases} Cn^{-2s/(2s+2\delta+2d+1)}, & \text{if } \ \pi \geq 2, \\ C(\log n/n)^{2s/(2s+2\delta+2d+1)}, & \text{if } \ \pi \in [1,2), \ \ s > (1/\pi - 1/2)(2\delta + 2d + 1), \end{cases}$$

where $C > 0$ is a constant and δ is a parameter which refers to the ordinary smooth assumption on g. We prove that w_n is near optimal via the determination of the lower bound. The proof of the upper bound uses a general theorem proved by Chesneau et al. (2008) and technical probability inequalities. The lower bound is proved by applying the Fano lemma.

The paper is organized as follows. In Section 2, we present wavelets and Besov balls. Section 3 clarifies the assumptions made on g and introduces some intermediate estimators. The BlockJS estimator is defined in Section 4. Section 5 is devoted to the results. The proofs are postponed in Section 6.

2 Wavelets and Besov balls

2.1 Wavelets

We consider an orthonormal wavelet basis generated by dilations and translations of a "father" Meyer-type wavelet ϕ and a "mother" Meyer-type wavelet ψ. The features of such wavelets are:

- the Fourier transforms of ϕ and ψ have bounded support. More precisely, we have

$$\begin{cases} \operatorname{supp}(\mathcal{F}(\phi)) \subset [-4\pi 3^{-1}, 4\pi 3^{-1}], \\ \operatorname{supp}(\mathcal{F}(\psi)) \subset [-8\pi 3^{-1}, -2\pi 3^{-1}] \cup [2\pi 3^{-1}, 8\pi 3^{-1}], \end{cases} \tag{2}$$

  where supp denotes the support and, for any $h \in \mathbb{L}^2_{per}([0,1])$, $\mathcal{F}(h)$ denotes the Fourier transform of h defined by

$$\mathcal{F}(h)(\ell) = \int_0^1 h(x) e^{-2i\pi\ell x} dx, \qquad \ell \in \mathbb{Z}.$$

- for any $\ell \in [-2\pi, -\pi] \cup [\pi, 2\pi]$, there exists a constant $c > 0$ such that

$$|\mathcal{F}(\psi)(\ell)| \geq c.$$

- (ϕ, ψ) is r-regular for a chosen $r \in \mathbb{N}$, i.e. $\phi \in \mathcal{C}^r$, $\psi \in \mathcal{C}^r$ and, for any $u \in \{0, \dots, r\}$,

$$\int_{-\infty}^{\infty} x^u \psi(x) dx = 0. \tag{3}$$

A consequence of (2) and (3) is that, for any $m \in \mathbb{N}$ and any $u \in \{0, \dots, r\}$,

$$\sup_{x\in\mathbb{R}} \left(\left|\phi^{(u)}(x)\right| \left(|x|^2+1\right)^m \right) < \infty, \qquad \sup_{x\in\mathbb{R}} \left(|\psi^{(u)}(x)| \left(|x|^2+1\right)^m \right) < \infty. \tag{4}$$

For the purposes of this paper, we use the periodised wavelet bases on the unit interval. For any $x \in [0,1]$, any integer j and any $k \in \{0, \dots, 2^j - 1\}$, let

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \qquad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

be the elements of the wavelet basis, and

$$\phi^{per}_{j,k}(x) = \sum_{\ell\in\mathbb{Z}} \phi_{j,k}(x - l), \qquad \psi^{per}_{j,k}(x) = \sum_{\ell\in\mathbb{Z}} \psi_{j,k}(x - l),$$

their periodised versions. There exists an integer τ such that the collection ζ defined by

$$\zeta = \left\{ \phi^{per}_{\tau,k}(.), k \in \{0, \dots, 2^\tau - 1\};\ \psi^{per}_{j,k}(.), \quad j \geq \tau,\ k \in \{0, \dots, 2^j - 1\} \right\}$$

constitutes an orthonormal basis of $\mathbb{L}^2_{per}([0,1])$. In what follows, the superscript "per" will be suppressed from the notations for convenience.

Then, for any $m \geq \tau$, a function $h \in \mathbb{L}^2_{per}([0,1])$ can be expanded into a wavelet series as

$$h(x) = \sum_{k=0}^{2^m-1} \alpha_{m,k}\phi_{m,k}(x) + \sum_{j=m}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x), \qquad x \in [0,1],$$

where

$$\alpha_{m,k} = \int_0^1 h(t)\overline{\phi_{m,k}(t)}dt, \qquad \beta_{j,k} = \int_0^1 h(t)\overline{\psi_{j,k}(t)}dt. \tag{5}$$

For further details about Meyer-type wavelets and wavelet decomposition, see Cohen et al. (1993), Walter (1994) and Zayed and Walter (1996).

2.2 Besov balls

Let $M \in (0,\infty)$, $s \in (0,\infty)$, $\pi \in [1,\infty)$ and $r \in [1,\infty)$. Let us set $\beta_{\tau-1,k} = \alpha_{\tau,k}$. We say that a function h belongs to the Besov balls $B^s_{\pi,r}(M)$ if and only if there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients (5) satisfy

$$\left(\sum_{j=\tau-1}^{\infty} \left(2^{j(s+1/2-1/\pi)} \left(\sum_{k=0}^{2^j-1} |\beta_{j,k}|^\pi \right)^{1/\pi} \right)^r \right)^{1/r} \leq M^*. \tag{6}$$

For a particular choice of parameters s, π and r, these sets contain the Hölder and Sobolev balls. See Meyer (1992).

3 Preliminary study

3.1 Ordinary smooth assumption on g

We suppose that there exist three constants, $c > 0$, $C > 0$ and $\delta > 1$, such that, for any $\ell \in \mathbb{Z}$, the Fourier coefficient of g, i.e. $F(g)(\ell)$, satisfies

$$c\left(1 + |\ell|^2\right)^{-\delta/2} \leq |F(g)(\ell)| \leq C\left(1 + |\ell|^2\right)^{-\delta/2}. \tag{7}$$

For example, consider the square integrable 1-periodic function g defined by

$$g(x) = \sum_{m\in\mathbb{Z}} e^{-|x+m|}, \qquad x \in [0,1].$$

Then, for any $\ell \in \mathbb{Z}$, $F(g)(\ell) = 2\left(1 + 4\pi^2|\ell|^2\right)^{-1}$ and (7) is satisfied with $\delta = 2$.

Further examples can be found in Pensky and Vidakovic (1999) and Fan and Koo (2002).

3.2 Preliminary to the estimation of $f^{(d)}$

As in Johnstone et al. (2004), we write the model (1) in the Fourier domain. First of all, notice that, for any $\ell \in \mathbb{Z}$, $F\left(f \star g\right)(\ell) = F(f)(\ell)F(g)(\ell)$. Therefore, if we set

$$y_\ell = \int_0^1 e^{-2\pi i \ell t} dY(t), \qquad e_\ell = \int_0^1 e^{-2\pi i \ell t} dW(t),$$

it follows from (1) that

$$y_\ell = \mathcal{F}(f)(\ell)\mathcal{F}(g)(\ell) + \epsilon n^{-1/2} e_\ell.$$

Assume that $f^{(d)} \in \mathbb{L}^2_{per}([0,1])$. Then, for any $m \geq \tau$, $f^{(d)}$ can be expanded into a wavelet series as

$$f^{(d)}(x) = \sum_{k=0}^{2^m-1} \alpha_{m,k}\phi_{m,k}(x) + \sum_{j=m}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x), \qquad x \in [0,1],$$

where

$$\alpha_{m,k} = \int_0^1 f^{(d)}(t)\overline{\phi_{m,k}(t)}dt, \qquad \beta_{j,k} = \int_0^1 f^{(d)}(t)\overline{\psi_{j,k}(t)}dt.$$

Let us now investigate the estimation of $\beta_{j,k}$. Since f is 1-periodic, for any $u \in \{0, \ldots, d\}$, $f^{(u)}$ is 1-periodic and $f^{(u)}(0) = f^{(u)}(1)$. By d integrations by parts, for any $\ell \in \mathbb{Z}$, we have

$$\mathcal{F}\left(f^{(d)}\right)(\ell) = (2\pi i \ell)^d \mathcal{F}(f)(\ell).$$

The Plancherel-Parseval theorem gives

$$\begin{aligned}\beta_{j,k} &= \int_0^1 f^{(d)}(t)\overline{\psi_{j,k}(t)}dt = \sum_{\ell\in\mathbb{Z}} \mathcal{F}\left(f^{(d)}\right)(\ell)\overline{\mathcal{F}\left(\psi_{j,k}\right)(\ell)} \\ &= \sum_{\ell\in\mathbb{Z}} (2\pi i \ell)^d \mathcal{F}(f)(\ell)\overline{\mathcal{F}\left(\psi_{j,k}\right)(\ell)}.\end{aligned}$$

Therefore, if we set

$$\widehat{\beta}_{j,k} = \sum_{\ell\in\mathbb{Z}} (2\pi i \ell)^d \frac{\overline{\mathcal{F}\left(\psi_{j,k}\right)(\ell)}}{\mathcal{F}(g)(\ell)} y_\ell,$$

then

$$\begin{aligned}\widehat{\beta}_{j,k} &= \sum_{\ell\in\mathbb{Z}} (2\pi i \ell)^d \mathcal{F}(f)(\ell)\overline{\mathcal{F}\left(\psi_{j,k}\right)(\ell)} + \epsilon n^{-1/2} \sum_{\ell\in\mathbb{Z}} (2\pi i \ell)^d \frac{\overline{\mathcal{F}\left(\psi_{j,k}\right)(\ell)}}{\mathcal{F}(g)(\ell)} e_\ell \\ &= \beta_{j,k} + \epsilon n^{-1/2} \sum_{\ell\in\mathbb{Z}} (2\pi i \ell)^d \frac{\overline{\mathcal{F}\left(\psi_{j,k}\right)(\ell)}}{\mathcal{F}(g)(\ell)} e_\ell.\end{aligned}$$

Since $(e_\ell)_{\ell\in\mathbb{Z}}$ are i.i.d. $\mathcal{N}(0,1)$, $\widehat{\beta}_{j,k}$ is an unbiased estimator of $\beta_{j,k}$ with distribution

$$\mathcal{N}\left(\beta_{j,k}, \epsilon^2 n^{-1}\sum_{\ell\in\mathbb{Z}}(2\pi\ell)^{2d}\frac{|\mathcal{F}\left(\psi_{j,k}\right)(\ell)|^2}{|\mathcal{F}(g)(\ell)|^2}\right).$$

4 BlockJS estimator

We use the notations introduced in subsection 3.2. We suppose that $f^{(d)} \in \mathbb{L}^2_{per}([0,1])$ and that (7) is satisfied (δ refers to this assumption). We now present the considered adaptive procedure for the estimation of $f^{(d)}$. Let j_1 and j_2 be the integers defined by

$$j_1 = \lfloor \log_2(\log n)\rfloor, \qquad j_2 = \lfloor (1/(2\delta+2d+1))\log_2(n/\log n)\rfloor,$$

where, for any $a\in\mathbb{R}$, $\lfloor a\rfloor$ denotes the whole number part of a. For any $j\in\{j_1,\ldots,j_2\}$, set $L=\lfloor \log n\rfloor$ and $A_j=\{1,\ldots,2^jL^{-1}\}$. For any $K\in A_j$, we consider the set

$$B_{j,K}=\{k\in\{0,\ldots,2^j-1\};\ (K-1)L\le k\le KL-1\}.$$

We define the Block James Stein estimator (BlockJS) by

$$\widehat{f}_{n,d}(x)=\sum_{k=0}^{2^{j_1}-1}\widehat{\alpha}_{j_1,k}\phi_{j_1,k}(x)+\sum_{j=j_1}^{j_2}\sum_{K\in A_j}\sum_{k\in B_{j,K}}\widehat{\beta}^*_{j,k}\psi_{j,k}(x), \qquad x\in[0,1], \quad (8)$$

where

$$\widehat{\beta}^*_{j,k}=\widehat{\beta}_{j,k}\left(1-\frac{\lambda\epsilon^2n^{-1}2^{2j(\delta+d)}}{\frac{1}{L}\sum_{k\in B_{j,K}}|\widehat{\beta}_{j,k}|^2}\right)_+,$$

with, for any $a\in\mathbb{R}$, $(a)_+=\max(a,0)$, $\lambda>0$, and

$$\widehat{\alpha}_{j_1,k}=\sum_{\ell\in\mathcal{D}_{j_1}}(2\pi i\ell)^d\frac{\overline{\mathcal{F}\left(\phi_{j_1,k}\right)(\ell)}}{\mathcal{F}(g)(\ell)}y_\ell, \qquad \widehat{\beta}_{j,k}=\sum_{\ell\in\mathcal{C}_j}(2\pi i\ell)^d\frac{\overline{\mathcal{F}\left(\psi_{j,k}\right)(\ell)}}{\mathcal{F}(g)(\ell)}y_\ell. \quad (9)$$

Here,

$$\mathcal{D}_{j_1}=\mathrm{supp}\left(\mathcal{F}(\phi_{j_1,0})\right)=\mathrm{supp}\left(\mathcal{F}\left(\phi_{j_1,k}\right)\right), \quad \mathcal{C}_j=\mathrm{supp}\left(\mathcal{F}(\psi_{j,0})\right)=\mathrm{supp}\left(\mathcal{F}\left(\psi_{j,k}\right)\right).$$

For the original construction of BlockJS (i.e. in the standard Gaussian noise model), we refer to Cai (1999).

Remark 1 The sets A_j and $B_{j,K}$ are chosen such that $\bigcup_{K\in A_j}B_{j,K}=\{0,\ldots,2^j-1\}$, for any $(K,K')\in A_j^2$ with $K\ne K'$, $B_{j,K}\cap B_{j,K'}=\varnothing$ and $\mathrm{Card}(B_{j,K})=L=\lfloor\log n\rfloor$.

Remark 2 Notice that, thanks to (2), for any $j\in\{j_1,\ldots,j_2\}$, we have

$$\begin{cases}\mathcal{D}_{j_1}\subset[-4\pi3^{-1}2^{j_1},4\pi3^{-1}2^{j_1}],\\ \mathcal{C}_j\subset[-8\pi3^{-1}2^j,-2\pi3^{-1}2^j]\cup[2\pi3^{-1}2^j,8\pi3^{-1}2^j].\end{cases} \quad (10)$$

5 Main results

Theorem 1 below determines the rates of convergence achieved by $\widehat{f}_{n,d}$ under the MISE over Besov balls.

Theorem 1 *Consider the model (1) and recall that we want to estimate* $f^{(d)}$ *with* $d \in \mathbb{N}$*. Assume that* (ϕ, ψ) *is* r*-regular for some* $r \geq d$ *and (7) is satisfied. Let* $\widehat{f}_{n,d}$ *be the estimator defined by (8) with a large enough* λ*. Then there exists a constant* $C > 0$ *such that, for any* $M \in (0, \infty)$, $\pi \in [1, \infty)$, $r \in [1, \infty)$, $s \in (1/\pi, \infty)$ *and* n *large enough, we have*

$$\sup_{f^{(d)} \in B^s_{\pi,r}(M)} \mathbb{E}\left(\int_0^1 \left(\widehat{f}_{n,d}(x) - f^{(d)}(x) \right)^2 dx \right) \leq C\varphi_n,$$

where

$$\varphi_n = \begin{cases} n^{-2s/(2s+2\delta+2d+1)}, & \text{if } \pi \geq 2, \\ (\log n/n)^{2s/(2s+2\delta+2d+1)}, & \text{if } \pi \in [1,2), \ s > (1/\pi - 1/2)(2\delta + 2d + 1). \end{cases}$$

It is natural to address the following question: is it φ_n the optimal rate of convergence ? Theorem 2 below gives the answer.

Theorem 2 *Consider the model (1) and recall that we want to estimate* $f^{(d)}$ *with* $d \in \mathbb{N}$*. Assume that (7) is satisfied. Then there exists a constant* $c > 0$ *such that, for any* $M \in (0, \infty)$, $\pi \in [1, \infty)$, $r \in [1, \infty)$, $s \in (1/\pi, \infty)$ *and* n *large enough, we have*

$$\inf_{\widetilde{f}_{n,d}} \sup_{f^{(d)} \in B^s_{\pi,r}(M)} \mathbb{E}\left(\int_0^1 \left(\widetilde{f}_{n,d}(x) - f^{(d)}(x) \right)^2 dx \right) \geq c\varphi_n^*,$$

where

$$\varphi_n^* = n^{-2s/(2s+2\delta+2d+1)}.$$

Theorem 2 shows that the rate of convergence φ_n achieved by $\widehat{f}_{n,d}$ is near optimal. Near is only due to the case $\pi \in [1,2)$ and $s > (1/\pi - 1/2)(2\delta + 2d + 1)$ where there is an extra logarithmic term.

Theorems 1 and 2 prove that $\widehat{f}_{n,d}$ is near optimal in the minimax sense.

6 Proofs

In the following proofs, c and C denote positive constants which can take different values for each mathematical term.

Proof of Theorem 1. Theorem 1 can be proved by using a more general theorem: (Chesneau et al. 2008, Theorem 3.1). To apply this result, two conditions on the estimators (9) are required: a moment condition and a concentration condition. They are presented in the two propositions below.

Proposition 1 (Moment condition) *Consider the framework of Theorem 1. Then*

- *there exists a constant* $C > 0$ *such that, for any* $k \in \{0, \ldots, 2^{j_1} - 1\}$*, the estimator* $\widehat{\alpha}_{j_1,k}$ *defined by (9) satisfies*

$$\mathbb{E}\left(|\widehat{\alpha}_{j_1,k} - \alpha_{j_1,k}|^2\right) \le C\epsilon^2 2^{2(\delta+d)j_1} n^{-1},$$

- *there exists a constant* $C > 0$ *such that, for any* $j \in \{j_1, \ldots, j_2\}$ *and any* $k \in \{0, \ldots, 2^j - 1\}$*, the estimator* $\widehat{\beta}_{j,k}$ *defined by (9) satisfies*

$$\mathbb{E}\left(|\widehat{\beta}_{j,k} - \beta_{j,k}|^4\right) \le C\epsilon^4 2^{4(\delta+d)j} n^{-2}.$$

Proof of Proposition 1. Let us prove the second point, the first one can be proved in a similar way. For any $j \in \{j_1, \ldots, j_2\}$ and any $k \in \{0, \ldots, 2^j - 1\}$, we have

$$\widehat{\beta}_{j,k} - \beta_{j,k} = \epsilon n^{-1/2} \sum_{\ell \in \mathcal{C}_j} (2\pi i \ell)^d \frac{\overline{\mathcal{F}(\psi_{j,k})(\ell)}}{\mathcal{F}(g)(\ell)} e_\ell \sim \mathcal{N}\left(0, n^{-1}\sigma_{j,k}^2\right), \tag{11}$$

where

$$\sigma_{j,k}^2 = \epsilon^2 \sum_{\ell \in \mathcal{C}_j} (2\pi\ell)^{2d} \frac{|\mathcal{F}(\psi_{j,k})(\ell)|^2}{|\mathcal{F}(g)(\ell)|^2}. \tag{12}$$

Due to (7) and (10), we have

$$\sup_{\ell \in \mathcal{C}_j} \left(\frac{(2\pi\ell)^{2d}}{|\mathcal{F}(g)(\ell)|^2}\right) \le C \sup_{\ell \in \mathcal{C}_j} \left((2\pi\ell)^{2d}\left(1 + |\ell|^2\right)^\delta\right) \le C 2^{2(\delta+d)j}. \tag{13}$$

It follows from (13) and the Plancherel-Parseval theorem that

$$\begin{aligned} \sigma_{j,k}^2 &\le \epsilon^2 \sup_{\ell \in \mathcal{C}_j} \left(\frac{(2\pi\ell)^{2d}}{|\mathcal{F}(g)(\ell)|^2}\right) \sum_{\ell \in \mathcal{C}_j} |\mathcal{F}(\psi_{j,k})(\ell)|^2 \\ &\le C\epsilon^2 2^{2(\delta+d)j} \sum_{\ell \in \mathcal{C}_j} |\mathcal{F}(\psi_{j,k})(\ell)|^2 = C\epsilon^2 2^{2(\delta+d)j} \int_{-\infty}^{\infty} |\mathcal{F}(\psi_{j,k})(y)|^2 dy \\ &= C\epsilon^2 2^{2(\delta+d)j} \int_0^1 |\psi_{j,k}(x)|^2 dx = C\epsilon^2 2^{2(\delta+d)j}. \end{aligned} \tag{14}$$

Putting (11), (12) and (14) together, we obtain

$$\mathbb{E}\left(|\widehat{\beta}_{j,k} - \beta_{j,k}|^4\right) \le C(\epsilon^2 2^{2(\delta+d)j} n^{-1})^2 \le C\epsilon^4 2^{4(\delta+d)j} n^{-2}.$$

Proposition 1 is proved.

□

Proposition 2 (Concentration condition) *Consider the framework of Theorem 1. Then there exists a constant* $\lambda > 0$ *such that, for any* $j \in \{j_1, \dots, j_2\}$*, any* $K \in \mathcal{A}_j$ *and* n *large enough, the estimators* $\left(\widehat{\beta}_{j,k}\right)_{k \in B_{j,K}}$ *defined by (9) satisfy*

$$\mathbb{P}\left(\left(\sum_{k \in B_{j,K}} |\widehat{\beta}_{j,k} - \beta_{j,k}|^2\right)^{1/2} \geq \lambda 2^{(\delta+d)j} (\log n/n)^{1/2}\right) \leq n^{-2}.$$

Proof of Proposition 2. We need the Cirelson inequality presented in Lemma 1 below.

Lemma 1 (Cirelson, Ibragimov and Sudakov (1976)) *Let* D *be a subset of* $\mathbb{R}$ *and* $(\vartheta_t)_{t \in D}$ *be a centered Gaussian process. If*

$$\mathbb{E}\left(\sup_{t \in D} \vartheta_t\right) \leq N, \qquad \sup_{t \in D} \mathbb{V}(\vartheta_t) \leq V$$

then, for any $x > 0$*, we have*

$$\mathbb{P}\left(\sup_{t \in D} \vartheta_t \geq x + N\right) \leq \exp\left(-\frac{x^2}{2V}\right).$$

For the sake of simplicity, set

$$V_{j,k} = \widehat{\beta}_{j,k} - \beta_{j,k} = \epsilon n^{-1/2} \sum_{\ell \in \mathcal{C}_j} (2\pi i \ell)^d \frac{\overline{\mathcal{F}(\psi_{j,k})(\ell)}}{\mathcal{F}(g)(\ell)} e_\ell.$$

Recall that $V_{j,k} \sim \mathcal{N}\left(0, n^{-1}\sigma_{j,k}^2\right)$, where $\sigma_{j,k}^2$ is defined by (12). Consider the set Ω defined by $\Omega = \left\{a = (a_k) \in \mathbb{R};\ \sum_{k \in B_{j,K}} a_k^2 \leq 1\right\}$. For any $a \in \Omega$, let $Z(a)$ be the centered Gaussian process defined by

$$Z(a) = \sum_{k \in B_{j,K}} a_k V_{j,k} = \epsilon n^{-1/2} \sum_{\ell \in \mathcal{C}_j} (2\pi i \ell)^d \frac{e_\ell}{\mathcal{F}(g)(\ell)} \sum_{k \in B_{j,K}} a_k \overline{\mathcal{F}(\psi_{j,k})(\ell)}.$$

By an argument of duality, we have

$$\sup_{a \in \Omega} Z(a) = \left(\sum_{k \in B_{j,K}} |V_{j,k}|^2\right)^{1/2} = \left(\sum_{k \in B_{j,K}} |\widehat{\beta}_{j,k} - \beta_{j,k}|^2\right)^{1/2}.$$

Now, let us determine the values of N and V which appeared in the Cirelson inequality.

Value of N. Using the Hölder inequality and (14), we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{a\in\Omega} Z(a)\right) &= \mathbb{E}\left(\left(\sum_{k\in B_{j,K}} |V_{j,k}|^2\right)^{1/2}\right) \leq \left(\sum_{k\in B_{j,K}} \mathbb{E}\left(|V_{j,k}|^2\right)\right)^{1/2} \\ &\leq C\left(n^{-1}\sum_{k\in B_{j,K}} \sigma_{j,k}^2\right)^{1/2} \leq C\epsilon 2^{(\delta+d)j} n^{-1/2}\left(\mathrm{Card}(B_{j,K})\right)^{1/2} \\ &= C\epsilon 2^{(\delta+d)j}(\log n/n)^{1/2}.\end{aligned}$$

Hence $N = C\epsilon 2^{(\delta+d)j}(\log n/n)^{1/2}$.

Value of V. Since, for any $(\ell,\ell')\in\mathbb{Z}^2$,

$$\mathbb{E}\left(e_\ell\overline{e_{\ell'}}\right) = \int_0^1 e^{-2i\pi(\ell-\ell')t}dt \begin{cases}1 & \text{if } \ell=\ell' \\ 0 & \text{otherwise,}\end{cases}$$

it comes

$$\begin{aligned}&\sup_{a\in\Omega} \mathbb{V}(Z(a)) = \sup_{a\in\Omega}\mathbb{E}\left(\left|\sum_{k\in B_{j,K}} a_k V_{j,k}\right|^2\right) \\ &= \sup_{a\in\Omega}\mathbb{E}\left(\sum_{k\in B_{j,K}}\sum_{k'\in B_{j,K}} a_k a_{k'} V_{j,k}\overline{V_{j,k'}}\right) \\ &= \epsilon^2 n^{-1}\sup_{a\in\Omega}\sum_{k\in B_{j,K}}\sum_{k'\in B_{j,K}} a_k a_{k'}\sum_{\ell\in\mathcal{C}_j}\sum_{\ell'\in\mathcal{C}_j}\frac{(2\pi i\ell)^d}{\mathcal{F}(g)(\ell)}\mathcal{F}(\psi_{j,k})(\ell)\times \\ &\quad \overline{\frac{(2\pi i\ell')^d}{\mathcal{F}(g)(\ell')}}\overline{\mathcal{F}(\psi_{j,k'})(\ell')}\mathbb{E}\left(e_\ell\overline{e_{\ell'}}\right) \\ &= \epsilon^2 n^{-1}\sup_{a\in\Omega}\sum_{k\in B_{j,K}}\sum_{k'\in B_{j,K}} a_k a_{k'}\sum_{\ell\in\mathcal{C}_j}\frac{(2\pi\ell)^{2d}}{|\mathcal{F}(g)(\ell)|^2}\mathcal{F}\left(\psi_{j,k}\right)(\ell)\overline{\mathcal{F}(\psi_{j,k'})(\ell)} \\ &= \epsilon^2 n^{-1}\sup_{a\in\Omega}\sum_{\ell\in\mathcal{C}_j}\frac{(2\pi\ell)^{2d}}{|\mathcal{F}(g)(\ell)|^2}\left|\sum_{k\in B_{j,K}} a_k\mathcal{F}\left(\psi_{j,k}\right)(\ell)\right|^2. \qquad (15)\end{aligned}$$

For any $a \in \Omega$, the Plancherel-Parseval theorem gives

$$\begin{aligned} \sum_{\ell \in \mathcal{C}_j} \left| \sum_{k \in B_{j,K}} a_k \mathcal{F}(\psi_{j,k})(\ell) \right|^2 &= \sum_{\ell \in \mathcal{C}_j} \left| \mathcal{F}\left(\sum_{k \in B_{j,K}} a_k \psi_{j,k} \right)(\ell) \right|^2 \\ = \int_{-\infty}^{\infty} \left| \mathcal{F}\left(\sum_{k \in B_{j,K}} a_k \psi_{j,k} \right)(y) \right|^2 dy &= \int_0^1 \left| \sum_{k \in B_{j,K}} a_k \psi_{j,k}(x) \right|^2 dx \\ = \sum_{k \in B_{j,K}} a_k^2 \le 1. \end{aligned} \tag{16}$$

Putting (15), (13) and (16) together, we have

$$\begin{aligned} \sup_{a \in \Omega} \mathbb{V}(Z(a)) &\le C\epsilon^2 n^{-1} 2^{2(\delta+d)j} \sup_{a \in \Omega} \sum_{\ell \in \mathcal{C}_j} \left| \sum_{k \in B_{j,K}} a_k \mathcal{F}\left(\psi_{j,k}\right)(\ell) \right|^2 \\ &\le C\epsilon^2 n^{-1} 2^{2(\delta+d)j}. \end{aligned}$$

Hence $V = C\epsilon^2 n^{-1} 2^{2(\delta+d)j}$.

Taking λ large enough and $x = 2^{-1}\lambda\epsilon 2^{(\delta+d)j}(\log n/n)^{1/2}$, the Cirelson inequality described in Lemma 1 yields

$$\begin{aligned} &\mathbb{P}\left(\left(\sum_{k \in B_{j,K}} |V_{j,k}|^2 \right)^{1/2} \ge \lambda\epsilon 2^{(\delta+d)j}(\log n/n)^{1/2} \right) \\ &\le \mathbb{P}\left(\left(\sum_{k \in B_{j,K}} |V_{j,k}|^2 \right)^{1/2} \ge 2^{-1}\lambda\epsilon 2^{(\delta+d)j}(\log n/n)^{1/2} + N \right) \\ &= \mathbb{P}\left(\sup_{a \in \Omega} Z(a) \ge x + N \right) \le \exp\left(-x^2/(2V)\right) \le \exp\left(-C\lambda^2 \log n\right) \\ &\le n^{-2}. \end{aligned}$$

Proposition 2 is proved.

□

Putting Propositions 1 and 2 in (Chesneau et al. 2008, Theorem 3.1), we end the proof of Theorem 1.

□

Proof of Theorem 2. Let us now present a consequence of the Fano lemma.

Lemma 2 *Let* $m \in \mathbb{N}^*$ *and* A *be a sigma algebra on the space* Ω*. For any* $i \in \{0, \ldots, m\}$*, let* $A_i \in A$ *such that, for any* $(i, j) \in \{0, \ldots, m\}^2$ *with* $i \neq j$*,*

$$A_i \cap A_j = \varnothing.$$

Let $(\mathbb{P}_i)_{i \in \{0,\ldots,m\}}$ *be* $m+1$ *probability measures on* (Ω, A)*. Then*

$$\sup_{i \in \{0,\ldots,m\}} \mathbb{P}_i\left(A_i^c\right) \geq \min\left(2^{-1}, \exp(-3e^{-1})\sqrt{m}\exp(-\chi_m)\right),$$

where

$$\chi_m = \inf_{v \in \{0,\ldots,m\}} \frac{1}{m} \sum_{\substack{k \in \{0,\ldots,m\} \\ k \neq v}} K(\mathbb{P}_k, \mathbb{P}_v),$$

and K *is the Kullbak-Leibler divergence defined by*

$$K(\mathbb{P}, \mathbb{Q}) = \begin{cases} \int \ln\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{P} & \text{if } \mathbb{P} << \mathbb{Q}, \\ \infty & \text{otherwise.} \end{cases}$$

The proof of Lemma 2 can be found in DeVore et al. (2006, Lemma 3.3). For further details and applications of the Fano lemma, see Tsybakov (2004).

Consider the Besov balls $B^s_{\pi,r}(M)$ (see (6)). Let j_0 be an integer suitably chosen below. For any $\varepsilon = (\varepsilon_k)_{k \in \{0,\ldots,2^{j_0}-1\}} \in \{0,1\}^{2^{j_0}}$ and $d \in \mathbb{N}^*$, set

$$h_\varepsilon(x) = M_* 2^{-j_0(s+1/2)} \sum_{k=0}^{2^{j_0}-1} \varepsilon_k \frac{1}{(d-1)!} \int_{-\infty}^{x} (x-y)^{d-1} \psi_{j_0,k}(y) dy, \\ x \in [0,1],$$

(and, if $d = 0$, set $h_\varepsilon(x) = M_* 2^{-j_0(s+1/2)} \sum_{k=0}^{2^{j_0}-1} \varepsilon_k \psi_{j_0,k}(x)$, $x \in [0,1]$). Notice that, due to (4), h_ε exists and, since $\psi_{j_0,k}$ is 1-periodic, h_ε is also 1-periodic. Using the Cauchy formula for repeated integration, we have

$$h_\varepsilon^{(d)}(x) = M_* 2^{-j_0(s+1/2)} \sum_{k=0}^{2^{j_0}-1} \varepsilon_k \psi_{j_0,k}(x), \qquad x \in [0,1].$$

So, for any $j \geq \tau$ and any $k \in \{0, \ldots, 2^j - 1\}$, the (mother) wavelet coefficient of $h_\varepsilon^{(d)}$ is

$$\beta_{j,k} = \int_0^1 h_\varepsilon^{(d)}(x) \psi_{j,k}(x) dx = \begin{cases} M_* \varepsilon_k 2^{-j_0(s+1/2)}, & \text{if } j = j_0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $h_\varepsilon^{(d)} \in B^s_{\pi,r}(M)$. Let us now recall the theorem of Varshamov-Gilbert (see, for instance, Tsybakov (2004, Lemma 2.7)): there exist a subset $E_{j_0} =$

$\left\{\varepsilon^{(0)},\ldots,\varepsilon^{(T_{j_0})}\right\}$ of $\{0,1\}^{2^{j_0}}$ and two constants, $c\in]0,1[$ and $\alpha\in]0,1[$, such that, for any $(u,v)\in\{0,\ldots,T_{j_0}\}^2$ with $u<v$,

$$\sum_{k=0}^{2^{j_0}-1}|\varepsilon_k^{(u)}-\varepsilon_k^{(v)}|\geq c2^{j_0},\qquad T_{j_0}\geq e^{\alpha 2^{j_0}}.$$

Considering such a E_{j_0}, for any $(u,v)\in\{0,\ldots,T_{j_0}\}^2$ with $u\neq v$, we have

$$\begin{aligned}\left(\int_0^1\left(h^{(d)}_{\varepsilon^{(u)}}(x)-h^{(d)}_{\varepsilon^{(v)}}(x)\right)^2dx\right)^{1/2}&=c2^{-j_0(s+1/2)}\left(\sum_{k=0}^{2^{j_0}-1}\left|\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right|\right)^{1/2}\\&\geq 2\delta_{j_0},\end{aligned}$$

where

$$\delta_{j_0}=c2^{j_0/2}2^{-j_0(s+1/2)}=c2^{-j_0 s}.$$

Using the Chebychev inequality, for any $\widetilde{f}_{n,p}$, we have

$$\delta_{j_0}^{-2}\sup_{f^{(d)}\in B^s_{\pi,r}(M)}\mathbb{E}\left(\int_0^1\left(\widetilde{f}_{n,d}(x)-f^{(d)}(x)\right)^2dx\right)\geq\sup_{u\in\{0,\ldots,T_{j_0}\}}\mathbb{P}_{h_{\varepsilon^{(u)}}}(A_u^c)=p,$$

where

$$A_u=\left\{\left(\int_0^1\left(\widetilde{f}_{n,d}(x)-h^{(d)}_{\varepsilon^{(u)}}(x)\right)^2dx\right)^{1/2}<\delta_{j_0}\right\}$$

and $\mathbb{P}_f$ is the distribution of (1). Notice that, for any $(u,v)\in\{0,\ldots,T_{j_0}\}^2$ with $u\neq v$, $A_u\cap A_v=\varnothing$. Lemma 2 applied to the probability measures $\left(\mathbb{P}_{h_{\varepsilon^{(u)}}}\right)_{u\in\{0,\ldots,T_{j_0}\}}$ gives

$$p\geq\min\left(2^{-1},\exp(-3e^{-1})\sqrt{T_{j_0}}\exp(-\chi_{T_{j_0}})\right),\tag{17}$$

where

$$\chi_{T_{j_0}}=\inf_{v\in\{0,\ldots,T_{j_0}\}}\frac{1}{T_{j_0}}\sum_{\substack{u\in\{0,\ldots,T_{j_0}\}\\u\neq v}}K\left(\mathbb{P}_{h_{\varepsilon^{(u)}}},\mathbb{P}_{h_{\varepsilon^{(v)}}}\right).$$

Let us now bound $\chi_{T_{j_0}}$. For any functions f_1 and f_2 in $\mathbb{L}^2_{per}([0,1])$, we have

$$\begin{aligned}K\left(\mathbb{P}_{f_1},\mathbb{P}_{f_2}\right)&=\frac{n}{2\epsilon^2}\int_0^1\left((f_1\star g)(x)-(f_2\star g)(x)\right)^2dx\\&=\frac{n}{2\epsilon^2}\int_0^1\left(((f_1-f_2)\star g)(x)\right)^2dx.\end{aligned}$$

The Plancherel-Parseval theorem yields

$$K\left(\mathbb{P}_{f_1},\mathbb{P}_{f_2}\right)=\frac{n}{2\epsilon^2}\sum_{\ell\in\mathbb{Z}}\left|\mathcal{F}(f_1-f_2)(\ell)\right|^2|\mathcal{F}(g)(\ell)|^2.$$

So, for any $(u,v)\in\{0,\ldots,T_{j_0}\}^2$ with $u\neq v$, we have

$$K\left(\mathbb{P}_{h_{\varepsilon^{(u)}}},\mathbb{P}_{h_{\varepsilon^{(v)}}}\right)=\frac{n}{2\epsilon^2}\sum_{\ell\in\mathbb{Z}}\left|\mathcal{F}\left(h_{\varepsilon^{(u)}}-h_{\varepsilon^{(v)}}\right)(\ell)\right|^2|\mathcal{F}(g)(\ell)|^2. \tag{18}$$

By definition, for any $(u,v)\in\{0,\ldots,T_{j_0}\}^2$ with $u\neq v$ and $\ell\in\mathbb{Z}$, we have

$$\begin{aligned}&\mathcal{F}\left(h_{\varepsilon^{(u)}}-h_{\varepsilon^{(v)}}\right)(\ell)\\&=M_*2^{-j_0(s+1/2)}\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)\times\\&\quad\frac{1}{(d-1)!}\mathcal{F}\left(\int_{-\infty}^{.}(.-y)^{d-1}\psi_{j_0,k}(y)dy\right)(\ell).\end{aligned} \tag{19}$$

Let us set, for any $k\in\{0,\ldots,2^{j_0}-1\}$,

$$\theta_k(x)=\int_{-\infty}^{x}(x-y)^{d-1}\psi_{j_0,k}(y)dy,\qquad x\in[0,1].$$

Then, for any $u\in\{0,\ldots,d\}$, $\theta_k^{(u)}$ is 1-periodic and $\theta_k^{(u)}(0)=\theta_k^{(u)}(1)$. Therefore, by d integrations by parts, for any $\ell\in\mathbb{Z}$, we have

$$\mathcal{F}\left(\theta_k^{(d)}\right)(\ell)=(2\pi i\ell)^d\mathcal{F}(\theta_k)(\ell).$$

Using again the Cauchy formula for repeated integration, we have $\theta_k^{(d)}(x)=\psi_{j_0,k}(x)$, $x\in[0,1]$. So, for any $\ell\in\mathcal{C}_{j_0}$ (excluding 0), (19) implies that

$$\begin{aligned}&\mathcal{F}(h_{\varepsilon^{(u)}}-h_{\varepsilon^{(v)}})(\ell)\\&=\frac{M_*}{(d-1)!}2^{-j_0(s+1/2)}\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)\frac{1}{(2\pi i\ell)^d}\mathcal{F}\left(\psi_{j_0,k}\right)(\ell).\end{aligned} \tag{20}$$

The equalities (18) and (20) imply that

$$\begin{aligned}&K\left(\mathbb{P}_{h_{\varepsilon^{(u)}}},\mathbb{P}_{h_{\varepsilon^{(v)}}}\right)\\&=Cn2^{-2j_0(s+1/2)}\sum_{\ell\in\mathcal{C}_{j_0}}\left|\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)\mathcal{F}\left(\psi_{j_0,k}\right)(\ell)\right|^2\frac{1}{(2\pi\ell)^{2d}}|\mathcal{F}(g)(\ell)|^2.\end{aligned} \tag{21}$$

By (7) and (10),

$$\sup_{\ell\in\mathcal{C}_{j_0}}\left(\frac{1}{(2\pi\ell)^{2d}}|\mathcal{F}(g)(\ell)|^2\right)\leq C\sup_{\ell\in\mathcal{C}_{j_0}}\left(\frac{1}{(2\pi\ell)^{2d}}\left(1+|\ell|^2\right)^{-\delta}\right)\leq C2^{-2j_0(\delta+d)}. \tag{22}$$

Moreover, the Plancherel-Parseval theorem implies that

$$\begin{aligned}
&\sum_{\ell\in\mathcal{C}_{j_0}}\left|\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)\mathcal{F}\left(\psi_{j_0,k}\right)(\ell)\right|^2\\
&=\sum_{\ell\in\mathcal{C}_{j_0}}\left|\mathcal{F}\left(\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)\psi_{j_0,k}\right)(\ell)\right|^2\\
&=\int_{-\infty}^{\infty}\left|\mathcal{F}\left(\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)\psi_{j_0,k}\right)(y)\right|^2 dy\\
&=\int_0^1\left|\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)\psi_{j_0,k}(x)\right|^2 dx=\sum_{k=0}^{2^{j_0}-1}\left(\varepsilon_k^{(u)}-\varepsilon_k^{(v)}\right)^2\leq C2^{j_0}.
\end{aligned} \tag{23}$$

It follows from (21), (22) and (23) that

$$K\left(\mathbb{P}_{h_{\varepsilon^{(u)}}},\mathbb{P}_{h_{\varepsilon^{(v)}}}\right)\leq Cn2^{-2j_0(s+1/2)}2^{-2j_0(\delta+d)}2^{j_0}=Cn2^{-2j_0(s+1/2+\delta+d)}2^{j_0}.$$

Hence

$$\begin{aligned}
\chi_{T_{j_0}}&=\inf_{v\in\{0,\ldots,T_{j_0}\}}\frac{1}{T_{j_0}}\sum_{\substack{u\in\{0,\ldots,T_{j_0}\}\\ u\neq v}}K\left(\mathbb{P}_{h_{\varepsilon^{(u)}}},\mathbb{P}_{h_{\varepsilon^{(v)}}}\right)\\
&\leq Cn2^{-2j_0(s+1/2+\delta+d)}2^{j_0}.
\end{aligned} \tag{24}$$

Putting (17) and (24) together and choosing j_0 such that

$$2^{-j_0(s+1/2+\delta+d)}=c_0n^{-1/2},$$

where c_0 denotes a well chosen constant, for any estimator $\widetilde{f}_{n,d}$ of $f^{(d)}$, we have

$$\begin{aligned}
\delta_{j_0}^{-2}\sup_{f^{(d)}\in B^s_{\pi,r}(M)}\mathbb{E}\left(\int_0^1\left(\widetilde{f}_{n,d}(x)-f^{(d)}(x)\right)^2dx\right)&\geq c\exp\left((\alpha/2)2^{j_0}-Cc_0^22^{j_0}\right)\\
&\geq c,
\end{aligned}$$

where

$$\delta_{j_0}=c2^{-j_0s}=n^{-s/(2s+2\delta+2d+1)}.$$

This complete the proof of Theorem 2.

□

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