Valuation and Hedging of Credit Derivatives

Tomasz R. Bielecki
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Monique Jeanblanc
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Marek Rutkowski
School of Mathematics and Statistics
University of New South Wales
Sydney, NSW 2052, Australia

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Contents

Introduction

The goal of these lecture notes is to present a survey of recent developments in the area of mathematical modeling of *credit risk* and *credit derivatives.* They are largely based on the following papers by T.R. Bielecki, M. Jeanblanc and M. Rutkowski:

- Modelling and valuation of credit risk. In: *Stochastic Methods in Finance,* M. Frittelli and W. Runggaldier, eds., Springer-Verlag, 2004, 27–126,
- Hedging of defaultable claims. In: *Paris-Princeton Lectures on Mathematical Finance 2003,* R. Carmona et al., eds. Springer-Verlag, 2004, 1–132,
- PDE approach to valuation and hedging of credit derivatives. *Quantitative Finance* 5 (2005), 257–270,
- Hedging of credit derivatives in models with totally unexpected default. In: *Stochastic Processes and Applications to Mathematical Finance,* J. Akahori et al., eds., World Scientific, Singapore, 2006, 35–100,
- Hedging of basket credit derivatives in credit default swap market. *Journal of Credit Risk* 3 (2007).

and on some chapters from the book by T.R. Bielecki and M. Rutkowski: *Credit Risk: Modelling, Valuation and Hedging*, Springer-Verlag, 2001.

Our recent working papers by can be found on the websites:

- www.defaultrisk.com
- www.maths.unsw.edu.au/statistics/pubs/statspubs.html

A lot of other interesting information is provided on the websites listed at the end of the bibliography of this document.

Credit risk embedded in a financial transaction is the risk that at least one of the parties involved in the transaction will suffer a financial loss due to default or decline in the creditworthiness of the counter-party to the transaction, or perhaps of some third party. For example:

- A holder of a corporate bond bears a risk that the (market) value of the bond will decline due to decline in credit rating of the issuer.
- A bank may suffer a loss if a bank's debtor defaults on payment of the interest due and (or) the principal amount of the loan.
- A party involved in a trade of a credit derivative, such as a credit default swap (CDS), may suffer a loss if a reference credit event occurs.
- The market value of individual tranches constituting a collateralized debt obligation (CDO) may decline as a result of changes in the correlation between the default times of the underlying defaultable securities (i.e., of the collateral).

The most extensively studied form of credit risk is the *default risk* – that is, the risk that a counterparty in a financial contract will not fulfil a contractual commitment to meet her/his obligations stated in the contract. For this reason, the main tool in the area of credit risk modeling is a judicious specification of the random time of default. A large part of the present text is devoted to this issue.

Our main goal is to present the most important mathematical tools that are used for the arbitrage valuation of defaultable claims, which are also known under the name of credit derivatives. We also examine the important issue of hedging these claims.

These notes are organized as follows:

- In Chapter 1, we provide a concise summary of the main developments within the so-called *structural approach* to modeling and valuation of credit risk. We also study very briefly the case of a random barrier.
- Chapter 2 is devoted to the study of a simple model of credit risk within the *hazard function* framework. We also deal here with the issue of replication of single- and multi-name credit derivatives in the stylized CDS market.
- Chapter 3 deals with the so-called *reduced-form approach* in which the main tool is the *hazard rate* process. This approach is of a purely probabilistic nature and, technically speaking, it has a lot in common with the reliability theory.
- Chapter 4 studies hedging strategies for defaultable claims under assumption that some primary defaultable assets are traded. We discuss some general results in a semimartingale set-up and we develop the PDE approach in a Markovian set-up.
- Chapter 5 provides an introduction to the area of modeling dependent defaults and, more generally, to modeling of dependent credit migrations for a portfolio of reference names. We present some applications of these models to the valuation of real-life examples of credit derivatives, such as: CDSs and credit default swaptions, first-to-default CDSs, CDS indices and CDOs.

Let us mention that the proofs of most results can be found in Bielecki and Rutkowski [12], Bielecki et al. [5, 6, 9] and Jeanblanc and Rutkowski [59]. We quote some of the seminal papers; the reader can also refer to books by Bruyère [25], Bluhm et al. [18], Bielecki and Rutkowski [12], Cossin and Pirotte [33], Duffie and Singleton [43], Frey, McNeil and Embrechts [49], Lando [65], or Schönbucher [83] for more information. At the end of the bibliography, we also provide some web addresses where articles can be downloaded.

Finally, it should be acknowledged that several results (especially within the reduced-form approach) were obtained independently by various authors, who worked under different set of assumptions and/or within different set-ups. For this reason, we decided to omit detailed credentials in most cases. We hope that our colleagues will accept our apologies for this deficiency, and we stress that this by no means signifies that any result given in what follows that is not explicitly attributed is ours.

'Begin at the beginning, and go on till you come to the end: then stop.'

Lewis Carroll, *Alice's Adventures in Wonderland*

Chapter 1

Structural Approach

In this chapter, we present the so-called *structural approach* to modeling credit risk, which is also known as the *value-of-the-firm approach.* This methodology refers directly to economic fundamentals, such as the capital structure of a company, in order to model credit events (a default event, in particular). As we shall see in what follows, the two major driving concepts in the structural modeling are: the total value of the firm's assets and the default triggering barrier. It is worth noting that this was historically the first approach used in this area – it goes back to the fundamental papers by Black and Scholes [17] and Merton [76].

1.1 Basic Assumptions

We fix a finite horizon date $T^* > 0$, and we suppose that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with some (reference) filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$, is sufficiently rich to support the following objects:

- The *short-term interest rate* process r, and thus also a default-free term structure model.
- The *firm's value process* V, which is interpreted as a model for the total value of the firm's assets.
- The *barrier process* v, which will be used in the specification of the default time τ.
- The *promised contingent claim* X representing the firm's liabilities to be redeemed at maturity date $T \le T^*$.
- The process A, which models the *promised dividends,* i.e., the liabilities stream that is redeemed continuously or discretely over time to the holder of a defaultable claim.
- The *recovery claim* $\widetilde{X}$ representing the recovery payoff received at time T, if default occurs prior to or at the claim's maturity date T.
- The *recovery process* Z, which specifies the recovery payoff at time of default, if it occurs prior to or at the maturity date T.

1.1.1 Defaultable Claims

Technical assumptions. We postulate that the processes V, Z, A and v are progressively measurable with respect to the filtration $\mathbb{F}$, and that the random variables X and $\widetilde{X}$ are $\mathcal{F}_T$-measurable. In addition, A is assumed to be a process of finite variation, with $A_0 = 0$. We assume without mentioning that all random objects introduced above satisfy suitable integrability conditions.

Probabilities $\mathbb{P}$ and $\mathbb{Q}$. The probability $\mathbb{P}$ is assumed to represent the *real-world* (or *statistical*) probability, as opposed to a *martingale measure* (also known as a *risk-neutral probability*). Any *martingale measure* will be denoted by $\mathbb{Q}$ in what follows.

Default time. In the structural approach, the default time τ will be typically defined in terms of the firm's value process V and the barrier process v. We set

$$\tau = \inf \{ t > 0 : t \in \mathcal{T} \text{ and } V_t \leq v_t \}$$

with the usual convention that the infimum over the empty set equals $+\infty$. In main cases, the set $\mathcal{T}$ is an interval $[0, T]$ (or $[0, \infty)$ in the case of perpetual claims). In first passage structural models, the default time τ is usually given by the formula:

$$\tau = \inf \{ t > 0 : t \in [0, T] \text{ and } V_t \leq \bar{v}(t) \},$$

where $\bar{v} : [0, T] \to \mathbb{R}_+$ is some deterministic function, termed the *barrier*.

Predictability of default time. Since the underlying filtration $\mathbb{F}$ in most structural models is generated by a standard Brownian motion, τ will be an $\mathbb{F}$-predictable stopping time (as any stopping time with respect to a Brownian filtration): there exists a sequence of increasing stopping times announcing the default time.

Recovery rules. If default does not occur before or at time T, the promised claim X is paid in full at time T. Otherwise, depending on the market convention, either (1) the amount $\widetilde{X}$ is paid at the maturity date T, or (2) the amount Z_τ is paid at time τ. In the case when default occurs at maturity, i.e., on the event $\{\tau = T\}$, we postulate that only the recovery payment $\widetilde{X}$ is paid. In a general setting, we consider simultaneously both kinds of recovery payoff, and thus a generic defaultable claim is formally defined as a quintuple $(X, A, \widetilde{X}, Z, \tau)$.

1.1.2 Risk-Neutral Valuation Formula

Suppose that our financial market model is arbitrage-free, in the sense that there exists a *martingale measure* (*risk-neutral probability*) $\mathbb{Q}$, meaning that price process of any tradeable security, which pays no coupons or dividends, becomes an $\mathbb{F}$-martingale under $\mathbb{Q}$, when discounted by the *savings account* B, given as

$$B_t = \exp\Big(\int_0^t r_u \, du \Big).$$

We introduce the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$, and we denote by D the process that models all cash flows received by the owner of a defaultable claim. Let us denote

$$X^d(T) = X \mathbb{1}_{\{\tau > T\}} + \widetilde{X} \mathbb{1}_{\{\tau \leq T\}}.$$

Definition 1.1.1 *The dividend process D of a defaultable contingent claim $(X, A, \widetilde{X}, Z, \tau)$, which settles at time T, equals*

$$D_t = X^d(T) \mathbb{1}_{\{t \geq T\}} + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u.$$

It is apparent that D is a process of finite variation, and

$$\int_{]0,t]} (1 - H_u) \, dA_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}} \, dA_u = A_{\tau-} \mathbb{1}_{\{\tau \leq t\}} + A_t \mathbb{1}_{\{\tau > t\}}.$$

Note that if default occurs at some date t, the promised dividend $A_t - A_{t-}$, which is due to be paid at this date, is not received by the holder of a defaultable claim. Furthermore, if we set $\tau \wedge t = \min\{\tau, t\}$ then

$$\int_{]0,t]} Z_u \, dH_u = Z_{\tau \wedge t} \mathbb{1}_{\{\tau \leq t\}} = Z_\tau \mathbb{1}_{\{\tau \leq t\}}.$$

Remark 1.1.1 In principle, the promised payoff X could be incorporated into the promised dividends process A. However, this would be inconvenient, since in practice the recovery rules concerning the promised dividends A and the promised claim X are different, in general. For instance, in the case of a defaultable coupon bond, it is frequently postulated that in case of default the future coupons are lost, but a strictly positive fraction of the face value is usually received by the bondholder.

We are in the position to define the ex-dividend price S_t of a defaultable claim. At any time t, the random variable S_t represents the current value of all future cash flows associated with a given defaultable claim.

Definition 1.1.2 *For any date $t \in [0,T[$, the ex-dividend price of the defaultable claim $(X, A, \widetilde{X}, Z, \tau)$ is given as*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{F}_t \Big). \tag{1.1}$$

In addition, we always set $S_T = X^d(T)$. The discounted ex-dividend price S_t^*, $t \in [0,T]$, satisfies

$$S_t^* = S_t B_t^{-1} - \int_{]0,t]} B_u^{-1} \, dD_u, \quad \forall\, t \in [0,T],$$

and thus it follows a supermartingale under $\mathbb{Q}$ if and only if the dividend process D is increasing. The process $S_t + B_t \int_{]0,t]} B_u^{-1} \, dD_u$ is also called the cum-dividend process.

1.1.3 Defaultable Zero-Coupon Bond

Assume that $A \equiv 0$, $Z \equiv 0$ and $X = L$ for some positive constant $L > 0$. Then the value process S represents the arbitrage price of a *defaultable zero-coupon bond* (also known as the *corporate discount bond*) with the face value L and recovery at maturity only. In general, the price $D(t,T)$ of such a bond equals

$$D(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} (L \mathbb{1}_{\{\tau > T\}} + \widetilde{X} \mathbb{1}_{\{\tau \le T\}}) \,\big|\, \mathcal{F}_t \big).$$

It is convenient to rewrite the last formula as follows:

$$D(t,T) = L B_t \, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} (\mathbb{1}_{\{\tau > T\}} + \delta(T) \mathbb{1}_{\{\tau \le T\}}) \,\big|\, \mathcal{F}_t \big),$$

where the random variable $\delta(T) = \widetilde{X}/L$ represents the so-called *recovery rate upon default.* It is natural to assume that $0 \le \widetilde{X} \le L$ so that $\delta(T)$ satisfies $0 \le \delta(T) \le 1$. Alternatively, we may re-express the bond price as follows:

$$D(t,T) = L\Big(B(t,T) - B_t \, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} w(T) \mathbb{1}_{\{\tau \le T\}} \,\big|\, \mathcal{F}_t \big) \Big),$$

where

$$B(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}}(B_T^{-1} \,|\, \mathcal{F}_t)$$

is the price of a unit default-free zero-coupon bond, and $w(T) = 1 - \delta(T)$ is the *writedown rate upon default.* Generally speaking, the time-t value of a corporate bond depends on the joint probability distribution under $\mathbb{Q}$ of the three-dimensional random variable $(B_T, \delta(T), \tau)$ or, equivalently, $(B_T, w(T), \tau)$.

Example 1.1.1 Merton [76] postulates that the recovery payoff upon default (that is, when $V_T < L$, equals $\widetilde{X} = V_T$, where the random variable V_T is the firm's value at maturity date T of a corporate bond. Consequently, the random recovery rate upon default equals $\delta(T) = V_T/L$, and the writedown rate upon default equals $w(T) = 1 - V_T/L$.

Expected writedowns. For simplicity, we assume that the savings account B is non-random – that is, the short-term rate r is deterministic. Then the price of a default-free zero-coupon bond equals $B(t,T) = B_t B_T^{-1}$, and the price of a zero-coupon corporate bond satisfies

$$D(t,T) = L_t(1 - w^*(t,T)),$$

where $L_t = LB(t,T)$ is the present value of future liabilities, and $w^*(t,T)$ is the *conditional expected writedown rate* under $\mathbb{Q}$. It is given by the following equality:

$$w^*(t,T) = \mathbb{E}_{\mathbb{Q}}\big(w(T)\mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t\big).$$

The *conditional expected writedown rate upon default* equals, under $\mathbb{Q}$,

$$w_t^* = \frac{\mathbb{E}_{\mathbb{Q}}\big(w(T)\mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t\big)}{\mathbb{Q}\{\tau \le T \,|\, \mathcal{F}_t\}} = \frac{w^*(t,T)}{p_t^*},$$

where $p_t^* = \mathbb{Q}\{\tau \le T \,|\, \mathcal{F}_t\}$ is the *conditional risk-neutral probability of default.* Finally, let $\delta_t^* = 1 - w_t^*$ be the *conditional expected recovery rate upon default* under $\mathbb{Q}$. In terms of p_t^*, δ_t^* and p_t^*, we obtain

$$D(t,T) = L_t(1 - p_t^*) + L_t p_t^* \delta_t^* = L_t(1 - p_t^* w_t^*).$$

If the random variables $w(T)$ and τ are conditionally independent with respect to the σ-field $\mathcal{F}_t$ under $\mathbb{Q}$, then we have $w_t^* = \mathbb{E}_{\mathbb{Q}}(w(T) \,|\, \mathcal{F}_t)$.

Example 1.1.2 In practice, it is common to assume that the recovery rate is non-random. Let the recovery rate $\delta(T)$ be constant, specifically, $\delta(T) = \delta$ for some real number δ. In this case, the writedown rate $w(T) = w = 1 - \delta$ is non-random as well. Then $w^*(t,T) = wp_t^*$ and $w_t^* = w$ for every $0 \le t \le T$. Furthermore, the price of a defaultable bond has the following representation

$$D(t,T) = L_t(1 - p_t^*) + \delta L_t p_t^* = L_t(1 - wp_t^*).$$

We shall return to various recovery schemes later in the text.

1.2 Classic Structural Models

Classic structural models are based on the assumption that the risk-neutral dynamics of the value process of the assets of the firm V are given by the SDE:

$$dV_t = V_t\left((r - \kappa)\,dt + \sigma_V\,dW_t\right), \quad V_0 > 0,$$

where κ is the constant payout (dividend) ratio, and the process W is a standard Brownian motion under the martingale measure $\mathbb{Q}$.

1.2.1 Merton's Model

We present here the classic model due to Merton [76].

Basic assumptions. A firm has a single liability with promised terminal payoff L, interpreted as the zero-coupon bond with maturity T and face value $L > 0$. The ability of the firm to redeem its debt is determined by the total value V_T of firm's assets at time T. Default may occur at time T only, and the default event corresponds to the event $\{V_T < L\}$. Hence, the stopping time τ equals

$$\tau = T\mathbb{1}_{\{V_T < L\}} + \infty\mathbb{1}_{\{V_T \ge L\}}.$$

Moreover $A = 0$, $Z = 0$, and

$$X^d(T) = V_T\mathbb{1}_{\{V_T < L\}} + L\mathbb{1}_{\{V_T \ge L\}}$$

so that $\tilde{X} = V_T$. In other words, the payoff at maturity equals

$$D_T = \min(V_T, L) = L - \max(L - V_T, 0) = L - (L - V_T)^+.$$

The latter equality shows that the valuation of the corporate bond in Merton's setup is equivalent to the valuation of a European put option written on the firm's value with strike equal to the bond's face value. Let $D(t,T)$ be the price at time $t < T$ of the corporate bond. It is clear that the value $D(V_t)$ of the firm's debt equals

$$D(V_t) = D(t,T) = L\,B(t,T) - P_t,$$

where P_t is the price of a put option with strike L and expiration date T. It is apparent that the value $E(V_t)$ of the firm's equity at time t equals

$$E(V_t) = V_t - D(V_t) = V_t - LB(t,T) + P_t = C_t,$$

where C_t stands for the price at time t of a call option written on the firm's assets, with strike price L and exercise date T. To justify the last equality above, we may also observe that at time T we have

$$E(V_T) = V_T - D(V_T) = V_T - \min(V_T, L) = (V_T - L)^+.$$

We conclude that the firm's shareholders are in some sense the holders of a call option on the firm's assets.

Merton's formula. Using the option-like features of a corporate bond, Merton [76] derived a closed-form expression for its arbitrage price. Let N denote the standard Gaussian cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \quad \forall\, x \in \mathbb{R}.$$

Proposition 1.2.1 *For every $0 \le t < T$ the value $D(t,T)$ of a corporate bond equals*

$$D(t,T) = V_t e^{-\kappa(T-t)} N\big(-d_+(V_t, T-t)\big) + L\,B(t,T) N\big(d_-(V_t, T-t)\big)$$

where

$$d_\pm(V_t, T-t) = \frac{\ln(V_t/L) + \big(r - \kappa \pm \frac{1}{2}\sigma_V^2\big)(T-t)}{\sigma_V \sqrt{T-t}}.$$

The unique replicating strategy for a defaultable bond involves holding, at any time $0 \le t < T$, $\phi_t^1 V_t$ units of cash invested in the firm's value and $\phi_t^2 B(t,T)$ units of cash invested in default-free bonds, where

$$\phi_t^1 = e^{-\kappa(T-t)} N\big(-d_+(V_t, T-t)\big)$$

and

$$\phi_t^2 = \frac{D(t,T) - \phi_t^1 V_t}{B(t,T)} = LN\big(d_-(V_t, T-t)\big).$$

Credit spreads. For notational simplicity, we set $\kappa = 0$. Then Merton's formula becomes:

$$D(t,T) = LB(t,T)\big(\Gamma_t N(-d) + N(d - \sigma_V\sqrt{T-t})\big),$$

where we denote $\Gamma_t = V_t/LB(t,T)$ and

$$d = d(V_t, T-t) = \frac{\ln(V_t/L) + (r + \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}}.$$

Since $LB(t,T)$ represents the current value of the face value of the firm's debt, the quantity Γ_t can be seen as a proxy of the asset-to-debt ratio $V_t/D(t,T)$. It can be easily verified that the inequality

$D(t,T) < LB(t,T)$ is valid. This property is equivalent to the positivity of the corresponding *credit spread* (see below).

Observe that in the present setup the continuously compounded yield $r(t,T)$ at time t on the T-maturity Treasury zero-coupon bond is constant, and equal to the short-term rate r. Indeed, we have

$$B(t,T) = e^{-r(t,T)(T-t)} = e^{-r(T-t)}.$$

Let us denote by $r^d(t,T)$ the continuously compounded yield on the corporate bond at time $t < T$, so that

$$D(t,T) = Le^{-r^d(t,T)(T-t)}.$$

From the last equality, it follows that

$$r^d(t,T) = -\frac{\ln D(t,T) - \ln L}{T-t}.$$

For $t < T$ the *credit spread* $S(t,T)$ is defined as the excess return on a defaultable bond:

$$S(t,T) = r^d(t,T) - r(t,T) = \frac{1}{T-t} \ln \frac{LB(t,T)}{D(t,T)}.$$

In Merton's model, we have

$$S(t,T) = -\frac{\ln\left(N(d - \sigma_V\sqrt{T-t}) + \Gamma_t N(-d)\right)}{T-t} > 0.$$

This agrees with the well-known fact that risky bonds have an expected return in excess of the risk-free interest rate. In other words, the yields on corporate bonds are higher than yields on Treasury bonds with matching notional amounts. Notice, however, when t tends to T, the credit spread in Merton's model tends either to infinity or to 0, depending on whether $V_T < L$ or $V_T > L$. Formally, if we define the *forward short spread at time* T as

$$FSS_T = \lim_{t \uparrow T} S(t,T)$$

then

$$FSS_T(\omega) = \begin{cases} 0, & \text{if } \omega \in \{V_T > L\}, \\ \infty, & \text{if } \omega \in \{V_T < L\}. \end{cases}$$

1.2.2 Black and Cox Model

By construction, Merton's model does not allow for a premature default, in the sense that the default may only occur at the maturity of the claim. Several authors put forward structural-type models in which this restrictive and unrealistic feature is relaxed. In most of these models, the time of default is given as the *first passage time* of the value process V to either a deterministic or a random barrier. In principle, the bond's default may thus occur at any time before or on the maturity date T. The challenge is to appropriately specify the lower threshold v, the recovery process Z, and to explicitly evaluate the conditional expectation that appears on the right-hand side of the risk-neutral valuation formula

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big),$$

which is valid for $t \in [0,T[$. As one might easily guess, this is a non-trivial mathematical problem, in general. In addition, the practical problem of the lack of direct observations of the value process V largely limits the applicability of the first-passage-time models based on the value of the firm process V.

Corporate zero-coupon bond. Black and Cox [16] extend Merton's [76] research in several directions, by taking into account such specific features of real-life debt contracts as: safety covenants, debt subordination, and restrictions on the sale of assets. Following Merton [76], they assume that the firm's stockholders receive continuous dividend payments, which are proportional to the current value of firm's assets. Specifically, they postulate that

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma_V\,dW_t\big), \quad V_0 > 0,$$

where W is a Brownian motion (under the risk-neutral probability $\mathbb{Q}$), the constant $\kappa \geq 0$ represents the payout ratio, and $\sigma_V > 0$ is the constant volatility. The short-term interest rate r is assumed to be constant.

Safety covenants. Safety covenants provide the firm's bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to a set standard. The standard for a poor performance is set by Black and Cox in terms of a time-dependent deterministic barrier $\bar{v}(t) = Ke^{-\gamma(T-t)}$, $t \in [0,T[$, for some constant $K > 0$. As soon as the value of firm's assets crosses this lower threshold, the bondholders take over the firm. Otherwise, default takes place at debt's maturity or not depending on whether $V_T < L$ or not.

Default time. Let us set

$$v_t = \begin{cases} \bar{v}(t), & \text{for } t < T, \\ L, & \text{for } t = T. \end{cases}$$

The default event occurs at the first time $t \in [0,T]$ at which the firm's value V_t falls below the level v_t, or the default event does not occur at all. The default time equals ($\inf \emptyset = +\infty$)

$$\tau = \inf\{\, t \in [0,T] : V_t \leq v_t\}.$$

The recovery process Z and the recovery payoff $\widetilde{X}$ are proportional to the value process: $Z \equiv \beta_2 V$ and $\widetilde{X} = \beta_1 V_T$ for some constants $\beta_1, \beta_2 \in [0,1]$. The case examined by Black and Cox [16] corresponds to $\beta_1 = \beta_2 = 1$.

To summarize, we consider the following model:

$$X = L,\ A \equiv 0,\ Z \equiv \beta_2 V,\ \widetilde{X} = \beta_1 V_T,\ \tau = \bar{\tau} \wedge \widehat{\tau},$$

where the *early default time* $\bar{\tau}$ equals

$$\bar{\tau} = \inf\{\, t \in [0,T) : V_t \leq \bar{v}(t)\}$$

and $\widehat{\tau}$ stands for Merton's default time: $\widehat{\tau} = T\mathbb{1}_{\{V_T<L\}} + \infty\mathbb{1}_{\{V_T\geq L\}}$.

Bond valuation. Similarly as in Merton's model, it is assumed that the short term interest rate is deterministic and equal to a positive constant r. We postulate, in addition, that $\bar{v}(t) \leq LB(t,T)$ or, more explicitly,

$$Ke^{-\gamma(T-t)} \leq Le^{-r(T-t)}, \quad \forall\, t \in [0,T],$$

so that, in particular, $K \leq L$. This condition ensures that the payoff to the bondholder at the default time τ never exceeds the face value of debt, discounted at a risk-free rate.

PDE approach. Since the model for the value process V is given in terms of a Markovian diffusion, a suitable partial differential equation can be used to characterize the value process of the corporate bond. Let us write $D(t,T) = u(V_t, t)$. Then the pricing function $u = u(v,t)$ of a defaultable bond satisfies the following PDE:

$$u_t(v,t) + (r-\kappa)vu_v(v,t) + \frac{1}{2}\sigma_V^2 v^2 u_{vv}(v,t) - ru(v,t) = 0$$

on the domain

$$\{(v,t) \in \mathbb{R}_+ \times \mathbb{R}_+ :\, 0 < t < T,\, v > Ke^{-\gamma(T-t)}\},$$

with the boundary condition

$$u(Ke^{-\gamma(T-t)}, t) = \beta_2 Ke^{-\gamma(T-t)}$$

and the terminal condition $u(v,T) = \min(\beta_1 v, L)$.

Probabilistic approach. For any $t < T$ the price $D(t,T) = u(V_t, t)$ of a defaultable bond has the following probabilistic representation, on the set $\{\tau > t\} = \{\bar{\tau} > t\}$

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(L e^{-r(T-t)} \mathbb{1}_{\{\bar{\tau} \geq T,\, V_T \geq L\}} \,\Big|\, \mathcal{F}_t \Big) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\Big(\beta_1 V_T e^{-r(T-t)} \mathbb{1}_{\{\bar{\tau} \geq T,\, V_T < L\}} \,\Big|\, \mathcal{F}_t \Big) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\Big(K\beta_2 e^{-\gamma(T-\bar{\tau})} e^{-r(\bar{\tau}-t)} \mathbb{1}_{\{t < \bar{\tau} < T\}} \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

After default – that is, on the set $\{\tau \leq t\} = \{\bar{\tau} \leq t\}$, we clearly have

$$D(t,T) = \beta_2 \bar{v}(\tau) B^{-1}(\tau, T) B(t,T) = K\beta_2 e^{-\gamma(T-\tau)} e^{r(t-\tau)}.$$

To compute the expected values above, we observe that:

- the first two conditional expectations can be computed by using the formula for the conditional probability $\mathbb{Q}\{V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t\}$,
- to evaluate the third conditional expectation, it suffices employ the conditional probability law of the first passage time of the process V to the barrier $\bar{v}(t)$.

Black and Cox formula. Before we state the bond valuation result due to Black and Cox [16], we find it convenient to introduce some notation. We denote

$$\begin{aligned} \nu &= r - \kappa - \frac{1}{2}\sigma_V^2, \\ m &= \nu - \gamma = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2 \\ b &= m\sigma^{-2}. \end{aligned}$$

For the sake of brevity, in the statement of Proposition 1.2.2 we shall write σ instead of σ_V. As already mentioned, the probabilistic proof of this result is based on the knowledge of the probability law of the first passage time of the geometric (exponential) Brownian motion to an exponential barrier.

Proposition 1.2.2 *Assume that $m^2 + 2\sigma^2(r-\gamma) > 0$. Prior to bond's default, that is: on the set $\{\tau > t\}$, the price process $D(t,T) = u(V_t, t)$ of a defaultable bond equals*

$$\begin{aligned} D(t,T) &= LB(t,T)\Big(N\big(h_1(V_t, T-t)\big) - Z_t^{2b\sigma^{-2}} N\big(h_2(V_t, T-t)\big)\Big) \\ &\quad + \beta_1 V_t e^{-\kappa(T-t)} \Big(N\big(h_3(V_t, T-t)\big) - N\big(h_4(V_t, T-t)\big)\Big) \\ &\quad + \beta_1 V_t e^{-\kappa(T-t)} Z_t^{2b+2} \Big(N\big(h_5(V_t, T-t)\big) - N\big(h_6(V_t, T-t)\big)\Big) \\ &\quad + \beta_2 V_t \Big(Z_t^{\theta+\zeta} N\big(h_7(V_t, T-t)\big) + Z_t^{\theta-\zeta} N\big(h_8(V_t, T-t)\big)\Big), \end{aligned}$$

where $Z_t = \bar{v}(t)/V_t$, $\theta = b+1$, $\zeta = \sigma^{-2}\sqrt{m^2 + 2\sigma^2(r-\gamma)}$ and

$$\begin{aligned} h_1(V_t, T-t) &= \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\ h_2(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\ h_3(V_t, T-t) &= \frac{\ln(L/V_t) - (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ h_4(V_t, T-t) &= \frac{\ln(K/V_t) - (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \end{aligned}$$

$$h_5(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$h_6(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(KV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$h_7(V_t, T-t) = \frac{\ln(\bar{v}(t)/V_t) + \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$
$$h_8(V_t, T-t) = \frac{\ln(\bar{v}(t)/V_t) - \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Special cases. Assume that $\beta_1 = \beta_2 = 1$ and the barrier function $\bar{v}$ is such that $K = L$. Then necessarily $\gamma \geq r$. It can be checked that for $K = L$ we have $D(t,T) = D_1(t,T) + D_3(t,T)$ where:

$$D_1(t,T) = LB(t,T)\big(N\big(h_1(V_t, T-t)\big) - Z_t^{2\hat{a}} N\big(h_2(V_t, T-t)\big)\big)$$
$$D_3(t,T) = V_t\big(Z_t^{\theta+\zeta} N\big(h_7(V_t, T-t)\big) + Z_t^{\theta-\zeta} N\big(h_8(V_t, T-t)\big)\big).$$

• **Case $\gamma = r$.** If we also assume that $\gamma = r$ then $\zeta = -\sigma^{-2}\hat{\nu}$, and thus

$$V_t Z_t^{\theta+\zeta} = LB(t,T), \quad V_t Z_t^{\theta-\zeta} = V_t Z_t^{2\hat{a}+1} = LB(t,T) Z_t^{2\hat{a}}.$$

It is also easy to see that in this case

$$h_1(V_t, T-t) = \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}} = -h_7(V_t, T-t),$$

while

$$h_2(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}} = h_8(V_t, T-t).$$

We conclude that if $\bar{v}(t) = Le^{-r(T-t)} = LB(t,T)$ then $D(t,T) = LB(t,T)$. This result is quite intuitive. A corporate bond with a safety covenant represented by the barrier function, which equals the discounted value of the bond's face value, is equivalent to a default-free bond with the same face value and maturity.

• **Case $\gamma > r$.** For $K = L$ and $\gamma > r$, it is natural to expect that $D(t,T)$ would be smaller than $LB(t,T)$. It is also possible to show that when γ tends to infinity (all other parameters being fixed), then the Black and Cox price converges to Merton's price.

1.2.3 Further Developments

The Black and Cox first-passage-time approach was later developed by, among others: Brennan and Schwartz [21, 22] – an analysis of convertible bonds, Nielsen et al. [78] – a random barrier and random interest rates, Leland [69], Leland and Toft [70] – a study of an optimal capital structure, bankruptcy costs and tax benefits, Longstaff and Schwartz [72] – a constant barrier and random interest rates, Brigo [23].

Other stopping times. In general, one can study the bond valuation problem for the default time given as

$$\tau = \inf\{\, t \in \mathbb{R}_+ : V_t \leq L(t)\},$$

where $L(t)$ is a deterministic function and V is a geometric Brownian motion. However, there exists few explicit results.

Moraux's model. Moraux [77] propose to model the default time as a *Parisian stopping time*. For a continuous process V and a given $t > 0$, we introduce $g_t^b(V)$, the last time before t at which the process V was at level b, that is,

$$g_t^b(V) = \sup\{\, 0 \leq s \leq t : V_s = b\}.$$

The *Parisian stopping time* is the first time at which the process V is below the level b for a time period of length greater than D, that is,

$$G_D^{-,b}(V) = \inf\{\, t \in \mathbb{R}_+ : (t - g_t^b(V))\mathbb{1}_{\{V_t<b\}} \geq D\}.$$

Clearly, this time is a stopping time. Let $\tau = G_D^{-,b}(V)$. In the case of Black-Scholes dynamics, it is possible to find the joint law of (τ, V_τ)

Another default time is the first time where the process V has spend more than D time below a level, that is, $\tau = \inf\{t \in \mathbb{R}_+ : A_t^V > D\}$ where $A_t^V = \int_0^t \mathbb{1}_{\{V_s>b\}}\, ds$. The law of this time is related to *cumulative options*.

Campi and Sbuelz model. Campi and Sbuelz [26] assume that the default time is given by a first hitting time of 0 by a CEV process, and they study the difficult problem of pricing an equity default swap. More precisely, they assume that the dynamics of the firm are

$$dS_t = S_{t-}\Big((r-\kappa)\, dt + \sigma S_t^\beta\, dW_t - dM_t\Big)$$

where W is a Brownian motion and M the compensated martingale of a Poisson process (i.e., $M_t = N_t - \lambda t$), and they define

$$\tau = \inf\{\, t \in \mathbb{R}_+ : S_t \leq 0\}.$$

In other terms, Campi and Sbuelz [26] set $\tau = \tau^\beta \wedge \tau^N$, where τ^N is the first jump of the Poisson process and

$$\tau^\beta = \inf\{\, t \in \mathbb{R}_+ : X_t \leq 0\}$$

where in turn

$$dX_t = X_{t-}\Big((r-\kappa+\lambda)\, dt + \sigma X_t^\beta\, dW_t\Big).$$

Using that the CEV process can be expressed in terms of a time-changed Bessel process, and results on the hitting time of 0 for a Bessel process of dimension smaller than 2, they obtain closed from solutions.

Zhou's model. Zhou [85] studies the case where the dynamics of the firm is

$$dV_t = V_{t-}\Big(\big(\mu - \lambda\nu\big)\, dt + \sigma\, dW_t + dX_t\Big)$$

where W is a Brownian motion, X a compound Poisson process, that is, $X_t = \sum_1^{N_t} e^{Y_i} - 1$ where $\ln Y_i \overset{\text{law}}{=} N(a, b^2)$ with $\nu = \exp(a + b^2/2) - 1$. Note that for this choice of parameters the process $V_t e^{-\mu t}$ is a martingale. Zhou first studies Merton's problem in that setting. Next, he gives an approximation for the first passage problem when the default time is $\tau = \inf\{\, t \in \mathbb{R}_+ : V_t \leq L\}$.

1.2.4 Optimal Capital Structure

We consider a firm that has an interest paying bonds outstanding. We assume that it is a consol bond, which pays continuously coupon rate c. Assume that $r > 0$ and the payout rate κ is equal to zero. This condition can be given a financial interpretation as the restriction on the sale of assets, as opposed to issuing of new equity. Equivalently, we may think about a situation in which the stockholders will make payments to the firm to cover the interest payments. However, they have the right to stop making payments at any time and either turn the firm over to the bondholders or pay them a lump payment of c/r per unit of the bond's notional amount.

Recall that we denote by $E(V_t)$ ($D(V_t)$, resp.) the value at time t of the firm equity (debt, resp.), hence the total value of the firm's assets satisfies $V_t = E(V_t) + D(V_t)$.

Black and Cox [16] argue that there is a critical level of the value of the firm, denoted as v^*, below which no more equity can be sold. The critical value v^* will be chosen by stockholders, whose aim is to minimize the value of the bonds (equivalently, to maximize the value of the equity). Let us observe that v^* is nothing else than a constant default barrier in the problem under consideration; the optimal default time τ^* thus equals $\tau^* = \inf\{ t \in \mathbb{R}_+ : V_t \le v^* \}$.

To find the value of v^*, let us first fix the bankruptcy level $\bar{v}$. The ODE for the pricing function $u^\infty = u^\infty(V)$ of a consol bond takes the following form (recall that $\sigma = \sigma_V$)

$$\frac{1}{2} V^2 \sigma^2 u^\infty_{VV} + rV u^\infty_V + c - r u^\infty = 0,$$

subject to the lower boundary condition $u^\infty(\bar{v}) = \min(\bar{v}, c/r)$ and the upper boundary condition

$$\lim_{V \to \infty} u^\infty_V(V) = 0.$$

For the last condition, observe that when the firm's value grows to infinity, the possibility of default becomes meaningless, so that the value of the defaultable consol bond tends to the value c/r of the default-free consol bond. The general solution has the following form:

$$u^\infty(V) = \frac{c}{r} + K_1 V + K_2 V^{-\alpha},$$

where $\alpha = 2r/\sigma^2$ and K_1, K_2 are some constants, to be determined from boundary conditions. We find that $K_1 = 0$, and

$$K_2 = \begin{cases} \bar{v}^{\alpha+1} - (c/r)\bar{v}^\alpha, & \text{if } \bar{v} < c/r, \\ 0, & \text{if } \bar{v} \ge c/r. \end{cases}$$

Hence, if $\bar{v} < c/r$ then

$$u^\infty(V_t) = \frac{c}{r} + \left(\bar{v}^{\alpha+1} - \frac{c}{r}\bar{v}^\alpha \right) V_t^{-\alpha}$$

or, equivalently,

$$u^\infty(V_t) = \frac{c}{r}\left(1 - \left(\frac{\bar{v}}{V_t} \right)^\alpha \right) + \bar{v}\left(\frac{\bar{v}}{V_t} \right)^\alpha.$$

It is in the interest of the stockholders to select the bankruptcy level in such a way that the value of the debt, $D(V_t) = u^\infty(V_t)$, is minimized, and thus the value of firm's equity

$$E(V_t) = V_t - D(V_t) = V_t - \frac{c}{r}(1 - \bar{q}_t) - \bar{v}\bar{q}_t$$

is maximized. It is easy to check that the optimal level of the barrier does not depend on the current value of the firm, and it equals

$$v^* = \frac{c}{r}\frac{\alpha}{\alpha+1} = \frac{c}{r + \sigma^2/2}.$$

Given the optimal strategy of the stockholders, the price process of the firm's debt (i.e., of a consol bond) takes the form, on the set $\{\tau^* > t\}$,

$$D^*(V_t) = \frac{c}{r} - \frac{1}{\alpha V_t^\alpha}\left(\frac{c}{r + \sigma^2/2} \right)^{\alpha+1}$$

or, equivalently,

$$D^*(V_t) = \frac{c}{r}(1 - q_t^*) + v^* q_t^*,$$

where

$$q_t^* = \left(\frac{v^*}{V_t} \right)^\alpha = \frac{1}{V_t^\alpha}\left(\frac{c}{r + \sigma^2/2} \right)^\alpha.$$

Further developments. We end this section by mentioning that other important developments in the area of optimal capital structure were presented in the papers by Leland [69], Leland and Toft [70], Christensen et al. [31]. Chen and Kou [29], Dao [34], Hilberink and Rogers [53], LeCourtois and Quittard-Pinon [68] study the same problem, but they model the firm's value process as a diffusion with jumps. The reason for this extension was to eliminate an undesirable feature of previously examined models, in which short spreads tend to zero when a bond approaches maturity date.

1.3 Stochastic Interest Rates

In this section, we assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, supports the short-term interest rate process r and the value process V. The dynamics under the martingale measure $\mathbb{Q}$ of the firm's value and of the price of a default-free zero-coupon bond $B(t,T)$ are

$$dV_t = V_t\big((r_t - \kappa(t))\,dt + \sigma(t)\,dW_t\big)$$

and

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\,dW_t\big)$$

respectively, where W is a d-dimensional standard $\mathbb{Q}$-Brownian motion. Furthermore, $\kappa : [0,T] \to \mathbb{R}$, $\sigma : [0,T] \to \mathbb{R}^d$ and $b(\cdot,T) : [0,T] \to \mathbb{R}^d$ are assumed to be bounded functions. The *forward value* $F_V(t,T) = V_t/B(t,T)$ of the firm satisfies under the *forward martingale measure* $\mathbb{P}_T$

$$dF_V(t,T) = -\kappa(t)F_V(t,T)\,dt + F_V(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T$$

where the process $W_t^T = W_t - \int_0^t b(u,T)\,du$, $t \in [0,T]$, is a d-dimensional Brownian motion under $\mathbb{P}_T$. For any $t \in [0,T]$, we set

$$F_V^{\kappa}(t,T) = F_V(t,T)e^{-\int_t^T \kappa(u)\,du}.$$

Then

$$dF_V^{\kappa}(t,T) = F_V^{\kappa}(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T.$$

Furthermore, it is apparent that $F_V^{\kappa}(T,T) = F_V(T,T) = V_T$. We consider the following modification of the Black and Cox approach

$$X = L,\ Z_t = \beta_2 V_t,\ \tilde{X} = \beta_1 V_T,\ \tau = \inf\{\, t \in [0,T] : V_t < v_t\},$$

where β_2, $\beta_1 \in [0,1]$ are constants, and the barrier v is given by the formula

$$v_t = \begin{cases} KB(t,T)e^{\int_t^T \kappa(u)\,du} & \text{for } t < T, \\ L & \text{for } t = T, \end{cases}$$

with the constant K satisfying $0 < K \leq L$.

Let us denote, for any $t \leq T$,

$$\kappa(t,T) = \int_t^T \kappa(u)\,du, \quad \sigma^2(t,T) = \int_t^T |\sigma(u) - b(u,T)|^2\,du$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. For brevity, we write $F_t = F_V^{\kappa}(t,T)$, and we denote

$$\eta_+(t,T) = \kappa(t,T) + \frac{1}{2}\sigma^2(t,T), \quad \eta_-(t,T) = \kappa(t,T) - \frac{1}{2}\sigma^2(t,T).$$

The following result extends Black and Cox valuation formula for a corporate bond to the case of random interest rates.

Proposition 1.3.1 *For any $t < T$, the forward price of a defaultable bond $F_D(t,T) = D(t,T)/B(t,T)$ equals on the set $\{\tau > t\}$*

$$\begin{aligned} &L\big(N\big(\widehat{h}_1(F_t,t,T)\big) - (F_t/K)e^{-\kappa(t,T)}N\big(\widehat{h}_2(F_t,t,T)\big)\big) \\ &\quad + \beta_1 F_t e^{-\kappa(t,T)}\big(N\big(\widehat{h}_3(F_t,t,T)\big) - N\big(\widehat{h}_4(F_t,t,T)\big)\big) \\ &\quad + \beta_1 K\big(N\big(\widehat{h}_5(F_t,t,T)\big) - N\big(\widehat{h}_6(F_t,t,T)\big)\big) \\ &\quad + \beta_2 K J_+(F_t,t,T) + \beta_2 F_t e^{-\kappa(t,T)}J_-(F_t,t,T), \end{aligned}$$

where

$$\begin{aligned}
\widehat{h}_1(F_t,t,T) &= \frac{\ln(F_t/L)-\eta_+(t,T)}{\sigma(t,T)},\\
\widehat{h}_2(F_t,T,t) &= \frac{2\ln K-\ln(LF_t)+\eta_-(t,T)}{\sigma(t,T)},\\
\widehat{h}_3(F_t,t,T) &= \frac{\ln(L/F_t)+\eta_-(t,T)}{\sigma(t,T)},\\
\widehat{h}_4(F_t,t,T) &= \frac{\ln(K/F_t)+\eta_-(t,T)}{\sigma(t,T)},\\
\widehat{h}_5(F_t,t,T) &= \frac{2\ln K-\ln(LF_t)+\eta_+(t,T)}{\sigma(t,T)},\\
\widehat{h}_6(F_t,t,T) &= \frac{\ln(K/F_t)+\eta_+(t,T)}{\sigma(t,T)},
\end{aligned}$$

and for any fixed $0 \le t < T$ *and* $F_t > 0$ *we set*

$$J_\pm(F_t,t,T) = \int_t^T e^{\kappa(u,T)}\,dN\left(\frac{\ln(K/F_t)+\kappa(t,T)\pm\frac{1}{2}\sigma^2(t,u)}{\sigma(t,u)}\right).$$

In the special case when $\kappa \equiv 0$, the formula of Proposition 1.3.1 covers as a special case the valuation result established by Briys and de Varenne [24]. In some other recent studies of first passage time models, in which the triggering barrier is assumed to be either a constant or an unspecified stochastic process, typically no closed-form solution for the value of a corporate debt is available, and thus a numerical approach is required (see, for instance, Longstaff and Schwartz [72], Nielsen et al. [78], or Saá-Requejo and Santa-Clara [81]).

1.4 Random Barrier

In the case of full information and Brownian filtration, the first hitting time of a deterministic barrier is predictable. This is no longer the case when we deal with incomplete information (as in Duffie and Lando [41], see also Chapter 2, Section 2.2.7), or when an additional source of randomness is present. We present here a formula for credit spreads arising in a special case of a totally inaccessible time of default. For a more detailed study we refer to Babbs and Bielecki [2]. As we shall see, the method we use here is close to the general method presented in Chapter 3.

We suppose here that the default barrier is a random variable η defined on the underlying probability space $(\Omega, \mathbb{P})$. The default occurs at time τ where

$$\tau = \inf\{t : V_t \le \eta\},$$

where V is the value of the firm and, for simplicity, $V_0 = 1$. Note that

$$\{\tau > t\} = \{\inf_{u\le t} V_u > \eta\}.$$

We shall denote by m_t^V the running minimum of V, i.e. $m_t^V = \inf_{u\le t} V_u$. With this notation, $\{\tau > t\} = \{m_t^V > \eta\}$. Note that m^V is a decreasing process.

1.4.1 Independent Barrier

In a first step we assume that, under the risk-neutral probability $\mathbb{Q}$, a random variable η modelling is independent of the value of the firm. We denote by F_η the cumulative distribution function of η, i.e., $F_\eta(z) = \mathbb{Q}(\eta \le z)$. We assume that F_η is differentiable and we denote by f_η its derivative.

Lemma 1.4.1 *Let* $F_t = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t)$ *and* $\Gamma_t = -\ln(1 - F_t)$. *Then*

$$\Gamma_t = -\int_0^t \frac{f_\eta(m_u^V)}{F_\eta(m_u^V)}\, dm_u^V.$$

Proof. If η is independent of $\mathcal{F}_\infty$, then

$$F_t = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(m_t^V \le \eta \,|\, \mathcal{F}_t) = 1 - F_\eta(m_t^V).$$

The process m^V is decreasing. It follows that $\Gamma_t = -\ln F_\eta(m_t^V)$, hence $d\Gamma_t = -\frac{f_\eta(m_t^V)}{F_\eta(m_t^V)}\, dm_t^V$ and

$$\Gamma_t = -\int_0^t \frac{f_\eta(m_u^V)}{F_\eta(m_u^V)}\, dm_u^V$$

as expected. □

Example 1.4.1 Assume that η is uniformly distributed on the interval $[0,1]$. Then, $\Gamma_t = -\ln m_t^V$. The computation of the expected value $\mathbb{E}_{\mathbb{Q}}(e^{\Gamma_T} f(V_T))$ requires the knowledge of the joint law of the pair (V_T, m_T^V).

We postulate now that the value process V is a geometric Brownian motion with a drift, that is, we set $V_t = e^{\Psi_t}$, where $\Psi_t = \mu t + \sigma W_t$. It is clear that $\tau = \inf\{\, t \in \mathbb{R}_+ : \Psi_t^* \le \psi\}$, where Ψ^* is the *running minimum* of the process Ψ: $\Psi_t^* = \inf\{\, \Psi_s : 0 \le s \le t\}$.

We choose the Brownian filtration as the reference filtration, i.e., we set $\mathbb{F} = \mathbb{F}^W$. Let us denote by $G(z)$ the cumulative distribution function under $\mathbb{Q}$ of the barrier ψ. We assume that $G(z) > 0$ for $z < 0$ and that G admits the density g with respect to the Lebesgue measure (note that $g(z) = 0$ for $z > 0$). This means that we assume that the value process V (hence also the process Ψ) is perfectly observed.

In addition, we postulate that the bond investor can observe the occurrence of the default time. Thus, he can observe the process $H_t = \mathbb{1}_{\{\tau \le t\}} = \mathbb{1}_{\{\Psi_t^* \le \psi\}}$. We denote by $\mathbb{H}$ the natural filtration of the process H. The information available to the investor is represented by the (enlarged) filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

We assume that the default time τ and interest rates are independent under $\mathbb{Q}$. Then, it is possible to establish the following result (see Giesecke [50] or Babbs and Bielecki [2]). Note that the process Ψ^* is decreasing, so that the integral with respect to this process is a (pathwise) Stieltjes integral.

Proposition 1.4.1 *Under the assumptions stated above, and additionally assuming* $L = 1$, $Z \equiv 0$ *and* $\tilde{X} = 0$, *we have that for every* $t < T$

$$S(t,T) = -\mathbb{1}_{\{\tau > t\}} \frac{1}{T-t} \ln \mathbb{E}_{\mathbb{Q}}\Big(e^{\int_t^T \frac{f_\eta(\Psi_u^*)}{F_\eta(\Psi_u^*)}\, d\Psi_u^*} \,\Big|\, \mathcal{F}_t \Big).$$

Later on, we will introduce the notion of a hazard process of a random time. For the default time τ defined above, the $\mathbb{F}$-hazard process Γ exists and is given by the formula

$$\Gamma_t = -\int_0^t \frac{f_\eta(\Psi_u^*)}{F_\eta(\Psi_u^*)}\, d\Psi_u^*.$$

This process is continuous, and thus the default time τ is a totally inaccessible stopping time with respect to the filtration $\mathbb{G}$.

Chapter 2

Hazard Function Approach

We provide in this chapter a detailed analysis of the relatively simple case of the *reduced form* methodology, when the flow of information available to an agent reduces to the observations of the random time which models the default event. The focus is on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the hazard function. We also study hedging strategies based on credit default swaps and/or defaultable zero-coupon bonds. Finally, we also present a credit risk model with several default times.

2.1 The Toy Model

We begin with the simple case where a riskless asset, with deterministic interest rate $(r(s); s \geq 0)$ is the only asset available in the default-free market. The price at time t of a risk-free zero-coupon bond with maturity T equals

$$B(t,T) = \exp\Big(-\int_t^T r(s)\,ds\Big).$$

Default occurs at time τ, where τ is assumed to be a positive random variable with density f, constructed on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. We denote by F the cumulative function of the random varible τ defined as $F(t) = \mathbb{Q}(\tau \leq t) = \int_0^t f(s)\,ds$ and we assume that $F(t) < 1$ for any $t > 0$. Otherwise, there would exists a date t_0 for which $F(t_0) = 1$, so that the default would occurs before or at t_0 with probability 1.

We emphasize that the random payoff of the form $\mathbb{1}_{\{T<\tau\}}$ cannot be perfectly hedged with deterministic zero-coupon bonds, which are the only tradeable primary assets in our model. To hedge the risk, we shall later postulate that some defaultable asset is traded, e.g., a defaultable zero-coupon bond or a credit default swap.

It is not difficult to generalize the study presented in what follows to the case where τ does not admit a density, by dealing with the right-continuous version of the cumulative function. The case where τ is bounded can also be studied along the same method. We leave the details to the reader.

2.1.1 Defaultable Zero-Coupon Bond with Payment at Maturity

A *defaultable zero-coupon bond* (DZC in short), or a *corporate zero-coupon bond*, with maturity T and the rebate (recovery) δ paid at maturity, consists of:

- The payment of one monetary unit at time T if default has not occurred before time T, i.e., if $\tau > T$,
- A payment of δ monetary units, made at maturity, if $\tau \leq T$, where $0 < \delta < 1$.

Value of the Defaultable Zero-Coupon Bond

The "fair value" of the defaultable zero-coupon bond is defined as the expectation of discounted payoffs

$$\begin{aligned} D^{(\delta)}(0,T) &= B(0,T)\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\le T\}}\big) \\ &= B(0,T)\,\mathbb{E}_{\mathbb{Q}}\big(1-(1-\delta)\mathbb{1}_{\{\tau\le T\}}\big) \\ &= B(0,T)\big(1-(1-\delta)F(T)\big). \end{aligned} \tag{2.1}$$

In fact, this quantity is a net present value and is equal to the value of the default free zero-coupon bond minus the expected loss, computed under the historical probability. Obviously, this value is not a hedging price.

The time-t value depends whether or not default has happened before this time. If default has occurred before time t, the payment of δ will be made at time T, and the price of the DZC is $\delta B(t,T)$.

If the default has not yet occurred, the holder does not know when it will occur. The value $D^{(\delta)}(t,T)$ of the DZC is the conditional expectation of the discounted payoff

$$B(t,T)\,\big(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\le T\}}\big)$$

given the information available at time t. We obtain

$$D^{(\delta)}(t,T) = \mathbb{1}_{\{\tau\le t\}}B(t,T)\delta + \mathbb{1}_{\{t<\tau\}}\widetilde{D}^{(\delta)}(t,T)$$

where the pre-default value $\widetilde{D}^{(\delta)}$ is defined as

$$\begin{aligned} \widetilde{D}^{(\delta)}(t,T) &= \mathbb{E}_{\mathbb{Q}}\big(B(t,T)\,(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\le T\}})\,\big|\,t<\tau\big) \\ &= B(t,T)\Big(1-(1-\delta)\mathbb{Q}(\tau\le T\,\big|\,t<\tau)\Big) \\ &= B(t,T)\left(1-(1-\delta)\frac{\mathbb{Q}(t<\tau\le T)}{\mathbb{Q}(t<\tau)}\right) \\ &= B(t,T)\left(1-(1-\delta)\frac{F(T)-F(t)}{1-F(t)}\right). \end{aligned} \tag{2.2}$$

Note that the value of the DZC is discontinuous at time τ, unless $F(T)=1$ (or $\delta=1$). In the case $F(T)=1$, the default appears with probability one before maturity and the DZC is equivalent to a payment of δ at maturity. If $\delta=1$, the DZC is simply a default-free zero coupon bond.

Formula (2.2) can be rewritten as follows

$$D^{(\delta)}(t,T) = B(t,T) - \text{EDLGD} \times \text{DP}$$

where the *expected discounted loss given default* (EDLGD) is defined as $B(t,T)(1-\delta)$ and the conditional *default probability* (DP) is defined as follows

$$DP = \frac{\mathbb{Q}(t<\tau\le T)}{\mathbb{Q}(t<\tau)} = \mathbb{Q}(\tau\le T\,|\,t<\tau)\;.$$

In case the payment is a function of the default time, say $\delta(\tau)$, the value of this defaultable zero-coupon is

$$\begin{aligned} D^{(\delta)}(0,T) &= \mathbb{E}_{\mathbb{Q}}\big(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,T)\delta(\tau)\mathbb{1}_{\{\tau\le T\}}\big) \\ &= B(0,T)\Big(\mathbb{Q}(T<\tau) + \int_0^T \delta(s)f(s)\,ds\Big). \end{aligned}$$

If the default has not occurred before t, the pre-default time-t value $\widetilde{D}^{(\delta)}(t,T)$ satisfies

$$\begin{aligned}\widetilde{D}^{(\delta)}(t,T) &= B(t,T)\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T<\tau\}} + \delta(\tau)\mathbb{1}_{\{\tau\le T\}}\,\big|\,t<\tau\big)\\ &= B(t,T)\Big(\frac{\mathbb{Q}(T<\tau)}{\mathbb{Q}(t<\tau)} + \frac{1}{\mathbb{Q}(t<\tau)}\int_t^T \delta(s)f(s)\,ds\Big).\end{aligned}$$

To summarize, we have

$$D^{(\delta)}(t,T) = \mathbb{1}_{\{t<\tau\}}\,\widetilde{D}^{(\delta)}(t,T) + \mathbb{1}_{\{\tau\le t\}}\,\delta(\tau)\,B(t,T).$$

Hazard Function

Let us recall the standing assumption that $F(t) < 1$ for any $t \in \mathbb{R}_+$. We introduce the *hazard function* Γ by setting

$$\Gamma(t) = -\ln(1-F(t))$$

for any $t \in \mathbb{R}_+$. Since we assumed that F is differentiable, the derivative $\Gamma'(t) = \gamma(t) = \dfrac{f(t)}{1-F(t)}$, where $f(t) = F'(t)$. This means that

$$1 - F(t) = e^{-\Gamma(t)} = \exp\left(-\int_0^t \gamma(s)\,ds\right) = \mathbb{Q}(\tau > t).$$

The quantity $\gamma(t)$ is the *hazard rate*. The interpretation of the hazard rate is the probability that the default occurs in a small interval dt given that the default did not occur before time t

$$\gamma(t) = \lim_{h\to 0}\frac{1}{h}\,\mathbb{Q}(\tau \le t+h\,|\,\tau > t).$$

Note that Γ is increasing.

Then, formula (2.2) reads

$$\begin{aligned}\widetilde{D}^{(\delta)}(t,T) &= B(t,T)\left(\frac{1-F(T)}{1-F(t)} + \delta\frac{F(T)-F(t)}{1-F(t)}\right)\\ &= R_T^{t,d} + \delta\big(B(t,T) - R_T^{t,d}\big),\end{aligned}$$

where we denote

$$R_T^{t,d} = \exp\Big(-\int_t^T (r(s)+\gamma(s))\,ds\Big).$$

In particular, for $\delta = 0$, we obtain $\widetilde{D}(t,T) = R_T^{t,d}$. Hence the spot rate has simply to be adjusted by means of the *credit spread* (equal to γ) in order to evaluate DZCs with zero recovery.

The dynamics of $\widetilde{D}^{(\delta)}$ can be easily written in terms of the function γ as

$$d\widetilde{D}^{(\delta)}(t,T) = (r(t)+\gamma(t))\,\widetilde{D}^{(\delta)}(t,T)\,dt - B(t,T)\gamma(t)\delta(t)\,dt.$$

The dynamics of $D^{(\delta)}(t,T)$ will be derived in the next section.

If γ and δ are constant, the credit spread equals

$$\frac{1}{T-t}\ln\frac{B(t,T)}{\widetilde{D}^{(\delta)}(t,T)} = \gamma - \frac{1}{T-t}\ln\Big(1+\delta(e^{\gamma(T-t)}-1)\Big)$$

and it converges to $\gamma(1-\delta)$ when t goes to T.

For any $t < T$, the quantity $\gamma(t,T) = \frac{f(t,T)}{1-F(t,T)}$ where

$$F(t,T) = \mathbb{Q}(\tau \le T \mid \tau > t)$$

and $f(t,T)\,dT = \mathbb{Q}(\tau \in dT \,|\, \tau > t)$ is called the *conditional hazard rate.* It is easily seen that

$$F(t,T) = 1 - \exp\Big(-\int_t^T \gamma(s,T)\,ds\Big).$$

Note, however, that in the present setting, we have that

$$1 - F(t,T) = \frac{\mathbb{Q}(\tau > T)}{\mathbb{Q}(\tau > t)} = \exp\Big(-\int_t^T \gamma(s)\,ds\Big)$$

and thus $\gamma(s,T) = \gamma(s)$.

Remark 2.1.1 In case τ is the first jump of an inhomogeneous Poisson process with deterministic intensity $(\lambda(t), t \geq 0)$

$$f(t) = \frac{\mathbb{Q}(\tau \in dt)}{dt} = \lambda(t)\exp\Big(-\int_0^t \lambda(s)\,ds\Big) = \lambda(t)e^{-\Lambda(t)}$$

where $\Lambda(t) = \int_0^t \lambda(s)\,ds$ and $\mathbb{Q}(\tau \leq t) = F(t) = 1 - e^{-\Lambda(t)}$. Hence the hazard function is equal to the compensator of the Poisson process, i.e., $\Gamma(t) = \Lambda(t)$. Conversely, if τ is a random time with density f, setting $\Lambda(t) = -\ln(1 - F(t))$ allows us to interpret τ as the first jump time of an inhomogeneous Poisson process with the intensity equal to the derivative of Λ.

2.1.2 Defaultable Zero-Coupon with Payment at Default

Here, a defaultable zero-coupon bond with maturity T consists of:

- The payment of one monetary unit at time T if default has not yet occurred,
- The payment of $\delta(\tau)$ monetary units, where δ is a deterministic function, made at time τ if $\tau \leq T$.

Value of the Defaultable Zero-Coupon

The value of this defaultable zero-coupon bond is

$$\begin{aligned} D^{(\delta)}(0,T) &= \mathbb{E}_{\mathbb{Q}}\big(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,\tau)\delta(\tau)\mathbb{1}_{\{\tau\leq T\}}\big) \\ &= \mathbb{Q}(T<\tau)B(0,T) + \int_0^T B(0,s)\delta(s)\,dF(s) \\ &= G(T)B(0,T) - \int_0^T B(0,s)\delta(s)\,dG(s), \end{aligned} \tag{2.3}$$

where $G(t) = 1 - F(t) = \mathbb{Q}(t < \tau)$ is the survival probability. Obviously, if the default has occurred before time t, the value of the DZC is null (this was not the case for the recovery payment made at bond's maturity), and $D^{(\delta)}(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}^{(\delta)}(t,T)$ where $\widetilde{D}^{(\delta)}(t,T)$ is a deterministic function (the predefault price). The pre-default time-t value $\widetilde{D}^{(\delta)}(t,T)$ satisfies

$$\begin{aligned} B(0,t)\widetilde{D}^{(\delta)}(t,T) &= \mathbb{E}_{\mathbb{Q}}\big(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,\tau)\delta(\tau)\mathbb{1}_{\{\tau\leq T\}} \,|\, t<\tau\big) \\ &= \frac{\mathbb{Q}(T<\tau)}{\mathbb{Q}(t<\tau)}B(0,T) + \frac{1}{\mathbb{Q}(t<\tau)}\int_t^T B(0,s)\delta(s)\,dF(s). \end{aligned}$$

Hence

$$R(t)G(t)\widetilde{D}^{(\delta)}(t,T) = G(T)B(0,T) - \int_t^T B(0,s)\delta(s)\,dG(s).$$

In terms of the hazard function Γ, we get

$$\widetilde{D}^{(\delta)}(0,T) = e^{-\Gamma(T)}B(0,T) + \int_0^T B(0,s)e^{-\Gamma(s)}\delta(s)\,d\Gamma(s). \tag{2.4}$$

The time-t value $\widetilde{D}^{(\delta)}(t,T)$ satisfies

$$B(0,t)e^{-\Gamma(t)}\widetilde{D}^{(\delta)}(t,T) = e^{-\Gamma(T)}B(0,T) + \int_t^T B(0,s)e^{-\Gamma(s)}\delta(s)\,d\Gamma(s).$$

Note that the process $t \to D^{(\delta)}(t,T)$ admits a discontinuity at time τ.

A Particular Case

If F is differentiable then the function $\gamma = \Gamma'$ satisfies $f(t) = \gamma(t)e^{-\Gamma(t)}$. Then,

$$\begin{aligned}\widetilde{D}^{(\delta)}(0,T) &= e^{-\Gamma(T)}B(0,T) + \int_0^T B(0,s)\gamma(s)e^{-\Gamma(s)}\delta(s)\,ds, \\ &= R^d(T) + \int_0^T R^d(s)\gamma(s)\delta(s)\,ds,\end{aligned} \tag{2.5}$$

and

$$R^d(t)\widetilde{D}^{(\delta)}(t,T) = R^d(T) + \int_t^T R^d(s)\gamma(s)\delta(s)\,ds$$

with

$$R^d(t) = \exp\Big(-\int_0^t (r(s)+\gamma(s))\,ds\Big).$$

The 'defaultable interest rate' is $r+\gamma$ and is, as expected, greater than r (the value of a DZC with $\delta = 0$ is smaller than the value of a default-free zero-coupon). The dynamics of $\widetilde{D}^{(\delta)}(t,T)$ are

$$d\widetilde{D}^{(\delta)}(t,T) = \Big((r(t)+\gamma(t))\widetilde{D}^{(\delta)}(t,T) - \delta(t)\gamma(t)\Big)dt.$$

The dynamics of $D^{(\delta)}(t,T)$ include a jump at time τ (see the next section).

Fractional Recovery of Treasury Value

This case corresponds to the the following recovery $\delta(t) = \delta B(t,T)$ at the moment of default. Under this convention, we have that

$$D^{(\delta)}(t,T) = \mathbb{1}_{\{t<\tau\}}\left(e^{-\int_t^T (r(s)+\gamma(s)\,ds} + \delta B(t,T)\int_t^T \gamma(s)e^{\int_t^s \gamma(u)\,du}\,ds\right).$$

Fractional Recovery of Market Value

Let us assume here that the recovery is $\delta(t) = \delta\widetilde{D}^{(\delta)}(t,T)$ where δ is a constant, that is, the recovery is $\delta D^{(\delta)}(\tau-,T)$. The dynamics of $\widetilde{D}^{(\delta)}$ are

$$d\widetilde{D}^{(\delta)}(t,T) = \big(r(t)+\gamma(t)(1-\delta(t))\big)\widetilde{D}^{(\delta)}(t,T)\,dt,$$

hence

$$\widetilde{D}^{(\delta)}(t,T) = \exp\left(-\int_t^T r(s)ds - \int_t^T \gamma(s)(1-\delta(s))\,ds\right).$$

2.1.3 Implied Default Probabilities

If defaultable zero-coupon bonds with zero recovery are traded in the market at price $D^{(\delta,*)}(t,T)$, the implied survival probability is $\mathbb{Q}^*$ such that

$$\mathbb{Q}^*(\tau > T \,|\, \tau > t) = \frac{D^{(\delta,*)}(t,T)}{B(t,T)}.$$

Of course, this probability may differ from the historical probability. The implied hazard rate is the function $\lambda(t,T)$ such that

$$\lambda(t,T) = -\frac{\partial}{\partial T} \ln \frac{D^{(\delta,*)}(t,T)}{B(t,T)} = \gamma^*(T).$$

In the toy model, the implied hazard rate is not very interesting. The aim is to obtain

$$\widetilde{D}^{(\delta,*)}(t,T) = B(t,T) \exp\Big(-\int_t^T \lambda(t,s)\, ds\Big).$$

This approach will be useful when the pre-default price is stochastic, rather than deterministic.

2.1.4 Credit Spreads

A term structure of credit spreads associated with the zero-coupon bonds $S(t,T)$ is defined as

$$S(t,T) = -\frac{1}{T-t} \ln \frac{D^{(\delta,*)}(t,T)}{B(t,T)}.$$

In our setting, on the set $\{\tau > t\}$

$$S(t,T) = -\frac{1}{T-t} \ln \mathbb{Q}^*(\tau > T \,|\, \tau > t),$$

whereas $S(t,T) = \infty$ on the set $\{\tau \le t\}$.

2.2 Martingale Approach

We shall now present the results of the previous section in a different form, following rather closely Dellacherie ([36], page 122). We keep the standing assumption that $F(t) < 1$ for any $t \in \mathbb{R}_+$, but we do impose any further assumptions on the c.d.f. F of τ under $\mathbb{Q}$ at this stage.

Definition 2.2.1 *The hazard function* Γ *by setting*

$$\Gamma(t) = -\ln(1 - F(t))$$

for any $t \in \mathbb{R}_+$.

We denote by $(H_t, t \ge 0)$ the right-continuous increasing process $H_t = \mathbb{1}_{\{t \ge \tau\}}$ and by $(\mathcal{H}_t)$ its natural filtration. The filtration $\mathbb{H}$ is the smallest filtration which makes τ a stopping time. The σ-algebra $\mathcal{H}_t$ is generated by the sets $\{\tau \le s\}$ for $s \le t$. The key point is that any integrable $\mathcal{H}_t$-measurable r.v. H has the form

$$H = h(\tau)\mathbb{1}_{\{\tau \le t\}} + h(t)\mathbb{1}_{\{t < \tau\}}$$

where h is a Borel function.

We now give some elementary formula for the computation of a conditional expectation with respect to $\mathcal{H}_t$, as presented, for instance, in Brémaud [19], Dellacherie [36], or Elliott [44].

Remark 2.2.1 Note that if the cumulative distribution function F is continuous then τ is known to be a $\mathbb{H}$-totally inaccessible stopping time (see Dellacherie and Meyer [39] IV, Page 107). We will not use this property explicitly.

2.2.1 Key Lemma

Lemma 2.2.1 *For any integrable, $\mathcal{G}$-measurable r.v. X we have that*

$$\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{H}_s)\mathbb{1}_{\{s<\tau\}} = \mathbb{1}_{\{s<\tau\}} \frac{\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{s<\tau\}})}{\mathbb{Q}(s<\tau)}\,. \tag{2.6}$$

Proof. The conditional expectation $\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{H}_s)$ is clearly $\mathcal{H}_s$-measurable. Therefore, it can be written in the form

$$\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{H}_s) = h(\tau)\mathbb{1}_{\{s\geq\tau\}} + h(s)\mathbb{1}_{\{s<\tau\}}$$

for some Borel function h. By multiplying both members by $\mathbb{1}_{\{s<\tau\}}$, and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{s<\tau\}}\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{H}_s)] &= \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}}X \,|\, \mathcal{H}_s)] = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}}X) \\
&= \mathbb{E}_{\mathbb{Q}}(h(s)\mathbb{1}_{\{s<\tau\}}) = h(s)\mathbb{Q}(s<\tau).
\end{aligned}$$

Hence $h(s) = \dfrac{\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{s<\tau\}})}{\mathbb{Q}(s<\tau)}$, which yields the desired result. □

Corollary 2.2.1 *Assume that Y is $\mathcal{H}_\infty$-measurable, so that $Y = h(\tau)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$. If the hazard function Γ of τ is continuous then*

$$\mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau\leq t\}}h(\tau) + \mathbb{1}_{\{t<\tau\}} \int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\, d\Gamma(u). \tag{2.7}$$

If τ admits the intensity function γ then

$$\mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau\leq t\}}h(\tau) + \mathbb{1}_{\{t<\tau\}} \int_t^\infty h(u)\gamma(u)e^{-\int_t^u \gamma(v)\,dv}\, du.$$

In particular, for any $t \leq s$ we have

$$\mathbb{Q}(\tau > s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}}e^{-\int_t^s \gamma(v)\,dv}$$

and

$$\mathbb{Q}(t<\tau<s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}}\Big(1 - e^{-\int_t^s \gamma(v)\,dv}\Big).$$

2.2.2 Martingales Associated with Default Time

Proposition 2.2.1 *The process $(M_t, t \geq 0)$ defined as*

$$M_t = H_t - \int_0^{\tau\wedge t} \frac{dF(s)}{1-F(s)} = H_t - \int_0^t (1-H_{s-})\frac{dF(s)}{1-F(s)}$$

is an $\mathbb{H}$-martingale.

Proof. Let $s < t$. Then:

$$\mathbb{E}_{\mathbb{Q}}(H_t - H_s \,|\, \mathcal{H}_s) = \mathbb{1}_{\{s<\tau\}}\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\leq t\}} \,|\, \mathcal{H}_s) = \mathbb{1}_{\{s<\tau\}}\frac{F(t)-F(s)}{1-F(s)}, \tag{2.8}$$

which follows from (2.6) with $X = \mathbb{1}_{\{\tau\leq t\}}$.

On the other hand, the quantity

$$C \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{Q}}\left[\int_s^t (1-H_{u-})\frac{dF(u)}{1-F(u)} \,\Big|\, \mathcal{H}_s\right],$$

is equal to

$$\begin{aligned} C &= \int_s^t \frac{dF(u)}{1-F(u)} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>u\}} | \mathcal{H}_s\right] \\ &= \mathbb{1}_{\{\tau>s\}} \int_s^t \frac{dF(u)}{1-F(u)} \left(1 - \frac{F(u)-F(s)}{1-F(s)}\right) \\ &= \mathbb{1}_{\{\tau>s\}} \left(\frac{F(t)-F(s)}{1-F(s)}\right) \end{aligned}$$

which, in view of (2.8), proves the result. □

The function

$$\int_0^t \frac{dF(s)}{1-F(s)} = -\ln(1-F(t)) = \Gamma(t)$$

is the *hazard function*.

From Proposition 2.2.1, we obtain the Doob-Meyer decomposition of the submartingale H_t as $M_t + \Gamma(t \wedge \tau)$. The predictable process $A_t = \Gamma(t \wedge \tau)$ is called the *compensator* of H.

In particular, if F is differentiable, the process

$$M_t = H_t - \int_0^{\tau \wedge t} \gamma(s)\, ds = H_t - \int_0^t \gamma(s)(1-H_s)\, ds$$

is a martingale, where $\gamma(s) = \dfrac{f(s)}{1-F(s)}$ is a deterministic, non-negative function, called *the intensity of* τ.

Proposition 2.2.2 *Assume that F (and thus also Γ) is a continuous function. Then the process $M_t = H_t - \Gamma(t \wedge \tau)$ follows a $\mathbb{D}$-martingale.*

We can now write the dynamics of a defaultable zero-coupon bond with recovery δ paid at hit, assuming that M is a martingale under the risk-neutral probability.

Proposition 2.2.3 *The risk-neutral dynamics of a DZC with recovery paid at hit is*

$$dD^{(\delta)}(t,T) = \Big(r(t)D^{(\delta)}(t,T) - \delta(t)\gamma(t)(1-H_t)\Big)\, dt - \widetilde{D}^{(\delta)}(t,T)\, dM_t \tag{2.9}$$

where M is the risk-neutral martingale $M_t = H_t - \int_0^t (1-H_s)\gamma_s\, ds$.

Proof. Combining the equality

$$D^{(\delta)}(t,T) = \mathbb{1}_{t<\tau} \widetilde{D}^{(\delta)}(t,T) = (1-H_t)\widetilde{D}^{(\delta)}(t,T)$$

with the dynamics of $\widetilde{D}^{(\delta)}(t,T)$, we obtain

$$\begin{aligned} dD^{(\delta)}(t,T) &= (1-H_t) d\widetilde{D}^{(\delta)}(t,T) - \widetilde{D}^{(\delta)}(t,T)\, dH_t \\ &= (1-H_t)\Big((r(t)+\gamma(t))\widetilde{D}^{(\delta)}(t,T) - \delta(t)\gamma(t)\Big)\, dt - \widetilde{D}^{(\delta)}(t,T))\, dH_t \\ &= \Big(r(t)D^{(\delta)}(t,T) - \delta(t)\gamma(t)(1-H_t)\Big)\, dt - \widetilde{D}^{(\delta)}(t,T)\, dM_t \end{aligned}$$

We emphasize that we are working here under a risk-neutral probability. We shall see further on how to compute the risk-neutral default intensity from historical one, using a suitable Radon-Nikodým density process. □

Proposition 2.2.4 *The process* $L_t \stackrel{\text{def}}{=} \mathbb{1}_{\{\tau>t\}} \exp\left(\int_0^t \gamma(s)ds\right)$ *is an* $\mathbb{H}$*-martingale and it satisfies*

$$L_t = 1 - \int_{]0,t]} L_{u-}\, dM_u. \tag{2.10}$$

In particular, for $t \in [0,T]$,

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau>t\}} \exp\left(-\int_t^T \gamma(s)ds\right).$$

Proof. Let us first show that L is an $\mathbb{H}$-martingale. Since the function γ is deterministic, for $t > s$

$$\mathbb{E}_{\mathbb{Q}}(L_t \,|\, \mathcal{H}_s) = \exp\left(\int_0^t \gamma(u)du\right) \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} \,|\, \mathcal{H}_s).$$

From the equality (2.6)

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} \,|\, \mathcal{H}_s) = \mathbb{1}_{\{\tau>s\}} \frac{1-F(t)}{1-F(s)} = \mathbb{1}_{\{\tau>s\}} \exp\left(-\Gamma(t) + \Gamma(s)\right).$$

Hence

$$\mathbb{E}_{\mathbb{Q}}(L_t \,|\, \mathcal{H}_s) = \mathbb{1}_{\{\tau>s\}} \exp\left(\int_0^s \gamma(u)\, du\right) = L_s.$$

To establish (2.10), it suffices to apply the integration by parts formula to the process

$$L_t = (1-H_t) \exp\left(\int_0^t \gamma(s)\, ds\right).$$

We obtain

$$\begin{aligned} dL_t &= -\exp\left(\int_0^t \gamma(s)\, ds\right) dH_t + \gamma(t) \exp\left(\int_0^t \gamma(s)\, ds\right)(1-H_t)\, dt \\ &= -\exp\left(\int_0^t \gamma(s)\, ds\right) dM_t. \end{aligned}$$

An alternative method is to show that L is the exponential martingale of M, i.e., L is the unique solution of the SDE

$$dL_t = -L_{t-}\, dM_t, \quad L_0 = 1.$$

This equation can be solved pathwise. □

Proposition 2.2.5 *Assume that* Γ *is a continuous function. Then for any (bounded) Borel measurable function* $h : \mathbb{R}_+ \to \mathbb{R}$, *the process*

$$M_t^h = \mathbb{1}_{\{\tau \le t\}} h(\tau) - \int_0^{t\wedge\tau} h(u)\, d\Gamma(u) \tag{2.11}$$

is an $\mathbb{H}$*-martingale.*

Proof. The proof given below provides an alternative proof of Proposition 2.2.2. We wish to establish through direct calculations the martingale property of the process M^h given by formula (2.11). To this end, notice that formula (2.7) in Corollary 2.2.1 gives

$$E\big(h(\tau)\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{H}_t\big) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)} \int_t^s h(u) e^{-\Gamma(u)}\, d\Gamma(u).$$

On the other hand, using the same formula, we get

$$J \stackrel{\text{def}}{=} E\Big(\int_{t\wedge\tau}^{s\wedge\tau} h(u)\,d\Gamma(u)\Big) = E\big(\tilde{h}(\tau)\mathbb{1}_{\{t<\tau\le s\}} + \tilde{h}(s)\mathbb{1}_{\{\tau>s\}}\,|\,\mathcal{H}_t\big)$$

where we set $\tilde{h}(s) = \int_t^s h(u)\,d\Gamma(u)$. Consequently,

$$J = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)}\Big(\int_t^s \tilde{h}(u)e^{-\Gamma(u)}\,d\Gamma(u) + e^{-\Gamma(s)}\tilde{h}(s)\Big).$$

To conclude the proof, it is enough to observe that Fubini's theorem yields

$$\begin{aligned}
&\int_t^s e^{-\Gamma(u)}\int_t^u h(v)\,d\Gamma(v)\,d\Gamma(u) + e^{-\Gamma(s)}\tilde{h}(s)\\
&= \int_t^s h(u)\int_u^s e^{-\Gamma(v)}\,d\Gamma(v)\,d\Gamma(u) + e^{-\Gamma(s)}\int_t^s h(u)\,d\Gamma(u)\\
&= \int_t^s h(u)e^{-\Gamma(u)}\,d\Gamma(u),
\end{aligned}$$

as expected. □

Corollary 2.2.2 *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a (bounded) Borel measurable function. Then the process*

$$\widetilde{M}_t^h = \exp\big(\mathbb{1}_{\{\tau\le t\}}h(\tau)\big) - \int_0^{t\wedge\tau}(e^{h(u)}-1)\,d\Gamma(u) \tag{2.12}$$

is an $\mathbb{H}$-martingale.

Proof. It is enough to observe that

$$\exp\big(\mathbb{1}_{\{\tau\le t\}}h(\tau)\big) = \mathbb{1}_{\{\tau\le t\}}e^{h(\tau)} + \mathbb{1}_{\{t\ge\tau\}} = \mathbb{1}_{\{\tau\le t\}}(e^{h(\tau)}-1)+1$$

and to apply the preceding result to $e^h - 1$. □

Proposition 2.2.6 *Assume that Γ is a continuous function. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Then the process*

$$\widehat{M}_t = (1+\mathbb{1}_{\{\tau\le t\}}h(\tau))\exp\Big(-\int_0^{t\wedge\tau} h(u)\,d\Gamma(u)\Big) \tag{2.13}$$

is an $\mathbb{H}$-martingale.

Proof. Observe that

$$\begin{aligned}
\widehat{M}_t &= \exp\Big(-\int_0^t (1-H_u)h(u)\,d\Gamma(u)\Big) + \mathbb{1}_{\{\tau\le t\}}h(\tau)\exp\Big(-\int_0^\tau (1-H_u)h(u)\,d\Gamma(u)\Big)\\
&= \exp\Big(-\int_0^t (1-H_u)h(u)\,d\Gamma(u)\Big) + \int_0^t h(u)\exp\Big(-\int_0^u (1-H_s)h(s)\,d\Gamma(s)\Big)dH_u
\end{aligned}$$

From Itô's calculus,

$$\begin{aligned}
d\widehat{M}_t &= \exp\Big(-\int_0^t (1-H_u)h(u)\,d\Gamma(u)\Big)(-(1-H_t)h(t)\,d\Gamma(t) + h(t)\,dH_t)\\
&= h(t)\exp\Big(-\int_0^t (1-H_u)h(u)\,d\Gamma(u)\Big)dM_t.
\end{aligned}$$

□

It is instructive to compare this result with the Doléans-Dade exponential of the process hM.

Example 2.2.1 In the case where N is an inhomogeneous Poisson process with deterministic intensity λ and τ is the moment of the first jump of N, let $H_t = N_{t\wedge\tau}$. It is well known that $N_t - \int_0^t \lambda(s)\,ds$ is a martingale. Therefore, the process stopped at time τ is also a martingale, i.e., $H_t - \int_0^{t\wedge\tau} \lambda(s)\,ds$ is a martingale. Furthermore, we have seen in Remark 2.1.1 that we can reduce our attention to this case, since any random time can be viewed as the first time where an inhomogeneous Poisson process jumps.

Exercise 2.2.1 Assume that F is only right-continuous, and let $F(t-)$ be the left-hand side limit of F at t. Show that the process $(M_t, t \geq 0)$ defined as

$$M_t = H_t - \int_0^{\tau\wedge t} \frac{dF(s)}{1-F(s-)} = H_t - \int_0^t (1-H_{s-})\frac{dF(s)}{1-F(s-)}$$

is an $\mathbb{H}$-martingale.

2.2.3 Representation Theorem

Proposition 2.2.7 *Let h be a (bounded) Borel function. Then, the martingale $M_t^h = \mathbb{E}_\mathbb{Q}(h(\tau)\,|\,\mathcal{H}_t)$ admits the representation*

$$\mathbb{E}_\mathbb{Q}(h(\tau)\,|\,\mathcal{H}_t) = \mathbb{E}_\mathbb{Q}(h(\tau)) - \int_0^{t\wedge\tau} (g(s)-h(s))\,dM_s,$$

where $M_t = H_t - \Gamma(t\wedge\tau)$ and

$$g(t) = -\frac{1}{G(t)}\int_t^\infty h(u)\,dG(u) = \frac{1}{G(t)}\,\mathbb{E}_\mathbb{Q}(h(\tau)\mathbb{1}_{\tau>t}). \tag{2.14}$$

Note that $g(t) = M_t^h$ on $\{t<\tau\}$. In particular, any square-integrable $\mathbb{H}$-martingale $(X_t, t\geq 0)$ can be written as $X_t = X_0 + \int_0^t x_s\,dM_s$ where $(x_t, t\geq 0)$ is an $\mathbb{H}$-predictable process.

Proof. We give below two different proofs.
a) From Lemma 2.2.1

$$\begin{aligned} M_t^h &= h(\tau)\mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{t<\tau\}}\frac{\mathbb{E}_\mathbb{Q}(h(\tau)\mathbb{1}_{\{t<\tau\}})}{\mathbb{Q}(t<\tau)} \\ &= h(\tau)\mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{t<\tau\}}e^{\Gamma(t)}\mathbb{E}_\mathbb{Q}(h(\tau)\mathbb{1}_{\{t<\tau\}}). \end{aligned}$$

An integration by parts leads to

$$\begin{aligned} &e^{\Gamma_t}\mathbb{E}_\mathbb{Q}\big(h(\tau)\mathbb{1}_{\{t<\tau\}}\big) = e^{\Gamma_t}\int_t^\infty h(s)dF(s) = g(t) \\ &= \int_0^\infty h(s)dF(s) - \int_0^t e^{\Gamma(s)}h(s)dF(s) + \int_0^t \mathbb{E}_\mathbb{Q}(h(\tau)\mathbb{1}_{\{s<\tau\}})e^{\Gamma(s)}d\Gamma(s) \end{aligned}$$

Therefore, since $\mathbb{E}_\mathbb{Q}(h(\tau)) = \int_0^\infty h(s)dF(s)$ and $M_s^h = e^{\Gamma(s)}\mathbb{E}_\mathbb{Q}(h(\tau)\mathbb{1}_{\{s<\tau\}}) = g(s)$ on $\{s<\tau\}$, the following equality holds on the set $\{t<\tau\}$:

$$e^{\Gamma_t}\mathbb{E}_\mathbb{Q}\big(h(\tau)\mathbb{1}_{\{t<\tau\}}\big) = \mathbb{E}_\mathbb{Q}(h(\tau)) - \int_0^t e^{\Gamma(s)}h(s)dF(s) + \int_0^t g(s)d\Gamma(s).$$

Hence

$$\begin{aligned} \mathbb{1}_{\{t<\tau\}}\mathbb{E}_\mathbb{Q}(h(\tau)\,|\,\mathcal{H}_t) &= \mathbb{1}_{\{t<\tau\}}\left(\mathbb{E}_\mathbb{Q}(h(\tau)) + \int_0^{t\wedge\tau}(g(s)-h(s))\frac{dF(s)}{1-F(s)}\right) \\ &= \mathbb{1}_{\{t<\tau\}}\left(\mathbb{E}_\mathbb{Q}(h(\tau)) - \int_0^{t\wedge\tau}(g(s)-h(s))(dH_s - d\Gamma(s))\right), \end{aligned}$$

where the last equality is due to $\mathbb{1}_{\{t<\tau\}} \int_0^{t\wedge\tau}(g(s)-h(s))dH_s = 0$.

On the complementary set $\{t \geq \tau\}$, we have seen that $\mathbb{E}_{\mathbb{Q}}(h(\tau)\,|\,\mathcal{H}_t) = h(\tau)$, whereas

$$\begin{aligned}\int_0^{t\wedge\tau}(g(s)-h(s))(dH_s - d\Gamma(s)) &= \int_{]0,\tau]}(g(s)-h(s))(dH_s - d\Gamma(s))\\ &= \int_{]0,\tau[}(g(s)-h(s))(dH_s - d\Gamma(s)) + (g(\tau^-)-h(\tau)).\end{aligned}$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_0^{t\wedge\tau}(g(s)-h(s))(dH_s - d\Gamma(s)) = M^H_{\tau^-} - (M^H_{\tau^-} - h(\tau)) = h(\tau).$$

The predictable representation theorem follows immediately.

b) An alternative proof consists in computing the conditional expectation

$$\begin{aligned}M^h_t &= \mathbb{E}_{\mathbb{Q}}(h(\tau)\,|\,\mathcal{H}_t) = h(\tau)\mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}}e^{-\Gamma(t)}\int_t^\infty h(u)dF(u)\\ &= \int_0^t h(s)\,dH_s + (1-H_t)e^{-\Gamma(t)}\int_t^\infty h(u)\,dF(u) = \int_0^t h(s)\,dH_s + (1-H_t)g(t)\end{aligned}$$

and to use Itô's formula and that $dM_t = dH_t - \gamma(t)(1-H_t)\,dt$. Using that

$$dF(t) = e^{\Gamma(t)}d\Gamma(t) = e^{\Gamma(t)}\gamma(t)\,dt = -dG(t)$$

we obtain

$$\begin{aligned}dM^h_t &= h(t)\,dH_t + (1-H_t)h(t)\gamma(t)dt - g(t)\,dH_t - (1-H_t)g(t)\gamma(t)\,dt\\ &= (h(t)-g(t))\,dH_t + (1-H_t)(h(t)-g(t))\gamma(t)\,dt = (h(t)-g(t))\,dM_t.\end{aligned}$$

This complete the proof. □

Exercise 2.2.2 Assume that Γ is right-continuous. Show that

$$\mathbb{E}_{\mathbb{Q}}(h(\tau)\,|\,\mathcal{H}_t) = \mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_0^{t\wedge\tau} e^{\Delta\Gamma(s)}(g(s)-h(s))\,dM_s.$$

2.2.4 Change of a Probability Measure

Let $\mathbb{Q}$ be an arbitrary probability measure on $(\Omega, \mathcal{H}_\infty)$, which is absolutely continuous with respect to $\mathbb{P}$. We denote by F the c.d.f. of τ under $\mathbb{P}$. Let η stand for the $\mathcal{H}_\infty$-measurable density of $\mathbb{Q}$ with respect to $\mathbb{P}$

$$\eta \stackrel{\text{def}}{=} \frac{d\mathbb{Q}}{d\mathbb{P}} = h(\tau) \geq 0, \quad \mathbb{P}\text{-a.s.}, \tag{2.15}$$

where $h : \mathbb{R} \to \mathbb{R}_+$ is a Borel measurable function satisfying

$$\mathbb{E}_{\mathbb{P}}(h(\tau)) = \int_0^\infty h(u)\,dF(u) = 1.$$

We can use Girsanov's theorem. Nevertheless, we prefer here to establish this theorem in our particular setting. Of course, the probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ if and only if the inequality in (2.15) is strict $\mathbb{P}$-a.s. Furthermore, we shall assume that $\mathbb{Q}(\tau = 0) = 0$ and $\mathbb{Q}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. Actually the first condition is satisfied for any $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$. For the second condition to hold, it is sufficient and necessary to assume that for every t

$$\mathbb{Q}(\tau > t) = 1 - F^*(t) = \int_{]t,\infty[} h(u)\,dF(u) > 0,$$

where F^* is the c.d.f. of τ under $\mathbb{Q}$

$$F^*(t) \stackrel{\text{def}}{=} \mathbb{Q}(\tau \leq t) = \int_{[0,t]} h(u)\, dF(u). \tag{2.16}$$

Put another way, we assume that

$$g(t) \stackrel{\text{def}}{=} e^{\Gamma(t)} \mathbb{E}\big(\mathbb{1}_{\{\tau > t\}} h(\tau)\big) = e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u) = e^{\Gamma(t)}\, \mathbb{Q}(\tau > t) > 0.$$

We assume throughout that this is the case, so that the hazard function Γ^* of τ with respect to $\mathbb{Q}$ is well defined. Our goal is to examine relationships between hazard functions Γ^* and Γ. It is easily seen that in general we have

$$\frac{\Gamma^*(t)}{\Gamma(t)} = \frac{\ln\Big(\int_{]t,\infty[} h(u)\, dF(u)\Big)}{\ln(1 - F(t))}, \tag{2.17}$$

since by definition $\Gamma^*(t) = -\ln(1 - F^*(t))$.

Assume first that F is an absolutely continuous function, so that the intensity function γ of τ under $\mathbb{P}$ is well defined. Recall that γ is given by the formula

$$\gamma(t) = \frac{f(t)}{1 - F(t)}.$$

On the other hand, the c.d.f. F^* of τ under $\mathbb{Q}$ now equals

$$F^*(t) \stackrel{\text{def}}{=} \mathbb{Q}(\tau \leq t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \leq t\}} h(\tau)) = \int_0^t h(u) f(u)\, du.$$

so that F^* follows an absolutely continuous function. Therefore, the intensity function γ^* of the random time τ under $\mathbb{Q}$ exists, and it is given by the formula

$$\gamma^*(t) = \frac{h(t)f(t)}{1 - F^*(t)} = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\, du}.$$

To derive a more straightforward relationship between the intensities γ and γ^*, let us introduce an auxiliary function $h^* : \mathbb{R}_+ \to \mathbb{R}$, given by the formula $h^*(t) = h(t)/g(t)$.

Notice that

$$\gamma^*(t) = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\, du} = \frac{h(t)f(t)}{\int_t^\infty h(u)f(u)\, du} = \frac{h(t)f(t)}{e^{-\Gamma(t)} g(t)} = h^*(t)\, \frac{f(t)}{1 - F(t)} = h^*(t)\gamma(t).$$

This means also that $d\Gamma^*(t) = h^*(t)\, d\Gamma(t)$. It appears that the last equality holds true if F is merely a continuous function. Indeed, if F (and thus F^*) is continuous, we get

$$d\Gamma^*(t) = \frac{dF^*(t)}{1 - F^*(t)} = \frac{d(1 - e^{-\Gamma(t)} g(t))}{e^{-\Gamma(t)} g(t)} = \frac{g(t) d\Gamma(t) - dg(t)}{g(t)} = h^*(t)\, d\Gamma(t).$$

To summarize, if the hazard function Γ is continuous then Γ^* is also continuous and $d\Gamma^*(t) = h^*(t)\, d\Gamma(t)$.

To understand better the origin of the function h^*, let us introduce the following non-negative $\mathbb{P}$-martingale (which is strictly positive when the probability measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent)

$$\eta_t \stackrel{\text{def}}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{H}_t} = \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{H}_t) = \mathbb{E}_{\mathbb{P}}(h(\tau) \,|\, \mathcal{H}_t), \tag{2.18}$$

so that $\eta_t = M_t^h$. The general formula for η_t reads (cf. (2.2.1))

$$\eta_t = \mathbb{1}_{\{\tau \leq t\}} h(\tau) + \mathbb{1}_{\tau > t}\, e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u) = \mathbb{1}_{\{\tau \leq t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} g(t).$$

Assume now that F is a continuous function. Then

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t)-\Gamma(u)} \, d\Gamma(u).$$

On the other hand, using the representation theorem, we get

$$M_t^h = M_0^h + \int_{]0,t]} M_{u-}^h (h^*(u) - 1) \, dM_u$$

where $h^*(u) = h(u)/g(u)$. We conclude that

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} (h^*(u) - 1) \, dM_u. \tag{2.19}$$

It is thus easily seen that

$$\eta_t = \big(1 + \mathbb{1}_{\{\tau \le t\}} v(\tau)\big) \exp\Big(- \int_0^{t\wedge\tau} v(u) \, d\Gamma(u)\Big), \tag{2.20}$$

where we write $v(t) = h^*(t) - 1$. Therefore, the martingale property of the process η, which is obvious from (2.18), is also a consequence of Proposition 2.2.6.

Remark 2.2.2 In view of (2.19), we have

$$\eta_t = \mathcal{E}_t\Big(\int_0^{\cdot} (h^*(u) - 1) \, dM_u \Big),$$

where $\mathcal{E}$ stands for the Doléans exponential. Representation (2.20) for the random variable η_t can thus be obtained from the general formula for the Doléans exponential.

We are in the position to formulate the following result (all statements were already established above).

Proposition 2.2.8 *Let $\mathbb{Q}$ be any probability measure on $(\Omega, \mathcal{H}_\infty)$ absolutely continuous with respect to $\mathbb{P}$, so that (2.15) holds for some function h. Assume that $\mathbb{Q}(\tau > t) > 0$ for every $t \in \mathbb{R}_+$. Then*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{H}_t} = \mathcal{E}_t\Big(\int_0^{\cdot} (h^*(u) - 1) \, dM_u \Big), \tag{2.21}$$

where

$$h^*(t) = h(t)/g(t), \quad g(t) = e^{\Gamma(t)} \int_t^\infty h(u) \, dF(u),$$

and $\Gamma^*(t) = g^*(t)\Gamma(t)$ with

$$g^*(t) = \frac{\ln\Big(\int_{]t,\infty[} h(u) \, dF(u) \Big)}{\ln(1 - F(t))}. \tag{2.22}$$

If, in addition, the random time τ admits the intensity function γ under $\mathbb{P}$, then the intensity function γ^* of τ under $\mathbb{Q}$ satisfies $\gamma^*(t) = h^*(t)\gamma(t)$ a.e. on $\mathbb{R}_+$. More generally, if the hazard function Γ of τ under $\mathbb{P}$ is continuous, then the hazard function Γ^* of τ under $\mathbb{Q}$ is also continuous, and it satisfies $d\Gamma^*(t) = h^*(t) \, d\Gamma(t)$.

Corollary 2.2.3 *If F is continuous then $M_t^* = H_t - \Gamma^*(t \wedge \tau)$ is an $\mathbb{H}$-martingale under $\mathbb{Q}$.*

Proof. In view Proposition 2.2.2, the corollary is an immediate consequence of the continuity of Γ^*. Alternatively, we may check directly that the product $U_t = \eta_t M^*_t = \eta_t(H_t - \Gamma^*(t\wedge\tau))$ follows a $\mathbb{H}$-martingale under $\mathbb{P}$. To this end, observe that the integration by parts formula for functions of finite variation yields

$$\begin{aligned} U_t &= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_t\, d\eta_t \\ &= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_{t-}\, d\eta_t + \sum_{u\le t} \Delta M^*_u \Delta\eta_u \\ &= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_{t-}\, d\eta_t + \mathbb{1}_{\{\tau\le t\}}(\eta_\tau - \eta_{\tau-}). \end{aligned}$$

Using (2.19), we obtain

$$\begin{aligned} U_t &= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_{t-}\, d\eta_t + \eta_{\tau-}\mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1) \\ &= \int_{]0,t]} \eta_{t-}\, d\big(\Gamma(t\wedge\tau) - \Gamma^*(t\wedge\tau) + \mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1)\big) + N_t, \end{aligned}$$

where the process N, which equals

$$N_t = \int_{]0,t]} \eta_{t-}\, dM_t + \int_{]0,t]} M^*_{t-}\, d\eta_t$$

is manifestly an $\mathbb{H}$-martingale with respect to $\mathbb{P}$. It remains to show that the process

$$N^*_t \stackrel{\text{def}}{=} \Gamma(t\wedge\tau) - \Gamma^*(t\wedge\tau) + \mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1)$$

follows an $\mathbb{H}$-martingale with respect to $\mathbb{P}$. By virtue of Proposition 2.2.5, the process

$$\mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1) + \Gamma(t\wedge\tau) - \int_0^{t\wedge\tau} h^*(u)\, d\Gamma(u)$$

is an $\mathbb{H}$-martingale. Therefore, to conclude the proof it is enough to notice that

$$\int_0^{t\wedge\tau} h^*(u)\, d\Gamma(u) - \Gamma^*(t\wedge\tau) = \int_0^{t\wedge\tau} (h^*(u)\, d\Gamma(u) - d\Gamma^*(u)) = 0,$$

where the last equality is a consequence of the relationship $d\Gamma^*(t) = h^*(t)\, d\Gamma(t)$ established in Proposition 2.2.8. □

2.2.5 Incompleteness of the Toy Model

In order to study the completeness of the financial market, we first need to define the tradeable assets. If the market consists only of the risk-free zero-coupon bond, there exists infinitely many equivalent martingale measures (EMMs). The discounted asset prices are constant, hence the set $\mathcal{Q}$ of all EMMs is the set of all probability measures equivalent to the historical one. For any $\mathbb{Q}\in\mathcal{Q}$, we denote by F_Q the cumulative distribution function of τ under $\mathbb{Q}$, i.e.,

$$F_Q(t) = \mathbb{Q}(\tau\le t).$$

The range of prices is defined as the set of prices which do not induce arbitrage opportunities. For a DZC with a constant rebate δ paid at maturity, the range of prices is thus equal to the set

$$\{\mathbb{E}_{\mathbb{Q}}(R_T(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau<T\}})),\ \mathbb{Q}\in\mathcal{Q}\}.$$

This set is exactly the interval $]\delta R_T, R_T[$. Indeed, it is obvious that the range of prices is included in the interval $]\delta R_T, R_T[$. Now, in the set $\mathcal{Q}$, one can select a sequence of probabilities $\mathbb{Q}_n$ that converges weakly to the Dirac measure at point 0 (resp. at point T) (the bounds are obtained as limit cases: the default appears at time 0^+, or never). Obviously, this range of prices is too large to be useful.

2.2.6 Risk-Neutral Probability Measures

It is usual to interpret the absence of arbitrage opportunities as the existence of an EMM. If DZCs are traded, their prices are *given by the market*, and the equivalent martingale measure $\mathbb{Q}$, *chosen by the market,* is such that, on the set $\{t < \tau\}$,

$$D^{(\delta)}(t,T) = B(t,T)\mathbb{E}_{\mathbb{Q}}\big([\mathbb{1}_{\{T<\tau\}} + \delta \mathbb{1}_{\{t<\tau \le T\}}] \,\big|t < \tau\big).$$

Therefore, we can derive the cumulative function of τ under $\mathbb{Q}$ from the market prices of the DZC as shown below.

Case of Zero Recovery

If a DZC with zero recovery of maturity T is traded at some price $D^{(\delta)}(t,T)$ belonging to the interval $]0, B(t,T)[$ then, under any risk-neutral probability $\mathbb{Q}$, the process $B(0,t)D^{(\delta)}(t,T)$ is a martingale (for the moment, we do not know whether the market model is complete, so we do not claim that an EMM is unique). The following equalities thus hold

$$D^{(\delta)}(t,T)B(0,t) = \mathbb{E}_{\mathbb{Q}}(B(0,T)\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{H}_t) = B(0,T)\mathbb{1}_{\{t<\tau\}} \exp\Big(- \int_t^T \lambda^Q(s)\,ds\Big)$$

where $\lambda^Q(s) = \dfrac{dF_Q(s)/ds}{1 - F_Q(s)}$. It is easily seen that if $D^{(\delta)}(0,T)$ belongs to the range of viable prices $]0, B(0,T)[$ for any T then the function λ^Q is strictly positive (and the converse holds true). The process λ^Q is the implied default intensity, specifically, the $\mathbb{Q}$-intensity of τ. Therefore, the value of the integral $\int_t^T \lambda^Q(s)\,ds$ is known for any t as soon as there DZC bonds will all maturities are traded at time 0. The unique risk-neutral intensity can be obtained from the prices of DZCs, specifically,

$$r(t) + \lambda^Q(t) = -\partial_T \ln D^{(\delta)}(t,T)\,|_{\,T=t}.$$

Remark 2.2.3 It is important to note that there is no relation between the risk-neutral intensity and the historical one. The risk-neutral intensity can be greater (resp. smaller) than the historical one. The historical intensity can be deduced from observation of default time, the risk-neutral one is obtained from the prices of traded defaultable claims.

Fixed Recovery at Maturity

If the prices of DZCs with different maturities are known, then from (2.1)

$$F_Q(T) = \frac{B(0,T) - D^{(\delta)}(0,T)}{B(0,T)(1-\delta)}$$

where $F_Q(t) = \mathbb{Q}(\tau \le t)$, so that the law of τ is known under the 'market' EMM. However, as observed by Hull and White [54], "extracting default probabilities from bond prices [is] in practice, usually more complicated. First, the recovery rate is usually non-zero. Second, most corporate bonds are not zero-coupon bonds".

Recovery at Default

In this case the cumulative function can be obtained using the derivative of the defaultable zero-coupon price with respect to the maturity. Indeed, denoting by $\partial_T D^{(\delta)}$ the derivative of the value of the DZC at time 0 with respect to the maturity, and assuming that $G = 1 - F$ is differentiable, we obtain from (2.3)

$$\partial_T D^{(\delta)}(0,T) = g(T)B(0,T) - G(T)B(0,T)r(T) - \delta(T)g(T)B(0,T),$$

where $g(t) = G'(t)$. Therefore, solving this equation leads to

$$\mathbb{Q}(\tau > t) = G(t) = \Delta(t)\left[1 + \int_0^t \partial_T D^{(\delta)}(0,s)\frac{1}{B(0,s)(1-\delta(s))}(\Delta(s))^{-1}ds\right],$$

where $\Delta(t) = \exp\left(\int_0^t \frac{r(u)}{1-\delta(u)}\,du\right)$.

2.2.7 Partial Information: Duffie and Lando's Model

Duffie and Lando [41] study the case where $\tau = \inf\{t : V_t \le m\}$ where V satisfies

$$dV_t = \mu(t, V_t)\,dt + \sigma(t, V_t)\,dW_t.$$

Here the process W is a Brownian motion. If the information is the Brownian filtration, the time τ is a stopping time with respect to a Brownian filtration, therefore is predictable and admits no intensity. We will discuss this point latter on. If the agents do not know the behavior of V, but only the minimal information $\mathcal{H}_t$, i.e. he knows when the default appears, the price of a zero-coupon is, in the case where the default is not yet occurred, $\exp\left(-\int_t^T \lambda(s)\,ds\right)$ where $\lambda(s) = \frac{f(s)}{G(s)}$ and $G(s) = \mathbb{P}(\tau > s), f = -G'$, as soon as the cumulative function of τ is differentiable. Duffie and Lando have obtained that the intensity is

$$\lambda(t) = \frac{1}{2}\sigma^2(t,0)\frac{\partial f}{\partial x}(t,0)$$

where $f(t,x)$ is the conditional density of V_t when $T_0 > t$, i.e., the differential with respect to x of

$$\frac{\mathbb{Q}(V_t \le x, \tau_0 > t)}{\mathbb{Q}(T_0 > t)},$$

where $\tau_0 = \inf\{t \in \mathbb{R}_+ : V_t = 0\}$. In the case where V is a time-homogenous diffusion, that is,

$$dV_t = \mu(V_t)\,dt + \sigma(V_t)\,dW_t,$$

the equality between Duffie and Lando's result and our result is less obvious. See Elliott et al. [45] for comments.

2.3 Pricing and Trading Defaultable Claims

This section gives a summary of basic results concerning the valuation and trading of generic defaultable claims. We start by analyzing the valuation of recovery payoffs.

2.3.1 Recovery at Maturity

Let S be the price of an asset which delivers only a recovery Z_τ at time T. We know already that the process

$$M_t = H_t - \int_0^t (1-H_s)\gamma_s\,ds$$

is an $\mathbb{H}$-martingale. Recall that $\gamma(t) = f(t)/G(t)$, f is the probability density function of τ and $G(t) = \mathbb{Q}(\tau > t)$. Observe that

$$\begin{aligned} e^{-rt}S_t &= \mathbb{E}_{\mathbb{Q}}(Z_\tau e^{-rT}\,|\,\mathcal{G}_t) = e^{-rT}\mathbb{1}_{\{\tau<t\}}Z_\tau + e^{-rT}\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{Q}}(Z_\tau\mathbb{1}_{\{t<\tau<T\}})}{G(t)} \\ &= e^{-rT}\int_0^t Z_u\,dH_u + e^{-rT}\mathbb{1}_{\{\tau>t\}}\widetilde{S}_t \end{aligned}$$

where $\widetilde{S}_t$ is the pre-default price, which is given here by the deterministic function

$$\widetilde{S}_t = \frac{\mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{t<\tau<T\}})}{G(t)} = \frac{\int_t^T Z_u f_u \, du}{G(t)} \, .$$

Hence

$$d\widetilde{S}_t = f(t)\frac{\int_t^T Z_u f_u du}{G^2(t)}\, dt - \frac{Z_t f_t}{G(t)}\, dt = \widetilde{S}_t \frac{f(t)}{G(t)}\, dt - \frac{Z_t f_t}{G(t)}\, dt \, .$$

It follows that

$$\begin{aligned} d(e^{-rt}S_t) &= e^{-rT}\left(Z_t\, dH_t + (1-H_t)\frac{f(t)}{G(t)}\big(\widetilde{S}_t - Z_t\big)dt - \widetilde{S}_{t-}\, dH_t\right) \\ &= \big(e^{-rT}Z_t - e^{-rt}S_{t-}\big)\big(dH_t - (1-H_t)\gamma_t\, dt\big) \\ &= e^{-rt}\big(e^{-r(T-t)}Z_t - S_{t-}\big)\, dM_t. \end{aligned}$$

In that case, the discounted price is a martingale under the risk-neutral probability $\mathbb{Q}$, and the price S does not vanishes (so long as δ does not)

2.3.2 Recovery at Default

Assume now that the recovery is paid at default time. Then the price of the derivative is obviously equal to 0 after the default time, and

$$e^{-rt}S_t = \mathbb{E}_{\mathbb{Q}}(Z_\tau e^{-r\tau}\mathbb{1}_{\{t<\tau\leq T\}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{Q}}(e^{-r\tau}Z_\tau \mathbb{1}_{\{t<\tau<T\}})}{G(t)} = \mathbb{1}_{\{\tau>t\}}\widetilde{S}_t$$

where the pre-default price is the deterministic function

$$\widetilde{S}_t = \frac{1}{G(t)}\int_t^T Z_u e^{-ru} f(u)\, du.$$

Consequently

$$\begin{aligned} d\widetilde{S}_t &= -Z_t e^{-rt}\frac{f(t)}{G(t)}\, dt + f(t)\frac{\int_t^T Z_u e^{-ru} f(u) du}{(\mathbb{Q}(\tau>t)^2}\, dt \\ &= -Z_t e^{-rt}\frac{f(t)}{G(t)}\, dt + \widetilde{S}_t\frac{f(t)}{G(t)}\, dt \\ &= \frac{f(t)}{G(t)}\big(-Z_t e^{-rt} + \widetilde{S}_t\big)\, dt \end{aligned}$$

and thus

$$\begin{aligned} d(e^{-rt}S_t) &= (1-H_t)\frac{f(t)}{G(t)}(-Z_t e^{-rt} + \widetilde{S}_t)\, dt - \widetilde{S}_t\, dH_t \\ &= -\widetilde{S}_t(dH_t - (1-H_t)\gamma_t\, dt) = (Z_t e^{-rt} - \widetilde{S}_t)\, dM_t - Z_t e^{-rt}(1-H_t)\gamma_t\, dt \\ &= e^{-rt}(Z_t - S_{t-})\, dM_t - Z_t e^{-rt}(1-H_t)\gamma_t\, dt. \end{aligned}$$

In that case, the discounted process is not an $\mathbb{H}$-martingale under the risk-neutral probability. By contrast, the process

$$S_t e^{-rt} + \int_0^t Z_s e^{-rs}(1-H_s)\gamma_s\, ds$$

follows an $\mathbb{H}$-martingale. The recovery can be seen as a dividend process, paid up time τ, at rate $Z\gamma$.

2.3.3 Generic Defaultable Claims

Let us first recall the notation. A strictly positive random variable τ, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, is termed a *random time*. In view of its interpretation, it will be later referred to as a *default time*. We introduce the jump process $H_t = \mathbb{1}_{\{\tau \le t\}}$ associated with τ, and we denote by $\mathbb{H}$ the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$, and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that we have $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$.

Definition 2.3.1 *By a defaultable claim maturing at T we mean the quadruple (X, A, Z, τ), where X is an $\mathcal{F}_T$-measurable random variable, A is an $\mathbb{F}$-adapted process of finite variation, Z is an $\mathbb{F}$-predictable process, and τ is a random time.*

The financial interpretation of the components of a defaultable claim becomes clear from the following definition of the dividend process D, which describes all cash flows associated with a defaultable claim over the lifespan $]0, T]$, that is, after the contract was initiated at time 0. Of course, the choice of 0 as the date of inception is arbitrary.

Definition 2.3.2 *The dividend process D of a defaultable claim maturing at T equals, for every $t \in [0, T]$,*

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u)\, dA_u + \int_{]0,t]} Z_u\, dH_u.$$

The financial interpretation of the definition above justifies the following terminology: X is the *promised payoff*, A represents the process of *promised dividends*, and the process Z, termed the *recovery process*, specifies the recovery payoff at default. It is worth stressing that, according to our convention, the cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim.

When dealing with a credit default swap, it is natural to assume that the premium paid at time 0 equals zero, and the process A represents the fee (annuity) paid in instalments up to maturity date or default, whichever comes first. For instance, if $A_t = -\kappa t$ for some constant $\kappa > 0$, then the 'price' of a stylized credit default swap is formally represented by this constant, referred to as the continuously paid *credit default rate* or *premium* (see Section 2.4.1 for details).

If the other covenants of the contract are known (i.e., the payoffs X and Z are given), the valuation of a swap is equivalent to finding the level of the rate κ that makes the swap valueless at inception. Typically, in a credit default swap we have $X = 0$, and Z is determined in reference to recovery rate of a reference credit-risky entity. In a more realistic approach, the process A is discontinuous, with jumps occurring at the premium payment dates. In this note, we shall only deal with a stylized CDS with a continuously paid premium.

Let us return to the general set-up. It is clear that the dividend process D follows a process of finite variation on $[0, T]$. Since

$$\int_{]0,t]} (1 - H_u)\, dA_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}}\, dA_u = A_{\tau-} \mathbb{1}_{\{\tau \le t\}} + A_t \mathbb{1}_{\{\tau > t\}},$$

it is also apparent that if default occurs at some date t, the 'promised dividend' $A_t - A_{t-}$ that is due to be received or paid at this date is disregarded. If we denote $\tau \wedge t = \min(\tau, t)$ then we have

$$\int_{]0,t]} Z_u\, dH_u = Z_{\tau \wedge t} \mathbb{1}_{\{\tau \le t\}} = Z_\tau \mathbb{1}_{\{\tau \le t\}}.$$

Let us stress that the process $D_u - D_t$, $u \in [t, T]$, represents all cash flows from a defaultable claim received by an investor who purchases it at time t. Of course, the process $D_u - D_t$ may depend on the past behavior of the claim (e.g., through some intrinsic parameters, such as credit spreads) as well as on the history of the market prior to t. The past dividends are not valued by the market, however, so that the current market value at time t of a claim (i.e., the price at which it trades at time t) depends only on future dividends to be paid or received over the time interval $]t,T]$.

Suppose that our underlying financial market model is arbitrage-free, in the sense that there exists a *spot martingale measure* $\mathbb{Q}$ (also referred to as a *risk-neutral probability*), meaning that $\mathbb{Q}$ is equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$, and the price process of any tradeable security, paying no coupons or dividends, follows a $\mathbb{G}$-martingale under $\mathbb{Q}$, when discounted by the *savings account* B, given by

$$B_t = \exp\left(\int_0^t r_u\,du\right), \quad \forall\, t \in \mathbb{R}_+. \tag{2.23}$$

2.3.4 Buy-and-Hold Strategy

We write S^i, $i = 1,\dots,k$ to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes S^i, $i = 1,\dots,k-1$ follow semimartingales. In addition, we set $S^k_t = B_t$ so that S^k represents the value process of the savings account. The last assumption is not necessary, however. We can assume, for instance, that S^k is the price of a T-maturity risk-free zero-coupon bond, or choose any other strictly positive price process as as numéraire.

For the sake of convenience, we assume that S^i, $i = 1,\dots,k-1$ are non-dividend-paying assets, and we introduce the discounted price processes S^{i*} by setting $S^{i*}_t = S^i_t/B_t$. All processes are assumed to be given on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{Q})$, where $\mathbb{Q}$ is interpreted as the real-life (i.e., statistical) probability measure.

Let us now assume that we have an additional traded security that pays dividends during its lifespan, assumed to be the time interval $[0,T]$, according to a process of finite variation D, with $D_0 = 0$. Let S denote a (yet unspecified) price process of this security. In particular, we do not postulate a priori that S follows a semimartingale. It is not necessary to interpret S as a price process of a defaultable claim, though we have here this particular interpretation in mind.

Let a $\mathbb{G}$-predictable, $\mathbb{R}^{k+1}$-valued process $\phi = (\phi^0, \phi^1, \dots, \phi^k)$ represent a generic trading strategy, where ϕ^j_t represents the number of shares of the j^{th} asset held at time t. We identify here S^0 with S, so that S is the 0^{th} asset. In order to derive a pricing formula for this asset, it suffices to examine a simple trading strategy involving S, namely, the buy-and-hold strategy.

Suppose that one unit of the 0^{th} asset was purchased at time 0, at the initial price S_0, and it was hold until time T. We assume all the proceeds from dividends are re-invested in the savings account B. More specifically, we consider a *buy-and-hold* strategy $\psi = (1, 0, \dots, 0, \psi^k)$, where ψ^k is a $\mathbb{G}$-predictable process. The associated *wealth process* $V(\psi)$ equals

$$V_t(\psi) = S_t + \psi^k_t B_t, \quad \forall\, t \in [0,T], \tag{2.24}$$

so that its initial value equals $V_0(\psi) = S_0 + \psi^k_0$.

Definition 2.3.3 *We say that a strategy $\psi = (1, 0, \dots, 0, \psi^k)$ is self-financing if*

$$dV_t(\psi) = dS_t + dD_t + \psi^k_t\,dB_t,$$

or more explicitly, for every $t \in [0,T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi^k_u\,dB_u. \tag{2.25}$$

We assume from now on that the process ψ^k is chosen in such a way (with respect to S, D and B) that a buy-and-hold strategy ψ is self-financing. Also, we make a standing assumption that the random variable $Y = \int_{]0,T]} B_u^{-1} dD_u$ is $\mathbb{Q}$-integrable.

Lemma 2.3.1 *The discounted wealth $V_t^*(\psi) = B_t^{-1} V_t(\psi)$ of any self-financing buy-and-hold trading strategy ψ satisfies, for every $t \in [0,T]$,*

$$V_t^*(\psi) = V_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} B_u^{-1} dD_u. \tag{2.26}$$

Hence we have, for every $t \in [0,T]$,

$$V_T^*(\psi) - V_t^*(\psi) = S_T^* - S_t^* + \int_{]t,T]} B_u^{-1}\, dD_u. \tag{2.27}$$

Proof. We define an auxiliary process $\widehat{V}(\psi)$ by setting $\widehat{V}_t(\psi) = V_t(\psi) - S_t = \psi_t^k B_t$ for $t \in [0,T]$. In view of (2.25), we have

$$\widehat{V}_t(\psi) = \widehat{V}_0(\psi) + D_t + \int_{]0,t]} \psi_u^k \, dB_u,$$

and so the process $\widehat{V}(\psi)$ follows a semimartingale. An application of Itô's product rule yields

$$\begin{aligned} d\big(B_t^{-1}\widehat{V}_t(\psi)\big) &= B_t^{-1} d\widehat{V}_t(\psi) + \widehat{V}_t(\psi)\, dB_t^{-1} \\ &= B_t^{-1} dD_t + \psi_t^k B_t^{-1} dB_t + \psi_t^k B_t \, dB_t^{-1} \\ &= B_t^{-1} dD_t, \end{aligned}$$

where we have used the obvious identity: $B_t^{-1} dB_t + B_t\, dB_t^{-1} = 0$. Integrating the last equality, we obtain

$$B_t^{-1}\big(V_t(\psi) - S_t\big) = B_0^{-1}\big(V_0(\psi) - S_0\big) + \int_{]0,t]} B_u^{-1} dD_u,$$

and this immediately yields (2.26). □

It is worth noting that Lemma 2.3.1 remains valid if the assumption that S^k represents the savings account B is relaxed. It suffices to assume that the price process S^k is a numéraire, that is, a strictly positive continuous semimartingale. For the sake of brevity, let us write $S^k = \beta$. We say that $\psi = (1, 0, \dots, 0, \psi^k)$ is self-financing it the wealth process

$$V_t(\psi) = S_t + \psi_t^k \beta_t, \quad \forall\, t \in [0,T],$$

satisfies, for every $t \in [0,T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi_u^k \, d\beta_u.$$

Lemma 2.3.2 *The relative wealth $V_t^*(\psi) = \beta_t^{-1} V_t(\psi)$ of a self-financing trading strategy ψ satisfies, for every $t \in [0,T]$,*

$$V_t^*(\psi) = V_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} \beta_u^{-1} dD_u,$$

where $S^* = \beta_t^{-1} S_t$.

Proof. The proof proceeds along the same lines as before, noting that $\beta^1 d\beta + \beta d\beta^1 + d\langle \beta, \beta^1 \rangle = 0$. □

2.3.5 Spot Martingale Measure

Our next goal is to derive the risk-neutral valuation formula for the ex-dividend price S_t. To this end, we assume that our market model is arbitrage-free, meaning that it admits a (not necessarily unique) martingale measure $\mathbb{Q}$, equivalent to $\mathbb{Q}$, which is associated with the choice of B as a numéraire.

Definition 2.3.4 *We say that $\mathbb{Q}$ is a spot martingale measure if the discounted price S^{i*} of any non-dividend paying traded security follows a $\mathbb{Q}$-martingale with respect to $\mathbb{G}$.*

It is well known that the discounted wealth process $V^*(\phi)$ of any self-financing trading strategy $\phi = (0, \phi^1, \phi^2, \dots, \phi^k)$ is a local martingale under $\mathbb{Q}$. In what follows, we shall only consider *admissible* trading strategies, that is, strategies for which the discounted wealth process $V^*(\phi)$ is a martingale under $\mathbb{Q}$. A market model in which only admissible trading strategies are allowed is *arbitrage-free*, that is, there are no arbitrage opportunities in this model.

Following this line of arguments, we postulate that the trading strategy ψ introduced in Section 2.3.4 is also *admissible*, so that its discounted wealth process $V^*(\psi)$ follows a martingale under $\mathbb{Q}$ with respect to $\mathbb{G}$. This assumption is quite natural if we wish to prevent arbitrage opportunities to appear in the extended model of the financial market. Indeed, since we postulate that S is traded, the wealth process $V(\psi)$ can be formally seen as an additional non-dividend paying tradeable security.

To derive a pricing formula for a defaultable claim, we make a natural assumption that the market value at time t of the 0^{th} security comes exclusively from the future dividends stream, that is, from the cash flows occurring in the open interval $]t, T[$. Since the lifespan of S is $[0, T]$, this amounts to postulate that $S_T = S^*_T = 0$. To emphasize this property, we shall refer to S as the *ex-dividend price* of the 0^{th} asset.

Definition 2.3.5 *A process S with $S_T = 0$ is the ex-dividend price of the 0^{th} asset if the discounted wealth process $V^*(\psi)$ of any self-financing buy-and-hold strategy ψ follows a $\mathbb{G}$-martingale under $\mathbb{Q}$.*

As a special case, we obtain the ex-dividend price a defaultable claim with maturity T.

Proposition 2.3.1 *The ex-dividend price process S associated with the dividend process D satisfies, for every $t \in [0, T]$,*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big). \tag{2.28}$$

Proof. The postulated martingale property of the discounted wealth process $V^*(\psi)$ yields, for every $t \in [0, T]$,

$$\mathbb{E}_{\mathbb{Q}}\big(V^*_T(\psi) - V^*_t(\psi) \,\big|\, \mathcal{G}_t \big) = 0.$$

Taking into account (2.27), we thus obtain

$$S^*_t = \mathbb{E}_{\mathbb{Q}}\Big(S^*_T + \int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big).$$

Since, by virtue of the definition of the ex-dividend price we have $S_T = S^*_T = 0$, the last formula yields (2.28). □

It is not difficult to show that the ex-dividend price S satisfies, for every $t \in [0, T]$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t, \tag{2.29}$$

where the process $\widetilde{S}$ represents the *ex-dividend pre-default price* of a defaultable claim.

The *cum-dividend price* process $\bar{S}$ associated with the dividend process D is given by the formula, for every $t \in [0, T]$,

$$\bar{S}_t = B_t \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big). \tag{2.30}$$

The corresponding discounted cum-dividend price process, $\widehat{S} \stackrel{\text{def}}{=} B^{-1} \bar{S}$, is a $\mathbb{G}$-martingale under $\mathbb{Q}$.

The savings account B can be replaced by an arbitrary numéraire β. The corresponding valuation formula becomes, for every $t \in [0, T]$,

$$S_t = \beta_t \, \mathbb{E}_{\mathbb{Q}^\beta}\Big(\int_{]t,T]} \beta_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big), \tag{2.31}$$

where $\mathbb{Q}^\beta$ is a martingale measure on $(\Omega, \mathcal{G}_T)$ associated with a numéraire β, that is, a probability measure on $(\Omega, \mathcal{G}_T)$ given by the formula

$$\frac{d\mathbb{Q}^\beta}{d\mathbb{Q}} = \frac{\beta_T}{\beta_0 B_T}, \quad \mathbb{Q}\text{-a.s.}$$

2.3.6 Self-Financing Trading Strategies

Let us now examine a general trading strategy $\phi = (\phi^0, \phi^1, \dots, \phi^k)$ with $\mathbb{G}$-predictable components. The associated *wealth process* $V(\phi)$ equals $V_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$, where, as before $S^0 = S$. A strategy ϕ is said to be *self-financing* if $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0, T]$, where the *gains process* $G(\phi)$ is defined as follows:

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 \, dD_u + \sum_{i=0}^k \int_{]0,t]} \phi_u^i \, dS_u^i.$$

Corollary 2.3.1 *Let $S^k = B$. Then for any self-financing trading strategy ϕ, the discounted wealth process $V^*(\phi) = B_t^{-1} V_t(\phi)$ follows a martingale under $\mathbb{Q}$.*

Proof. Since B is a continuous process of finite variation, Itô's product rule gives

$$dS_t^{i*} = S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i$$

for $i = 0, 1, \dots, k$, and so

$$\begin{aligned} dV_t^*(\phi) &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} dV_t(\phi) \\ &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} \Big(\sum_{i=0}^k \phi_t^i \, dS_t^i + \phi_t^0 \, dD_t \Big) \\ &= \sum_{i=0}^k \phi_t^i \big(S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i \big) + \phi_t^0 B_t^{-1} dD_t \\ &= \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \big(dS_t^* + B_t^{-1} dD_t \big) = \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \, d\widehat{S}_t, \end{aligned}$$

where the auxiliary process $\widehat{S}$ is given by the following expression:

$$\widehat{S}_t = S_t^* + \int_{]0,t]} B_u^{-1} \, dD_u.$$

To conclude, it suffices to observe that in view of (2.28) the process $\widehat{S}$ satisfies

$$\widehat{S}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big), \tag{2.32}$$

and thus it follows a martingale under $\mathbb{Q}$. □

It is worth noting that $\widehat{S}_t$, given by formula (2.32), represents the discounted *cum-dividend price* at time t of the 0^{th} asset, that is, the arbitrage price at time t of all past and future dividends associated with the 0^{th} asset over its lifespan. To check this, let us consider a buy-and-hold strategy such that $\psi_0^k = 0$. Then, in view of (2.27), the terminal wealth at time T of this strategy equals

$$V_T(\psi) = B_T \int_{]0,T]} B_u^{-1}\, dD_u. \tag{2.33}$$

It is clear that $V_T(\psi)$ represents all dividends from S in the form of a single payoff at time T. The arbitrage price $\pi_t(\widehat{Y})$ at time $t < T$ of a claim $\widehat{Y} = V_T(\psi)$ equals (under the assumption that this claim is attainable)

$$\pi_t(\widehat{Y}) = B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big)$$

and thus $\widehat{S}_t = B_t^{-1}\pi_t(\widehat{Y})$. It is clear that discounted cum-dividend price follows a martingale under $\mathbb{Q}$ (under the standard integrability assumption).

Remarks 2.3.1 (i) Under the assumption of uniqueness of a spot martingale measure $\mathbb{Q}$, any $\mathbb{Q}$-integrable contingent claim is attainable, and the valuation formula established above can be justified by means of replication.
(ii) Otherwise – that is, when a martingale probability measure $\mathbb{Q}$ is not uniquely determined by the model $(S^1, S^2, \dots, S^k)$ – the right-hand side of (2.28) may depend on the choice of a particular martingale probability, in general. In this case, a process defined by (2.28) for an arbitrarily chosen spot martingale measure $\mathbb{Q}$ can be taken as the no-arbitrage price process of a defaultable claim. In some cases, a market model can be completed by postulating that S is also a traded asset.

2.3.7 Martingale Properties of Prices of Defaultable Claims

In the next result, we summarize the martingale properties of prices of a generic defaultable claim.

Corollary 2.3.2 *The discounted cum-dividend price $\widehat{S}_t$, $t \in [0,T]$, of a defaultable claim is a $\mathbb{Q}$-martingale with respect to $\mathbb{G}$. The discounted ex-dividend price S_t^*, $t \in [0,T]$, satisfies*

$$S_t^* = \widehat{S}_t - \int_{]0,t]} B_u^{-1}\, dD_u, \quad \forall\, t \in [0,T],$$

and thus it follows a supermartingale under $\mathbb{Q}$ if and only if the dividend process D is increasing.

In an application considered in Section 2.4, the finite variation process A is interpreted as the positive premium paid in instalments by the claim-holder to the counterparty in exchange for a positive recovery (received by the claim-holder either at maturity or at default). It is thus natural to assume that A is a decreasing process, and all other components of the dividend process are increasing processes (that is, we postulate that $X \geq 0$, and $Z \geq 0$). It is rather clear that, under these assumptions, the discounted ex-dividend price S^* is neither a super- or submartingale under $\mathbb{Q}$, in general.

Assume now that $A \equiv 0$, so that the premium for a defaultable claim is paid upfront at time 0, and it is not accounted for in the dividend process D. We postulate, as before, that $X \geq 0$, and $Z \geq 0$. In this case, the dividend process D is manifestly increasing, and thus the discounted ex-dividend price S^* is a supermartingale under $\mathbb{Q}$. This feature is quite natural since the discounted expected value of future dividends decreases when time elapses.

The final conclusion is that the martingale properties of the price of a defaultable claim depend on the specification of a claim and conventions regarding the prices (ex-dividend price or cum-dividend price). This point will be illustrated below by means of a detailed analysis of prices of credit default swaps.

2.4 Hedging of Single Name Credit Derivatives

Following Bielecki et al. [11], we shall now apply the general theory to a particular class of contracts, namely, to credit default swaps. We do not need to specify the underlying market model at this stage, but we make the following standing assumptions.

Assumptions (A). We assume throughout that:
(i) $\mathbb{Q}$ is a spot martingale measure on $(\Omega, \mathcal{G}_T)$,
(ii) the interest rate $r = 0$, so that the price of a savings account $B_t = 1$ for every $t \in \mathbb{R}_+$.

For the sake of simplicity, these restrictions are maintained in Section 2.5 of the present work, but they will be relaxed in a follow-up paper.

2.4.1 Stylized Credit Default Swap

A stylized T-maturity credit default swap is formally introduced through the following definition.

Definition 2.4.1 *A credit default swap (CDS) with a constant rate κ and recovery at default is a defaultable claim $(0, A, Z, \tau)$ where $Z(t) = \delta(t)$ and $A(t) = -\kappa t$ for every $t \in [0, T]$. A function $\delta : [0, T] \to \mathbb{R}$ represents the default protection, and κ is the CDS rate (also termed the spread, premium or annuity of a CDS).*

We denote by F the cumulative distribution function of the default time τ under $\mathbb{Q}$, and we assume that F is a continuous function, with $F(0) = 0$ and $F(T) < 1$. Also, we write $G = 1 - F$ to denote the *survival probability function* of τ, so that $G(t) > 0$ for every $t \in [0, T]$.

Since we start with only one tradeable asset in our model (the savings account), it is clear that any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{H}_T)$ equivalent to $\mathbb{Q}$ can be chosen as a spot martingale measure. The choice of $\mathbb{Q}$ is reflected in the cumulative distribution function F (in particular, in the default intensity if F admits a density function). In practical applications of reduced-form models, the choice of F is done by calibration.

2.4.2 Pricing of a CDS

Since the ex-dividend price of a CDS is the price at which it is actually traded, we shall refer to the ex-dividend price as the *price* in what follows. Recall that we also introduced the so-called *cumulative price*, which encompasses also past dividends reinvested in the savings account.

Let $s \in [0, T]$ stands for some fixed date. We consider a stylized T-maturity CDS contract with a constant rate κ and default protection function δ, initiated at time s and maturing at T. The dividend process of a CDS equals

$$D_t = \int_{]0,t]} \delta(u)\, dH_u - \kappa \int_{]0,t]} (1 - H_u)\, du \tag{2.34}$$

and thus, in view of (2.28), the price of this CDS is given by the formula

$$S_t(\kappa, \delta, T) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\le T\}}\delta(\tau)\,\Big|\,\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau\wedge T) - t\big)\,\Big|\,\mathcal{H}_t\Big) \tag{2.35}$$

where the first conditional expectation represents the current value of the *default protection stream* (or the *protection leg*), and the second is the value of the *survival annuity stream* (or the *fee leg*). To alleviate notation, we shall write $S_t(\kappa)$ instead of $S_t(\kappa, \delta, T)$ in what follows.

Lemma 2.4.1 *The price at time $t \in [s,T]$ of a credit default swap started at s, with rate κ and protection payment $\delta(\tau)$ at default, equals*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \left(-\int_t^T \delta(u)\, dG(u) - \kappa \int_t^T G(u)\, du \right). \tag{2.36}$$

Proof. We have, on the set $\{t < \tau\}$,

$$\begin{aligned} S_t(\kappa) &= -\frac{\int_t^T \delta(u)\, dG(u)}{G(t)} - \kappa \left(\frac{-\int_t^T u\, dG(u) + TG(T)}{G(t)} - t \right) \\ &= \frac{1}{G(t)} \left(-\int_t^T \delta(u)\, dG(u) - \kappa \Big(TG(T) - tG(t) - \int_t^T u\, dG(u) \Big) \right). \end{aligned}$$

Since

$$\int_t^T G(u)\, du = TG(T) - tG(t) - \int_t^T u\, dG(u), \tag{2.37}$$

we conclude that (2.36) holds. □

The *pre-default price* is defined as the unique function $\widetilde{S}(\kappa)$ such that we have (see Lemma 2.5.1 with $n = 1$)

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa), \quad \forall\, t \in [0,T]. \tag{2.38}$$

Combining (2.36) with (2.38), we find that the pre-default price of the CDS equals, for $t \in [s,T]$,

$$\widetilde{S}_t(\kappa) = \frac{1}{G(t)} \left(-\int_t^T \delta(u)\, dG(u) - \kappa \int_t^T G(u)\, du \right) = \widetilde{\delta}(t,T) - \kappa \widetilde{A}(t,T) \tag{2.39}$$

where

$$\widetilde{\delta}(t,T) = -\frac{1}{G(t)} \int_t^T \delta(u)\, dG(u)$$

is the pre-default price at time t of the protection leg, and

$$\widetilde{A}(t,T) = \frac{1}{G(t)} \int_t^T G(u)\, du$$

represents the pre-default price at time t of the fee leg for the period $[t,T]$ per one unit of spread κ. We shall refer to $\widetilde{A}(t,T)$ as the *CDS annuity*. Note that $\widetilde{S}(\kappa)$ is a continuous function, under our assumption that G is continuous.

2.4.3 Market CDS Rate

A CDS that has null value at its inception plays an important role as a benchmark CDS, and thus we introduce a formal definition, in which it is implicitly assumed that a recovery function δ of a CDS is given, and that we are on the event $\{\tau > s\}$.

Definition 2.4.2 *A market CDS started at s is the CDS initiated at time s whose initial value is equal to zero. The T-maturity market CDS rate (also known as the fair CDS spread) at time s is the fixed level of the rate $\kappa = \kappa(s,T)$ that makes the T-maturity CDS started at s valueless at its inception. The market CDS rate at time s is thus determined by the equation $\widetilde{S}_s(\kappa(s,T)) = 0$ where $\widetilde{S}_s(\kappa)$ is given by (2.39).*

Under the present assumptions, by virtue of (2.39), the T-maturity market CDS rate $\kappa(s,T)$ equals, for every $s \in [0,T]$,

$$\kappa(s,T) = \frac{\widetilde{\delta}(s,T)}{\widetilde{A}(s,T)} = -\frac{\int_s^T \delta(u)\, dG(u)}{\int_s^T G(u)\, du}. \tag{2.40}$$

Example 2.4.1 Assume that $\delta(t) = \delta$ is constant, and $F(t) = 1 - e^{-\gamma t}$ for some constant default intensity $\gamma > 0$ under $\mathbb{Q}$. In that case, the valuation formulae for a CDS can be further simplified. In view of Lemma 2.4.1, the ex-dividend price of a (spot) CDS with rate κ equals, for every $t \in [0,T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\delta\gamma - \kappa)\gamma^{-1}\Big(1 - e^{-\gamma(T-t)}\Big).$$

The last formula (or the general formula (2.40)) yields $\kappa(s,T) = \delta\gamma$ for every $s < T$, so that the market rate $\kappa(s,T)$ is here independent of s. As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s, but indeed at any time $t \in [s,T]$, both prior to and after default. Hence this process follows a trivial martingale under $\mathbb{Q}$. As we shall see in what follows, this martingale property the ex-dividend price of a market CDS is an exception, in the sense so that it fails to hold if the default intensity varies over time.

In what follows, we fix a maturity date T and we assume that credit default swaps with different inception dates have a common recovery function δ. We shall write briefly $\kappa(s)$ instead of $\kappa(s,T)$. Then we have the following result, in which the quantity $\nu(t,s) = \kappa(t) - \kappa(s)$ represents the *calendar CDS market spread* (for a given maturity T).

Proposition 2.4.1 *The price of a market CDS started at s with recovery δ at default and maturity T equals, for every $t \in [s,T]$,*

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}}\,(\kappa(t) - \kappa(s))\,\widetilde{A}(t,T) = \mathbb{1}_{\{t<\tau\}}\,\nu(t,s)\widetilde{A}(t,T). \tag{2.41}$$

Proof. To establish (2.41), it suffices to observe that $S_t(\kappa(s)) = S_t(\kappa(s)) - S_t(\kappa(t))$ since $S_t(\kappa(t)) = 0$, and to use (2.39) with $\kappa = \kappa(t)$ and $\kappa = \kappa(s)$. □

Note that formula (2.41) can be extended to any value of κ, specifically, we have that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\kappa(t) - \kappa)\widetilde{A}(t,T), \tag{2.42}$$

assuming that the CDS with rate κ was initiated at some date $s \in [0,t]$. The last representation shows that the price of a CDS can take negative values. The negative value occurs whenever the current market spread is lower than the contracted spread.

2.4.4 Price Dynamics of a CDS

In the remainder of Section 2.4, we assume that

$$G(t) = \mathbb{Q}(\tau > t) = \exp\left(-\int_0^t \gamma(u)\, du\right), \quad \forall\, t \in [0,T],$$

where the default intensity $\gamma(t)$ under $\mathbb{Q}$ is a strictly positive deterministic function. Recall that the process M, given by the formula

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du, \quad \forall\, t \in [0,T], \tag{2.43}$$

is an $\mathbb{H}$-martingale under $\mathbb{Q}$.

We first focus on dynamics of the price of a CDS with rate κ started at some date $s < T$.

Lemma 2.4.2 (i) *The dynamics of the price* $S_t(\kappa)$, $t \in [s,T]$, *are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt. \tag{2.44}$$

(ii) *The cumulative price process* $\widehat{S}_t(\kappa)$, $t \in [s,T]$, *is an* $\mathbb{H}$*-martingale under* $\mathbb{Q}$*, specifically,*

$$d\widehat{S}_t(\kappa) = \big(\delta(t) - S_{t-}(\kappa)\big)\, dM_t. \tag{2.45}$$

Proof. To prove (i), it suffices to recall that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa) = (1 - H_t)\widetilde{S}_t(\kappa)$$

so that the integration by parts formula yields

$$dS_t(\kappa) = (1 - H_t)\, d\widetilde{S}_t(\kappa) - \widetilde{S}_{t-}(\kappa)\, dH_t.$$

Using formula (2.36), we find easily that

$$d\widetilde{S}_t(\kappa) = \gamma(t)\widetilde{S}_t(\kappa)\, dt + (\kappa - \delta(t)\gamma(t))\, dt. \tag{2.46}$$

In view of (2.43) and the fact that $S_{\tau-}(\kappa) = \widetilde{S}_{\tau-}(\kappa)$ and $S_t(\kappa) = 0$ for $t \ge \tau$, the proof of (2.44) is complete.

To prove part (ii), we note that (2.28) and (2.30) yield

$$\widehat{S}_t(\kappa) - \widehat{S}_s(\kappa) = S_t(\kappa) - S_s(\kappa) + D_t - D_s. \tag{2.47}$$

Consequently,

$$\begin{aligned} \widehat{S}_t(\kappa) - \widehat{S}_s(\kappa) &= S_t(\kappa) - S_s(\kappa) + \int_s^t \delta(u)\, dH_u - \kappa \int_s^t (1 - H_u)\, du \\ &= S_t(\kappa) - S_s(\kappa) + \int_s^t \delta(u)\, dM_u - \int_s^t (1 - H_u)(\kappa - \delta(u)\gamma(u))\, du \\ &= \int_s^t \big(\delta(u) - S_{u-}(\kappa)\big)\, dM_u \end{aligned}$$

where the last equality follows from (2.44). □

Equality (2.44) emphasizes the fact that a single cash flow of $\delta(\tau)$ occurring at time τ can be formally treated as a dividend stream at the rate $\delta(t)\gamma(t)$ paid continuously prior to default. It is clear that we also have

$$dS_t(\kappa) = -\widetilde{S}_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt. \tag{2.48}$$

2.4.5 Dynamic Replication of a Defaultable Claim

Our goal is to show that in order to replicate a general defaultable claim, it suffices to trade dynamically in two assets: a CDS maturing at T, and the savings account B, assumed here to be constant. Since one may always work with discounted values, the last assumption is not restrictive. Moreover, it is also possible to take a CDS with any maturity $U \ge T$.

Let ϕ^0, ϕ^1 be $\mathbb{H}$-predictable processes and let $C : [0,T] \to \mathbb{R}$ be a function of finite variation with $C(0) = 0$. We say that $(\phi, C) = (\phi^0, \phi^1, C)$ is a *self-financing trading strategy with dividend stream* C if the wealth process $V(\phi, C)$, defined as

$$V_t(\phi, C) = \phi^0_t + \phi^1_t S_t(\kappa) \tag{2.49}$$

where $S_t(\kappa)$ is the price of a CDS at time t, satisfies

$$dV_t(\phi, C) = \phi^1_t\big(dS_t(\kappa) + dD_t\big) - dC(t) = \phi^1_t \, d\widehat{S}_t(\kappa) - dC(t) \tag{2.50}$$

where the dividend process D of a CDS is in turn given by (2.34). Note that C represents both outflows and infusions of funds. It will be used to cover the running cashflows associated with a claim we wish to replicate.

Consider a defaultable claim (X, A, Z, τ) where X is a constant, A is a function of finite variation, and Z is some recovery function. In order to define replication of a defaultable claim (X, A, Z, τ), it suffices to consider trading strategies on the random interval $[0, \tau \wedge T]$.

Definition 2.4.3 *We say that a trading strategy* (ϕ, C) replicates a defaultable claim (X, A, Z, τ) *if:*
(i) the processes $\phi = (\phi^0, \phi^1)$ *and* $V(\phi, C)$ *are stopped at* $\tau \wedge T$*,*
(ii) $C(\tau \wedge t) = A(\tau \wedge t)$ *for every* $t \in [0, T]$*,*
(iii) the equality $V_{\tau\wedge T}(\phi, C) = Y$ *holds, where the random variable* Y *equals*

$$Y = X\mathbb{1}_{\{\tau>T\}} + Z(\tau)\mathbb{1}_{\{\tau\leq T\}}. \tag{2.51}$$

Remark 2.4.1 Alternatively, one may say that a self-financing trading strategy $\phi = (\phi, 0)$ (i.e., a trading strategy with $C = 0$) replicates a defaultable claim (X, A, Z, τ) if and only if $V_{\tau\wedge T}(\phi) = \widehat{Y}$, where we set

$$\widehat{Y} = X\mathbb{1}_{\{\tau>T\}} + A(\tau \wedge T) + Z(\tau)\mathbb{1}_{\{\tau\leq T\}}. \tag{2.52}$$

However, in the case of non-zero (possibly random) interest rates, it is more convenient to define replication of a defaultable claim via Definition 2.4.3, since the running payoffs specified by A are distributed over time and thus, in principle, they need to be discounted accordingly (this does not show in (2.52), since it is assumed here that $r = 0$).

Let us denote, for every $t \in [0, T]$,

$$\widetilde{Z}(t) = \frac{1}{G(t)}\left(XG(T) - \int_t^T Z(u)\, dG(u)\right) \tag{2.53}$$

and

$$\widetilde{A}(t) = \frac{1}{G(t)}\int_{]t,T]} G(u)\, dA(u). \tag{2.54}$$

Let π and $\widetilde{\pi}$ be the risk-neutral value and the pre-default risk-neutral value of a defaultable claim under $\mathbb{Q}$, so that $\pi_t = \mathbb{1}_{\{t<\tau\}}\widetilde{\pi}(t)$ for every $t \in [0, T]$. Also, let $\widehat{\pi}$ stand for its risk-neutral cumulative price. It is clear that $\widetilde{\pi}(0) = \pi(0) = \widehat{\pi}(0) = \mathbb{E}_{\mathbb{Q}}(\widehat{Y})$

Proposition 2.4.2 *The pre-default risk-neutral value of a defaultable claim* (X, A, Z, τ) *equals* $\widetilde{\pi}(t) = \widetilde{Z}(t) + \widetilde{A}(t)$ *and thus*

$$d\widetilde{\pi}(t) = \gamma(t)(\widetilde{\pi}(t) - Z(t))\, dt - dA(t). \tag{2.55}$$

Moreover

$$d\pi_t = (Z(t) - \widetilde{\pi}(t-))\, dM_t - dA(t \wedge \tau) \tag{2.56}$$

and

$$d\widehat{\pi}_t = (Z(t) - \widetilde{\pi}(t-))\, dM_t. \tag{2.57}$$

Proof. The proof of equality $\widetilde{\pi}(t) = \widetilde{Z}(t) + \widetilde{A}(t)$ is similar to the derivation of formula (2.39). We have

$$\begin{aligned}
\pi_t &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\}}Y + A(\tau\wedge T) - A(\tau\wedge t)\,\Big|\,\mathcal{H}_t\Big)\\
&= \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\Big(XG(T) - \int_t^T Z(u)\,dG(u)\Big) + \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\int_{]t,T]} G(u)\,dA(u)\\
&= \mathbb{1}_{\{t<\tau\}}(\widetilde{Z}(t) + \widetilde{A}(t)) = \mathbb{1}_{\{t<\tau\}}\widetilde{\pi}(t).
\end{aligned}$$

By elementary computation, we obtain

$$d\widetilde{Z}(t) = \gamma(t)(\widetilde{Z}(t) - Z(t))\,dt, \quad d\widetilde{A}(t) = \gamma(t)\widetilde{A}(t)\,dt - dA(t),$$

and thus (2.55) holds. Finally, (2.56) follows easily from (2.55) and the integration by parts formula applied to the equality $\pi_t = (1-H_t)\widetilde{\pi}(t)$ (see the proof of Lemma 2.4.2 for similar computations). The last formula is also clear. □

The next proposition shows that the risk-neutral value of a defaultable claim is also its replication price, that is, a defaultable claim derives its value from the price of the traded CDS.

Theorem 2.1 *Assume that the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ holds for every $t \in [0,T]$. Let $\phi^1_t = \widetilde{\phi}_1(\tau\wedge t)$, where the function $\widetilde{\phi}_1 : [0,T] \to \mathbb{R}$ is given by the formula*

$$\widetilde{\phi}_1(t) = \frac{Z(t) - \widetilde{\pi}(t-)}{\delta(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0,T], \tag{2.58}$$

and let $\phi^0_t = V_t(\phi, A) - \phi^1_t S_t(\kappa)$, where the process $V(\phi, A)$ is given by the formula

$$V_t(\phi, A) = \widetilde{\pi}(0) + \int_{]0,\tau\wedge t]} \widetilde{\phi}_1(u)\,d\widehat{S}_u(\kappa) - A(t\wedge\tau). \tag{2.59}$$

Then the trading strategy (ϕ^0, ϕ^1, A) replicates a defaultable claim (X, A, Z, τ).

Proof. Assume first that a trading strategy $\phi = (\phi^0, \phi^1, C)$ is a replicating strategy for (X, A, Z, τ). By virtue of condition (i) in Definition 2.4.3 we have $C = A$ and thus, by combining (2.59) with (2.45), we obtain

$$dV_t(\phi, A) = \phi^1_t(\delta(t) - \widetilde{S}_t(\kappa))\,dM_t - dA(\tau\wedge t)$$

For ϕ^1 given by (2.58), we thus obtain

$$dV_t(\phi, A) = (Z(t) - \widetilde{\pi}(t-))\,dM_t - dA(\tau\wedge t).$$

It is thus clear that if we take $\phi^1_t = \widetilde{\phi}_1(\tau\wedge t)$ with $\widetilde{\phi}_1$ given by (2.58), and the initial condition $V_0(\phi, A) = \widetilde{\pi}(0) = \pi_0$, then we have that $V_t(\phi, A) = \pi(t)$ for every $t \in [0,T]$. It is now easily seen that all conditions of Definition 2.4.3 are satisfied since, in particular, $\pi_{\tau\wedge T} = Y$ with Y given by (2.51). □

Remark 2.4.2 Of course, if we take as (X, A, Z, τ) a CDS with rate κ and recovery function δ, then we have $Z(t) = \delta(t)$ and $\widetilde{\pi}(t-) = \widetilde{\pi}(t) = \widetilde{S}_t(\kappa)$, so that $\phi^1_t = 1$ for every $t \in [0,T]$.

2.5 Dynamic Hedging of Basket Credit Derivatives

In this section, we shall examine hedging of first-to-default basket claims with single name credit default swaps on the underlying n credit names, denoted as $1, 2, \ldots, n$. Our standing assumption (A) is maintained throughout this section.

Let the random times $\tau_1, \tau_2, \ldots, \tau_n$ given on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ represent the default times of with n credit names. We denote by $\tau_{(1)} = \tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_n = \min(\tau_1, \tau_2, \ldots, \tau_n)$ the moment of the first default, so that no defaults are observed on the event $\{\tau_{(1)} > t\}$.

Let

$$F(t_1, t_2, \ldots, t_n) = \mathbb{Q}(\tau_1 \le t_1, \tau_2 \le t_2, \ldots, \tau_n \le t_n)$$

be the joint probability distribution function of default times. We assume that the probability distribution of default times is jointly continuous, and we write $f(t_1, t_2, \ldots, t_n)$ to denote the joint probability density function. Also, let

$$G(t_1, t_2, \ldots, t_n) = \mathbb{Q}(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_n > t_n)$$

stand for the joint probability that the names $1, 2, \ldots, n$ have survived up to times $t_1, t_2, \ldots, t_n$. In particular, the joint survival function equals

$$G(t, \ldots, t) = \mathbb{Q}(\tau_1 > t, \tau_2 > t, \ldots, \tau_n > t) = \mathbb{Q}(\tau_{(1)} > t) = G_{(1)}(t).$$

For each $i = 1, 2, \ldots, n$, we introduce the default indicator process $H^i_t = \mathbb{1}_{\{\tau_i \le t\}}$ and the corresponding filtration $\mathbb{H}^i = (\mathcal{H}^i_t)_{t \in \mathbb{R}_+}$ where $\mathcal{H}^i_t = \sigma(H^i_u : u \le t)$. We denote by $\mathbb{G}$ the joint filtration generated by default indicator processes $H^1, H^2, \ldots, H^n$, so that $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2 \vee \ldots \vee \mathbb{H}^n$. It is clear that $\tau_{(1)}$ is a $\mathbb{G}$-stopping time as the infimum of $\mathbb{G}$-stopping times.

Finally, we write $H^{(1)}_t = \mathbb{1}_{\{\tau_{(1)} \le t\}}$ and $\mathbb{H}^{(1)} = (\mathcal{H}^{(1)}_t)_{t \in \mathbb{R}_+}$ where $\mathcal{H}^{(1)}_t = \sigma(H^{(1)}_u : u \le t)$.

Since we assume that $\mathbb{Q}(\tau_i = \tau_j) = 0$ for any $i \ne j$, $i, j = 1, 2, \ldots, n$, we also have that

$$H^{(1)}_t = H^{(1)}_{t \wedge \tau_{(1)}} = \sum_{i=1}^n H^i_{t \wedge \tau_{(1)}}.$$

We make the standing assumption $\mathbb{Q}(\tau_{(1)} > T) = G_{(1)}(T) > 0$.

For any $t \in [0, T]$, the event $\{\tau_{(1)} > t\}$ is an atom of the σ-field $\mathcal{G}_t$. Hence the following simple, but useful, result.

Lemma 2.5.1 *Let X be a $\mathbb{Q}$-integrable stochastic process. Then*

$$\mathbb{E}_{\mathbb{Q}}(X_t \,|\, \mathcal{G}_t) \mathbb{1}_{\{\tau_{(1)} > t\}} = \widetilde{X}(t) \mathbb{1}_{\{\tau_{(1)} > t\}}$$

where the function $\widetilde{X} : [0, T] \to \mathbb{R}$ is given by the formula

$$\widetilde{X}(t) = \frac{\mathbb{E}_{\mathbb{Q}}\big(X_t \mathbb{1}_{\{\tau_{(1)} > t\}}\big)}{G_{(1)}(t)}, \quad \forall\, t \in [0, T].$$

If X is a $\mathbb{G}$-adapted, $\mathbb{Q}$-integrable stochastic process then

$$X_t = X_t \mathbb{1}_{\{\tau_{(1)} \le t\}} + \widetilde{X}(t) \mathbb{1}_{\{\tau_{(1)} > t\}}, \quad \forall\, t \in [0, T].$$

By convention, the function $\widetilde{X} : [0, T] \to \mathbb{R}$ is called the *pre-default value* of X.

2.5.1 First-to-Default Intensities

In this section, we introduce the so-called *first-to-default intensities.* This natural concept will prove useful in the valuation and hedging of the first-to-default basket claims.

Definition 2.5.1 *The function $\widetilde{\lambda}_i : \mathbb{R}_+ \to \mathbb{R}_+$ given by*

$$\widetilde{\lambda}_i(t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{Q}(t < \tau_i \le t + h \,|\, \tau_{(1)} > t) \tag{2.60}$$

is called the i^{th} first-to-default intensity. *The function* $\widetilde{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$\widetilde{\lambda}(t) = \lim_{h \downarrow 0} \frac{1}{h} \, \mathbb{Q}(t < \tau_{(1)} \leq t + h \,|\, \tau_{(1)} > t) \tag{2.61}$$

is called the first-to-default intensity.

Let us denote

$$\partial_i G(t, \dots, t) = \frac{\partial G(t_1, t_2, \dots, t_n)}{\partial t_i}\Big|_{t_1 = t_2 = \dots = t_n = t}.$$

Then we have the following elementary lemma summarizing the properties of the first-to-default intensity.

Lemma 2.5.2 *The* i^{th} *first-to-default intensity* $\widetilde{\lambda}_i$ *satisfies, for* $i = 1, 2, \dots, n$,

$$\begin{aligned} \widetilde{\lambda}_i(t) &= \frac{\int_t^\infty \dots \int_t^\infty f(u_1, \dots, u_{i-1}, t, u_{i+1}, \dots, u_n) \, du_1 \dots du_{i-1} du_{i+1} \dots du_n}{G(t, \dots, t)} \\ &= \frac{\int_t^\infty \dots \int_t^\infty F(du_1, \dots, du_{i-1}, t, du_{i+1}, \dots, du_n)}{G_{(1)}(t)} = - \frac{\partial_i G(t, \dots, t)}{G_{(1)}(t)}. \end{aligned}$$

The first-to-default intensity $\widetilde{\lambda}$ *satisfies*

$$\widetilde{\lambda}(t) = - \frac{1}{G_{(1)}(t)} \frac{dG_{(1)}(t)}{dt} = \frac{f_{(1)}(t)}{G_{(1)}(t)} \tag{2.62}$$

where $f_{(1)}(t)$ *is the probability density function of* $\tau_{(1)}$. *The equality* $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$ *holds.*

Proof. Clearly

$$\widetilde{\lambda}_i(t) = \lim_{h \downarrow 0} \frac{1}{h} \frac{\int_t^\infty \dots \int_t^{t+h} \dots \int_t^\infty f(u_1, \dots, u_i, \dots, u_n) \, du_1 \dots du_i \dots du_n}{G(t, \dots, t)}$$

and thus the first asserted equality follows. The second equality follows directly from (2.61) and the definition of $G_{(1)}$. Finally, equality $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$ is equivalent to the equality

$$\lim_{h \downarrow 0} \frac{1}{h} \sum_{i=1}^n \mathbb{Q}(t < \tau_i \leq t + h \,|\, \tau_{(1)} > t) = \lim_{h \downarrow 0} \frac{1}{h} \, \mathbb{Q}(t < \tau_{(1)} \leq t + h \,|\, \tau_{(1)} > t),$$

which in turn is easy to establish. □

Remarks 2.5.1 The i^{th} first-to-default intensity $\widetilde{\lambda}_i$ should not be confused with the (marginal) intensity function λ_i of τ_i, which is defined as

$$\lambda_i(t) = \frac{f_i(t)}{G_i(t)}, \quad \forall \, t \in \mathbb{R}_+,$$

where f_i is the (marginal) probability density function of τ_i, that is,

$$f_i(t) = \int_0^\infty \dots \int_0^\infty f(u_1, \dots, u_{i-1}, t, u_{i+1}, \dots, u_n) \, du_1 \dots du_{i-1} du_{i+1} \dots du_n,$$

and $G_i(t) = 1 - F_i(t) = \int_t^\infty f_i(u) \, du$. Indeed, we have that $\widetilde{\lambda}_i \neq \lambda_i$, in general. However, if $\tau_1, \dots, \tau_n$ are mutually independent under $\mathbb{Q}$ then $\widetilde{\lambda}_i = \lambda_i$, that is, the first-to-default and marginal default intensities coincide.

It is also rather clear that the first-to-default intensity $\widetilde{\lambda}$ is not equal to the sum of marginal default intensities, that is, we have that $\widetilde{\lambda}(t) \neq \sum_{i=1}^n \lambda_i(t)$, in general.

2.5.2 First-to-Default Martingale Representation Theorem

We now state an integral representation theorem for a $\mathbb{G}$-martingale stopped at $\tau_{(1)}$ with respect to some basic processes. To this end, we define, for $i = 1, 2, \dots n$,

$$\widehat{M}_t^i = H^i_{t\wedge\tau_{(1)}} - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}_i(u)\, du, \quad \forall\, t \in \mathbb{R}_+. \tag{2.63}$$

Then we have the following *first-to-default martingale representation theorem.*

Proposition 2.5.1 *Consider the $\mathbb{G}$-martingale $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{G}_t)$, $t \in [0,T]$, where Y is a $\mathbb{Q}$-integrable random variable given by the expression*

$$Y = \sum_{i=1}^n Z_i(\tau_i)\mathbb{1}_{\{\tau_i \le T,\, \tau_i = \tau_{(1)}\}} + X\mathbb{1}_{\{\tau_{(1)} > T\}} \tag{2.64}$$

for some functions $Z_i : [0,T] \to \mathbb{R}$, $i = 1, 2, \dots, n$ and some constant X. Then $\widehat{M}$ admits the following representation

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y) + \sum_{i=1}^n \int_{]0,t]} h_i(u)\, d\widehat{M}_u^i \tag{2.65}$$

where the functions h_i, $i = 1, 2, \dots, n$ are given by

$$h_i(t) = Z_i(t) - \widehat{M}_{t-} = Z_i(t) - \widetilde{M}(t-), \quad \forall\, t \in [0,T], \tag{2.66}$$

where $\widetilde{M}$ is the unique function such that $\widehat{M}_t\mathbb{1}_{\{\tau_{(1)}>t\}} = \widetilde{M}(t)\mathbb{1}_{\{\tau_{(1)}>t\}}$ for every $t \in [0,T]$. The function $\widetilde{M}$ satisfies $\widetilde{M}_0 = \mathbb{E}_{\mathbb{Q}}(Y)$ and

$$d\widetilde{M}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)\big(\widetilde{M}(t) - Z_i(t)\big)\, dt. \tag{2.67}$$

More explicitly

$$\widetilde{M}(t) = \mathbb{E}_{\mathbb{Q}}(Y)\exp\left\{\int_0^t \widetilde{\lambda}(s)\, ds\right\} - \int_0^t \sum_{i=1}^n \widetilde{\lambda}_i(s) Z_i(s) \exp\left\{\int_s^t \widetilde{\lambda}(u)\, du\right\} ds.$$

Proof. To alleviate notation, we provide the proof of this result in a bivariate setting only. In that case, $\tau_{(1)} = \tau_1 \wedge \tau_2$ and $\mathcal{G}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2$. We start by noting that

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Z_1(\tau_1)\mathbb{1}_{\{\tau_1\le T,\,\tau_2>\tau_1\}} \,|\, \mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}}(Z_2(\tau_2)\mathbb{1}_{\{\tau_2\le T,\,\tau_1>\tau_2\}} \,|\, \mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{\tau_{(1)}>T\}} \,|\, \mathcal{G}_t),$$

and thus (see Lemma 2.5.1)

$$\mathbb{1}_{\{\tau_{(1)}>t\}}\widehat{M}_t = \mathbb{1}_{\{\tau_{(1)}>t\}}\widetilde{M}(t) = \mathbb{1}_{\{\tau_{(1)}>t\}}\sum_{i=1}^3 \widetilde{Y}^i(t)$$

where the auxiliary functions $\widetilde{Y}^i : [0,T] \to \mathbb{R}$, $i = 1, 2, 3$, are given by

$$\widetilde{Y}^1(t) = \frac{\int_t^T du Z_1(u) \int_u^\infty dv f(u,v)}{G_{(1)}(t)},\ \widetilde{Y}^2(t) = \frac{\int_t^T dv Z_2(v) \int_v^\infty du f(u,v)}{G_{(1)}(t)},\ \widetilde{Y}^3(t) = \frac{XG_{(1)}(T)}{G_{(1)}(t)}.$$

By elementary calculations and using Lemma 2.5.2, we obtain

$$\begin{aligned}
\frac{d\widetilde{Y}^1(t)}{dt} &= -\frac{Z_1(t)\int_t^\infty dv f(t,v)}{G_{(1)}(t)} - \frac{\int_t^T du Z_1(u)\int_u^\infty dv f(u,v)}{G_{(1)}^2(t)}\,\frac{dG_{(1)}(t)}{dt} \\
&= -Z_1(t)\,\frac{\int_t^\infty dv f(t,v)}{G_{(1)}(t)} - \frac{\widetilde{Y}^1(t)}{G_{(1)}(t)}\,\frac{dG_{(1)}(t)}{dt} \\
&= -Z_1(t)\widetilde{\lambda}_1(t) + \widetilde{Y}^1(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)),
\end{aligned} \tag{2.68}$$

and thus, by symmetry,

$$\frac{d\widetilde{Y}^2(t)}{dt} = -Z_2(t)\widetilde{\lambda}_2(t) + \widetilde{Y}^2(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)). \tag{2.69}$$

Moreover

$$\frac{d\widetilde{Y}^3(t)}{dt} = -\frac{XG_{(1)}(T)}{G^2_{(1)}(t)}\frac{dG_{(1)}(t)}{dt} = \widetilde{Y}^3(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)). \tag{2.70}$$

Hence recalling that $\widetilde{M}(t) = \sum_{i=1}^3 \widetilde{Y}^i(t)$, we obtain from (2.68)-(2.70)

$$d\widetilde{M}(t) = -\widetilde{\lambda}_1(t)\big(Z_1(t) - \widetilde{M}(t)\big)\,dt - \widetilde{\lambda}_2(t)\big(Z_2(t) - \widetilde{M}(t)\big)\,dt \tag{2.71}$$

Consequently, since the function $\widetilde{M}$ is continuous, we have, on the event $\{\tau_{(1)} > t\}$,

$$d\widehat{M}_t = -\widetilde{\lambda}_1(t)\big(Z_1(t) - \widehat{M}_{t-}\big)\,dt - \widetilde{\lambda}_2(t)\big(Z_2(t) - \widehat{M}_{t-}\big)\,dt.$$

We shall now check that both sides of equality (2.65) coincide at time $\tau_{(1)}$ on the event $\{\tau_{(1)} \le T\}$. To this end, we observe that we have, on the event $\{\tau_{(1)} \le T\}$,

$$\widehat{M}_{\tau_{(1)}} = Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}},$$

whereas the right-hand side in (2.65) is equal to

$$\begin{aligned}
&\widehat{M}_0 + \int_{]0,\tau_{(1)}[} h_1(u)\,d\widehat{M}^1_u + \int_{]0,\tau_{(1)}[} h_2(u)\,d\widehat{M}^2_u \\
&\quad + \mathbb{1}_{\{\tau_{(1)}=\tau_1\}}\int_{[\tau_{(1)}]} h_1(u)\,dH^1_u + \mathbb{1}_{\{\tau_{(1)}=\tau_2\}}\int_{[\tau_{(1)}]} h_2(u)\,dH^2_u \\
&\quad = \widetilde{M}(\tau_{(1)}-) + \big(Z_1(\tau_1) - \widetilde{M}(\tau_{(1)}-)\big)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + \big(Z_2(\tau_2) - \widetilde{M}(\tau_{(1)}-)\big)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}} \\
&\quad = Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}}
\end{aligned}$$

as $\widetilde{M}(\tau_{(1)}-) = \widehat{M}_{\tau_{(1)}-}$. Since the processes on both sides of equality (2.65) are stopped at $\tau_{(1)}$, we conclude that equality (2.65) is valid for every $t \in [0,T]$. Formula (2.67) was also established in the proof (see formula (2.71)). □

The next result shows that the basic processes $\widehat{M}^i$ are in fact $\mathbb{G}$-martingales. They will be referred to as the *basic first-to-default martingales.*

Corollary 2.5.1 *For each $i = 1, 2, \dots, n$, the process $\widehat{M}^i$ given by the formula* (2.63) *is a $\mathbb{G}$-martingale stopped at $\tau_{(1)}$.*

Proof. Let us fix $k \in \{1, 2, \dots, n\}$. It is clear that the process $\widehat{M}^k$ is stopped at $\tau_{(1)}$. Let $\widetilde{M}^k(t) = \int_0^t \widetilde{\lambda}_i(u)\,du$ be the unique function such that

$$\mathbb{1}_{\{\tau_{(1)}>t\}}\widehat{M}^i_t = \mathbb{1}_{\{\tau_{(1)}>t\}}\widetilde{M}^k(t), \quad \forall\, t \in [0,T].$$

Let us take $h_k(t) = 1$ and $h_i(t) = 0$ for any $i \neq k$ in formula (2.65), or equivalently, let us set

$$Z_k(t) = 1 + \widetilde{M}^k(t), \quad Z_i(t) = \widetilde{M}^k(t),\ i \neq k,$$

in the definition (2.64) of the random variable Y. Finally, the constant X in (2.64) is chosen in such a way that the random variable Y satisfies $\mathbb{E}_{\mathbb{Q}}(Y) = \widehat{M}^k_0$. Then we may deduce from (2.65) that $\widehat{M}^k = \widehat{M}$, and thus $\widehat{M}^k$ is manifestly a $\mathbb{G}$-martingale. □

2.5.3 Price Dynamics of the i^{th} CDS

As traded assets in our model, we take the constant savings account and a family of single-name CDSs with default protections δ_i and rates κ_i. For convenience, we assume that the CDSs have the same maturity T, but this assumption can be easily relaxed. The i^{th} traded CDS is formally defined by its dividend process

$$D_t^i = \int_{(0,t]} \delta_i(u)\, dH_u^i - \kappa_i(t \wedge \tau_i), \quad \forall\, t \in [0,T].$$

Consequently, the price at time t of the i^{th} CDS equals

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau_i \le T\}}\delta_i(\tau_i) \,|\, \mathcal{G}_t) - \kappa_i\, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau_i\}}\big((\tau_i \wedge T) - t\big) \,\big|\, \mathcal{G}_t\big). \tag{2.72}$$

To replicate a first-to-default claim, we only need to examine the dynamics of each CDS on the interval $[0, \tau_{(1)} \wedge T]$. The following lemma will prove useful in this regard.

Lemma 2.5.3 *We have, on the event* $\{\tau_{(1)} > t\}$,

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}}\delta_i(\tau_{(1)}) + \sum_{j\neq i} \mathbb{1}_{\{t<\tau_{(1)}=\tau_j\le T\}} S^i_{\tau_{(1)}}(\kappa_i) - \kappa_i \mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)} \wedge T - t) \,\Big|\, \mathcal{G}_t\Big).$$

Proof. We first note that the price $S_t^i(\kappa_i)$ can be represented as follows, on the event $\{\tau_{(1)} > t\}$,

$$\begin{aligned} S_t^i(\kappa_i) \;&=\; \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}}\delta_i(\tau_{(1)}) + \sum_{j\neq i} \mathbb{1}_{\{t<\tau_{(1)}=\tau_j\le T\}}(\mathbb{1}_{\{\tau_{(1)}<\tau_i\le T\}}\delta_i(\tau_i \wedge T) \\ &\quad - \kappa_i \mathbb{1}_{\{\tau_{(1)}<\tau_i\}}(\tau_i - \tau_{(1)})) \,\Big|\, \mathcal{G}_t\Big) - \kappa_i\, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)} \wedge T - t) \,\big|\, \mathcal{G}_t\big). \end{aligned}$$

By conditioning first on the σ-field $\mathcal{G}_{\tau_{(1)}}$, we obtain the claimed formula. □

Representation established in Lemma 2.5.3 is by no means surprising; it merely shows that in order to compute the price of a CDS prior to the first default, we can either do the computations in a single step, by considering the cash flows occurring on $]t, \tau_i \wedge T]$, or we can compute first the price of the contract at time $\tau_{(1)} \wedge T$, and subsequently value all cash flows occurring on $]t, \tau_{(1)} \wedge T]$. However, it also shows that we can consider from now on not the original i^{th} CDS but the associated CDS contract with random maturity $\tau_i \wedge T$.

Similarly as in Section 2.4.2, we write $S_t^i(\kappa_i) = \mathbb{1}_{\{t<\tau_{(1)}\}}\widetilde{S}_t^i(\kappa_i)$ where the pre-default price $\widetilde{S}_t^i(\kappa_i)$ satisfies

$$\widetilde{S}_t^i(\kappa_i) = \widetilde{\delta}^i(t,T) - \kappa_i \widetilde{A}^i(t,T) \tag{2.73}$$

where $\widetilde{\delta}^i(t,T)$ and $\kappa \widetilde{A}^i(t,T)$ are pre-default values of the protection leg and the fee leg respectively.

For any $j \neq i$, we define a function $S^i_{t|j}(\kappa_i) : [0,T] \to \mathbb{R}$, which represents the price of the i^{th} CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. Formally, this quantity is defined as the unique function satisfying

$$\mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}} S^i_{\tau_{(1)}|j}(\kappa_i) = \mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}} S^i_{\tau_{(1)}}(\kappa_i)$$

so that

$$\mathbb{1}_{\{\tau_{(1)}\le T\}} S^i_{\tau_{(1)}}(\kappa_i) = \sum_{j\neq i} \mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}} S^i_{\tau_{(1)}|j}(\kappa_i).$$

Let us examine the case of two names. Then the function $S^1_{t|2}(\kappa_1)$, $t \in [0,T]$, represents the price of the first CDS at time t on the event $\{\tau_{(1)} = \tau_2 = t\}$.

Lemma 2.5.4 *The function* $S^1_{v|2}(\kappa_1)$, $v \in [0,T]$, *equals*

$$S^1_{v|2}(\kappa_1) = \frac{\int_v^T \delta_1(u) f(u,v) du}{\int_v^\infty f(u,v)\, du} - \kappa_1 \frac{\int_v^T du \int_u^\infty dz f(z,v)}{\int_v^\infty f(u,v)\, du}. \tag{2.74}$$

Proof. Note that the conditional c.d.f. of τ_1 given that $\tau_1 > \tau_2 = v$ equals

$$\mathbb{Q}(\tau_1 \le u \,|\, \tau_1 > \tau_2 = v) = F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_v^u f(z,v)\, dz}{\int_v^\infty f(z,v)\, dz}, \quad \forall\, u \in [v,\infty],$$

so that the conditional tail equals

$$G_{\tau_1|\tau_1>\tau_2=v}(u) = 1 - F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_u^\infty f(z,v)\, dz}{\int_v^\infty f(z,v)\, dz}, \quad \forall\, u \in [v,\infty]. \tag{2.75}$$

Let J be the right-hand side of (2.74). It is clear that

$$J = -\int_v^T \delta_1(u)\, dG_{\tau_1|\tau_1>\tau_2=v}(u) - \kappa_1 \int_v^T G_{\tau_1|\tau_1>\tau_2=v}(u)\, du.$$

Combining Lemma 2.4.1 with the fact that $S^1_{\tau_{(1)}}(\kappa_i)$ is equal to the conditional expectation with respect to σ-field $\mathcal{G}_{\tau_{(1)}}$ of the cash flows of the i^{th} CDS on $]\tau_{(1)} \vee \tau_i, \tau_i \wedge T]$, we conclude that J coincides with $S^1_{v|2}(\kappa_1)$, the price of the first CDS on the event $\{\tau_{(1)} = \tau_2 = v\}$. □

The following result extends Lemma 2.4.2.

Lemma 2.5.5 *The dynamics of the pre-default price* $\widetilde{S}^i_t(\kappa_i)$ *are*

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\lambda}(t)\widetilde{S}^i_t(\kappa_i)\, dt + \Big(\kappa_i - \delta_i(t)\widetilde{\lambda}_i(t) - \sum_{j\neq i}^n S^i_{t|j}(\kappa_i)\widetilde{\lambda}_i(t)\Big) dt \tag{2.76}$$

where $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$, *or equivalently,*

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\lambda}_i(t)\big(\widetilde{S}^i_t(\kappa_i) - \delta_i(t)\big)\, dt + \sum_{j\neq i} \widetilde{\lambda}_j(t)\big(\widetilde{S}^i_t(\kappa_i) - S^i_{t|j}(\kappa_i)\big)\, dt + \kappa_i dt. \tag{2.77}$$

The cumulative price of the i^{th} *CDS stopped at* $\tau_{(1)}$ *satisfies*

$$\widehat{S}^i_t(\kappa_i) = S^i_t(\kappa_i) + \int_0^t \delta_i(u)\, dH^i_{u\wedge\tau_{(1)}} + \sum_{j\neq i} \int_0^t S^i_{u|j}(\kappa_i)\, dH^j_{u\wedge\tau_{(1)}} - \kappa_i(\tau_{(1)} \wedge t), \tag{2.78}$$

and thus

$$d\widehat{S}^i_t(\kappa_i) = \big(\delta_i(t) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^i_t + \sum_{j\neq i} \big(S^i_{t|j}(\kappa_i) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^j_t. \tag{2.79}$$

Proof. We shall consider the case $n = 2$. Using the formula derived in Lemma 2.5.3, we obtain

$$\widetilde{\delta}^1(t,T) = \frac{\int_t^T du\, \delta_1(u) \int_u^\infty dv f(u,v)}{G_{(1)}(t)} + \frac{\int_t^T dv\, S^1_{v|2}(\kappa_1) \int_v^\infty du f(u,v)}{G_{(1)}(t)}. \tag{2.80}$$

By adapting equality (2.68), we get

$$d\widetilde{\delta}^1(t,T) = \Big((\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t))\widetilde{g}_1(t) - \widetilde{\lambda}_1(t)\delta_1(t) - \widetilde{\lambda}_2(t) S^1_{t|2}(\kappa_1)\Big) dt. \tag{2.81}$$

To establish (2.76)-(2.77), we need also to examine the fee leg. Its price equals

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}\}}\kappa_1\big((\tau_{(1)}\wedge T)-t\big)\,\Big|\,\mathcal{G}_t\Big)=\mathbb{1}_{\{t<\tau_{(1)}\}}\kappa_1\widetilde{A}^i(t,T),$$

To evaluate the conditional expectation above, it suffices to use the c.d.f. $F_{(1)}$ of the random time $\tau_{(1)}$. As in Section 2.4.1 (see the proof of Lemma 2.4.1), we obtain

$$\widetilde{A}^i(t,T)=\frac{1}{G_{(1)}(t)}\int_t^T G_{(1)}(u)\,du, \tag{2.82}$$

and thus

$$d\widetilde{A}^i(t,T)=\big(1+(\widetilde{\lambda}_1(t)+\widetilde{\lambda}_2(t))\widetilde{A}^i(t,T)\big)\,dt.$$

Since $\widetilde{S}^1_t(\kappa_1)=\widetilde{\delta}^i(t,T)-\kappa_i\widetilde{A}^i(t,T)$, the formulae (2.76)-(2.77) follow. Formula (2.78) is rather clear. Finally, dynamics (2.79) can be easily deduced from (2.77) and (2.78) □

2.5.4 Risk-Neutral Valuation of a First-to-Default Claim

In this section, we shall analyze the risk-neutral valuation of first-to-default claims on a basket of n credit names.

Definition 2.5.2 *A first-to-default claim (FTDC) with maturity T is a defaultable claim $(X,A,Z,\tau_{(1)})$ where X is a constant amount payable at maturity if no default occurs, $A:[0,T]\to\mathbb{R}$ with $A_0=0$ is a function of bounded variation representing the dividend stream up to $\tau_{(1)}$, and $Z=(Z_1,Z_2,\dots,Z_n)$ is the vector of functions $Z_i:[0,T]\to\mathbb{R}$ where $Z_i(\tau_{(1)})$ specifies the recovery received at time $\tau_{(1)}$ if the i^{th} name is the first defaulted name, that is, on the event $\{\tau_i=\tau_{(1)}\le T\}$.*

We define the *risk-neutral value* π of an FTDC by setting

$$\pi_t=\sum_{i=1}^n\mathbb{E}_{\mathbb{Q}}\Big(Z_i(\tau_i)\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}}+\mathbb{1}_{\{t<\tau_{(1)}\}}\int_t^T(1-H^{(1)}_u)\,dA(u)+X\mathbb{1}_{\{\tau_{(1)}>T\}}\,\Big|\,\mathcal{G}_t\Big),$$

and the *risk-neutral cumulative value* $\widehat{\pi}$ of an FTDC by the formula

$$\begin{aligned}\widehat{\pi}_t &= \sum_{i=1}^n\mathbb{E}_{\mathbb{Q}}\Big(Z_i(\tau_i)\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}}+\mathbb{1}_{\{t<\tau_{(1)}\}}\int_t^T(1-H^{(1)}_u)\,dA(u)\,\Big|\,\mathcal{G}_t\Big)\\ &\quad+\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{\tau_{(1)}>T\}}|\mathcal{G}_t)+\sum_{i=1}^n\int_0^t Z_i(u)\,dH^i_{u\wedge\tau_{(1)}}+\int_0^t(1-H^{(1)}_u)\,dA(u)\end{aligned}$$

where the last two terms represent the past dividends. Let us stress that the risk-neutral valuation of an FTDC will be later supported by replication arguments (see Theorem 2.2), and thus risk-neutral value π of an FTDC will be shown to be its replication price.

By the *pre-default risk-neutral value* associated with a $\mathbb{G}$-adapted process π, we mean the function $\widetilde{\pi}$ such that $\pi_t\mathbb{1}_{\{\tau_{(1)}>t\}}=\widetilde{\pi}(t)\mathbb{1}_{\{\tau_{(1)}>t\}}$ for every $t\in[0,T]$. Direct calculations lead to the following result, which can also be deduced from Proposition 2.5.1.

Lemma 2.5.6 *The pre-default risk-neutral value of an FTDC equals*

$$\widetilde{\pi}(t)=\sum_{i=1}^n\frac{\Psi_i(t)}{G_{(1)}(t)}+\frac{1}{G_{(1)}(t)}\int_t^T G_{(1)}(u)\,dA(u)+X\frac{G_{(1)}(T)}{G_{(1)}(t)} \tag{2.83}$$

where

$$\Psi_i(t)=\int_{u_i=t}^T\int_{u_1=u_i}^\infty\dots\int_{u_{i-1}=u_i}^\infty\int_{u_{i+1}=u_i}^\infty\dots\int_{u_n=u_i}^\infty Z_i(u_i)\,F(du_1,\dots,du_{i-1},du_i,du_{i+1},\dots,du_n).$$

The next result extends Proposition 2.4.2 to the multi-name set-up. Its proof is similar to the proof of Lemma 2.5.5, and thus it is omitted.

Proposition 2.5.2 *The pre-default risk-neutral value of an FTDC satisfies*

$$d\widetilde{\pi}(t) = \sum_{i=1} \widetilde{\lambda}_i(t)\big(\widetilde{\pi}(t) - Z_i(t)\big)\,dt - dA(t). \tag{2.84}$$

Moreover, the risk-neutral value of an FTDC satisfies

$$d\pi_t = \sum_{i=1}^{n} \left(Z_i(t) - \widetilde{\pi}(t-)\right) d\widehat{M}^i_u - dA(\tau_{(1)} \wedge t), \tag{2.85}$$

and the risk-neutral cumulative value $\widehat{\pi}$ *of an FTDC satisfies*

$$d\widehat{\pi}_t = \sum_{i=1}^{n} \left(Z_i(t) - \widetilde{\pi}(t-)\right) d\widehat{M}^i_u. \tag{2.86}$$

2.5.5 Dynamic Replication of a First-to-Default Claim

Let $B = 1$ and single-name CDSs with prices $S^1(\kappa_1), \dots, S^n(\kappa_n)$ be traded assets. We say that a $\mathbb{G}$-predictable process $\phi = (\phi^0, \phi^1, \dots, \phi^n)$ and a function C of finite variation with $C(0) = 0$ define a *self-financing strategy with dividend stream* C if the wealth process $V(\phi, C)$, defined as

$$V_t(\phi, C) = \phi^0_t + \sum_{i=1}^{n} \phi^i_t S^i_t(\kappa_i), \tag{2.87}$$

satisfies

$$dV_t(\phi, C) = \sum_{i=1}^{n} \phi^i_t\big(dS^i_t(\kappa_i) + dD^i_t\big) - dC(t) = \sum_{i=1}^{n} \phi^i_t\, d\widehat{S}^i_t(\kappa_i) - dC(t) \tag{2.88}$$

where $S^i(\kappa_i)$ ($\widehat{S}^i(\kappa_i)$, respectively) is the price (cumulative price, respectively) of the i^{th} CDS.

Definition 2.5.3 *We say that a trading strategy* (ϕ, C) replicates an FTDC $(X, A, Z, \tau_{(1)})$ *if:*
(i) the processes $\phi = (\phi^0, \phi^1, \dots, \phi^n)$ *and* $V(\phi, C)$ *are stopped at* $\tau_{(1)} \wedge T$,
(ii) $C(\tau_{(1)} \wedge t) = A(\tau_{(1)} \wedge t)$ *for every* $t \in [0, T]$,
(iii) the equality $V_{\tau_{(1)} \wedge T}(\phi, C) = Y$ *holds, where the random variable* Y *equals*

$$Y = X \mathbb{1}_{\{\tau_{(1)} > T\}} + \sum_{i=1}^{n} Z_i(\tau_{(1)}) \mathbb{1}_{\{\tau_i = \tau_{(1)} \le T\}}. \tag{2.89}$$

We are now in a position to extend Theorem 2.1 to the case of a first-to-default claim on a basket of n credit names.

Theorem 2.2 *Assume that* $\det N(t) \neq 0$ *for every* $t \in [0, T]$, *where*

$$N(t) = \begin{bmatrix} \delta_1(t) - \widetilde{S}^1_t(\kappa_1) & S^2_{t|1}(\kappa_2) - \widetilde{S}^2_t(\kappa_2) & \cdot & S^n_{t|1}(\kappa_n) - \widetilde{S}^n_t(\kappa_n) \\ S^1_{t|2}(\kappa_1) - \widetilde{S}^1_t(\kappa_1) & \delta_2(t) - \widetilde{S}^2_t(\kappa_2) & \cdot & S^n_{t|2}(\kappa_n) - \widetilde{S}^n_t(\kappa_n) \\ \dots & \cdot & \cdot & \cdot \\ S^1_{t|n}(\kappa_1) - \widetilde{S}^1_t(\kappa_1) & S^2_{t|n}(\kappa_1) - \widetilde{S}^2_t(\kappa_1) & \cdot & \delta_n(t) - \widetilde{S}^n_t(\kappa_n) \end{bmatrix}$$

Let $\widetilde{\phi}(t) = (\widetilde{\phi}_1(t), \widetilde{\phi}_2(t), \dots, \widetilde{\phi}_n(t))$ *be the unique solution to the equation* $N(t)\widetilde{\phi}(t) = h(t)$ *where* $h(t) = (h_1(t), h_2(t), \dots, h_n(t))$ *with* $h_i(t) = Z_i(t) - \widetilde{\pi}(t-)$ *and where* $\widetilde{\pi}$ *is given by Lemma 2.5.6. More explicitly, the functions* $\widetilde{\phi}_1, \widetilde{\phi}_2, \dots, \widetilde{\phi}_n$ *satisfy, for* $t \in [0, T]$ *and* $i = 1, 2, \dots, n$,

$$\widetilde{\phi}_i(t)\big(\delta_i(t) - \widetilde{S}^i_t(\kappa_i)\big) + \sum_{j \neq i} \widetilde{\phi}_j(t)\big(S^j_{t|i}(\kappa_j) - \widetilde{S}^j_t(\kappa_j)\big) = Z_i(t) - \widetilde{\pi}(t-). \tag{2.90}$$

Let us set $\phi^i_t = \widetilde{\phi}_i(\tau_{(1)} \wedge t)$ *for* $i = 1, 2, \ldots, n$ *and let*

$$\phi^0_t = V_t(\phi, A) - \sum_{i=1}^{n} \phi^i_t S^i_t(\kappa_i), \quad \forall\, t \in [0, T], \tag{2.91}$$

where the process $V(\phi, A)$ *is given by the formula*

$$V_t(\phi, A) = \widetilde{\pi}(0) + \sum_{i=1}^{n} \int_{]0,\tau_{(1)}\wedge t]} \widetilde{\phi}_i(u)\, d\widehat{S}^i_u(\kappa_i) - A(\tau_{(1)} \wedge t). \tag{2.92}$$

Then the trading strategy (ϕ, A) *replicates an FTDC* $(X, A, Z, \tau_{(1)})$.

Proof. The proof is based on similar arguments as the proof of Theorem 2.1. It suffices to check that under the assumption of the theorem, for a trading strategy (ϕ, A) stopped at $\tau_{(1)}$, we obtain from (2.88) and (2.79) that

$$dV_t(\phi, A) = \sum_{i=1}^{n} \phi^i_t \Big(\big(\delta_i(t) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^i_t + \sum_{j \neq i} \big(S^i_{t|j}(\kappa_i) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^j_t \Big) - dA(\tau_{(1)} \wedge t).$$

For $\phi^i_t = \widetilde{\phi}_i(\tau_{(1)} \wedge t)$, where the functions $\widetilde{\phi}_1, \widetilde{\phi}_2, \ldots, \widetilde{\phi}_n$ solve (2.90), we thus obtain

$$dV_t(\phi, A) = \sum_{i=1}^{n} (Z_i(t) - \widetilde{\pi}(t-))\, d\widehat{M}^i_t - dA(\tau_{(1)} \wedge t).$$

By comparing the last formula with (2.85), we conclude that if, in addition, $V_0(\phi, A) = \pi_0 = \widetilde{\pi}_0$ and ϕ^0 is given by (2.91), then the strategy (ϕ, A) replicates an FTDC $(X, A, Z, \tau_{(1)})$. □

2.5.6 Conditional Default Distributions

In the case of first-to-default claims, it was enough to consider the unconditional distribution of default times. As expected, in order to deal with a general basket defaultable claim, we need to analyze conditional distributions of default times. It is possible to follow the approach presented in preceding sections, and to explicitly derive the dynamics of all processes of interest on the time interval $[0, T]$. However, since we deal here with a simple model of joint defaults, it suffices to make a non-restrictive assumption that we work on the canonical space $\Omega = \mathbb{R}^n$, and to use simple arguments based on conditioning with respect to past defaults.

Suppose that k names out of a total of n names have already defaulted. To introduce a convenient notation, we adopt the convention that the $n - k$ non-defaulted names are in their original order $j_1 < \ldots < j_{n-k}$, and the k defaulted names $i_1, \ldots, i_k$ are ordered in such a way that $u_1 < \ldots < u_k$. For the sake of brevity, we write $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\}$ to denote the *information structure* of the past k defaults.

Definition 2.5.4 *The joint conditional distribution function of default times* $\tau_{j_1}, \ldots, \tau_{j_{n-k}}$ *equals, for every* $t_1, \ldots, t_{n-k} > u_k$,

$$F(t_1, \ldots, t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k) = \mathbb{Q}\big(\tau_{j_1} \leq t_1, \ldots, \tau_{j_{n-k}} \leq t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\big).$$

The joint conditional survival function of default times $\tau_{j_1}, \ldots, \tau_{j_{n-k}}$ *is given by the expression*

$$G(t_1, \ldots, t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k) = \mathbb{Q}\big(\tau_{j_1} > t_1, \ldots, \tau_{j_{n-k}} > t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\big)$$

for every $t_1, \ldots, t_{n-k} > u_k$.

As expected, the conditional first-to-default intensities are defined using the joint conditional distributions, instead of the joint unconditional distribution. We write $G_{(1)}(t \,|D_k) = G(t, \ldots, t \,|D_k)$.

Definition 2.5.5 *Given the event D_k, for any $j_l \in \{j_1, \ldots, j_{n-k}\}$, the conditional first-to-default intensity of a surviving name j_l is denoted by $\widetilde{\lambda}_{j_l}(t\,|D_k) = \widetilde{\lambda}_{j_l}(t\,|\,\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k)$, and is given by the formula*

$$\widetilde{\lambda}_{j_l}(t\,|D_k) = \frac{\int_t^\infty \int_t^\infty \ldots \int_t^\infty dF(t_1, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{n-k}|D_k)}{G_{(1)}(t\,|D_k)}, \quad \forall\, t \in [u_k, T].$$

In Section 2.5.3, we introduced the processes $S^i_{t|j}(\kappa_j)$ representing the value of the i^{th} CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. According to the notation introduced above, we thus dealt with the conditional value of the i^{th} CDS with respect to $D_1 = \{\tau_j = t\}$. It is clear that to value a CDS for each surviving name we can proceed as prior to the first default, except that now we should use the conditional distribution

$$F(t_1, \ldots, t_{n-1}\,|\,D_1) = F(t_1, \ldots, t_{n-1}\,|\,\tau_j = j), \quad \forall\, t_1, \ldots, t_{n-1} \in [t, T],$$

rather than the unconditional distribution $F(t_1, \ldots, t_n)$ employed in Proposition 2.5.6. The same argument can be applied to any default event D_k. The corresponding conditional version of Proposition 2.5.6 is rather easy to formulate and prove, and thus we feel there is no need to provide an explicit conditional pricing formula here.

The conditional first-to-default intensities introduced in Definition 2.5.5 will allow us to construct the conditional first-to-default martingales in a similar way as we defined the first-to-default martingales M^i associated with the first-to-default intensities $\widetilde{\lambda}_i$. However, since any name can default at any time, we need to introduce an entire family of conditional martingales, whose compensators are based on intensities conditioned on the information structure of past defaults.

Definition 2.5.6 *Given the default event $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\}$, for each surviving name $j_l \in \{j_1, \ldots, j_{n-k}\}$, we define the basic conditional first-to-default martingale $\widehat{M}^{j_l}_{t|D_k}$ by setting*

$$\widehat{M}^{j_l}_{t|D_k} = H^{j_l}_{t\wedge\tau_{(k+1)}} - \int_{u_k}^t \mathbb{1}_{\{u<\tau_{(k+1)}\}} \widetilde{\lambda}_{j_l}(u\,|D_k)\, du, \quad \forall\, t \in [u_k, T]. \tag{2.93}$$

The process $\widehat{M}^{j_l}_{t|D_k}$, $t \in [u_k, T]$, is a martingale under the conditioned probability measure $\mathbb{Q}|D_k$, *that is, the probability measure $\mathbb{Q}$ conditioned on the event D_k, and with respect to the filtration generated by default processes of the surviving names, that is, the filtration $\mathcal{G}^{D_k}_t \stackrel{\text{def}}{=} \mathcal{H}^{j_1}_t \vee \ldots \vee \mathcal{H}^{j_{n-k}}_t$ for $t \in [u_k, T]$.*

Since we condition on the event D_k, we have $\tau_{(k+1)} = \tau_{j_1} \wedge \tau_{j_2} \wedge \ldots \wedge \tau_{j_{n-k}}$, so that $\tau_{(k+1)}$ is the first default for all surviving names. Formula (2.93) is thus a rather straightforward generalization of formula (2.63). In particular, for $k = 0$ we obtain $\widehat{M}^i_{t|D_0} = \widehat{M}^i_t$, $t \in [0, T]$, for any $i = 1, 2, \ldots, n$. The martingale property of the process $\widehat{M}^{j_l}_{t|D_k}$, stated in Definition 2.5.6, follows from Proposition 2.5.3 (it can also be seen as a conditional version of Corollary 2.5.1).

We are in the position to state the conditional version of the first-to-default martingale representation theorem of Section 2.5.2. Formally, this result is nothing else than a restatement of the martingale representation formula of Proposition 2.5.1 in terms of conditional first-to-default intensities and conditional first-to-default martingales.

Let us fix an event D_k write $\mathbb{G}^{D_k} = \mathbb{H}^{j_1} \vee \ldots \vee \mathbb{H}^{j_{n-k}}$.

Proposition 2.5.3 *Let Y be a random variable given by the formula*

$$Y = \sum_{l=1}^{n-k} Z_{j_l|D_k}(\tau_{j_l}) \mathbb{1}_{\{\tau_{j_l} \le T,\, \tau_{j_l} = \tau_{(k+1)}\}} + X \mathbb{1}_{\{\tau_{(k+1)} > T\}} \tag{2.94}$$

for some functions $Z_{j_l|D_k} : [u_k, T] \to \mathbb{R}$, $l = 1, 2, \ldots, n-k$, *and some constant* X *(possibly dependent on* D_k*). Let us define*

$$\widehat{M}_{t|D_k} = \mathbb{E}_{\mathbb{Q}|D_k}(Y \mid \mathcal{G}_t^{D_k}), \quad \forall\, t \in [u_k, T]. \tag{2.95}$$

Then $\widehat{M}_{t|D_k}$, $t \in [u_k, T]$, *is a* $\mathbb{G}^{D_k}$*-martingale with respect to the conditioned probability measure* $\mathbb{Q}|D_k$ *and it admits the following representation, for* $t \in [u_k, T]$,

$$\widehat{M}_{t|D_k} = \widehat{M}_{0|D_k} + \sum_{l=1}^{n-k} \int_{]u_k,t]} h_{j_l}(u|D_k)\, d\widehat{M}^{j_l}_{u|D_k}$$

where the processes h_{j_l} *are given by*

$$h_{j_l}(t\,|D_k) = Z_{j_l|D_k}(t) - \widehat{M}_{t-|D_k}, \quad \forall\, t \in [u_k, T].$$

Proof. The proof relies on a direct extension of arguments used in the proof of Proposition 2.5.1 to the context of conditional default distributions. Therefore, it is left to the reader. □

2.5.7 Recursive Valuation of a Basket Claim

We are ready extend the results developed in the context of first-to-default claims to value and hedge general basket claims. A generic basket claim is any contingent claim that pays a specified amount on each default from a basket of n credit names and a constant amount at maturity T if no defaults have occurred prior to or at T.

Definition 2.5.7 *A basket claim associated with a family of* n *credit names is given as* $(X, A, \bar{Z}, \bar{\tau})$ *where* X *is a constant amount payable at maturity only if no default occurs prior to or at* T*, the vector* $\bar{\tau} = (\tau_1, \ldots, \tau_n)$ *represents default times, and the time-dependent matrix* $\bar{Z}$ *represents the payoffs at defaults, specifically,*

$$\bar{Z} = \begin{bmatrix} Z_1(t\,|D_0) & Z_2(t\,|D_0) & . & Z_n(t\,|D_0) \\ Z_1(t\,|D_1) & Z_2(t\,|D_1) & . & Z_n(t\,|D_1) \\ . & . & . & . \\ Z_1(t\,|D_{n-1}) & Z_2(t\,|D_{n-1}) & . & Z_n(t\,|D_{n-1}) \end{bmatrix}.$$

Note that the above matrix $\bar{Z}$ is presented in the shorthand notation. In fact, in each row we need to specify, for an arbitrary choice of the event $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\}$ and any name $j_l \notin \{i_1, \ldots, i_k\}$, the *conditional payoff function* at the moment of the $(k+1)^{\text{th}}$ default:

$$Z_{j_l}(t\,|D_k) = Z_{j_l}(t \mid \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k), \quad \forall\, t \in [u_k, T].$$

In the financial interpretation, the function $Z_{j_l}(t\,|D_k)$ determines the recovery payment at the default of the name j_l, conditional on the event D_k, on the event $\{\tau_{j_l} = \tau_{(k+1)} = t\}$, that is, assuming that the name j_l is the first defaulting name among all surviving names. In particular, $Z_i(t\,|D_0) \stackrel{\text{def}}{=} Z_i(t)$ represents the recovery payment at the default of the i^{th} name at time $t \in [0, T]$, given that no defaults have occurred prior to t, that is, at the moment of the first default (note that the symbol D_0 means merely that we consider a situation of no defaults prior to t).

Example 2.5.1 Let us consider the k^{th} -to-default claim for some fixed $k \in \{1, 2, \ldots, n\}$. Assume that the payoff at the k^{th} default depends only on the moment of the k^{th} default and the identity of the k^{th} -to-default name. Then all rows of the matrix $\bar{Z}$ are equal to zero, except for the k^{th} row, which is $[Z_1(t \mid k-1), Z_2(t \mid k-1), \ldots, Z_n(t \mid k-1)]$ for $t \in [0, T]$. We write here $k-1$, rather than D_{k-1}, in order to emphasize that the knowledge of timings and identities of the k defaulted names is not relevant under the present assumptions.

More generally, for a generic basket claim in which the payoff at the i^{th} default depends on the time of the i^{th} default and identity of the i^{th} defaulting name only, the recovery matrix $\bar{Z}$ reads

$$\bar{Z} = \begin{bmatrix} Z_1(t) & Z_2(t) & . & Z_n(t) \\ Z_1(t\,|1) & Z_2(t\,|1) & . & Z_n(t\,|1) \\ . & . & . & . \\ Z_1(t\,|n-1) & Z_2(t\,|n-1) & . & Z_n(t\,|n-1) \end{bmatrix}$$

where $Z_j(t\,|k-1)$ represents the payoff at the moment $\tau_{(k)} = t$ of the k^{th} default if j is the k^{th} defaulting name, that is, on the event $\{\tau_j = \tau_{(k)} = t\}$. This shows that in several practically important examples of basket credit derivatives, the matrix $\bar{Z}$ will have a simple structure.

It is clear that any basket claim can be represented as a static portfolio of k^{th} -to-default claims for $k = 1, 2, \dots, n$. However, this decomposition does not seem to be advantageous for our purposes. In what follows, we prefer to represent a basket claim as a sequence of *conditional first-to-default claims*, with the same value between any two defaults as our basket claim. In that way, we will be able to directly apply results developed for the case of first-to-default claims and thus to produce a simple iterative algorithm for the valuation and hedging of a basket claim.

Instead of stating a formal result, using a rather heavy notation, we prefer to first focus on the computational procedure for valuation and hedging of a basket claim. The important concept in this procedure is the *conditional pre-default price*

$$\widetilde{Z}(t\,|D_k) = \widetilde{Z}(t\,|\,\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k), \quad \forall\, t \in [u_k, T],$$

of a "conditional first-to-default claim". The function $\widetilde{Z}(t\,|D_k)$, $t \in [u_k, T]$, is defined as the risk-neutral value of a conditional FTDC on $n-k$ surviving names, with the following recovery payoffs upon the first default at any date $t \in [u_k, T]$

$$\widehat{Z}_{j_l}(t\,|\,D_k) = Z_{j_l}(t\,|\,D_k) + \widetilde{Z}(t\,|\,D_k, \tau_{j_l} = t). \tag{2.96}$$

Assume for the moment that for any name $j_m \notin \{i_1, \dots, i_k, j_l\}$ the conditional recovery payoff $\widehat{Z}_{j_m}(t\,|\,\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1})$ upon the first default after date u_{k+1} is known. Then we can compute the function

$$\widetilde{Z}(t\,|\,\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1}), \quad \forall\, t \in [u_{k+1}, T],$$

as in Lemma 2.5.6, but using conditional default distribution. The assumption that the conditional payoffs are known is in fact not restrictive, since the functions appearing in right-hand side of (2.96) are known from the previous step in the following recursive pricing algorithm.

- *First step.* We first derive the value of a basket claim assuming that all but one defaults have already occurred. Let $D_{n-1} = \{\tau_{i_1} = u_1, \dots, \tau_{i_{n-1}} = u_{n-1}\}$. For any $t \in [u_{n-1}, T]$, we deal with the payoffs

  $$\widehat{Z}_{j_1}(t\,|D_{n-1}) = Z_{j_1}(t\,|D_{n-1}) = Z_{j_1}(t\,|\,\tau_{i_1} = u_1, \dots, \tau_{i_{n-1}} = u_{n-1}),$$

  for $j_1 \notin \{i_1, \dots, i_{n-1}\}$ where the recovery payment function $Z_{j_1}(t\,|D_{n-1})$, $t \in [u_{n-1}, T]$, is given by the specification of the basket claim. Hence we can evaluate the pre-default value $\widetilde{Z}(t\,|D_{n-1})$ at any time $t \in [u_{n-1}, T]$, as a value of a conditional first-to-default claim with the said payoff, using the conditional distribution under $\mathbb{Q}|D_{n-1}$ of the random time $\tau_{j_1} = \tau_{i_n}$ on the interval $[u_{n-1}, T]$.

- *Second step.* In this step, we assume that all but two names have already defaulted. Let $D_{n-2} = \{\tau_{i_1} = u_1, \dots, \tau_{i_{n-2}} = u_{n-2}\}$. For each surviving name $j_1, j_2 \notin \{i_1, \dots, i_{n-2}\}$, the payoff $\widehat{Z}_{j_l}(t\,|D_{n-2})$, $t \in [u_{n-2}, T]$, of a basket claim at the moment of the next default formally comprises the recovery payoff from the defaulted name j_l which is $Z_{j_l}(t\,|D_{n-2})$ and the pre-default value $\widetilde{Z}(t \,|\, D_{n-2}, \tau_{j_l} = t)$, $t \in [u_{n-2}, T]$, which was computed in the first step. Therefore, we have

$$\widehat{Z}_{j_l}(t \,|\, D_{n-2}) = Z_{j_l}(t \,|\, D_{n-2}) + \widetilde{Z}(t \,|\, D_{n-2}, \tau_{j_l} = t), \quad \forall\, t \in [u_{n-2}, T].$$

To find the value of a basket claim between the $(n-2)^{\text{th}}$ and $(n-1)^{\text{th}}$ default, it suffices to compute the pre-default value of the conditional FTDC associated with the two surviving names, $j_1, j_2 \notin \{i_1, \ldots, i_{n-2}\}$. Since the conditional payoffs $\widehat{Z}_{j_1}(t \,|D_{n-2})$ and $\widehat{Z}_{j_2}(t \,|D_{n-2})$ are known, we may compute the expectation under the conditional probability measure $\mathbb{Q}|D_{n-2}$ in order to find the pre-default value of this conditional FTDC for any $t \in [u_{n-2}, T]$.

- *General induction step.* We now assume that exactly k default have occurred, that is, we assume that the event $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\}$ is given. ¿From the preceding step, we know the function $\widetilde{Z}(t \,|D_{k+1})$ where $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1}\}$. In order to compute $\widetilde{Z}(t \,|D_k)$, we set

$$\widehat{Z}_{j_l}(t \,|\, D_k) = Z_{j_l}(t \,|\, D_k) + \widetilde{Z}(t \,|\, D_k, \tau_{j_l} = t), \quad \forall\, t \in [u_k, T], \tag{2.97}$$

for any $j_1, \ldots, j_{n-k} \notin \{i_1, \ldots, i_k\}$, and we compute $\widetilde{Z}(t \,|D_k), t \in [u_k, T]$, as the risk-neutral value under $\mathbb{Q}|D_k$ at time of the conditional FTDC with the payoffs given by (2.97).

We are in the position state the valuation result for a basket claim, which can be formally proved using the reasoning presented above.

Proposition 2.5.4 *The risk-neutral value at time $t \in [0, T]$ of a basket claim $(X, A, \bar{Z}, \bar{\tau})$ equals*

$$\pi_t = \sum_{k=0}^{n-1} \widetilde{Z}(t \,|D_k) \mathbb{1}_{[\tau_{(k)} \wedge T, \tau_{(k+1)} \wedge T[}(t), \quad \forall\, t \in [0, T],$$

where $D_k = D_k(\omega) = \{\tau_{i_1}(\omega) = u_1, \ldots, \tau_{i_k}(\omega) = u_k\}$ for $k = 1, 2, \ldots, n$, and D_0 means that no defaults have yet occurred.

2.5.8 Recursive Replication of a Basket Claim

From the discussion of the preceding section, it is clear that a basket claim can be conveniently interpreted as a specific sequence of conditional first-to-default claims. Hence it is easy to guess that the replication of a basket claim should refer to hedging of the underlying sequence of conditional first-to-default claims. In the next result, we denote $\tau_{(0)} = 0$.

Theorem 2.3 *For any $k = 0, 1, \ldots, n$, the replicating strategy ϕ for a basket claim $(X, A, \bar{Z}, \bar{\tau})$ on the time interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ coincides with the replicating strategy for the conditional FTDC with payoffs $\widehat{Z}(t \,|D_k)$ given by (2.97). The replicating strategy $\phi = (\phi^0, \phi^{j1}, \ldots, \phi^{j_{n-k}}, A)$, corresponding to the units of savings account and units of CDS on each surviving name at time t, has the wealth process*

$$V_t(\phi, A) = \phi^0_t + \sum_{l=1}^{n-i} \phi^{j_l}_t S^{j_l}_t(\kappa_{j_l})$$

where processes ϕ^{j_l}, $l = 1, 2, \ldots, n-k$ can be computed by the conditional version of Theorem 2.2.

Proof. We know that the basket claim can be decomposed into a series of conditional first-to-default claims. So, at any given moment of time $t \in [0, T]$, assuming that k defaults have already occurred, our basket claim is equivalent to the conditional FTDC with payoffs $\widehat{Z}(t \,|\, D_k)$ and the pre-default value $\widetilde{Z}(t \,|D_k)$. This conditional FTDC is alive up to the next default $\tau_{(k+1)}$ or maturity T, whichever comes first. Hence it is clear that the replicating strategy of a basket claim over the random interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ need to coincide with the replicating strategy for this conditional first-to-default claim, and thus it can be found along the same lines as in Theorem 2.2, using the conditional distribution under $\mathbb{Q}|D_k$ of defaults for surviving names. □

2.6 Applications to Copula-Based Credit Risk Models

In this section, we will apply our previous results to some specific models, in which some common copulas are used to model dependence between default times (see, for instance, Cherubini et al. [30], Embrechts et al. [47], Laurent and Gregory [67], Li [71] or McNeil et al. [75]). It is fair to admit that copula-based credit risk models are not fully suitable for a dynamical approach to credit risk, since the future behavior of credit spreads can be predicted with certainty, up to the observations of default times. Hence they are unsuitable for hedging of option-like contracts on credit spreads. On the other hand, however, these models are of a common use in practical valuation credit derivatives and thus we decided to present them here. Of course, our results are more general, so that they can be applied to an arbitrary joint distribution of default times (i.e., not necessarily given by some copula function). Also, in a follow-up work, we will extend the results of this work to a fully dynamical set-up.

For simplicity of exposition and in order to get more explicit formulae, we only consider the bivariate situation and we make the following standing assumptions.

Assumptions (B). We assume from now on that:
(i) we are given an FTDC $(X, A, Z, \tau_{(1)})$ where $Z = (Z_1, Z_2)$ for some constants Z_1, Z_2 and X,
(ii) the default times τ_1 and τ_2 have exponential marginal distributions with parameters λ_1 and λ_2,
(ii) the recovery δ_i of the i^{th} CDS is constant and $\kappa_i = \lambda_i \delta_i$ for $i = 1, 2$ (see Example 2.4.1).

Before proceeding to computations, let us note that

$$\int_{u=t}^{T} \int_{v=u}^{\infty} G(du, dv) = -\int_t^T G(du, u)$$

and thus, assuming that the pair (τ_1, τ_2) has the joint probability density function $f(u, v)$,

$$\int_t^T du \int_u^\infty dv f(u, v) = -\int_t^T \partial_1 G(u, u)\, du$$

and

$$dv \int_a^b f(u, v)\, du = G(a, dv) - G(b, dv) = dv\big(\partial_2 G(b, v) - \partial_2 G(a, v)\big)$$

$$\int_v^T du \int_u^\infty dz f(z, v) = -\int_v^T \partial_2 G(u, v)\, du.$$

2.6.1 Independent Default Times

Let us first consider the case where the default times τ_1 and τ_2 are independent (this corresponds to the product copula $C(u, v) = uv$). In view of independence, the marginal intensities and the first-to-default intensities can be easily shown to coincide. We have, for $i = 1, 2$

$$G_i(u) = \mathbb{Q}(\tau_i > u) = e^{-\lambda_i u}$$

and thus the joint survival function equals

$$G(u, v) = G_1(u) G_2(v) = e^{-\lambda_1 u} e^{-\lambda_2 v}.$$

Consequently

$$F(du, dv) = G(du, dv) = \lambda_1 \lambda_2 e^{-\lambda_1 u} e^{-\lambda_2 v}\, du dv = f(u, v)\, du dv$$

and $G(du, u) = -\lambda_1 e^{-(\lambda_1 + \lambda_2) u}\, du$.

Proposition 2.6.1 *Assume that the default times τ_1 and τ_2 are independent. Then the replicating strategy for an FTDC $(X, 0, Z, \tau_{(1)})$ is given as*

$$\widetilde{\phi}^1(t) = \frac{Z_1 - \widetilde{\pi}(t)}{\delta_1}, \quad \widetilde{\phi}^2(t) = \frac{Z_2 - \widetilde{\pi}(t)}{\delta_2}$$

where

$$\widetilde{\pi}(t) = \frac{(Z_1\lambda_1 + Z_2\lambda_2)}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + Xe^{-(\lambda_1+\lambda_2)(T-t)}.$$

Proof. From the previous remarks, we obtain

$$\begin{aligned}
\widetilde{\pi}(t) &= \frac{Z_1 \int_t^T \int_u^\infty dF(u,v)}{G(t,t)} + \frac{Z_2 \int_t^T \int_v^\infty dF(u,v)}{G(t,t)} + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{Z_1\lambda_1 \int_t^T e^{-(\lambda_1+\lambda_2)u} du}{e^{-(\lambda_1+\lambda_2)t}} + \frac{Z_2\lambda_2 \int_t^T e^{-(\lambda_1+\lambda_2)v} dv}{e^{-(\lambda_1+\lambda_2)t}} + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{Z_1\lambda_1}{(\lambda_1+\lambda_2)}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + \frac{Z_2\lambda_2}{(\lambda_1+\lambda_2)}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{(Z_1\lambda_1 + Z_2\lambda_2)}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + Xe^{-(\lambda_1+\lambda_2)(T-t)}.
\end{aligned}$$

Under the assumption of independence of default times, we also have that $S^i_{t|j}(\kappa_i) = \widetilde{S}^i_t(\kappa_i)$ for $i, j = 1, 2$ and $i \neq j$. Furthermore from Example 2.4.1, we have that $\widetilde{S}^i_t(\kappa_i) = 0$ for $t \in [0, T]$ and thus the matrix $N(t)$ in Theorem 2.2 reduces to

$$N(t) = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}.$$

The replicating strategy can be found easily by solving the linear equation $N(t)\widetilde{\phi}(t) = h(t)$ where $h(t) = (h_1(t), h_2(t))$ with $h_i(t) = Z_i - \widetilde{\pi}(t-) = Z_i - \widetilde{\pi}(t)$ for $i = 1, 2$. □

As another important case of a first-to-default claim, we take a first-to-default swap (FTDS). For a stylized FTDS we have $X = 0$, $A(t) = -\kappa_{(1)}t$ where $\kappa_{(1)}$ is the *swap spread*, and $Z_i(t) = \delta_i \in [0, 1)$ for some constants δ_i, $i = 1, 2$. Hence an FTDS is formally given as an FTDC $(0, -\kappa_{(1)}t, (\delta_1, \delta_2), \tau_{(1)})$.

Under the present assumptions, we easily obtain

$$\pi_0 = \widetilde{\pi}(0) = \frac{1 - e^{\lambda T}}{\lambda}\Big((\lambda_1\delta_1 + \lambda_2\delta_2) - \kappa_{(1)}\Big)$$

where $\lambda = \lambda_1 + \lambda_2$. The *FTDS market spread* is the level of $\kappa_{(1)}$ that makes the FTDS valueless at initiation. Hence in this elementary example this spread equals $\lambda_1\delta_1 + \lambda_2\delta_2$. In addition, it can be shown that under the present assumptions we have that $\widetilde{\pi}(t) = 0$ for every $t \in [0, T]$.

Suppose that we wish to hedge the short position in the FTDS using two CDSs, say CDS^i, $i = 1, 2$, with respective default times τ_i, protection payments δ_i and spreads $\kappa_i = \lambda_i\delta_i$. Recall that in the present set-up we have that, for $t \in [0, T]$,

$$S^i_{t|j}(\kappa_i) = \widetilde{S}^i_t(\kappa_i) = 0, \quad i, j = 1, 2, \ i \neq j. \tag{2.98}$$

Consequently, we have here that $h_i(t) = -Z_i(t) = -\delta_i$ for every $t \in [0, T]$. It then follows from equation $N(t)\widetilde{\phi}(t) = h(t)$ that $\widetilde{\phi}_1(t) = \widetilde{\phi}_2(t) = 1$ for every $t \in [0, T]$ and thus $\phi^0_t = 0$ for every $t \in [0, T]$. This result is by no means surprising: we hedge a short position in the FTDS by holding a static portfolio of two single-name CDSs since, under the present assumptions, the FTDS is equivalent to such a portfolio of corresponding single name CDSs. Of course, one would not expect that this feature will still hold in a general case of dependent default times.

The first equality in (2.98) is due to the standing assumption of independence of default times τ_1 and τ_2 and thus it will no longer be true for other copulas (see foregoing subsections). The second equality follows from the postulate that the risk-neutral marginal distributions of default times are exponential. In practice, the risk-neutral marginal distributions of default times will be obtained by fitting the model to market data (i.e., market prices of single name CDSs) and thus typically they will not be exponential. To conclude, both equalities in (2.98) are unlikely to hold in any real-life implementation. Hence this example show be seen as the simplest illustration of general results and we do not pretend that it has any practical merits. Nevertheless, we believe that it might be useful to give a few more comments on the hedging strategy considered in this example.

Suppose that a dealer sells one FTDS and hedges his short position by holding a portfolio composed of one CDS^1 contract and one CDS^2 contract. Let us consider the event $\{\tau_{(1)} = \tau_1 < T\}$. The cumulative premium the dealer collects on the time interval $[0, t],\ t \le \tau_{(1)}$, for selling the FTDS equals $(\lambda_1\delta_1 + \lambda_2\delta_2)t$. The protection coverage that the dealer has to pay at time $\tau_{(1)}$ equals δ_1 and the FTDS is terminated at τ_1. Now, the cumulative premium the dealer pays on the time interval $[0, t],\ t \le \tau_{(1)}$, for holding the portfolio of one CDS^1 contract and one CDS^2 contract is $(\lambda_1\delta_1 + \lambda_2\delta_2)t$. At time τ_1, the dealer receives the protection payment of δ_1. The CDS^1 is terminated at time τ_1; the dealer still holds the CDS^2 contract, however. Recall, though, that the ex-dividend price (i.e., the market price) of this contract is zero. Hence the dealer may unwind the contract at time $\tau_{(1)}$ at no cost (again, this only holds under the assumption of independence and exponential marginals). Consequently the dealer's P/L is flat (zero) over the lifetime of the FTDS and the dealer has no positions in the remaining CDS at the first default time. Though we consider here the simplest set-up, it is plausible that a similar interpretation of a hedging strategy will also apply in a more general framework.

2.6.2 Archimedean Copulas

We now proceed to the case of exponentially distributed, but dependent, default times. Their interdependence will be specified by a choice of some *Archimedean copula*. Recall that a bivariate Archimedean copula is defined as

$$C(u, v) = \varphi^{-1}(\varphi(u), \varphi(v))$$

where φ is called the *generator* of a copula.

Clayton Copula

Recall that the generator of the *Clayton copula* is given as

$$\varphi(s) = s^{-\theta} - 1, \quad s \in \mathbb{R}_+,$$

for some strictly positive parameter θ. Hence the bivariate Clayton copula can be represented as follows

$$C(u, v) = C_\theta^{Clayton}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}.$$

Under the present assumptions, the corresponding joint survival function $G(u, v)$ equals

$$G(u, v) = C(G_1(u), G_2(v)) = (e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}}$$

so that

$$\frac{G(u, dv)}{dv} = -\lambda_2 e^{\lambda_2 v\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}-1}$$

and

$$f(u, v) = \frac{G(du, dv)}{du dv} = (\theta + 1)\lambda_1\lambda_2 e^{\lambda_1 u\theta + \lambda_2 v\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}-2}.$$

Proposition 2.6.2 *Let the joint distribution of* (τ_1, τ_2) *be given by the Clayton copula with* $\theta > 0$. *Then the replicating strategy for an FTDC* $(X, 0, Z, \tau_{(1)})$ *is given by the expressions*

$$\widetilde{\phi}_1(t) = \frac{\delta_2(Z_1 - \widetilde{\pi}(t)) + S^2_{t|1}(\kappa_2)(Z_2 - \widetilde{\pi}(t))}{\delta_1\delta_2 - S^1_{t|2}(\kappa_1)S^2_{t|1}(\kappa_2)}, \tag{2.99}$$

$$\widetilde{\phi}_2(t) = \frac{\delta_1(Z_2 - \widetilde{\pi}(t)) + S^1_{t|2}(\kappa_1)(Z_1 - \widetilde{\pi}(t))}{\delta_1\delta_2 - S^1_{t|2}(\kappa_1)S^2_{t|1}(\kappa_2)}, \tag{2.100}$$

where

$$\begin{aligned}
\widetilde{\pi}(t) \;&=\; Z_1 \frac{\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} + Z_2 \frac{\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} \\
&\quad + X \frac{(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}}}{(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}},
\end{aligned}$$

$$\begin{aligned}
S^1_{v|2}(\kappa_1) \;&=\; \delta_1 \frac{[(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}} \\
&\quad - \kappa_1 \frac{\int_v^T (e^{\lambda_1\theta u} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1} du}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}},
\end{aligned}$$

and

$$\begin{aligned}
S^2_{u|1}(\kappa_2) \;&=\; \delta_2 \frac{[(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta u} + e^{\lambda_2\theta u} - 1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta u} + e^{\lambda_2\theta u} - 1)^{-1/\theta-1}} \\
&\quad - \kappa_2 \frac{\int_u^T (e^{\lambda_1\theta u} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1} dv}{(e^{\lambda_1\theta u} + e^{\lambda_2\theta u} - 1)^{-\frac{1}{\theta}-1}}.
\end{aligned}$$

Proof. Using the observation that

$$\begin{aligned}
\int_t^T du \int_u^\infty f(u,v)dv \;&=\; \int_t^T \lambda_1 e^{\lambda_1 u\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 u\theta} - 1)^{-\frac{1}{\theta}-1}\, du \\
&=\; \frac{1}{\theta}\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds
\end{aligned}$$

and thus by symmetry

$$\int_t^T dv \int_v^\infty f(u,v)du = \frac{1}{\theta}\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds.$$

We thus obtain

$$\begin{aligned}
\widetilde{\pi}(t) \;&=\; \frac{Z_1\int_t^T\int_u^\infty dG(u,v)}{G(t,t)} + \frac{Z_2\int_t^T\int_v^\infty dG(u,v)}{G(t,t)} + X\frac{G(T,T)}{G(t,t)} \\
&=\; Z_1 \frac{\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} + Z_2 \frac{\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} \\
&\quad + X \frac{(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}}}{(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}}.
\end{aligned}$$

We are in a position to determine the replicating strategy. Under the standing assumption that $\kappa_i = \lambda_i\delta_i$ for $i = 1, 2$ we still have that $\widetilde{S}^i_t(\kappa_i) = 0$ for $i = 1, 2$ and for $t \in [0, T]$. Hence the matrix $N(t)$ reduces to

$$N(t) = \begin{bmatrix} \delta_1 & -S^2_{t|1}(\kappa_2) \\ -S^1_{t|2}(\kappa_1) & \delta_2 \end{bmatrix}$$

where

$$
\begin{aligned}
S^1_{v|2}(\kappa_1) &= \delta_1 \frac{\int_v^T f(u,v)\,du}{\int_v^\infty f(u,v)\,du} - \kappa_1 \frac{\int_v^T \int_u^\infty f(z,v)\,dzdu}{\int_v^\infty f(u,v)\,du} \\
&= \delta_1 \frac{G(T,dv) - G(v,dv)}{G(v,dv)} + \kappa_1 \frac{\int_t^T G(u,dv)}{G(v,dv)} \\
&= \delta_1 \frac{[(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}} \\
&\quad - \kappa_1 \frac{\int_v^T (e^{\lambda_1\theta u} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}\,du}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}}.
\end{aligned}
$$

The expression for $S^2_{u|1}(\kappa_2)$ can be found by analogous computations. By solving the equation $N(t)\widetilde{\phi}(t) = h(t)$, we obtain the desired expressions (2.99)-(2.100). □

Similar computations can be done for the valuation and hedging of a first-to-default swap.

Gumbel Copula

As another of an Archimedean copula, we consider the *Gumbel copula* with the generator

$$\varphi(s) = (-\ln s)^\theta, \quad s \in \mathbb{R}_+,$$

for some $\theta \geq 1$. The bivariate Gumbel copula can thus be written as

$$C(u,v) = C_\theta^{Gumbel}(u,v) = e^{-[(-\ln u)^\theta + (-\ln v)^\theta]^{\frac{1}{\theta}}}.$$

Under our standing assumptions, the corresponding joint survival function $G(u,v)$ equals

$$G(u,v) = C(G_1(u), G_2(v)) = e^{-(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}.$$

Consequently

$$\frac{dG(u,v)}{dv} = -G(u,v)\lambda_2^\theta v^{\theta-1}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}$$

and

$$\frac{dG(u,v)}{dudv} = G(u,v)(\lambda_1\lambda_2)^\theta (uv)^{\theta-1}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-2}\big((\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}} + \theta - 1\big).$$

Proposition 2.6.3 *Let the joint distribution of (τ_1, τ_2) be given by the Gumbel copula with $\theta \geq 1$. Then the replicating strategy for an FTDC $(X, 0, Z, \tau_{(1)})$ is given by (2.99)-(2.100) with*

$$\widetilde{\pi}(t) = (Z_1\lambda_1^\theta + Z_2\lambda_2^\theta)\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + Xe^{-\lambda(T-t)},$$

$$
\begin{aligned}
S^1_{v|2}(\kappa_1) &= \delta_1 \frac{e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1} - e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}} \\
&\quad - \kappa_1 \frac{\int_v^T e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}\,du}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}},
\end{aligned}
$$

$$
\begin{aligned}
S^2_{u|1}(\kappa_2) &= \delta_2 \frac{e^{-(\lambda_1^\theta u^\theta + \lambda_2^\theta T^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta T^\theta)^{\frac{1}{\theta}-1} - e^{-\lambda v}\lambda^{1-\theta}u^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta}u^{1-\theta}} \\
&\quad - \kappa_2 \frac{\int_u^T e^{-(\lambda_1^\theta u^\theta + \lambda_2^\theta T^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}\,dv}{e^{-\lambda v}\lambda^{1-\theta}u^{1-\theta}}.
\end{aligned}
$$

Proof. We have

$$\begin{aligned}\int_t^T\int_u^\infty dG(u,v) &= \int_t^T \lambda_1^\theta(\lambda_1^\theta+\lambda_2^\theta)^{\frac{1}{\theta}-1}e^{-(\lambda_1^\theta+\lambda_2^\theta)^{\frac{1}{\theta}}u}\,du\\ &= (-\lambda_1^\theta\lambda^{-\theta}e^{-\lambda u})|_{u=t}^{u=T} = \lambda_1^\theta\lambda^{-\theta}(e^{-\lambda t}-e^{-\lambda T})\end{aligned}$$

where $\lambda=(\lambda_1^\theta+\lambda_2^\theta)^{\frac{1}{\theta}}$. Similarly

$$\int_t^T\int_v^\infty dG(u,v)=\lambda_2^\theta\lambda^{-\theta}(e^{-\lambda t}-e^{-\lambda T}).$$

Furthermore $G(T,T)=e^{-\lambda T}$ and $G(t,t)=e^{-\lambda t}$. Hence

$$\begin{aligned}\widetilde{\pi}(t) &= Z_1\frac{\int_t^T\int_u^\infty dG(u,v)}{G(t,t)}+Z_2\frac{\int_t^T\int_v^\infty dG(u,v)}{G(t,t)}+X\frac{G(T,T)}{G(t,t)}\\ &= Z_1\lambda_1^\theta\lambda^{-\theta}(e^{-\lambda t}-e^{-\lambda T})+Z_2\lambda_2^\theta\lambda^{-\theta}(e^{-\lambda t}-e^{-\lambda T})+Xe^{-\lambda(T-t)}\\ &= (Z_1\lambda_1^\theta+\delta_2Z_2^\theta)\lambda^{-\theta}(e^{-\lambda t}-e^{-\lambda T})+Xe^{-\lambda(T-t)}.\end{aligned}$$

In order to find the replicating strategy, we proceed as in the proof of Proposition 2.6.2. Under the present assumptions, we have

$$\begin{aligned}S^1_{v|2}(\kappa_1) &= \delta_1\frac{\int_v^T f(u,v)du}{\int_v^\infty f(u,v)du}-\kappa_1\frac{\int_v^T\int_u^\infty f(z,v)dzdu}{\int_v^\infty f(u,v)du}\\ &= \delta_1\frac{e^{-(\lambda_1^\theta T^\theta+\lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta T^\theta+\lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}-e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}\\ &\quad-\kappa_1\frac{\int_v^T e^{-(\lambda_1^\theta T^\theta+\lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta+\lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}du}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}.\end{aligned}$$

This completes the proof. □

2.6.3 One-Factor Gaussian Copula

Let us finally consider the industry standard one-factor Gaussian copula model proposed by Li [71]. We no longer postulate that the marginal distributions of default times are exponential.

Let Y_i, $i=0,1,\dots,n$ be $n+1$ independent Gaussian random variables with zero mean and unit variance. The random variable X_i is given as

$$X_i=\rho_iY_0+\sqrt{1-\rho_i^2}\,Y_i$$

where the random variable Y_0 represents the common factor and Y_i denotes the idiosyncratic factor. The i^{th} default time is given by the formula

$$\tau_i=\inf\{\,t\in\mathbb{R}_+ : F_{X_i}^{-1}(F_i(t))\geq X_i\}$$

where F_{X_i} is the c.d.f. of X_i and F_i is the marginal distribution function of τ_i. We have

$$\begin{aligned}\mathbb{Q}(\tau_i\geq t\,|\,Y_0=y) &= \mathbb{Q}\Big(X_i\geq F_{X_i}^{-1}(F_i(t))\,\Big|\,Y_0=y\Big)\\ &= \mathbb{Q}\Big(\rho_iY_0+\sqrt{1-\rho_i^2}\,Y_i\geq F_{X_i}^{-1}(F_i(t))\,\Big|\,Y_0=y\Big)\\ &= \mathbb{Q}\left(Y_i\geq\frac{F_{X_i}^{-1}(F_i(t))-\rho_iY_0}{\sqrt{1-\rho_i^2}}\,\middle|\,Y_0=y\right)\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{Q}\Big(Y_i \geq \frac{F_{X_i}^{-1}(F_i(t)) - \rho_i y}{\sqrt{1-\rho_i^2}}\Big) \\
&= 1 - F_{Y_i}\Big(\frac{F_{X_i}^{-1}(F_i(t)) - \rho_i y}{\sqrt{1-\rho_i^2}}\Big)
\end{aligned}
$$

where F_{Y_i} is the c.d.f. of Y_i. In the bivariate case, we obtain

$$
\begin{aligned}
&\mathbb{Q}(\tau_1 \geq u, \tau_2 \geq v \,|\, Y_0 = y) \\
&\quad = \mathbb{Q}\Big(X_1 \geq F_{X_1}^{-1}(F_1(u)), X_2 \geq F_{X_2}^{-1}(F_2(v)) \,\Big|\, Y_0 = y\Big) \\
&\quad = \mathbb{Q}\Big(\rho_1 Y_0 + \sqrt{1-\rho_1^2}\, Y_1 \geq F_{X_1}^{-1}(F_1(u)), \\
&\qquad\quad \rho_2 Y_0 + \sqrt{1-\rho_2^2}\, Y_2 \geq F_{X_2}^{-1}(F_2(v)) \,\Big|\, Y_0 = y\Big) \\
&\quad = \mathbb{Q}\Big(Y_1 \geq \frac{F_{X_1}^{-1}(F_1(u)) - \rho_1 y}{\sqrt{1-\rho_1^2}}, Y_2 \geq \frac{F_{X_2}^{-1}(F_2(v)) - \rho_2 y}{\sqrt{1-\rho_2^2}}\Big) \\
&\quad = \mathbb{Q}\Big(Y_1 \geq \frac{F_{X_1}^{-1}(F_1(u)) - \rho_1 y}{\sqrt{1-\rho_1^2}}\Big)\mathbb{Q}\Big(Y_2 \geq \frac{F_{X_2}^{-1}(F_2(v)) - \rho_2 y}{\sqrt{1-\rho_2^2}}\Big) \\
&\quad = \Big(1 - F_{Y_1}\Big(\frac{F_{X_1}^{-1}(F_1(u)) - \rho_1 y}{\sqrt{1-\rho_1^2}}\Big)\Big)\Big(1 - F_{Y_2}\Big(\frac{F_{X_2}^{-1}(F_2(v)) - \rho_2 y}{\sqrt{1-\rho_2^2}}\Big)\Big).
\end{aligned}
$$

Hence the joint survival function $G(u,v)$ of τ_1, τ_2 equals

$$
\begin{aligned}
G(u,v) &= \mathbb{Q}(\tau_1 \geq u, \tau_2 \geq v) \\
&= \int_{\mathbb{R}} \mathbb{Q}(\tau_1 \geq u, \tau_2 \geq v \,|\, Y_0 = y) f_{Y_0}(y)\, dy \\
&= \int_{\mathbb{R}} \Big(1 - F_{Y_1}\Big(\frac{F_{X_1}^{-1}(F_1(u)) - \rho_1 y}{\sqrt{1-\rho_1^2}}\Big)\Big)\Big(1 - F_{Y_2}\Big(\frac{F_{X_2}^{-1}(F_2(v)) - \rho_2 y}{\sqrt{1-\rho_2^2}}\Big)\Big) f_{Y_0}(y)\, dy
\end{aligned}
$$

where f_{Y_0} is the probability density function of Y_0. Therefore

$$
\begin{aligned}
\frac{dG(u,v)}{dv} &= -\int_{\mathbb{R}_+} \Big(1 - F_{Y_1}\Big(\frac{F_{X_1}^{-1}(F_1(u)) - \rho_1 y}{\sqrt{1-\rho_1^2}}\Big)\Big) f_{Y_2}\Big(\frac{F_{X_2}^{-1}(F_2(v)) - \rho_2 y}{\sqrt{1-\rho_2^2}}\Big) \\
&\quad \times \frac{1}{\sqrt{1-\rho_2^2}} \frac{1}{f_2[F_{X_2}^{-1}(F_2(v))]}\, f_2(v) f_{Y_0}(y)\, dy
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dG(u,v)}{dudv} &= \int_{\mathbb{R}_+} f_{Y_2}\Big(\frac{F_{X_2}^{-1}(F_2(v)) - \rho_2 y}{\sqrt{1-\rho_2^2}}\Big) \frac{1}{\sqrt{1-\rho_2^2}} \frac{1}{f_2[F_{X_2}^{-1}(F_2(v))]}\, f_2(u) \\
&\quad \times f_{Y_1}\Big(\frac{F_{X_1}^{-1}(F_1(u)) - \rho_1 y}{\sqrt{1-\rho_1^2}}\Big) \frac{1}{\sqrt{1-\rho_1^2}} \frac{1}{f_1[F_{X_1}^{-1}(F_1(u))]}\, f_1(u) f_{Y_0}(y)\, dy
\end{aligned}
$$

where f_1 and f_2 are the probability density functions of τ_1 and τ_2 respectively. In principle, we are now in a position to combine these results with Lemma 2.5.6 and Theorem 2.2. The expressions for the prices and replicating strategy will be less explicit then in previously studied cases, however.

Chapter 3

Hazard Process Approach

In the general *reduced-form approach*, we deal with two kinds of information: the information from the assets prices, denoted as $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$, and the information from the default time, that is, the knowledge of the time where the default occurred in the past, if the default has indeed already appeared. As we already know, the latter information is modeled by the filtration $\mathbb{H}$ generated by the default process H.

At the intuitive level, the *reference filtration* $\mathbb{F}$ is generated by prices of some assets, or by other economic factors (such as, e.g., interest rates). This filtration can also be a subfiltration of the prices. The case where $\mathbb{F}$ is the trivial filtration is exactly what we have studied in the toy example. Though in typical examples $\mathbb{F}$ is chosen to be the Brownian filtration, most theoretical results do not rely on such a specification of the filtration $\mathbb{F}$. We denote by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ the *full filtration* (also known as the *enlarged filtration*).

Special attention will be paid in this chapter to the so-called hypothesis (H) , which, in the present context, postulates the invariance of the martingale property with respect to the enlargement of $\mathbb{F}$ by the observations of a default time. In order to examine the exact meaning of market completeness in a defaultable world and to deduce the hedging strategies for credit derivatives, we shall establish a suitable version of a representation theorem. Most results from this chapter can be found, for instance, in survey papers by Jeanblanc and Rutkowski [59, 60].

3.1 General Case

The concepts introduced in the previous chapter will now be extended to a more general set-up, when allowance for a larger flow of information – formally represented hereafter by some reference filtration $\mathbb{F}$ – is made.

We denote by τ a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, satisfying: $\mathbb{Q}\{\tau = 0\} = 0$ and $\mathbb{Q}\{\tau > t\} > 0$ for any $t \in \mathbb{R}_+$. We introduce a right-continuous process H by setting $H_t = \mathbb{1}_{\{\tau \le t\}}$ and we denote by $\mathbb{H}$ the associated filtration: $\mathcal{H}_t = \sigma(H_u : u \le t)$. Let $\mathbb{G} = (\mathcal{G}_t)_{t \ge 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{Q})$. All filtrations considered in what follows are implicitly assumed to satisfy the 'usual conditions' of right-continuity and completeness. For each $t \in \mathbb{R}_+$, the total information available at time t is captured by the σ-field $\mathcal{G}_t$.

We shall focus on the case described in the following assumption. We assume that we are given an auxiliary filtration $\mathbb{F}$ such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$; that is, $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in \mathbb{R}_+$. For the sake of simplicity, we assume that the σ-field $\mathcal{F}_0$ is trivial (so that $\mathcal{G}_0$ is a trivial σ-field as well).

The process H is obviously $\mathbb{G}$-adapted, but it is not necessarily $\mathbb{F}$-adapted. In other words, the random time τ is a $\mathbb{G}$-stopping time, but it may fail to be an $\mathbb{F}$-stopping time.

Lemma 3.1.1 *Assume that the filtration* $\mathbb{G}$ *satisfies* $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$*. Then* $\mathbb{G} \subseteq \mathbb{G}^*$*, where* $\mathbb{G}^* = (\mathcal{G}_t^*)_{t \geq 0}$ *with*

$$\mathcal{G}_t^* \stackrel{\text{def}}{=} \big\{ A \in \mathcal{G} \,:\, \exists\, B \in \mathcal{F}_t, \;\; A \cap \{\tau > t\} = B \cap \{\tau > t\} \big\}.$$

Proof. It is rather clear that the class $\mathcal{G}_t^*$ is a sub-σ-field of $\mathcal{G}$. Therefore, it is enough to check that $\mathcal{H}_t \subseteq \mathcal{G}_t^*$ and $\mathcal{F}_t \subseteq \mathcal{G}_t^*$ for every $t \in \mathbb{R}_+$. Put another way, we need to verify that if either $A = \{\tau \leq u\}$ for some $u \leq t$ or $A \in \mathcal{F}_t$, then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. In the former case we may take $B = \emptyset$, and in the latter $B = A$. □

For any $t \in \mathbb{R}_+$, we write $F_t = \mathbb{Q}\{\tau \leq t \,|\, \mathcal{F}_t\}$, and we denote by G the $\mathbb{F}$-*survival process* of τ with respect to the filtration $\mathbb{F}$, given as:

$$G_t \stackrel{\text{def}}{=} 1 - F_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in \mathbb{R}_+.$$

Notice that for any $0 \leq t \leq s$ we have $\{\tau \leq t\} \subseteq \{\tau \leq s\}$, and so

$$\mathbb{E}_{\mathbb{Q}}(F_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{Q}\{\tau \leq s \,|\, \mathcal{F}_s\} \,|\, \mathcal{F}_t) = \mathbb{Q}\{\tau \leq s \,|\, \mathcal{F}_t\} \geq \mathbb{Q}\{\tau \leq t \,|\, \mathcal{F}_t\} = F_t.$$

This shows that the process F (G, resp.) follows a bounded, non-negative $\mathbb{F}$-submartingale ($\mathbb{F}$-supermartingale, resp.) under $\mathbb{Q}$. We may thus deal with the right-continuous modification of F (of G) with finite left-hand limits. The next definition is a rather straightforward generalization of the concept of the hazard function (see Definition 2.2.1).

Definition 3.1.1 *Assume that* $F_t < 1$ *for* $t \in \mathbb{R}_+$*. The* $\mathbb{F}$*-hazard process of* τ *under* $\mathbb{Q}$*, denoted by* Γ*, is defined through the formula* $1 - F_t = e^{-\Gamma_t}$*. Equivalently,* $\Gamma_t = -\ln G_t = -\ln(1 - F_t)$ *for every* $t \in \mathbb{R}_+$.

Since $G_0 = 1$, it is clear that $\Gamma_0 = 0$. For the sake of conciseness, we shall refer briefly to Γ as the $\mathbb{F}$-hazard process, rather than the $\mathbb{F}$-hazard process under $\mathbb{Q}$, unless there is a danger of confusion.

Throughout this chapter, we will work under the standing assumption that the inequality $F_t < 1$ holds for every $t \in \mathbb{R}_+$, so that the $\mathbb{F}$-hazard process Γ is well defined. Hence the case when τ is an $\mathbb{F}$-stopping time (that is, the case when $\mathbb{F} = \mathbb{G}$) is not dealt with here.

3.1.1 Key Lemma

We shall first focus on the conditional expectation $\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{G}_t)$, where Y is a $\mathbb{Q}$-integrable random variable. We start by the following result, which is a direct counterpart of Lemma 2.2.1.

Lemma 3.1.2 *For any* $\mathcal{G}$*-measurable, integrable random variable* Y *and any* $t \in \mathbb{R}_+$ *we have*

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{F}_t)}{\mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}}. \tag{3.1}$$

In particular, for any $t \leq s$

$$\mathbb{Q}\{t < \tau \leq s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{Q}\{t < \tau \leq s \,|\, \mathcal{F}_t\}}{\mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}}. \tag{3.2}$$

Proof. Let us denote $C = \{\tau > t\}$. We need to verify that (recall that $\mathcal{F}_t \subseteq \mathcal{G}_t$)

$$\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_C Y \mathbb{Q}(C \,|\, \mathcal{F}_t) \,\big|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t) \,\big|\, \mathcal{G}_t\big).$$

Put another way, we need to show that for any $A \in \mathcal{G}_t$ we have

$$\int_A \mathbb{1}_C Y \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_A \mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{Q}.$$

In view of Lemma 3.1.1, for any $A \in \mathcal{G}_t$ we have $A \cap C = B \cap C$ for some event $B \in \mathcal{F}_t$, and so

$$\begin{aligned}\int_A \mathbb{1}_C Y \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} &= \int_{A\cap C} Y \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_{B\cap C} Y \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} \\ &= \int_B \mathbb{1}_C Y \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_B \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} \\ &= \int_B \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t) \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_{B\cap C} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{Q} \\ &= \int_{A\cap C} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_A \mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{Q}.\end{aligned}$$

This ends the proof. □

The following corollary is straightforward.

Corollary 3.1.1 *Let Y be an $\mathcal{G}_T$-measurable, integrable random variable. Then*

$$\mathbb{E}_{\mathbb{Q}}(Y\mathbb{1}_{T<\tau} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{Q}}(Y\mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} \,|\, \mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}(Y e^{-\Gamma_T} \,|\, \mathcal{F}_t). \tag{3.3}$$

Lemma 3.1.3 *Let h be an $\mathbb{F}$-predictable process. Then,*

$$\mathbb{E}_{\mathbb{Q}}(h_\tau \mathbb{1}_{\tau<T} \,|\, \mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T h_u \, dF_u \,\Big|\, \mathcal{F}_t \Big) \tag{3.4}$$

We are not interested in $\mathbb{G}$-predictable processes, mainly because any $\mathbb{G}$-predictable process is equal, on $\{t \le \tau\}$ to an $\mathbb{F}$-predictable process. As we shall see, this elementary result will allow us to compute the value of credit derivatives, as soon as some elementary defaultable assets are priced by the market.

3.1.2 Martingales

Proposition 3.1.1 *(i) The process $L_t = (1 - H_t)e^{\Gamma(t)}$ is a $\mathbb{G}$-martingale.*
(ii) If X is an $\mathbb{F}$-martingale then XL is a $\mathbb{G}$-martingale.
(iii) If the process Γ is increasing and continuous, then the process $M_t = H_t - \Gamma(t \wedge \tau)$ is a $\mathbb{G}$-martingale.

Proof. (i) From Lemma 3.1.2, for any $t > s$,

$$\mathbb{E}_{\mathbb{Q}}(L_t \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \,|\, \mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}} e^{\Gamma_s} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \,|\, \mathcal{F}_s) = \mathbb{1}_{\{\tau>s\}} e^{\Gamma_s} = L_s$$

since

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} | \mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} \,|\, \mathcal{F}_t) e^{\Gamma_t} | \mathcal{F}_s) = 1.$$

(ii) From Lemma 3.1.2,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(L_t X_t \,|\, \mathcal{G}_s) &= \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} L_t X_t \,|\, \mathcal{G}_s) \\ &= \mathbb{1}_{\{\tau>s\}} e^{\Gamma_s} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} e^{-\Gamma_t} X_t \,|\, \mathcal{F}_s) \\ &= \mathbb{1}_{\{\tau>s\}} e^{\Gamma_s} \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} \,|\, \mathcal{F}_t) e^{-\Gamma_t} X_t \,|\, \mathcal{F}_s) \\ &= L_s X_s.\end{aligned}$$

(iii) From integration by parts formula (H is a finite variation process, and Γ an increasing continuous process):

$$dL_t = (1 - H_t)e^{\Gamma_t} d\Gamma_t - e^{\Gamma_t} dH_t$$

and the process $M_t = H_t - \Gamma(t \wedge \tau)$ can be written

$$M_t \equiv \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u) d\Gamma_u = - \int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a $\mathbb{G}$-local martingale since L is $\mathbb{G}$-martingale. It should be noted that, if Γ is not increasing, the differential of e^Γ is more complicated. □

3.1.3 Interpretation of the Intensity

The submartingale property of F implies, from the Doob-Meyer decomposition, that $F = Z + A$ where Z is a $\mathbb{F}$-martingale and A a $\mathbb{F}$-predictable increasing process.

Lemma 3.1.4 *We have*

$$\mathbb{E}_{\mathbb{Q}}(h_\tau \mathbb{1}_{\{\tau<T\}} \,|\, \mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T h_u \, dA_u \,\Big|\, \mathcal{F}_t \Big).$$

In this general setting, the process Γ is not with finite variation. Hence, part (iii) in Proposition 3.1.1 does not yield the Doob-Meyer decomposition of H. We shall assume, for simplicity, that F is continuous.

Proposition 3.1.2 *Assume that F is a continuous process. Then the process*

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dA_u}{1 - F_u}, \quad \forall \, t \in \mathbb{R}_+,$$

is a $\mathbb{G}$-martingale.

Proof. Let $s < t$. We give the proof in two steps, using the Doob-Meyer decomposition $F = Z + A$ of F.
First step. We shall prove that

$$\mathbb{E}_{\mathbb{Q}}(H_t \,|\, \mathcal{G}_s) = H_s + \mathbb{1}_{\{s<\tau\}} \frac{1}{1 - F_s} \mathbb{E}_{\mathbb{Q}}(A_t - A_s \,|\, \mathcal{F}_s)$$

Indeed,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(H_t \,|\, \mathcal{G}_s) &= 1 - \mathbb{Q}(t < \tau \,|\, \mathcal{G}_s) = 1 - \mathbb{1}_{\{s<\tau\}} \frac{1}{1 - F_s} \, \mathbb{E}_{\mathbb{Q}}(1 - F_t \,|\, \mathcal{F}_s) \\
&= 1 - \mathbb{1}_{\{s<\tau\}} \frac{1}{1 - F_s} \, \mathbb{E}_{\mathbb{Q}}(1 - Z_t - A_t \,|\, \mathcal{F}_s) \\
&= 1 - \mathbb{1}_{\{s<\tau\}} \frac{1}{1 - F_s} (1 - Z_s - A_s - \mathbb{E}_{\mathbb{Q}}(A_t - A_s \,|\, \mathcal{F}_s) \\
&= 1 - \mathbb{1}_{\{s<\tau\}} \frac{1}{1 - F_s} (1 - F_s - \mathbb{E}_{\mathbb{Q}}(A_t - A_s \,|\, \mathcal{F}_s) \\
&= \mathbb{1}_{\{\tau \le s\}} + \mathbb{1}_{\{s<\tau\}} \frac{1}{1 - F_s} \, \mathbb{E}_{\mathbb{Q}}(A_t - A_s \,|\, \mathcal{F}_s)
\end{aligned}$$

Second step. Let us

$$\Lambda_t = \int_0^t (1 - H_s) \frac{dA_s}{1 - F_s}.$$

We shall prove that

$$\mathbb{E}_{\mathbb{Q}}(\Lambda_{t\wedge\tau} \,|\, \mathcal{G}_s) = \Lambda_{s\wedge\tau} + \mathbb{1}_{\{s<\tau\}} \frac{1}{1 - F_s} \, \mathbb{E}_{\mathbb{Q}}(A_t - A_s \,|\, \mathcal{F}_s).$$

From the key lemma, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\Lambda_{t\wedge\tau}\,|\,\mathcal{G}_s) &= \Lambda_{s\wedge\tau}\mathbb{1}_{\{\tau\le s\}} + \mathbb{1}_{\{s<\tau\}}\frac{1}{1-F_s}\,\mathbb{E}_{\mathbb{Q}}\left(\int_s^\infty \Lambda_{t\wedge u}\,dF_u\,|\,\mathcal{F}_s\right)\\
&= \Lambda_{s\wedge\tau}\mathbb{1}_{\{\tau\le s\}} + \mathbb{1}_{\{s<\tau\}}\frac{1}{1-F_s}\,\mathbb{E}_{\mathbb{Q}}\left(\int_s^t \Lambda_u\,dF_u + \int_t^\infty \Lambda_t\,dF_u\,|\,\mathcal{F}_s\right)\\
&= \Lambda_{s\wedge\tau}\mathbb{1}_{\{\tau\le s\}} + \mathbb{1}_{\{s<\tau\}}\frac{1}{1-F_s}\,\mathbb{E}_{\mathbb{Q}}\left(\int_s^t \Lambda_u\,dF_u + \Lambda_t(1-F_t)\,|\,\mathcal{F}_s\right).
\end{aligned}$$

Using the integration by parts formula and the fact that Λ is of bounded variation and continuous, we obtain

$$d(\lambda_t(1-F_t)) = -\Lambda_t\,dF_t + (1-F_t)d\Lambda_t = -\Lambda_t\,dF_t + dA_t.$$

Hence

$$\int_s^t \Lambda_u\,dF_u + \Lambda_t(1-F_t) = -\Lambda_t(1-F_t) + \Lambda_s(1-F_s) + A_t - A_s + \Lambda_t(1-F_t) = \Lambda_s(1-F_s) + A_t - A_s.$$

It follows that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\Lambda_{t\wedge\tau}\,|\,\mathcal{G}_s) &= \Lambda_{s\wedge\tau}\mathbb{1}_{\{\tau\le s\}} + \mathbb{1}_{\{s<\tau\}}\frac{1}{1-F_s}\,\mathbb{E}_{\mathbb{Q}}\left(\Lambda_s(1-F_s) + A_t - A_s\,|\,\mathcal{F}_s\right)\\
&= \Lambda_{s\wedge\tau} + \mathbb{1}_{\{s<\tau\}}\frac{1}{1-F_s}\,\mathbb{E}_{\mathbb{Q}}\left(A_t - A_s\,|\,\mathcal{F}_s\right).
\end{aligned}$$

This completes the proof. □

Let us assume that A is absolutely continuous with respect to the Lebesgue measure and let us denote by a its derivative. We have proved the existence of a $\mathbb{F}$-adapted process γ, called the intensity, such that the process

$$H_t - \int_0^{t\wedge\tau} \gamma_u\,du = H_t - \int_0^t (1-H_u)\gamma_u\,du$$

is a $\mathbb{G}$-martingale. More precisely, $\gamma_t = \frac{a_t}{1-F_t}$ for $t \in \mathbb{R}_+$.

Lemma 3.1.5 *The intensity process γ satisfies*

$$\gamma_t = \lim_{h\to 0}\frac{1}{h}\frac{\mathbb{Q}(t<\tau<t+h\,|\,\mathcal{F}_t)}{\mathbb{Q}(t<\tau\,|\,\mathcal{F}_t)}.$$

Proof. The martingale property of M implies that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau<t+h\}}\,|\,\mathcal{G}_t) - \int_t^{t+h}\mathbb{E}_{\mathbb{Q}}((1-H_s)\lambda_s\,|\,\mathcal{G}_t)\,ds = 0.$$

It follows that, by the projection on $\mathcal{F}_t$,

$$\mathbb{Q}(t<\tau<t+h\,|\,\mathcal{F}_t) = \int_t^{t+h}\lambda_s\,\mathbb{Q}(s<\tau\,|\,\mathcal{F}_t)\,ds.$$

□

3.1.4 Reduction of the Reference Filtration

Suppose from now on that $\widetilde{\mathcal{F}}_t \subset \mathcal{F}_t$ and define $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$. The associated hazard process is given by $\widetilde{\Gamma}_t = -\ln(\widetilde{G}_t)$ with $\widetilde{G}_t = \mathbb{Q}(t<\tau\,|\,\widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(G_t\,|\,\widetilde{\mathcal{F}}_t)$. Then the key lemma implies that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}}Y\,|\,\widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tau>t\}}e^{\widetilde{\Gamma}_t}\,\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}}Y\,|\,\widetilde{\mathcal{F}}_t).$$

If Y is a $\widetilde{\mathcal{F}}_T$-measurable variable, then

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tau>t\}}e^{\widetilde{\Gamma}_t}\,\mathbb{E}_{\mathbb{Q}}(\widetilde{G}_T Y \,|\, \widetilde{\mathcal{F}}_t).$$

From

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y \,|\, \mathcal{G}_t) \,|\, \widetilde{\mathcal{G}}_t),$$

we deduce that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y|\widetilde{\mathcal{G}}_t) &= \mathbb{E}_{\mathbb{Q}}\left(e^{\Gamma_t}\mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}(G_TY|\mathcal{F}_t)\,|\,\widetilde{\mathcal{G}}_t\right)\\
&= \mathbb{1}_{\{\tau>t\}}e^{\widetilde{\Gamma}_t}\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{Q}}(G_TY|\mathcal{F}_t)\,|\,\widetilde{\mathcal{F}}_t\right).
\end{aligned}$$

It can be noted that, from the uniqueness of the pre-default $\mathbb{F}$-adapted value, for any t,

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{G}_TY|\widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{Q}}(G_TY|\mathcal{F}_t)\,|\,\widetilde{\mathcal{F}}_t\right).$$

As a check, a simple computation shows

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}(G_TY|\mathcal{F}_t)e^{\Gamma_t}\,|\,\widetilde{\mathcal{F}}_t\right) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)e^{\Gamma_t}\mathbb{E}_{\mathbb{Q}}(G_TY|\mathcal{F}_t)\,|\,\widetilde{\mathcal{F}}_t\right)\\
&= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(G_TY|\mathcal{F}_t)\,|\,\widetilde{\mathcal{F}}_t\right) = \mathbb{E}_{\mathbb{Q}}(G_TY|\widetilde{\mathcal{F}}_t)\\
&= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(G_T|\widetilde{\mathcal{F}}_T)Y|\widetilde{\mathcal{F}}_t\right) = \mathbb{E}_{\mathbb{Q}}(\widetilde{G}_TY|\widetilde{\mathcal{F}}_t)
\end{aligned}$$

since Y we assumed that is $\widetilde{\mathcal{F}}_T$-measurable.

Let $F = Z + A$ be the Doob-Meyer decomposition of the submartingale F with respect to $\mathbb{F}$, and let us assume that A is differentiable with respect to t, that is, $A_t = \int_0^t a_s\,ds$. Then the process $\widetilde{A}_t = \mathbb{E}_{\mathbb{Q}}(A_t|\widetilde{\mathcal{F}}_t)$ is a submartingale with respect to $\widetilde{\mathbb{F}}$ and its Doob-Meyer decomposition is $\widetilde{A} = \widetilde{z}+\widetilde{\alpha}$. Hence, setting $\widetilde{Z}_t = \mathbb{E}_{\mathbb{Q}}(Z_t|\widetilde{\mathcal{F}}_t)$, the submartingale

$$\widetilde{F}_t = \mathbb{Q}(t \geq \tau \,|\, \widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(F_t \,|\, \widetilde{\mathcal{F}}_t)$$

admits the Doob-Meyer decomposition $\widetilde{F} = \widetilde{Z} + \widetilde{z} + \widetilde{\alpha}$. The next lemma furnishes the link between a and $\widetilde{\alpha}$.

Lemma 3.1.6 *The compensator of $\widetilde{F}$ equals*

$$\widetilde{\alpha}_t = \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s)\,ds.$$

Proof. Let us prove that the process

$$M^F_t = \mathbb{E}_{\mathbb{Q}}(F_t \,|\, \widetilde{\mathcal{F}}_t) - \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s)\,ds$$

is an $\widetilde{\mathbb{F}}$-martingale. Clearly, it is integrable and $\widetilde{\mathbb{F}}$-adapted. Moreover

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(M^F_T|\widetilde{\mathcal{F}}_t) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(F_T|\widetilde{\mathcal{F}}_T) - \int_0^T \mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s)\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right)\\
&= \mathbb{E}_{\mathbb{Q}}(F_T|\widetilde{\mathcal{F}}_t) - \mathbb{E}_{\mathbb{Q}}\left(\int_0^t \mathbb{E}_{\mathbb{Q}}(a_s|\widetilde{\mathcal{F}}_s)\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(\int_t^T \mathbb{E}_{\mathbb{Q}}(a_s|\widetilde{\mathcal{F}}_s)\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right)\\
&= \widetilde{Z}_t + \mathbb{E}_{\mathbb{Q}}\left(\int_0^t a_s\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right) + \mathbb{E}_{\mathbb{Q}}\left(\int_t^T a_s\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right)
\end{aligned}$$

$$
\begin{aligned}
& -\mathbb{E}_{\mathbb{Q}}\left(\int_0^t \mathbb{E}_{\mathbb{Q}}(a_s|\widetilde{\mathcal{F}}_s)\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(\int_t^T \mathbb{E}_{\mathbb{Q}}(a_s|\widetilde{\mathcal{F}}_s)\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right) \\
= \;& M_t^F + \mathbb{E}_{\mathbb{Q}}\left(\int_t^T f_s\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(\int_t^T \mathbb{E}_{\mathbb{Q}}(f_s|\widetilde{\mathcal{F}}_s)\,ds \,\Big|\, \widetilde{\mathcal{F}}_t\right) \\
= \;& M_t^F + \int_t^T \mathbb{E}_{\mathbb{Q}}(f_s\,|\,\widetilde{\mathcal{F}}_t)\,ds - \int_t^T \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(f_s|\widetilde{\mathcal{F}}_s) \,\Big|\, \widetilde{\mathcal{F}}_t\right) ds \\
= \;& M_t^F + \int_t^T \mathbb{E}_{\mathbb{Q}}(a_s\,|\,\widetilde{\mathcal{F}}_t)\,ds - \int_t^T \mathbb{E}_{\mathbb{Q}}(a_s|\widetilde{\mathcal{F}}_t)\,ds = M_t^F.
\end{aligned}
$$

Hence the process

$$
\left(\widetilde{F}_t - \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s|\widetilde{\mathcal{F}}_s)\,ds,\; t \geq 0\right)
$$

is a $\widetilde{\mathbb{F}}$-martingale and the process $\int_0^\cdot \mathbb{E}_{\mathbb{Q}}(a_s\,|\,\widetilde{\mathcal{F}}_s)\,ds$ is predictable. The uniqueness of the Doob-Meyer decomposition implies that

$$
\widetilde{\alpha}_t = \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s\,|\,\widetilde{\mathcal{F}}_s)\,ds
$$

as expected. □

Remark 3.1.1 It follows that

$$
H_t - \int_0^{t\wedge\tau} \frac{\widetilde{f}_s}{1-\widetilde{F}_s}\,ds
$$

is a $\widetilde{\mathbb{G}}$-martingale and that the $\widetilde{\mathbb{F}}$-intensity of τ is equal to $\mathbb{E}_{\mathbb{Q}}(a_s|\widetilde{\mathcal{F}}_s)/\widetilde{G}_s$, and not, as one might have expected, to $\mathbb{E}_{\mathbb{Q}}(a_s/G_s|\widetilde{\mathcal{F}}_s)$. Note that even if the hypothesis (H) holds between $\widetilde{\mathbb{F}}$ and $\mathbb{F}$, this proof cannot be simplified, since the process $\widetilde{F}_t$ is increasing but not $\widetilde{\mathbb{F}}$-predictable (there is no reason for $\widetilde{F}_t$ to admit an intensity).

This result can also be proved directly, thanks to the following result (due to Brémaud):

$$
H_t - \int_0^{t\wedge\tau} \lambda_s\,ds
$$

is a $\mathbb{G}$-martingale and thus

$$
H_t - \int_0^{t\wedge\tau} \mathbb{E}_{\mathbb{Q}}(\lambda_s|\widetilde{\mathcal{G}}_s)\,ds
$$

is an $\widetilde{\mathbb{G}}$-martingale. Note that

$$
\int_0^{t\wedge\tau} \mathbb{E}_{\mathbb{Q}}(\lambda_s\,|\,\widetilde{\mathcal{G}}_s)\,ds = \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{E}_{\mathbb{Q}}(\lambda_s\,|\,\widetilde{\mathcal{G}}_s)\,ds = \int_0^t \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s\,|\,\widetilde{\mathcal{G}}_s)\,ds
$$

and

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s\,|\,\widetilde{\mathcal{G}}_s) &= \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{G}_s}\,\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s\,|\,\widetilde{\mathcal{F}}_s) \\
&= \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{G}_s}\,\mathbb{E}_{\mathbb{Q}}(G_s\lambda_s\,|\,\widetilde{\mathcal{F}}_s) = \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{G}_s}\,\mathbb{E}_{\mathbb{Q}}(a_s\,|\,\widetilde{\mathcal{F}}_s).
\end{aligned}
$$

We thus conclude that

$$
H_t - \int_0^{t\wedge\tau} \frac{\mathbb{E}_{\mathbb{Q}}(a_s\,|\,\widetilde{\mathcal{F}}_s)}{\widetilde{G}_s}\,ds
$$

is an $\widetilde{\mathbb{G}}$-martingale, which is the desired result.

3.1.5 Enlargement of Filtration

We may work directly with the filtration $\mathbb{G}$, provided that the decomposition of any $\mathbb{F}$-martingale in this filtration is known up to time τ. For example, if B is an $\mathbb{F}$-Brownian motion, its decomposition in the $\mathbb{G}$ filtration up to time τ is

$$B_{t\wedge\tau} = \beta_{t\wedge\tau} + \int_0^{t\wedge\tau} \frac{d\langle B, G\rangle_s}{G_{s-}},$$

where $(\beta_{t\wedge\tau},\ t \geq 0)$ is a continuous $\mathbb{G}$-martingale with the increasing process $t\wedge\tau$. If the dynamics of an asset S are given by

$$dS_t = S_t\big(r_t\,dt + \sigma_t\,dB_t\big)$$

in a default-free framework, where B is a Brownian motion, then its dynamics are

$$dS_t = S_t\Big(r_t\,dt + \sigma_t\frac{d\langle B, G\rangle_t}{G_{t-}} + \sigma_t\,d\beta_t\Big)$$

in the default filtration, if we restrict our attention to time before default. Therefore, the default will act as a change of drift term on the prices.

3.2 Hypothesis (H)

In a general setting, $\mathbb{F}$ martingales do not remains $\mathbb{G}$-martingales. We study here a specific case.

3.2.1 Equivalent Formulations

We shall now examine the hypothesis (H) which reads:

(H) Every $\mathbb{F}$ local martingale is a $\mathbb{G}$ local martingale.

This hypothesis implies, for instance, that any $\mathbb{F}$-Brownian motion remains a Brownian motion in the enlarged filtration $\mathbb{G}$. It was studied by Brémaud and Yor [20], Mazziotto and Szpirglas [74], and for financial purpose by Kusuoka [63]. This can be written in any of the equivalent forms (see, e.g., Dellacherie and Meyer [37]).

Lemma 3.2.1 *Assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the process $H_t = 1\!\!1_{\{\tau \leq t\}}$. Then the following conditions are equivalent to the hypothesis (H) .*
(i) For any $t, h \in \mathbb{R}_+$, we have

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_{t+h}). \tag{3.5}$$

(i′) For any $t \in \mathbb{R}_+$, we have

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty). \tag{3.6}$$

(ii) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{Q}$, that is,

$$\mathbb{E}_{\mathbb{Q}}(\xi\,\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{F}_t)\,\mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_t)$$

for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and bounded, $\mathcal{G}_t$-measurable random variable η.
(iii) For any $t \in \mathbb{R}_+$, and any $u \geq t$ the σ-fields $\mathcal{F}_u$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.
(iv) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ: $\mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{F}_t)$.
(v) For any $t \in \mathbb{R}_+$, and any bounded, $\mathcal{G}_t$-measurable random variable η: $\mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_\infty)$.

Proof. If the hypothesis (H) holds then (3.6) is valid as well. If (3.6) holds, the the fact that $\mathcal{H}_t$ is generated by the sets $\{\tau \le s\}, s \le t$ proves that $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. The desired property now follows. This result can be also found in [38]. The equivalence between (3.6) and (3.5) is left to the reader.

Using the monotone class theorem, it can be shown that conditions (i) and (i′) are equivalent. The proof of equivalence of conditions (i′)–(v) can be found, for instance, in Section 6.1.1 of Bielecki and Rutkowski [12] (for related results, see Elliott et al. [45]). Hence we shall only show that condition (iv) and the hypothesis (H) are equivalent.

Assume first that the hypothesis (H) holds. Consider any bounded, $\mathcal{F}_\infty$-measurable random variable ξ. Let $M_t = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{F}_t)$ be the martingale associated with ξ. Of course, M is a local martingale with respect to $\mathbb{F}$. Then the hypothesis (H) implies that M is also a local martingale with respect to $\mathbb{G}$, and thus a $\mathbb{G}$-martingale, since M is bounded (recall that any bounded local martingale is a martingale). We conclude that $M_t = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{G}_t)$ and thus (iv) holds.

Suppose now that (iv) holds. First, we note that the standard truncation argument shows that the boundedness of ξ in (iv) can be replaced by the assumption that ξ is $\mathbb{Q}$-integrable. Hence, any $\mathbb{F}$-martingale M is an $\mathbb{G}$-martingale, since M is clearly $\mathbb{G}$-adapted and we have, for every $t \le s$,

$$M_t = \mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{G}_t),$$

where the second equality follows from (iv). Suppose now that M is an $\mathbb{F}$-local martingale. Then there exists an increasing sequence of $\mathbb{F}$-stopping times τ_n such that $\lim_{n\to\infty} \tau_n = \infty$, for any n the stopped process M^{τ_n} follows a uniformly integrable $\mathbb{F}$-martingale. Hence M^{τ_n} is also a uniformly integrable $\mathbb{G}$-martingale, and this means that M is a $\mathbb{G}$-local martingale. $\square$

Remarks 3.2.1 (i) Equality (3.6) appears in several papers on default risk, typically without any reference to the hypothesis (H). For example, in Madan and Unal [73], the main theorem follows from the fact that (3.6) holds (see the proof of B9 in the appendix of [73]). This is also the case for Wong's model [84].
(ii) If τ is $\mathcal{F}_\infty$-measurable and (3.6) holds then τ is an $\mathbb{F}$-stopping time. If τ is an $\mathbb{F}$-stopping time then equality (3.5) holds. If $\mathbb{F}$ is the Brownian filtration, then τ is predictable and $\Lambda = H$.
(iii) Though condition (H) does not necessarily hold true, in general, it is satisfied when τ is constructed through the so-called canonical approach (or for Cox processes). This hypothesis is quite natural under the historical probability and it is stable under some changes of a probability measure. However, Kusuoka [63] provides an example where (H) holds under the historical probability, but it fails hold after an equivalent change of a probability measure. This counter-example is linked to modeling of dependent defaults.
(iv) Hypothesis (H) holds, in particular, if τ is independent from $\mathcal{F}_\infty$ (see Greenfield [51]).
(v) If hypothesis (H) holds then from the condition

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty), \quad \forall\, t \in \mathbb{R}_+,$$

we deduce easily that F is an increasing process.

Comments 3.2.1 See Elliott et al. [45] for more comments. The property that F is increasing is equivalent to the fact that any $\mathbb{F}$-martingale stopped at time τ is a $\mathbb{G}$ martingale. Nikeghbali and Yor [79] proved that this is equivalent to $\mathbb{E}_{\mathbb{Q}}(M_\tau) = M_0$ for any bounded $\mathbb{F}$-martingale M. The hypothesis (H) was also studied by Florens and Fougere [48], who coined the term *noncausality*.

Proposition 3.2.1 *Assume that the hypothesis (H) holds. If X is an $\mathbb{F}$-martingale then the processes XL and $[L, X]$ are $\mathbb{G}$-local martingales.*

Proof. We have seen in Proposition 3.1.1 that the process XL is a $\mathbb{G}$-martingale. Since $[L, X] = LX - \int L_-\, dX - \int X_-\, dL$ and X is an $\mathbb{F}$-martingale (and thus also a $\mathbb{G}$-martingale), the process $[L, X]$ is manifestly a $\mathbb{G}$-martingale as the sum of three $\mathbb{G}$-martingales. $\square$

3.2.2 Canonical Construction of a Default Time

We shall now briefly describe the most commonly used construction of a default time associated with a given hazard process Γ. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given $\mathbb{F}$-hazard process Γ. We assume that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{Q})$. As usual, we assume that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, Γ is given by the equality

$$\Gamma_t = \int_0^t \gamma_u \, du, \quad \forall \, t \in \mathbb{R}_+,$$

for some non-negative, $\mathbb{F}$-progressively measurable intensity process γ.

To construct a random time τ, we shall postulate that the underlying probability space $(\Omega, \mathbb{F}, \mathbb{Q})$ is sufficiently rich to support a random variable ξ, which is uniformly distributed on the interval $[0, 1]$ and independent of the filtration $\mathbb{F}$ under $\mathbb{Q}$. In this version of the canonical construction, Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}$.

We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf \{ \, t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \xi \, \} = \inf \{ \, t \in \mathbb{R}_+ : \Gamma_t \geq \eta \, \}, \tag{3.7}$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbb{Q}$. It is not difficult to find the process $F_t = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t)$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbb{Q}(\tau > t \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\xi < e^{-x})_{x = \Gamma_t} = e^{-\Gamma_t}. \tag{3.8}$$

Consequently, we have

$$1 - F_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{Q}(\tau > t \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t\big) = e^{-\Gamma_t}, \tag{3.9}$$

and so F is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that the process Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}$. As an immediate consequence of (3.8) and (3.9), we obtain the following property of the canonical construction of the default time (cf. (3.6))

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t), \quad \forall \, t \in \mathbb{R}_+. \tag{3.10}$$

To sum up, we have that

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_u) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = e^{-\Gamma_t} \tag{3.11}$$

for any two dates $0 \leq t \leq u$.

3.2.3 Stochastic Barrier

Suppose that

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is a continuous, strictly increasing, $\mathbb{F}$-adapted process. Our goal is to show that there exists a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential law of parameter 1, such that $\tau = \inf \{t \geq 0 \,:\, \Gamma_t > \Theta\}$. Let us set $\Theta \stackrel{\text{def}}{=} \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$\mathbb{Q}(\Theta > u \,|\, \mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$.

3.2.4 Change of a Probability Measure

Kusuoka [63] shows, by means of a counter-example, that the hypothesis (H) is not invariant with respect to an equivalent change of the underlying probability measure, in general. It is worth noting that his counter-example is based on two filtrations, $\mathbb{H}^1$ and $\mathbb{H}^2$, generated by the two random times τ^1 and τ^2, and he chooses $\mathbb{H}^1$ to play the role of the reference filtration $\mathbb{F}$. We shall argue that in the case where $\mathbb{F}$ is generated by a Brownian motion, the above-mentioned invariance property is valid under mild technical assumptions.

Girsanov's Theorem

From Proposition 3.1.2 we know that the process $M_t = H_t - \Gamma_{t\wedge\tau}$ is a $\mathbb{G}$-martingale. We fix $T > 0$. For a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ we introduce the $\mathbb{G}$-martingale η_t, $t \leq T$, by setting

$$\eta_t \stackrel{\text{def}}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}_{\,|\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}_t), \quad \mathbb{P}\text{-a.s.}, \tag{3.12}$$

where X is a $\mathcal{G}_T$-measurable integrable random variable, such that $\mathbb{P}(X > 0) = 1$.

The Radon-Nikodým density process η admits the following representation

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \int_{]0,t]} \zeta_u \, dM_u$$

where ξ and ζ are $\mathbb{G}$-predictable stochastic processes. Since η is a strictly positive process, we get

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \big(\beta_u \, dW_u + \kappa_u \, dM_u\big) \tag{3.13}$$

where β and κ are $\mathbb{G}$-predictable processes, with $\kappa > -1$.

Proposition 3.2.2 *Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{G}_T)$ equivalent to $\mathbb{P}$. If the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by* (3.12) *with η satisfying* (3.13)*, then the process*

$$W_t^* = W_t - \int_0^t \beta_u \, du, \quad \forall\, t \in [0, T], \tag{3.14}$$

follows a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{Q}$, and the process

$$M_t^* \stackrel{\text{def}}{=} M_t - \int_{]0,t\wedge\tau]} \kappa_u \, d\Gamma_u = H_t - \int_{]0,t\wedge\tau]} (1 + \kappa_u) \, d\Gamma_u, \quad \forall\, t \in [0, T], \tag{3.15}$$

is a $\mathbb{G}$-martingale orthogonal to W^*.

Proof. Notice first that for $t \leq T$ we have

$$\begin{aligned} d(\eta_t W_t^*) &= W_t^* \, d\eta_t + \eta_{t-} \, dW_t^* + d[W^*, \eta]_t \\ &= W_t^* \, d\eta_t + \eta_{t-} \, dW_t - \eta_{t-} \beta_t \, dt + \eta_{t-} \beta_t \, d[W, W]_t \\ &= W_t^* \, d\eta_t + \eta_{t-} \, dW_t. \end{aligned}$$

This shows that W^* is a $\mathbb{G}$-martingale under $\mathbb{Q}$. Since the quadratic variation of W^* under $\mathbb{Q}$ equals $[W^*, W^*]_t = t$ and W^* is continuous, by virtue of Lévy's theorem it is clear that W^* follows a Brownian motion under $\mathbb{Q}$. Similarly, for $t \leq T$

$$\begin{aligned} d(\eta_t M_t^*) &= M_t^* \, d\eta_t + \eta_{t-} \, dM_t^* + d[M^*, \eta]_t \\ &= M_t^* \, d\eta_t + \eta_{t-} \, dM_t - \eta_{t-} \kappa_t \, d\Gamma_{t\wedge\tau} + \eta_{t-} \kappa_t \, dH_t \\ &= M_t^* \, d\eta_t + \eta_{t-} (1 + \kappa_t) \, dM_t. \end{aligned}$$

We conclude that M^* is a $\mathbb{G}$-martingale under $\mathbb{Q}$. To conclude it is enough to observe that W^* is a continuous process and M^* follows a process of finite variation. □

Corollary 3.2.1 *Let Y be a $\mathbb{G}$-martingale with respect to $\mathbb{Q}$. Then Y admits the following decomposition*

$$Y_t = Y_0 + \int_0^t \xi^*_u \, dW^*_u + \int_{]0,t]} \zeta^*_u \, dM^*_u, \tag{3.16}$$

where ξ^* and ζ^* are $\mathbb{G}$-predictable stochastic processes.

Proof. Consider the process $\widetilde{Y}$ given by the formula

$$\widetilde{Y}_t = \int_{]0,t]} \eta^{-1}_{u-} \, d(\eta_u Y_u) - \int_{]0,t]} \eta^{-1}_{u-} Y_{u-} \, d\eta_u.$$

It is clear that $\widetilde{Y}$ is a $\mathbb{G}$-martingale under $\mathbb{P}$. Notice also that Itô's formula yields

$$\eta^{-1}_{u-} \, d(\eta_u Y_u) = dY_u + \eta^{-1}_{u-} Y_{u-} \, d\eta_u + \eta^{-1}_{u-} \, d[Y, \eta]_u,$$

and thus

$$Y_t = Y_0 + \widetilde{Y}_t - \int_{]0,t]} \eta^{-1}_{u-} \, d[Y, \eta]_u. \tag{3.17}$$

From the predictable representation theorem, we know that

$$\widetilde{Y}_t = Y_0 + \int_0^t \widetilde{\xi}_u \, dW_u + \int_{]0,t]} \widetilde{\zeta}_u \, dM_u \tag{3.18}$$

for some $\mathbb{G}$-predictable processes $\widetilde{\xi}$ and $\widetilde{\zeta}$. Therefore

$$\begin{aligned} dY_t &= \widetilde{\xi}_t \, dW_t + \widetilde{\zeta}_t \, dM_t - \eta^{-1}_{t-} \, d[Y, \eta]_t \\ &= \widetilde{\xi}_t \, dW^*_t + \widetilde{\zeta}_t (1 + \kappa_t)^{-1} \, dM^*_t \end{aligned}$$

since (3.13) combined with (3.17)-(3.18) yield

$$\eta^{-1}_{t-} \, d[Y, \eta]_t = \widetilde{\xi}_t \beta_t \, dt + \widetilde{\zeta}_t \kappa_t (1 + \kappa_t)^{-1} \, dH_t.$$

To derive the last equality we observe, in particular, that in view of (3.17) we have (we take into account continuity of Γ)

$$\Delta[Y, \eta]_t = \eta_{t-} \widetilde{\zeta}_t \kappa_t \, dH_t - \kappa_t \Delta[Y, \eta]_t.$$

We conclude that Y satisfies (3.16) with $\xi^* = \widetilde{\xi}$ and $\zeta^* = \widetilde{\zeta}(1 + \kappa)^{-1}$. □

Preliminary Lemma

Let us first examine a general set-up in which $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the default process H. We say that $\mathbb{Q}$ is locally equivalent to $\mathbb{P}$ if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$. Then there exists the Radon-Nikodým density process η such that

$$d\mathbb{Q} \,|_{\mathcal{G}_t} = \eta_t \, d\mathbb{P} \,|_{\mathcal{G}_t}, \quad \forall \, t \in \mathbb{R}_+. \tag{3.19}$$

Part (i) in the next lemma is well known (see Jamshidian [56]). We assume that the hypothesis (H) holds under $\mathbb{P}$.

Lemma 3.2.2 *(i) Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$, with the associated Radon-Nikodým density process η. If the density process η is $\mathbb{F}$-adapted then we have $\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. Hence, the hypothesis (H) is also valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensities of τ under $\mathbb{Q}$ and under $\mathbb{P}$ coincide.*
(ii) Assume that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ and $d\mathbb{Q} = \eta_\infty \, d\mathbb{P}$, so that $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{G}_t)$. Then the hypothesis (H) is valid under $\mathbb{Q}$ whenever we have, for every $t \in \mathbb{R}_+$,

$$\frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)}. \tag{3.20}$$

Proof. To prove (i), assume that the density process η is $\mathbb{F}$-adapted. We have for each $t \leq s \in \mathbb{R}_+$

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_s) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_s),$$

where the last equality follows by another application of the Bayes formula. The assertion now follows from part (i) in Lemma 3.2.1.

To prove part (ii), it suffices to establish the equality

$$\widehat{F}_t \stackrel{\text{def}}{=} \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty), \quad \forall\, t \in \mathbb{R}_+. \tag{3.21}$$

Note that since the random variables $\eta_t \mathbb{1}_{\{\tau \leq t\}}$ and η_t are $\mathbb{P}$-integrable and $\mathcal{G}_t$-measurable, using the Bayes formula, part (v) in Lemma 3.2.1, and assumed equality (3.20), we obtain the following chain of equalities

$$\begin{aligned}\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty).\end{aligned}$$

We conclude that the hypothesis (H) holds under $\mathbb{Q}$ if and only if (3.20) is valid. □

Unfortunately, straightforward verification of condition (3.20) is rather cumbersome. For this reason, we shall provide alternative sufficient conditions for the preservation of the hypothesis (H) under a locally equivalent probability measure.

Case of the Brownian Filtration

Let W be a Brownian motion under $\mathbb{P}$ and $\mathbb{F}$ its natural filtration. Since we work under the hypothesis (H), the process W is also a $\mathbb{G}$-martingale, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Hence, W is a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}$. Our goal is to show that the hypothesis (H) is still valid under $\mathbb{Q} \in \mathcal{Q}$ for a large class $\mathcal{Q}$ of (locally) equivalent probability measures on $(\Omega, \mathcal{G})$.

Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. Kusuoka [63] (see also Section 5.2.2 in Bielecki and Rutkowski [12]) proved that, under the hypothesis (H), any $\mathbb{G}$-martingale under $\mathbb{P}$ can be represented as the sum of stochastic integrals with respect to the Brownian motion W and the jump martingale M. In our set-up, Kusuoka's representation theorem implies that there exist $\mathbb{G}$-predictable processes θ and $\zeta > -1$, such that the Radon-Nikodým density η of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies the following SDE

$$d\eta_t = \eta_{t-}\big(\theta_t \, dW_t + \zeta_t \, dM_t\big) \tag{3.22}$$

with the initial value $\eta_0 = 1$. More explicitly, the process η equals

$$\eta_t = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u \, dW_u\right) \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) = \eta_t^{(1)} \eta_t^{(2)}, \tag{3.23}$$

where we write

$$\eta_t^{(1)} = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u \, dW_u\right) = \exp\left(\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \theta_u^2 \, du\right), \tag{3.24}$$

and

$$\eta_t^{(2)} = \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) = \exp\left(\int_0^t \ln(1+\zeta_u) \, dH_u - \int_0^{t \wedge \tau} \zeta_u \gamma_u \, du\right). \tag{3.25}$$

Moreover, by virtue of a suitable version of Girsanov's theorem, the following processes $\widehat{W}$ and $\widehat{M}$ are $\mathbb{G}$-martingales under $\mathbb{Q}$

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}} \gamma_u \zeta_u \, du. \tag{3.26}$$

Proposition 3.2.3 *Assume that the hypothesis (H) holds under $\mathbb{P}$. Let $\mathbb{Q}$ be a probability measure locally equivalent to $\mathbb{P}$ with the associated Radon-Nikodým density process η given by formula (3.23). If the process θ is $\mathbb{F}$-adapted then the hypothesis (H) is valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensity of τ under $\mathbb{Q}$ equals $\widehat{\gamma}_t = (1+\widetilde{\zeta}_t)\gamma_t$, where $\widetilde{\zeta}$ is the unique $\mathbb{F}$-predictable process such that the equality $\widetilde{\zeta}_t \mathbb{1}_{\{t\leq\tau\}} = \zeta_t \mathbb{1}_{\{t\leq\tau\}}$ holds for every $t\in\mathbb{R}_+$.*

Proof. Let $\widetilde{\mathbb{P}}$ be the probability measure locally equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G})$, given by

$$d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\Big(\int_0^{\cdot}\zeta_u\,dM_u\Big)\,d\mathbb{P}\,|\,_{\mathcal{G}_t} = \eta^{(2)}_t\,d\mathbb{P}\,|\,_{\mathcal{G}_t}. \tag{3.27}$$

We claim that the hypothesis (H) holds under $\widetilde{\mathbb{P}}$. From Girsanov's theorem, the process W follows a Brownian motion under $\widetilde{\mathbb{P}}$ with respect to both $\mathbb{F}$ and $\mathbb{G}$. Moreover, from the predictable representation property of W under $\widetilde{\mathbb{P}}$, we deduce that any $\mathbb{F}$-local martingale L under $\widetilde{\mathbb{P}}$ can be written as a stochastic integral with respect to W. Specifically, there exists an $\mathbb{F}$-predictable process ξ such that

$$L_t = L_0 + \int_0^t \xi_u\,dW_u.$$

This shows that L is also a $\mathbb{G}$-local martingale, and thus the hypothesis (H) holds under $\widetilde{\mathbb{P}}$. Since

$$d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\Big(\int_0^{\cdot}\theta_u\,dW_u\Big)\,d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t},$$

by virtue of part (i) in Lemma 3.2.2, the hypothesis (H) is valid under $\mathbb{Q}$ as well. The last claim in the statement of the lemma can be deduced from the fact that the hypothesis (H) holds under $\mathbb{Q}$ and, by Girsanov's theorem, the process

$$\widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\,du = H_t - \int_0^t \mathbb{1}_{\{u<\tau\}}(1+\widetilde{\zeta}_u)\gamma_u\,du$$

is a $\mathbb{Q}$-martingale. □

We claim that the equality $\widetilde{\mathbb{P}} = \mathbb{P}$ holds on the filtration $\mathbb{F}$. Indeed, we have $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{F}_t} = \widetilde{\eta}_t\,d\mathbb{P}\,|\,_{\mathcal{F}_t}$, where we write $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta^{(2)}_t\,|\,\mathcal{F}_t)$, and

$$\mathbb{E}_{\mathbb{P}}(\eta^{(2)}_t\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\Big(\int_0^{\cdot}\zeta_u\,dM_u\Big)\,\Big|\,\mathcal{F}_\infty\right) = 1, \quad \forall\, t\in\mathbb{R}_+, \tag{3.28}$$

where the first equality follows from part (v) in Lemma 3.2.1.

To establish the second equality in (3.28), we first note that since the process M is stopped at τ, we may assume, without loss of generality, that $\zeta = \widetilde{\zeta}$ where the process $\widetilde{\zeta}$ is $\mathbb{F}$-predictable. Moreover,the conditional cumulative distribution function of τ given $\mathcal{F}_\infty$ has the form $1-\exp(-\Gamma_t(\omega))$. Hence, for arbitrarily selected sample paths of processes ζ and Γ, the claimed equality can be seen as a consequence of the martingale property of the Doléans exponential.

Formally, it can be proved by following elementary calculations, where the first equality is a consequence of (3.25)),

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\Big(\int_0^{\cdot}\widetilde{\zeta}_u\,dM_u\Big)\,\Big|\,\mathcal{F}_\infty\right) &= \mathbb{E}_{\mathbb{P}}\left(\big(1+\mathbb{1}_{\{t\geq\tau\}}\widetilde{\zeta}_\tau\big)\exp\Big(-\int_0^{t\wedge\tau}\widetilde{\zeta}_u\gamma_u\,du\Big)\,\Big|\,\mathcal{F}_\infty\right)\\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty\big(1+\mathbb{1}_{\{t\geq u\}}\widetilde{\zeta}_u\big)\exp\Big(-\int_0^{t\wedge u}\widetilde{\zeta}_v\gamma_v\,dv\Big)\gamma_u e^{-\int_0^u\gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\right)\\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^t(1+\widetilde{\zeta}_u)\gamma_u\exp\Big(-\int_0^u(1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du\,\Big|\,\mathcal{F}_\infty\right)
\end{aligned}$$

$$
\begin{aligned}
&+\exp\Big(-\int_0^t \widetilde{\zeta}_v\gamma_v\,dv\Big)\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^\infty \gamma_u e^{-\int_0^u \gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\Big)\\
&=\int_0^t (1+\widetilde{\zeta}_u)\gamma_u \exp\Big(-\int_0^u (1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du\\
&+\exp\Big(-\int_0^t \widetilde{\zeta}_v\gamma_v\,dv\Big)\int_t^\infty \gamma_u e^{-\int_0^u \gamma_v\,dv}du\\
&=1-\exp\Big(-\int_0^t (1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)+\exp\Big(-\int_0^t \widetilde{\zeta}_v\gamma_v\,dv\Big)\exp\Big(-\int_0^t \gamma_v\,dv\Big)=1,
\end{aligned}
$$

where the second last equality follows by an application of the chain rule.

Extension to Orthogonal Martingales

Equality (3.28) suggests that Proposition 3.2.3 can be extended to the case of arbitrary orthogonal local martingales. Such a generalization is convenient, if we wish to cover the situation considered in Kusuoka's counterexample.

Let N be a local martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$. It is also a $\mathbb{G}$-local martingale, since we maintain the assumption that the hypothesis (H) holds under $\mathbb{P}$. Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. We assume that the Radon-Nikodým density process η of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$
d\eta_t = \eta_{t-}\big(\theta_t\,dN_t + \zeta_t\,dM_t\big) \tag{3.29}
$$

for some $\mathbb{G}$-predictable processes θ and $\zeta > -1$ (the properties of the process θ depend, of course, on the choice of the local martingale N). The next result covers the case where N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, so that the product MN follows a $\mathbb{G}$-local martingale.

Proposition 3.2.4 *Assume that the following conditions hold:*
(a) N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$,
(b) N has the predictable representation property under $\mathbb{P}$ with respect to $\mathbb{F}$, in the sense that any $\mathbb{F}$-local martingale L under $\mathbb{P}$ can be written as

$$
L_t = L_0 + \int_0^t \xi_u\,dN_u, \quad \forall\, t \in \mathbb{R}_+,
$$

for some $\mathbb{F}$-predictable process ξ,
(c) $\widetilde{\mathbb{P}}$ is a probability measure on $(\Omega, \mathcal{G})$ such that (3.27) holds.
Then we have:
(i) the hypothesis (H) is valid under $\widetilde{\mathbb{P}}$,
(ii) if the process θ is $\mathbb{F}$-adapted then the hypothesis (H) is valid under $\mathbb{Q}$.

The proof of the proposition hinges on the following simple lemma.

Lemma 3.2.3 *Under the assumptions of Proposition 3.2.4, we have:*
(i) N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$,
(ii) N has the predictable representation property for $\mathbb{F}$-local martingales under $\widetilde{\mathbb{P}}$.

Proof. In view of (c), we have $d\widetilde{\mathbb{P}}\,|_{\mathcal{G}_t} = \eta^{(2)}_t\,d\mathbb{P}\,|_{\mathcal{G}_t}$, where the density process $\eta^{(2)}$ is given by (3.25), so that $d\eta^{(2)}_t = \eta^{(2)}_{t-}\zeta_t\,dM_t$. From the assumed orthogonality of N and M, it follows that N and $\eta^{(2)}$ are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, and thus $N\eta^{(2)}$ is a $\mathbb{G}$-local martingale under $\mathbb{P}$ as well. This means that N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$, so that (i) holds.

To establish part (ii) in the lemma, we first define the auxiliary process $\widetilde{\eta}$ by setting $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \,|\, \mathcal{F}_t)$. Then manifestly $d\widetilde{\mathbb{P}}\,|_{\mathcal{F}_t} = \widetilde{\eta}_t \, d\mathbb{P}\,|_{\mathcal{F}_t}$, and thus in order to show that any $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$ follows an $\mathbb{F}$-local martingale under $\mathbb{P}$, it suffices to check that $\widetilde{\eta}_t = 1$ for every $t \in \mathbb{R}_+$, so that $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$. To this end, we note that

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}} \left(\mathcal{E}_t \left(\int_0^{\cdot} \zeta_u \, dM_u \right) \Big|\, \mathcal{F}_\infty \right) = 1, \quad \forall \, t \in \mathbb{R}_+,$$

where the first equality follows from part (v) in Lemma 3.2.1, and the second one can established similarly as the second equality in (3.28).

We are in a position to prove (ii). Let L be an $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$. Then it follows also an $\mathbb{F}$-local martingale under $\mathbb{P}$ and thus, by virtue of (b), it admits an integral representation with respect to N under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. This shows that N has the predictable representation property with respect to $\mathbb{F}$ under $\widetilde{\mathbb{P}}$. □

We now proceed to the proof of Proposition 3.2.4.

Proof of Proposition 3.2.4. We shall argue along the similar lines as in the proof of Proposition 3.2.3. To prove (i), note that by part (ii) in Lemma 3.2.3 we know that any $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$ admits the integral representation with respect to N. But, by part (i) in Lemma 3.2.3, N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$. We conclude that L is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$, and thus the hypothesis (H) is valid under $\widetilde{\mathbb{P}}$. Assertion (ii) now follows from part (i) in Lemma 3.2.2. □

Remark 3.2.1 It should be stressed that Proposition 3.2.4 is not directly employed in what follows. We decided to present it here, since it sheds some light on specific technical problems arising in the context of modeling dependent default times through an equivalent change of a probability measure (see Kusuoka [63]).

Example 3.2.1 Kusuoka [63] presents a counter-example based on the two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We write $M_t^i = H_t^i - \int_0^{t \wedge \tau_i} \gamma_i(u) \, du$, where $H_t^i = \mathbb{1}_{\{t \geq \tau_i\}}$ and γ_i is the deterministic intensity function of τ_i under $\mathbb{P}$. Let us set $d\mathbb{Q}\,|_{\mathcal{G}_t} = \eta_t \, d\mathbb{P}\,|_{\mathcal{G}_t}$, where $\eta_t = \eta_t^{(1)} \eta_t^{(2)}$ and, for $i = 1, 2$ and every $t \in \mathbb{R}_+$,

$$\eta_t^{(i)} = 1 + \int_0^t \eta_{u-}^{(i)} \zeta_u^{(i)} \, dM_u^i = \mathcal{E}_t \left(\int_0^{\cdot} \zeta_u^{(i)} \, dM_u^i \right)$$

for some $\mathbb{G}$-predictable processes $\zeta^{(i)}$, $i = 1, 2$, where $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$. We set $\mathbb{F} = \mathbb{H}^1$ and $\mathbb{H} = \mathbb{H}^2$. Manifestly, the hypothesis (H) holds under $\mathbb{P}$. Moreover, in view of Proposition 3.2.4, it is still valid under the equivalent probability measure $\widetilde{\mathbb{P}}$ given by

$$d\widetilde{\mathbb{P}}\,|_{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^{\cdot} \zeta_u^{(2)} \, dM_u^2 \right) d\mathbb{P}\,|_{\mathcal{G}_t}.$$

It is clear that $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$, since

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}} \left(\mathcal{E}_t \left(\int_0^{\cdot} \zeta_u^{(2)} \, dM_u^2 \right) \Big|\, \mathcal{H}_t^1 \right) = 1, \quad \forall \, t \in \mathbb{R}_+.$$

However, the hypothesis (H) is not necessarily valid under $\mathbb{Q}$ if the process $\zeta^{(1)}$ fails to be $\mathbb{F}$-adapted. In Kusuoka's counter-example, the process $\zeta^{(1)}$ was chosen to be explicitly dependent on both random times, and it was shown that the hypothesis (H) does not hold under $\mathbb{Q}$. For an alternative approach to Kusuoka's example, through an absolutely continuous change of a probability measure, the interested reader may consult Collin-Dufresne et al. [32].

3.3 Representation Theorem

Kusuoka [63] establishes the following representation theorem.

Theorem 3.1 *Assume that the hypothesis (H) holds. Then any $\mathbb{G}$-square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M associated with τ.*

We assume, for simplicity, that F is continuous and $F_t < 1$ for every $t \in \mathbb{R}^+$. Since the hypothesis (H) holds, F is an increasing process. Then

$$dF_t = e^{-\Gamma_t} d\Gamma_t$$

and

$$d(e^{\Gamma_t}) = e^{\Gamma_t} d\Gamma_t = e^{\Gamma_t} \frac{dF_t}{1-F_t}\,. \tag{3.30}$$

Proposition 3.3.1 *Suppose that hypothesis (H) holds under $\mathbb{Q}$ and that any $\mathbb{F}$-martingale is continuous. Then, the martingale $M^h_t = \mathbb{E}_{\mathbb{Q}}(h_\tau | \mathcal{G}_t)$, where h is an $\mathbb{F}$-predictable process such that $\mathbb{E}_{\mathbb{Q}}(h_\tau) < \infty$, admits the following decomposition as the sum of a continuous martingale and a discontinuous martingale*

$$M^h_t = m^h_0 + \int_0^{t\wedge\tau} e^{\Gamma_u} dm^h_u + \int_{]0,t\wedge\tau]} (h_u - J_u)\, dM_u, \tag{3.31}$$

where m^h is the continuous $\mathbb{F}$-martingale

$$m^h_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^\infty h_u dF_u \mid \mathcal{F}_t\Big),$$

J is the process

$$J_t = e^{\Gamma_t}\Big(m^h_t - \int_0^t h_u dF_u\Big)$$

and M is the discontinuous $\mathbb{G}$-martingale $M_t = H_t - \Gamma_{t\wedge\tau}$ where $d\Gamma_u = \dfrac{dF_u}{1-F_u}$.

Proof. We know that

$$\begin{aligned} M^h_t &= \mathbb{E}_{\mathbb{Q}}(h_\tau \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau\le t\}} h_\tau + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^\infty h_u\, dF_u \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{1}_{\{\tau\le t\}} h_\tau + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\Big(m^h_t - \int_0^t h_u\, dF_u\Big). \end{aligned} \tag{3.32}$$

We will give two different proofs.

First proof. Noting that Γ is an increasing process and m^h a continuous martingale, and using the integration by parts formula, we deduce that

$$\begin{aligned} dJ_t &= e^{\Gamma_t}\, dm^h_t + \Big(m^h_t - \int_0^t h_u\, dF_u\Big)\gamma_t e^{\Gamma_t}\, dt - e^{\Gamma_t} h_t\, dF_t \\ &= e^{\Gamma_t}\, dm^h_t + J_t \gamma_t e^{\Gamma_t}\, dt - e^{\Gamma_t} h_t\, dF_t. \end{aligned}$$

Therefore, from (3.30)

$$dJ_t = e^{\Gamma_t}\, dm^h_t + (J_t - h_t)\frac{dF_t}{1-F_t},$$

or, in an integrated form,

$$J_t = m_0 + \int_0^t e^{\Gamma_u}\, dm^h_u + \int_0^t (J_u - h_u)\, d\Gamma_u.$$

Note that $J_u = M^h_u$ for $u < \tau$. Therefore, on the event $\{t < \tau\}$,

$$M^h_t = m^h_0 + \int_0^{t\wedge\tau} e^{\Gamma_u}\, dm^h_u + \int_0^{t\wedge\tau} (J_u - h_u)\, d\Gamma_u.$$

From (3.32), the jump of M^h at time τ is $h_\tau - J_\tau = h_\tau - M^h_{\tau-}$. Then (3.31) follows.

Second proof. The equality (3.32) can be re-written as

$$M^h_t = \int_0^t h_u\, dH_u + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \left(m^h_t - \int_0^t h_u\, dF_u \right).$$

Hence the result can be obtained by the integration by parts formula. □

Remark 3.3.1 Since the hypothesis (H) holds and Γ is $\mathbb{F}$-adapted, the processes $(m_t, t \geq 0)$ and $(\int_0^{t\wedge\tau} e^{\Gamma_u} dm_u,\, t \geq 0)$ are also $\mathbb{G}$-martingales.

3.4 Case of a Partial Information

As pointed out by Jamshidian [55], "one may wish to apply the general theory perhaps as an intermediate step, to a subfiltration that is not equal to the default-free filtration. In that case, $\mathbb{F}$ rarely satisfies hypothesis (H) ". We present below a few simple cases when such a situation arises.

3.4.1 Information at Discrete Times

Assume that under $\mathbb{Q}$

$$dV_t = V_t(\mu\, dt + \sigma\, dW_t), \quad V_0 = v,$$

or explicitly

$$V_t = v e^{\sigma(W_t + \nu t)} = v e^{\sigma X_t}$$

where we denote $\nu = (\mu - \sigma^2/2)/\sigma$ and $X_t = W_t + \nu t$. The default time is assumed to be the first hitting time of α with $\alpha < v$. Specifically, we set

$$\tau = \inf\{t \in \mathbb{R}_+ :\, V_t \leq \alpha\} = \inf\{t \in \mathbb{R}_+ :\, X_t \leq a\}$$

where $a = \sigma^{-1} \ln(\alpha/v)$. Here $\mathbb{F}$ is the filtration of the observations of V at discrete times $t_1, \ldots, t_n$ where $t_n \leq t < t_{n+1}$, that is,

$$\mathcal{F}_t = \sigma(V_{t_1}, \ldots, V_{t_n},\, t_i \leq t).$$

Our goal is to compute $F_t = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t)$. Let us recall that

$$\mathbb{Q}(\inf_{s\leq t} X_s > z) = \Phi(\nu, t, z), \tag{3.33}$$

where

$$\begin{aligned}
\Phi(\nu, t, z) &= N\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} N\left(\frac{z + \nu t}{\sqrt{t}}\right), \qquad \text{for } z < 0,\ t > 0,\\
&= 0, \qquad \text{for } z \geq 0,\ t \geq 0,\\
\Phi(\nu, 0, z) &= 1, \qquad \text{for } z < 0.
\end{aligned}$$

- **Case:** $t < t_1$

In that case, F_t is the cumulative function of τ. Since $a < 0$, we obtain

$$\begin{aligned} F_t &= \mathbb{Q}(\tau \le t) = \mathbb{Q}(\inf_{s\le t} X_s \le a) \\ &= 1 - \Phi(\nu, t, a) = N\left(\frac{a-\nu t}{\sqrt{t}}\right) + e^{2\nu a} N\left(\frac{a+\nu t}{\sqrt{t}}\right). \end{aligned}$$

- **Case:** $t_1 < t < t_2$

We denote by $\mathcal{F}^W_t = \sigma(W_s, s \le t)$ the natural filtration of the Brownian motion (this is also the natural filtration of X)

$$\begin{aligned} F_t &= \mathbb{Q}(\tau \le t \,|\, X_{t_1}) = 1 - \mathbb{Q}(\tau > t \,|\, X_{t_1}) \\ &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\inf_{s<t_1} X_s > a\}}\, \mathbb{Q}(\inf_{t_1 \le s < t} X_s > a \,|\, \mathcal{F}^W_{t_1}) \,\Big|\, X_{t_1}\Big). \end{aligned}$$

The independence and stationarity of the increments of X yield

$$\mathbb{Q}(\inf_{t_1 \le s < t} X_s > a \,|\, \mathcal{F}^W_{t_1}) = \Phi(\nu, t - t_1, a - X_{t_1}).$$

Hence

$$F_t = 1 - \Phi(\nu, t - t_1, a - X_{t_1}) \mathbb{Q}(\inf_{s<t_1} X_s > a \,|\, X_{t_1}).$$

From results on Brownian bridges, for $X_{t_1} > a$, we obtain (we omit the parameter ν in the definition of Φ)

$$F_t = 1 - \Phi(t - t_1, a - X_{t_1}) \left[1 - \exp\left(-\frac{2\,a}{t_1}(a - X_{t_1})\right)\right]. \tag{3.34}$$

The case $X_{t_1} \le a$ corresponds to default, and thus for $X_{t_1} \le a$ we have $F_t = 1$.

The process F is continuous and increasing in $[t_1, t_2[$. When t approaches t_1 from above, for $X_{t_1} > a$,

$$F_{t_1^+} = \exp\left[-\frac{2a}{t_1}(a - X_{t_1})\right],$$

because $\lim_{t \to t_1^+} \Phi(t - t_1, a - X_{t_1}) = 1$.

For $X_{t_1} > a$, the jump of F at t_1 is

$$\Delta F^2_{t_1} = \exp\left[-\frac{2a}{t_1}(a - X_{t_1})\right] - 1 + \Phi(t_1, a).$$

For $X_{t_1} \le a$, $\Phi(t - t_1, a - X_{t_1}) = 0$ by the definition of Φ and

$$\Delta F_{t_1} = \Phi(t_1, a).$$

- **General case:** $t_i < t < t_{i+1} < T$, $i \ge 2$

For $t_i < t < t_{i+1}$, we have that

$$\begin{aligned} \mathbb{Q}(\tau > t \,|\, X_{t_1}, \dots, X_{t_i}) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\inf_{s\le t_i} X_s > a\}}\, \mathbb{Q}(\inf_{t_i \le s < t} X_s > a \,|\, \mathcal{F}_{t_i}) \,\Big|\, X_{t_1}, \dots, X_{t_i}\right) \\ &= \Phi(t - t_i, a - X_{t_i})\, \mathbb{Q}\left(\inf_{s \le t_i} X_s > a \,\Big|\, X_{t_1}, \dots, X_{t_i}\right). \end{aligned}$$

Write K_i for the second term on the right-hand-side

$$\begin{aligned} K_i &= \mathbb{Q}\left(\inf_{s \le t_i} X_s > a \,\Big|\, X_{t_1}, \dots, X_{t_i}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\inf_{s\le t_{i-1}} X_s > a\}}\, \mathbb{Q}(\inf_{t_{i-1} \le s < t_i} X_s > a \,|\, \mathcal{F}_{t_{i-1}} \vee X_{t_i}) \,\Big|\, X_{t_1}, \dots, X_{t_i}\right). \end{aligned}$$

Obviously,

$$\begin{aligned}\mathbb{Q}(\inf_{t_{i-1}\le s<t_i} X_s > a \,|\, \mathcal{F}_{t_{i-1}} \vee X_{t_i}) &= \mathbb{Q}(\inf_{t_{i-1}\le s<t_i} X_s > a \,|\, X_{t_{i-1}}, X_{t_i}) \\ &= \exp\left(-\frac{2}{t_i - t_{i-1}}(a - X_{t_{i-1}})(a - X_{t_i})\right).\end{aligned}$$

Therefore,

$$K_i = K_{i-1} \exp\left(-\frac{2}{t_i - t_{i-1}}(a - X_{t_{i-1}})(a - X_{t_i})\right). \tag{3.35}$$

Hence

$$\begin{aligned}\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) &= 1 \qquad \text{if } X_{t_j} < a \text{ for at least one } t_j \text{ such that } t_j < t, \\ &= 1 - \Phi(t - t_i, a - X_{t_i})K_i,\end{aligned}$$

where

$$K_i = k(t_1, X_{t_1}, 0)k(t_2 - t_1, X_{t_1}, X_{t_2}) \cdots k(t_i - t_{i-1}, X_{t_{i-1}}, X_{t_i})$$

and

$$k(s, x, y) = 1 - \exp\left(-\frac{2}{s}(a - x)(a - y)\right).$$

Lemma 3.4.1 *The process ζ defined by $\zeta_t = \sum_{t_i \le t} \Delta F_{t_i}$ is an $\mathbb{F}$-martingale.*

Proof. Consider first the times $t_i \le s < t \le t_{i+1}$. In this case, it is obvious that $\mathbb{E}_{\mathbb{Q}}(\zeta_t \,|\, \mathcal{H}_s) = \zeta_s$ since $\zeta_t = \zeta_s = \zeta_{t_i}$, which is $\mathcal{H}_s$-measurable.

It suffices to show that $\mathbb{E}_{\mathbb{Q}}(\zeta_t \,|\, \mathcal{F}_s) = \zeta_s$ for $t_i \le s < t_{i+1} \le t < t_{i+2}$. In this case, $\zeta_s = \zeta_{t_i}$ and $\zeta_t = \zeta_{t_i} + \Delta F_{t_{i+1}}$. Therefore,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\zeta_t \,|\, \mathcal{F}_s) &= \mathbb{E}_{\mathbb{Q}}(\zeta_{t_i} + \Delta F_{t_{i+1}} \,|\, \mathcal{F}_s) \\ &= \zeta_{t_i} + \mathbb{E}_{\mathbb{Q}}(\Delta F_{t_{i+1}} \,|\, \mathcal{F}_s),\end{aligned}$$

which shows that it is necessary to prove that $\mathbb{E}_{\mathbb{Q}}(\Delta F_{t_{i+1}} \,|\, \mathcal{F}_s) = 0$.

Let $s < u < t_{i+1} < v < t$. Then,

$$\mathbb{E}_{\mathbb{Q}}(F_v - F_u \,|\, \mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{u<\tau\le v\}} \,|\, \mathcal{F}_s).$$

When $v \to t_{i+1}, v > t_{i+1}$ and $u \to t_{i+1}, u < t_{i+1}$ we get $F_v - F_u \to \Delta F_{t_{i+1}}$. It follows that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\Delta F_{t_{i+1}} \,|\, \mathcal{F}_s) &= \lim_{u\to t_{i+1}, v\to t_{i+1}} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{u<\tau\le v\}} \,|\, \mathcal{F}_s) \\ &= \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau = t_{i+1}\}} \,|\, \mathcal{F}_s) = 0.\end{aligned}$$

□

The Doob-Meyer decomposition of F is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where ζ is an $\mathbb{F}$-martingale and $F_t - \zeta_t$ is a predictable increasing process.

The intensity of the default time would be the process λ defined as

$$\lambda_t \, dt = \frac{d(F_t - \zeta_t)}{1 - F_{t-}}.$$

Comments 3.4.1 It is also possible, as in Duffie and Lando [41], to assume that the observation at time $[t]$ is only $V_{[t]} + \epsilon$ where ϵ is a noise, modelled as a random variable independent of V. Another interesting example, related to Parisian stopping times, is presented in Çetin et al. [28]

3.4.2 Delayed Information

In Guo et al. [52], the authors study a structural model with delayed information. More precisely, they start from a structural model where τ is a $\mathcal{F}_t$-stopping time, and they set $\widetilde{\mathcal{F}}_t = \mathcal{F}_{t-\delta}$ where $\delta > 0$ and $\mathcal{F}_s$ is the trivial filtration for negative s. We set $\mathcal{G}_t = \mathcal{F}_t$ and $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$. We prove here that the process $\widetilde{F}$ is not increasing.

Let $\tau_b = \inf\{t \,:\, W_t = b\}$. Then, for $t > \delta$,

$$\begin{aligned}
\widetilde{F}_t &= \mathbb{Q}(\tau_b \le t \,|\, \widetilde{\mathcal{F}}_t) = \mathbb{Q}\big(\inf_{s\le t} W_s \ge b \,|\, \widetilde{\mathcal{F}}_t\big)\\
&= 1\!\!1_{\{\inf_{s\le t-\delta} W_s < b\}}\, \mathbb{Q}\big(\inf_{t-\delta<s\le t} W_s \ge b \,|\, \widetilde{\mathcal{F}}_t\big)\\
&= 1\!\!1_{\{\inf_{s\le t-\delta} W_s < b\}}\, \mathbb{Q}\big(\inf_{t-\delta<s\le t} W_s - W_{t-\delta} \ge b - W_{t-\delta} \,|\, \widetilde{\mathcal{F}}_t\big) = 1\!\!1_{\{\inf_{s\le t-\delta} W_s < b\}} \Phi(\delta, b - W_{t-\delta})
\end{aligned}$$

where

$$\Phi(u,x) = \mathbb{Q}\big(\inf_{s\le u} B_s \ge x\big) = \mathbb{Q}\big(\sup_{s\le u} W_s \le -x\big) = \mathbb{Q}\big(|W_u| \le -x\big) = N(-x) - N(x).$$

For $t < \delta$, we have $\widetilde{F}_t = \mathbb{Q}(\tau_b \le t)$.

3.5 Intensity Approach

In the so-called intensity approach, the starting point is the knowledge of default time τ and some filtration $\mathbb{G}$ such that τ is a $\mathbb{G}$-stopping time. The (martingale) intensity is then defined as any non-negative process λ, such that

$$M_t \stackrel{\text{def}}{=} H_t - \int_0^{t\wedge\tau} \lambda_s \, ds$$

is a $\mathbb{G}$-martingale. The existence of the intensity relies on the fact that H is an increasing process, therefore a sub-martingale, and thus it can be written as a martingale M plus a predictable, increasing process A. The increasing process A is such that $A_t 1\!\!1_{\{t\ge\tau\}} = A_\tau 1\!\!1_{\{t\ge\tau\}}$. In the case where τ is a predictable stopping time, obviously $A = H$. In fact, the intensity exists only if τ is a totally inaccessible stopping time.

We emphasize that, in that setting the intensity is not well defined after time τ. Specifically, if λ is an intensity then for any non-negative predictable process g the process $\tilde{\lambda}_t = \lambda_t 1\!\!1_{\{t\le\tau\}} + g_t 1\!\!1_{\{t>\tau\}}$ is also an intensity.

Lemma 3.5.1 *The process*

$$L_t = 1\!\!1_{\{t<\tau\}} \exp\left(\int_0^t \lambda_s \, ds\right)$$

is a $\mathbb{G}$*-martingale.*

Proof. From the integration by parts formula, we get

$$dL_t = \exp\Big(\int_0^t \lambda_s\, ds\Big)\big(-dH_t + (1-H_{t-})\lambda_t\, dt\big) = -\exp\Big(\int_0^t \lambda_s\, ds\Big)\, dM_t.$$

This shows that L is a $\mathbb{G}$-martingale. □

Proposition 3.5.1 *If the process*

$$Y_t = \mathbb{E}_{\mathbb{Q}}\Big(X \exp\Big(-\int_0^T \lambda_u\, du\Big) \,\Big|\, \mathcal{G}_t\Big)$$

is continuous at time τ then

$$\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\,\mathbb{E}_{\mathbb{Q}}\Big(X\exp\Big(-\int_t^T\lambda_u\,du\Big)\,\Big|\,\mathcal{G}_t\Big). \tag{3.36}$$

Proof. The process

$$U_t = \mathbb{1}_{\{t<\tau\}}\exp\Big(\int_0^t\lambda_s\,ds\Big)\,\mathbb{E}_{\mathbb{Q}}\Big(X\exp\Big(-\int_0^T\lambda_u\,du\Big)\,\Big|\,\mathcal{G}_t\Big) = L_tY_t$$

is a $\mathbb{G}$-martingale. Indeed, $dU_t = L_{t-}\,dY_t + Y_t\,dL_t$ and

$$\mathbb{E}_{\mathbb{Q}}(U_T\,|\,\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t) = U_t.$$

The result now follows. □

It should be stressed that the continuity of the process Y at time τ depends on the choice of λ after time τ. Let us mention that the jump size ΔY_τ is usually difficult to compute.

Proposition 3.5.2 *If the process Y is not continuous then*

$$\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\exp\Big(\int_0^t\lambda_s\,ds\Big)\,\mathbb{E}_{\mathbb{Q}}(Xe^{-\Lambda_T}\,|\,\mathcal{G}_t) - \mathbb{E}_{\mathbb{Q}}(\Delta Y_\tau e^{\Lambda_\tau}\,|\,\mathcal{G}_t).$$

Proof. We have

$$dU_t = L_{t-}\,dY_t + Y_{t-}\,dL_t + d[L,Y]_t = L_{t-}\,dY_t + Y_{t-}\,dL_t + \Delta L_t\Delta Y_t$$

and

$$\mathbb{E}_{\mathbb{Q}}(U_T\,|\,\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}\big(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t\big) = U_t - e^{\Lambda_\tau}\mathbb{E}_{\mathbb{Q}}\big(\Delta Y_\tau e^{\Lambda_\tau}\,|\,\mathcal{G}_t\big).$$

Then, for any $X\in\mathcal{G}_T$,

$$\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\Big(e^{\Lambda_t}\mathbb{E}_{\mathbb{Q}}\big(e^{-\Lambda_T}X\,|\,\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(e^{\Lambda_\tau}\,\Delta Y_\tau\,|\,\mathcal{G}_t\big)\Big)$$

where $Y_t = \mathbb{E}_{\mathbb{Q}}\big(Xe^{-\Lambda_T}\,|\,\mathcal{G}_t\big)$ and $\Lambda_t = \int_0^t\lambda_u\,du$. □

Aven's Lemma

We end this chapter by recalling Aven's lemma [1].

Lemma 3.5.2 *Let $(\Omega,\mathbb{G},\mathbb{Q})$ be a filtered probability space and N be a counting process. Assume that $\mathbb{E}_{\mathbb{Q}}(N_t)<\infty$ for any t. Let $(h_n, n\geq 1)$ be a sequence of real numbers converging to 0, and*

$$Y_t^{(n)} = \frac{1}{h_n}\,\mathbb{E}_{\mathbb{Q}}(N_{t+h_n} - N_t\,|\,\mathcal{G}_t).$$

Assume that there exists non-negative, $\mathbb{G}$-adapted processes λ_t and y_t such that:
(i) For any t, $\lim Y_t^{(n)} = \lambda_t$,
(ii) For any t, there exists for almost all ω an $n_0 = n_0(t,\omega)$ such that

$$|\,Y_s^{(n)} - \lambda_s(\omega)\,| \leq y_s(\omega),\ s\leq t,\ n\geq n_0(t,\omega),$$

(iii) For any t, $\int_0^t y_s\,ds<\infty$.
Then the process $N_t - \int_0^t\lambda_s\,ds$ is a $\mathbb{G}$-martingale.

We emphasize that, using this result when $N_t = H_t$ gives a value of the intensity which is equal to 0 after the default time. This is not convenient for Duffie's no-jump criteria since, for this choice of intensity, the process Y in Proposition 3.5.1 has a jump at time τ. We refer to Jeanblanc and LeCam [58] for a more detailed comparison between the intensity and the hazard process approaches.

Chapter 4

Hedging of Defaultable Claims

In this chapter, we shall study hedging strategies for credit derivatives under assumption that some primary defaultable (as well as non-defaultable) assets are traded, and thus they can be used in replication of non-traded contingent claims. We follow here the paper by Bielecki et al. [9].

4.1 Semimartingale Model with a Common Default

In what follows, we fix a finite horizon date $T > 0$. For the purpose of this chapter, it is enough to formally define a generic defaultable claim through the following definition.

Definition 4.1.1 A *defaultable claim* with maturity date T is represented by a triplet (X, Z, τ), where:
(i) the *default time* τ specifies the random time of default, and thus also the default events $\{\tau \le t\}$ for every $t \in [0, T]$,
(ii) the *promised payoff* $X \in \mathcal{F}_T$ represents the random payoff received by the owner of the claim at time T, provided that there was no default prior to or at time T; the actual payoff at time T associated with X thus equals $X\mathbb{1}_{\{T<\tau\}}$,
(iii) the $\mathbb{F}$-adapted *recovery process* Z specifies the recovery payoff Z_τ received by the owner of a claim at time of default (or at maturity), provided that the default occurred prior to or at maturity date T.

In practice, hedging of a credit derivative after default time is usually of minor interest. Also, in a model with a single default time, hedging after default reduces to replication of a non-defaultable claim. It is thus natural to define the replication of a defaultable claim in the following way.

Definition 4.1.2 *We say that a self-financing strategy ϕ replicates a defaultable claim (X, Z, τ) if its wealth process $V(\phi)$ satisfies $V_T(\phi)\mathbb{1}_{\{T<\tau\}} = X\mathbb{1}_{\{T<\tau\}}$ and $V_\tau(\phi)\mathbb{1}_{\{T\ge\tau\}} = Z_\tau\mathbb{1}_{\{T\ge\tau\}}$.*

When dealing with replicating strategies, in the sense of Definition 4.1.2, we will always assume, without loss of generality, that the components of the process ϕ are $\mathbb{F}$-predictable processes.

4.1.1 Dynamics of Asset Prices

We assume that we are given a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a (possibly multi-dimensional) standard Brownian motion W and a random time τ admitting an $\mathbb{F}$-intensity γ under $\mathbb{P}$, where $\mathbb{F}$ is the filtration generated by W. In addition, we assume that τ satisfies (3.6), so that the hypothesis (H) is valid under $\mathbb{P}$ for filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Since the default time admits an $\mathbb{F}$-intensity, it is not an $\mathbb{F}$-stopping time. Indeed, any stopping time with respect to a Brownian filtration is known to be predictable.

We interpret τ as the common default time for all defaultable assets in our model. For simplicity, we assume that only three primary assets are traded in the market, and the dynamics under the historical probability $\mathbb{P}$ of their prices are, for $i = 1, 2, 3$ and $t \in [0, T]$,

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dM_t \big), \tag{4.1}$$

where $M_t = H_t - \int_0^t (1 - H_s)\gamma_s\, ds$ is a martingale, or equivalently,

$$dY_t^i = Y_{t-}^i \big((\mu_{i,t} - \kappa_{i,t}\gamma_t \mathbb{1}_{\{t<\tau\}})\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dH_t \big). \tag{4.2}$$

The processes $(\mu_i, \sigma_i, \kappa_i) = (\mu_{i,t}, \sigma_{i,t}, \kappa_{i,t},\, t \geq 0)$, $i = 1, 2, 3$, are assumed to be $\mathbb{G}$-adapted, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. In addition, we assume that $\kappa_i \geq -1$ for any $i = 1, 2, 3$, so that Y^i are nonnegative processes, and they are strictly positive prior to τ. Note that, in the case of constant coefficients we have that

$$Y_t^i = Y_0^i e^{\mu_i t} e^{\sigma_i W_t - \sigma_i^2 t/2} e^{-\kappa_i \gamma_i (t \wedge \tau)} (1 + \kappa_i)^{H_t}.$$

According to Definition 4.1.2, replication refers to the behavior of the wealth process $V(\phi)$ on the random interval $[\![0, \tau \wedge T]\!]$ only. Hence, for the purpose of replication of defaultable claims of the form (X, Z, τ), it is sufficient to consider prices of primary assets stopped at $\tau \wedge T$. This implies that instead of dealing with $\mathbb{G}$-adapted coefficients in (4.1), it suffices to focus on $\mathbb{F}$-adapted coefficients of stopped price processes. However, for the sake of completeness, we shall also deal with T-maturity claims of the form $Y = G(Y_T^1, Y_T^2, Y_T^3, H_T)$ (see Section 4.4 below).

Pre-Default Values

As will become clear in what follows, when dealing with defaultable claims of the form (X, Z, τ), we will be mainly concerned with the so-called pre-default prices. The *pre-default price* $\widetilde{Y}^i$ of the ith asset is an $\mathbb{F}$-adapted, continuous process, given by the equation, for $i = 1, 2, 3$ and $t \in [0, T]$,

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_{i,t} - \kappa_{i,t}\gamma_t)\, dt + \sigma_{i,t}\, dW_t \big) \tag{4.3}$$

with $\widetilde{Y}_0^i = Y_0^i$. Put another way, $\widetilde{Y}^i$ is the unique $\mathbb{F}$-predictable process such that $\widetilde{Y}_t^i \mathbb{1}_{\{t\leq\tau\}} = Y_t^i \mathbb{1}_{\{t\leq\tau\}}$ for $t \in \mathbb{R}_+$. When dealing with the pre-default prices, we may and do assume, without loss of generality, that the processes μ_i, σ_i and κ_i are $\mathbb{F}$-predictable.

It is worth stressing that the historically observed drift coefficient equals $\mu_{i,t} - \kappa_{i,t}\gamma_t$, rather than $\mu_{i,t}$. The drift coefficient denoted by $\mu_{i,t}$ is already credit-risk adjusted in the sense of our model, and it is not directly observed. This convention was chosen here for the sake of simplicity of notation. It also lends itself to the following intuitive interpretation: if ϕ^i is the number of units of the ith asset held in our portfolio at time t then the gains/losses from trades in this asset, prior to default time, can be represented by the differential

$$\phi_t^i\, d\widetilde{Y}_t^i = \phi_t^i \widetilde{Y}_t^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t \big) - \phi_t^i \widetilde{Y}_t^i \kappa_{i,t}\gamma_t\, dt.$$

The last term may be here separated, and formally treated as an effect of continuously paid dividends at the dividend rate $\kappa_{i,t}\gamma_t$. However, this interpretation may be misleading, since this quantity is not directly observed. In fact, the mere estimation of the drift coefficient in dynamics (4.3) is not practical.

Still, if this formal interpretation is adopted, it is sometimes possible make use of the standard results concerning the valuation of derivatives of dividend-paying assets. It is, of course, a delicate issue how to separate in practice both components of the drift coefficient. We shall argue below that although the dividend-based approach is formally correct, a more pertinent and simpler way of dealing with hedging relies on the assumption that only the effective drift $\mu_{i,t} - \kappa_{i,t}\gamma_t$ is observable. In practical approach to hedging, the values of drift coefficients in dynamics of asset prices play no essential role, so that they are considered as market observables.

Market Observables

To summarize, we assume throughout that the *market observables* are: the pre-default market prices of primary assets, their volatilities and correlations, as well as the jump coefficients $\kappa_{i,t}$ (the financial interpretation of jump coefficients is examined in the next subsection). To summarize we postulate that under the statistical probability $\mathbb{P}$ we have

$$dY_t^i = Y_{t-}^i\big(\widetilde{\mu}_{i,t}\,dt + \sigma_{i,t}\,dW_t + \kappa_{i,t}\,dH_t\big) \tag{4.4}$$

where the drift terms $\widetilde{\mu}_{i,t}$ are not observable, but we can observe the volatilities $\sigma_{i,t}$ (and thus the assets correlations), and we have an a priori assessment of jump coefficients $\kappa_{i,t}$. In this general set-up, the most natural assumption is that the dimension of a driving Brownian motion W equals the number of tradable assets. However, for the sake of simplicity of presentation, we shall frequently assume that W is one-dimensional. One of our goals will be to derive closed-form solutions for replicating strategies for derivative securities in terms of market observables only (whenever replication of a given claim is actually feasible). To achieve this goal, we shall combine a general theory of hedging defaultable claims within a continuous semimartingale set-up, with a judicious specification of particular models with deterministic volatilities and correlations.

Recovery Schemes

It is clear that the sample paths of price processes Y^i are continuous, except for a possible discontinuity at time τ. Specifically, we have that

$$\Delta Y_\tau^i := Y_\tau^i - Y_{\tau-}^i = \kappa_{i,\tau} Y_{\tau-}^i,$$

so that $Y_\tau^i = Y_{\tau-}^i(1+\kappa_{i,\tau}) = \widetilde{Y}_{\tau-}^i(1+\kappa_{i,\tau})$.

A primary asset Y^i is termed a *default-free asset* (*defaultable asset*, respectively) if $\kappa_i = 0$ ($\kappa_i \neq 0$, respectively). In the special case when $\kappa_i = -1$, we say that a defaultable asset Y^i is subject to a *total default*, since its price drops to zero at time τ and stays there forever. Such an asset ceases to exist after default, in the sense that it is no longer traded after default. This feature makes the case of a total default quite different from other cases, as we shall see in our study below.

In market practice, it is common for a credit derivative to deliver a positive recovery (for instance, a *protection payment*) in case of default. Formally, the value of this recovery at default is determined as the value of some underlying process, that is, it is equal to the value at time τ of some $\mathbb{F}$-adapted recovery process Z.

For example, the process Z can be equal to δ, where δ is a constant, or to $g(t, \delta Y_t)$ where g is a deterministic function and $(Y_t,\, t \geq 0)$ is the price process of some default-free asset. Typically, the recovery is paid at default time, but it may also happen that it is postponed to the maturity date.

Let us observe that the case where a defaultable asset Y^i pays a pre-determined recovery at default is covered by our set-up defined in (4.1). For instance, the case of a constant recovery payoff $\delta_i \geq 0$ at default time τ corresponds to the process $\kappa_{i,t} = \delta_i (Y_{t-}^i)^{-1} - 1$. Under this convention, the price Y^i is governed under $\mathbb{P}$ by the SDE

$$dY_t^i = Y_{t-}^i\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t + (\delta_i (Y_{t-}^i)^{-1} - 1)\,dM_t\big). \tag{4.5}$$

If the recovery is proportional to the pre-default value $Y_{\tau-}^i$, and is paid at default time τ (this scheme is known as the *fractional recovery of market value*), we have $\kappa_{i,t} = \delta_i - 1$ and

$$dY_t^i = Y_{t-}^i\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t + (\delta_i - 1)\,dM_t\big). \tag{4.6}$$

4.2 Trading Strategies in a Semimartingale Set-up

We consider trading within the time interval $[0,T]$ for some finite horizon date $T > 0$. For the sake of expositional clarity, we restrict our attention to the case where only three primary assets are traded. The general case of k traded assets was examined by Bielecki et al. [8, 10].

In this section, we consider a fairly general set-up. In particular, processes Y^i, $i = 1,2,3$, are assumed to be nonnegative semi-martingales on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with some filtration $\mathbb{G}$. We assume that they represent spot prices of traded assets in our model of the financial market. Neither the existence of a savings account, nor the market completeness are assumed, in general.

Our goal is to characterize contingent claims which are *hedgeable*, in the sense that they can be replicated by continuously rebalanced portfolios consisting of primary assets. Here, by a contingent claim we mean an arbitrary $\mathcal{G}_T$-measurable random variable. We work under the standard assumptions of a frictionless market.

4.2.1 Unconstrained Strategies

Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a trading strategy; in particular, each process ϕ^i is predictable with respect to the filtration $\mathbb{G}$. The wealth of ϕ equals

$$V_t(\phi) = \sum_{i=1}^{3} \phi^i_t Y^i_t, \quad \forall\, t \in [0,T],$$

and a trading strategy ϕ is said to be *self-financing* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{3} \int_0^t \phi^i_u \, dY^i_u, \quad \forall\, t \in [0,T].$$

Let Φ stand for the class of all self-financing trading strategies. We shall first prove that a self-financing strategy is determined by its initial wealth and the two components ϕ^2, ϕ^3. To this end, we postulate that the price of Y^1 follows a strictly positive process, and we choose Y^1 as a numéraire asset. We shall now analyze the relative values:

$$V^1_t(\phi) := V_t(\phi)(Y^1_t)^{-1}, \quad Y^{i,1}_t := Y^i_t (Y^1_t)^{-1}.$$

Lemma 4.2.1 (i) *For any $\phi \in \Phi$, we have*

$$V^1_t(\phi) = V^1_0(\phi) + \sum_{i=2}^{3} \int_0^t \phi^i_u \, dY^{i,1}_u, \quad \forall\, t \in [0,T].$$

(ii) *Conversely, let X be a $\mathcal{G}_T$-measurable random variable, and let us assume that there exists $x \in \mathbb{R}$ and $\mathbb{G}$-predictable processes ϕ^i, $i = 2,3$ such that*

$$X = Y^1_T \left(x + \sum_{i=2}^{3} \int_0^T \phi^i_u \, dY^{i,1}_u \right). \tag{4.7}$$

Then there exists a $\mathbb{G}$-predictable process ϕ^1 such that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and replicates X. Moreover, the wealth process of ϕ (i.e. the time-t price of X) satisfies $V_t(\phi) = V^1_t Y^1_t$, where

$$V^1_t = x + \sum_{i=2}^{3} \int_0^t \phi^i_u \, dY^{i,1}_u, \quad \forall\, t \in [0,T]. \tag{4.8}$$

Proof. In the case of continuous semimartingales, (it is a well-known result; for discontinuous processes, the proof is not much different. We reproduce it here for the reader's convenience.

Let us first introduce some notation. As usual, $[X,Y]$ stands for the *quadratic covariation* of the two semi-martingales X and Y, as defined by the integration by parts formula:

$$X_tY_t = X_0Y_0 + \int_0^t X_{u-}\,dY_u + \int_0^t Y_{u-}\,dX_u + [X,Y]_t.$$

For any càdlàg (i.e., RCLL) process Y, we denote by $\Delta Y_t = Y_t - Y_{t-}$ the size of the jump at time t. Let $V = V(\phi)$ be the value of a self-financing strategy, and let $V^1 = V^1(\phi) = V(\phi)(Y^1)^{-1}$ be its value relative to the numéraire Y^1. The integration by parts formula yields

$$dV_t^1 = V_{t-}d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}dV_t + d[(Y^1)^{-1}, V]_t.$$

From the self-financing condition, we have $dV_t = \sum_{i=1}^3 \phi_t^i\,dY_t^i$. Hence, using elementary rules to compute the quadratic covariation $[X,Y]$ of the two semi-martingales X,Y, we obtain

$$\begin{aligned}
dV_t^1 &= \phi_t^1 Y_{t-}^1\,d(Y_t^1)^{-1} + \phi_t^2 Y_{t-}^2\,d(Y_t^1)^{-1} + \phi_t^3 Y_{t-}^3\,d(Y_t^1)^{-1} \\
&\quad + (Y_{t-}^1)^{-1}\phi_t^1\,dY_t^1 + (Y_{t-}^1)^{-1}\phi_t^2\,dY_t^1 + (Y_{t-}^1)^{-1}\phi_t^3\,dY_t^1 \\
&\quad + \phi_t^1\,d[(Y^1)^{-1}, Y^1]_t + \phi_t^2\,d[(Y^1)^{-1}, Y^2]_t + \phi_t^3\,d[(Y^1)^{-1}, Y^1]_t \\
&= \phi_t^1\big(Y_{t-}^1\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^1 + d[(Y^1)^{-1}, Y^1]_t\big) \\
&\quad + \phi_t^2\big(Y_{t-}^2\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_{t-}^1 + d[(Y^1)^{-1}, Y^2]_t\big) \\
&\quad + \phi_t^3\big(Y_{t-}^3\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_{t-}^1 + d[(Y^1)^{-1}, Y^3]_t\big).
\end{aligned}$$

We now observe that

$$Y_{t-}^1\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^1 + d[(Y^1)^{-1}, Y^1]_t = d(Y_t^1(Y_t^1)^{-1}) = 0$$

and

$$Y_{t-}^i\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^i + d[(Y^1)^{-1}, Y^i]_t = d((Y_t^1)^{-1}Y_t^i).$$

Consequently,

$$dV_t^1 = \phi_t^2\,dY_t^{2,1} + \phi_t^3\,dY_t^{3,1},$$

as was claimed in part (i). We now proceed to the proof of part (ii). We assume that (4.7) holds for some constant x and processes ϕ^2, ϕ^3, and we define the process V^1 by setting (cf. (4.8))

$$V_t^1 = x + \sum_{i=2}^3 \int_0^t \phi_u^i\,dY_u^{i,1}, \quad \forall\, t \in [0,T].$$

Next, we define the process ϕ^1 as follows:

$$\phi_t^1 = V_t^1 - \sum_{i=2}^3 \phi_t^i Y_t^{i,1} = (Y_t^1)^{-1}\Big(V_t - \sum_{i=2}^3 \phi_t^i Y_t^i\Big),$$

where $V_t = V_t^1 Y_t^1$. Since $dV_t^1 = \sum_{i=2}^3 \phi_t^i\,dY_t^{i,1}$, we obtain

$$\begin{aligned}
dV_t &= d(V_t^1Y_t^1) = V_{t-}^1 dY_t^1 + Y_{t-}^1 dV_t^1 + d[Y^1, V^1]_t \\
&= V_{t-}^1 dY_t^1 + \sum_{i=2}^3 \phi_t^i\big(Y_{t-}^1\,dY_t^{i,1} + d[Y^1, Y^{i,1}]_t\big).
\end{aligned}$$

From the equality

$$dY_t^i = d(Y_t^{i,1}Y_t^1) = Y_{t-}^{i,1}dY_t^1 + Y_{t-}^1 dY_t^{i,1} + d[Y^1, Y^{i,1}]_t,$$

it follows that

$$dV_t = V^1_{t-} dY^1_t + \sum_{i=2}^{3} \phi^i_t \big(dY^i_t - Y^{i,1}_{t-} dY^1_t\big) = \Big(V^1_{t-} - \sum_{i=2}^{3} \phi^i_t Y^{i,1}_{t-}\Big) dY^1_t + \sum_{i=2}^{3} \phi^i_t \, dY^i_t,$$

and our aim is to prove that $dV_t = \sum_{i=1}^{3} \phi^i_t \, dY^i_t$. The last equality holds if

$$\phi^1_t = V^1_t - \sum_{i=2}^{3} \phi^i_t Y^{i,1}_t = V^1_{t-} - \sum_{i=2}^{3} \phi^i_t Y^{i,1}_{t-}, \tag{4.9}$$

i.e., if $\Delta V^1_t = \sum_{i=2}^{3} \phi^i_t \Delta Y^{i,1}_t$, which is the case from the definition (4.8) of V^1. Note also that from the second equality in (4.9) it follows that the process ϕ^1 is indeed $\mathbb{G}$-predictable. Finally, the wealth process of ϕ satisfies $V_t(\phi) = V^1_t Y^1_t$ for every $t \in [0,T]$, and thus $V_T(\phi) = X$. $\square$

We say that a self-financing strategy ϕ *replicates* a claim $X \in \mathcal{G}_T$ if

$$X = \sum_{i=1}^{3} \phi^i_T Y^i_T = V_T(\phi),$$

or equivalently,

$$X = V_0(\phi) + \sum_{i=1}^{3} \int_0^T \phi^i_t \, dY^i_t.$$

Suppose that there exists an EMM for some choice of a numéraire asset, and let us restrict our attention to the class of all *admissible* trading strategies, so that our model is arbitrage-free.

Assume that a claim X can be replicated by some admissible trading strategy, so that it is *attainable* (or *hedgeable*). Then, by definition, the *arbitrage price* at time t of X, denoted as $\pi_t(X)$, equals $V_t(\phi)$ for any admissible trading strategy ϕ that replicates X.

In the context of Lemma 4.2.1, it is natural to choose as an EMM a probability measure $\mathbb{Q}^1$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ and such that the prices $Y^{i,1}$, $i = 2, 3$, are $\mathbb{G}$-martingales under $\mathbb{Q}^1$. If a contingent claim X is hedgeable, then its arbitrage price satisfies

$$\pi_t(X) = Y^1_t \mathbb{E}_{\mathbb{Q}^1}(X (Y^1_T)^{-1} \,|\, \mathcal{G}_t).$$

We emphasize that even if an EMM $\mathbb{Q}^1$ is not unique, the price of any hedgeable claim X is given by this conditional expectation. That is to say, in case of a hedgeable claim these conditional expectations under various equivalent martingale measures coincide.

In the special case where $Y^1_t = B(t,T)$ is the price of a default-free zero-coupon bond with maturity T (abbreviated as ZCB in what follows), $\mathbb{Q}^1$ is called *T-forward martingale measure*, and it is denoted by $\mathbb{Q}_T$. Since $B(T,T) = 1$, the price of any hedgeable claim X now equals $\pi_t(X) = B(t,T) \, \mathbb{E}_{\mathbb{Q}_T}(X \,|\, \mathcal{G}_t)$.

4.2.2 Constrained Strategies

In this section, we make an additional assumption that the price process Y^3 is strictly positive. Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a self-financing trading strategy satisfying the following constraint:

$$\sum_{i=1}^{2} \phi^i_t Y^i_{t-} = Z_t, \quad \forall \, t \in [0,T], \tag{4.10}$$

for a predetermined, $\mathbb{G}$-predictable process Z. In the financial interpretation, equality (4.10) means that a portfolio ϕ is rebalanced in such a way that the total wealth invested in assets Y^1, Y^2 matches a predetermined stochastic process Z. For this reason, the constraint given by (4.10) is referred to as the *balance condition*.

Our first goal is to extend part (i) in Lemma 4.2.1 to the case of constrained strategies. Let $\Phi(Z)$ stand for the class of all (admissible) self-financing trading strategies satisfying the balance condition (4.10). They will be sometimes referred to as *constrained strategies*. Since any strategy $\phi \in \Phi(Z)$ is self-financing, from $dV_t(\phi) = \sum_{i=1}^3 \phi_t^i \, dY_t^i$, we obtain

$$\Delta V_t(\phi) = \sum_{i=1}^3 \phi_t^i \Delta Y_t^i = V_t(\phi) - \sum_{i=1}^3 \phi_t^i Y_{t-}^i.$$

By combining this equality with (4.10), we deduce that

$$V_{t-}(\phi) = \sum_{i=1}^3 \phi_t^i Y_{t-}^i = Z_t + \phi_t^3 Y_{t-}^i.$$

Let us write $Y_t^{i,3} = Y_t^i (Y_t^3)^{-1}$, $Z_t^3 = Z_t (Y_t^3)^{-1}$. The following result extends Lemma 1.7 in Bielecki et al. [5] from the case of continuous semi-martingales to the general case (see also [8, 10]). It is apparent from Proposition 4.2.1 that the wealth process $V(\phi)$ of a strategy $\phi \in \Phi(Z)$ depends only on a single component of ϕ, namely, ϕ^2.

Proposition 4.2.1 *The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of any trading strategy $\phi \in \Phi(Z)$ satisfies*

$$V_t^3(\phi) = V_0^3(\phi) + \int_0^t \phi_u^2 \left(dY_u^{2,3} - \frac{Y_{u-}^{2,3}}{Y_{u-}^{1,3}} \, dY_u^{1,3} \right) + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3}. \tag{4.11}$$

Proof. Let us consider discounted values of price processes Y^1, Y^2, Y^3, with Y^3 taken as a numéraire asset. By virtue of part (i) in Lemma 4.2.1, we thus have

$$V_t^3(\phi) = V_0^3(\phi) + \sum_{i=1}^2 \int_0^t \phi_u^i \, dY_u^{i,3}. \tag{4.12}$$

The balance condition (4.10) implies that

$$\sum_{i=1}^2 \phi_t^i Y_{t-}^{i,3} = Z_t^3,$$

and thus

$$\phi_t^1 = (Y_{t-}^{1,3})^{-1} \Big(Z_t^3 - \phi_t^2 Y_{t-}^{2,3} \Big). \tag{4.13}$$

By inserting (4.13) into (4.12), we arrive at the desired formula (4.11). □

The next result will prove particularly useful for deriving replicating strategies for defaultable claims.

Proposition 4.2.2 *Let a $\mathcal{G}_T$-measurable random variable X represent a contingent claim that settles at time T. We set*

$$dY_t^* = dY_t^{2,3} - \frac{Y_{t-}^{2,3}}{Y_{t-}^{1,3}} \, dY_t^{1,3} = dY_t^{2,3} - Y_{t-}^{2,1} \, dY_t^{1,3}, \tag{4.14}$$

where, by convention, $Y_0^* = 0$. Assume that there exists a $\mathbb{G}$-predictable process ϕ^2, such that

$$X = Y_T^3 \left(x + \int_0^T \phi_t^2 \, dY_t^* + \int_0^T \frac{Z_t^3}{Y_{t-}^{1,3}} \, dY_t^{1,3} \right). \tag{4.15}$$

Then there exist $\mathbb{G}$-predictable processes ϕ^1 and ϕ^3 such that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ belongs to $\Phi(Z)$ and replicates X. The wealth process of ϕ equals, for every $t \in [0,T]$,

$$V_t(\phi) = Y_t^3 \left(x + \int_0^t \phi_u^2 \, dY_u^* + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3} \right). \tag{4.16}$$

Proof. As expected, we first set (note that the process ϕ^1 is a $\mathbb{G}$-predictable process)

$$\phi^1_t = \frac{1}{Y^1_{t-}}\Big(Z_t - \phi^2_t Y^2_{t-}\Big) \tag{4.17}$$

and

$$V^3_t = x + \int_0^t \phi^2_u \, dY^*_u + \int_0^t \frac{Z^3_u}{Y^{1,3}_{u-}} \, dY^{1,3}_u.$$

Arguing along the same lines as in the proof of Proposition 4.2.1, we obtain

$$V^3_t = V^3_0 + \sum_{i=1}^2 \int_0^t \phi^i_u \, dY^{i,3}_u.$$

Now, we define

$$\phi^3_t = V^3_t - \sum_{i=1}^2 \phi^i_t Y^{i,3}_t = (Y^3_t)^{-1}\Big(V_t - \sum_{i=1}^2 \phi^i_t Y^i_t\Big),$$

where $V_t = V^3_t Y^3_t$. As in the proof of Lemma 4.2.1, we check that

$$\phi^3_t = V^3_{t-} - \sum_{i=1}^2 \phi^i_t Y^{i,3}_{t-},$$

and thus the process ϕ^3 is $\mathbb{G}$-predictable. It is clear that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and its wealth process satisfies $V_t(\phi) = V_t$ for every $t \in [0,T]$. In particular, $V_T(\phi) = X$, so that ϕ replicates X. Finally, equality (4.17) implies (4.10), and thus ϕ belongs to the class $\Phi(Z)$. □

Note that equality (4.15) is a necessary (by Lemma 4.2.1) and sufficient (by Proposition 4.2.2) condition for the existence of a constrained strategy that replicates a given contingent claim X.

Synthetic Asset

Let us take $Z = 0$, so that $\phi \in \Phi(0)$. Then the balance condition becomes $\sum_{i=1}^2 \phi^i_t Y^i_{t-} = 0$, and formula (4.11) reduces to

$$dV^3_t(\phi) = \phi^2_t \left(dY^{2,3}_t - \frac{Y^{2,3}_{t-}}{Y^{1,3}_{t-}} \, dY^{1,3}_t \right). \tag{4.18}$$

The process $\bar{Y}^2 = Y^3 Y^*$, where Y^* is defined in (4.14) is called a *synthetic asset.* It corresponds to a particular self-financing portfolio, with the long position in Y^2 and the short position of $Y^{2,1}_{t-}$ number of shares of Y^1, and suitably re-balanced positions in the third asset so that the portfolio is self-financing, as in Lemma 4.2.1.

It can be shown (see Bielecki et al. [8, 10]) that trading in primary assets Y^1, Y^2, Y^3 is formally equivalent to trading in assets $Y^1, \bar{Y}^2, Y^3$. This observation supports the name synthetic asset attributed to the process $\bar{Y}^2$. Note, however, that the synthetic asset process may take negative values.

Case of Continuous Asset Prices

In the case of continuous asset prices, the relative price $Y^* = \bar{Y}^2 (Y^3)^{-1}$ of the synthetic asset can be given an alternative representation, as the following result shows. Recall that the *predictable bracket* of the two continuous semi-martingales X and Y, denoted as $\langle X, Y \rangle$, coincides with their quadratic covariation $[X, Y]$.

Proposition 4.2.3 *Assume that the price processes Y^1 and Y^2 are continuous. Then the relative price of the synthetic asset satisfies*

$$Y_t^* = \int_0^t (Y_u^{3,1})^{-1} e^{\alpha_u}\, d\widehat{Y}_u,$$

where $\widehat{Y}_t := Y_t^{2,1} e^{-\alpha_t}$ and

$$\alpha_t := \langle \ln Y^{2,1}, \ln Y^{3,1} \rangle_t = \int_0^t (Y_u^{2,1})^{-1} (Y_u^{3,1})^{-1}\, d\langle Y^{2,1}, Y^{3,1}\rangle_u. \tag{4.19}$$

In terms of the auxiliary process $\widehat{Y}$, formula (4.11) *becomes*

$$V_t^3(\phi) = V_0^3(\phi) + \int_0^t \widehat{\phi}_u\, d\widehat{Y}_u + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}}\, dY_u^{1,3}, \tag{4.20}$$

where $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$.

Proof. It suffices to give the proof for $Z = 0$. The proof relies on the integration by parts formula stating that for any two continuous semi-martingales, say X and Y, we have

$$Y_t^{-1}\big(dX_t - Y_t^{-1} d\langle X, Y\rangle_t\big) = d(X_t Y_t^{-1}) - X_t\, dY_t^{-1},$$

provided that Y is strictly positive. An application of this formula to processes $X = Y^{2,1}$ and $Y = Y^{3,1}$ leads to

$$(Y_t^{3,1})^{-1}\big(dY_t^{2,1} - (Y_t^{3,1})^{-1} d\langle Y^{2,1}, Y^{3,1}\rangle_t\big) = d(Y_t^{2,1}(Y_t^{3,1})^{-1}) - Y_t^{2,1}\, d(Y^{3,1})_t^{-1}.$$

The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of a strategy $\phi \in \Phi(0)$ satisfies

$$\begin{aligned} V_t^3(\phi) &= V_0^3(\phi) + \int_0^t \phi_u^2\, dY_u^* \\ &= V_0^3(\phi) + \int_0^t \phi_u^2 (Y_u^{3,1})^{-1} e^{\alpha_u}\, d\widehat{Y}_u, \\ &= V_0^3(\phi) + \int_0^t \widehat{\phi}_u\, d\widehat{Y}_u \end{aligned}$$

where we denote $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$.

Remark 4.2.1 The financial interpretation of the auxiliary process $\widehat{Y}$ will be studied below. Let us only observe here that if Y^* is a local martingale under some probability $\mathbb{Q}$ then $\widehat{Y}$ is a $\mathbb{Q}$-local martingale (and vice versa, if $\widehat{Y}$ is a $\widehat{\mathbb{Q}}$-local martingale under some probability $\widehat{\mathbb{Q}}$ then Y^* is a $\widehat{\mathbb{Q}}$-local martingale). Nevertheless, for the reader's convenience, we shall use two symbols $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$, since this equivalence holds for continuous processes only.
It is thus worth stressing that we will apply Proposition 4.2.3 to pre-default values of assets, rather than directly to asset prices, within the set-up of a semimartingale model with a common default, as described in Section 4.1.1. In this model, the asset prices may have discontinuities, but their pre-default values follow continuous processes.

4.3 Martingale Approach to Valuation and Hedging

Our goal is to derive quasi-explicit conditions for replicating strategies for a defaultable claim in a fairly general set-up introduced in Section 4.1.1. In this section, we only deal with trading strategies based on the reference filtration $\mathbb{F}$, and the underlying price processes (that is, prices of default-free assets and pre-default values of defaultable assets) are assumed to be continuous. Hence, our arguments will hinge on Proposition 4.2.3, rather than on a more general Proposition 4.2.1. We shall also adapt Proposition 4.2.2 to our current purposes.

To simplify the presentation, we make a standing assumption that all coefficient processes are such that the SDEs appearing below admit unique strong solutions, and all stochastic exponentials (used as Radon-Nikodým derivatives) are true martingales under respective probabilities.

4.3.1 Defaultable Asset with Total Default

In this section, we shall examine in some detail a particular model where the two assets, Y^1 and Y^2, are default-free and satisfy

$$dY_t^i = Y_t^i\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t\big), \quad i = 1, 2,$$

where W is a one-dimensional Brownian motion. The third asset is a defaultable asset with total default, so that

$$dY_t^3 = Y_{t-}^3\big(\mu_{3,t}\,dt + \sigma_{3,t}\,dW_t - dM_t\big).$$

Since we will be interested in replicating strategies in the sense of Definition 4.1.2, we may and do assume, without loss of generality, that the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbb{F}$-predictable, rather than $\mathbb{G}$-predictable. Recall that, in general, there exist $\mathbb{F}$-predictable processes $\widetilde{\mu}_3$ and $\widetilde{\sigma}_3$ such that

$$\widetilde{\mu}_{3,t}\mathbb{1}_{\{t\le\tau\}} = \mu_{3,t}\mathbb{1}_{\{t\le\tau\}}, \quad \widetilde{\sigma}_{3,t}\mathbb{1}_{\{t\le\tau\}} = \sigma_{3,t}\mathbb{1}_{\{t\le\tau\}}. \tag{4.21}$$

We assume throughout that $Y_0^i > 0$ for every i, so that the price processes Y^1, Y^2 are strictly positive, and the process Y^3 is nonnegative, and has strictly positive pre-default value.

Default-Free Market

It is natural to postulate that the default-free market with the two traded assets, Y^1 and Y^2, is arbitrage-free. More precisely, we choose Y^1 as a numéraire, and we require that there exists a probability measure $\mathbb{P}^1$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the process $Y^{2,1}$ is a $\mathbb{P}^1$-martingale. The dynamics of processes $(Y^1)^{-1}$ and $Y^{2,1}$ are

$$d(Y_t^1)^{-1} = (Y_t^1)^{-1}\big((\sigma_{1,t}^2 - \mu_{1,t})\,dt - \sigma_{1,t}\,dW_t\big), \tag{4.22}$$

and

$$dY_t^{2,1} = Y_t^{2,1}\big((\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}))\,dt + (\sigma_{2,t} - \sigma_{1,t})\,dW_t\big),$$

respectively. Hence, the necessary condition for the existence of an EMM $\mathbb{P}^1$ is the inclusion $A \subseteq B$, where $A = \{(t,\omega) \in [0,T]\times\Omega :\ \sigma_{1,t}(\omega) = \sigma_{2,t}(\omega)\}$ and $B = \{(t,\omega) \in [0,T]\times\Omega :\ \mu_{1,t}(\omega) = \mu_{2,t}(\omega)\}$. The necessary and sufficient condition for the existence and uniqueness of an EMM $\mathbb{P}^1$ reads

$$\mathbb{E}_{\mathbb{P}}\left\{\mathcal{E}_T\left(\int_0^{\cdot}\theta_u\,dW_u\right)\right\} = 1 \tag{4.23}$$

where the process θ is given by the formula (by convention, $0/0 = 0$)

$$\theta_t = \sigma_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}, \quad \forall\, t \in [0,T]. \tag{4.24}$$

Note that in the case of constant coefficients, if $\sigma_1 = \sigma_2$ then the model is arbitrage-free only in the trivial case when $\mu_2 = \mu_1$.

Remark 4.3.1 Since the martingale measure $\mathbb{P}^1$ is unique, the default-free model (Y^1, Y^2) is complete. However, this is not a necessary assumption and thus it can be relaxed. As we shall see in what follows, it is typically more natural to assume that the driving Brownian motion W is multi-dimensional.

Arbitrage-Free Property

Let us now consider also a defaultable asset Y^3. Our goal is now to find a martingale measure $\mathbb{Q}^1$ (if it exists) for relative prices $Y^{2,1}$ and $Y^{3,1}$. Recall that we postulate that the hypothesis (H) holds under $\mathbb{P}$ for filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. The dynamics of $Y^{3,1}$ under $\mathbb{P}$ are

$$dY_t^{3,1} = Y_{t-}^{3,1}\Big\{\big(\mu_{3,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{3,t})\big)dt + (\sigma_{3,t} - \sigma_{1,t})\, dW_t - dM_t\Big\}.$$

Let $\mathbb{Q}^1$ be any probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and let η be the associated Radon-Nikodým density process, so that

$$d\mathbb{Q}^1\,|\,_{\mathcal{G}_t} = \eta_t\, d\mathbb{P}\,|\,_{\mathcal{G}_t}, \tag{4.25}$$

where the process η satisfies

$$d\eta_t = \eta_{t-}(\theta_t\, dW_t + \zeta_t\, dM_t) \tag{4.26}$$

for some $\mathbb{G}$-predictable processes θ and ζ, and η is a $\mathbb{G}$-martingale under $\mathbb{P}$.

From Girsanov's theorem, the processes $\widehat{W}$ and $\widehat{M}$, given by

$$\widehat{W}_t = W_t - \int_0^t \theta_u\, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\, du, \tag{4.27}$$

are $\mathbb{G}$-martingales under $\mathbb{Q}^1$. To ensure that $Y^{2,1}$ is a $\mathbb{Q}^1$-martingale, we postulate that (4.23) and (4.24) are valid. Consequently, for the process $Y^{3,1}$ to be a $\mathbb{Q}^1$-martingale, it is necessary and sufficient that ζ satisfies

$$\gamma_t\zeta_t = \mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}(\sigma_{3,t} - \sigma_{1,t}).$$

To ensure that $\mathbb{Q}^1$ is a probability measure equivalent to $\mathbb{P}$, we require that $\zeta_t > -1$. The unique martingale measure $\mathbb{Q}^1$ is then given by the formula (4.25) where η solves (4.26), so that

$$\eta_t = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u\, dW_u\right)\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u\, dM_u\right).$$

We are in a position to formulate the following result.

Proposition 4.3.1 *Assume that the process θ given by (4.24) satisfies (4.23), and*

$$\zeta_t = \frac{1}{\gamma_t}\left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}(\sigma_{3,t} - \sigma_{1,t})\right) > -1. \tag{4.28}$$

Then the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free and complete. The dynamics of relative prices under the unique martingale measure $\mathbb{Q}^1$ are

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\, d\widehat{W}_t, \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\, d\widehat{W}_t - d\widehat{M}_t\big). \end{aligned}$$

Since the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbb{F}$-adapted, the process $\widehat{W}$ is an $\mathbb{F}$-martingale (hence, a Brownian motion) under $\mathbb{Q}^1$. Hence, by virtue of Proposition 3.2.3, the hypothesis (H) holds under $\mathbb{Q}^1$, and the $\mathbb{F}$-intensity of default under $\mathbb{Q}^1$ equals

$$\widehat{\gamma}_t = \gamma_t(1 + \zeta_t) = \gamma_t + \left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}(\sigma_{3,t} - \sigma_{1,t})\right).$$

Example 4.3.1 We present an example where the condition (4.28) does not hold, and thus arbitrage opportunities arise. Assume the coefficients are constant and satisfy: $\mu_1 = \mu_2 = \sigma_1 = 0$, $\mu_3 < -\gamma$ for a constant default intensity $\gamma > 0$. Then

$$Y_t^3 = \mathbb{1}_{\{t<\tau\}} Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t + (\mu_3 + \gamma)t\right) \leq Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t\right) = V_t(\phi),$$

where $V(\phi)$ represents the wealth of a self-financing strategy $(\phi^1, \phi^2, 0)$ with $\phi^2 = \frac{\sigma_3}{\sigma_2}$. Hence, the arbitrage strategy would be to sell the asset Y^3, and to follow the strategy ϕ.

Remark 4.3.2 Let us stress once again, that the existence of an EMM is a necessary condition for viability of a financial model, but the uniqueness of an EMM is not always a convenient condition to impose on a model. In fact, when constructing a model, we should be mostly concerned with its flexibility and ability to reflect the pertinent risk factors, rather than with its mathematical completeness. In the present context, it is natural to postulate that the dimension of the underlying Brownian motion equals the number of tradeable risky assets. In addition, each particular model should be tailored to provide intuitive and handy solutions for a predetermined family of contingent claims that will be priced and hedged within its framework.

Hedging a Survival Claim

We first focus on replication of a *survival claim* $(X, 0, \tau)$, that is, a defaultable claim represented by the terminal payoff $X\mathbb{1}_{\{T<\tau\}}$, where X is an $\mathcal{F}_T$-measurable random variable. For the moment, we maintain the simplifying assumption that W is one-dimensional. As we shall see in what follows, it may lead to certain pathological features of a model. If, on the contrary, the driving noise is multi-dimensional, most of the analysis remains valid, except that the model completeness is no longer ensured, in general.

Recall that $\widetilde{Y}^3$ stands for the pre-default price of Y^3, defined as (see (4.3))

$$d\widetilde{Y}_t^3 = \widetilde{Y}_t^3 \big((\widetilde{\mu}_{3,t} + \gamma_t)\, dt + \widetilde{\sigma}_{3,t}\, dW_t\big) \tag{4.29}$$

with $\widetilde{Y}_0^3 = Y_0^3$. This strictly positive, continuous, $\mathbb{F}$-adapted process enjoys the property that $Y_t^3 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^3$. Let us denote the pre-default values in the numéraire $\widetilde{Y}^3$ by $\widetilde{Y}_t^{i,3} = Y_t^i(\widetilde{Y}_t^3)^{-1}$, $i = 1, 2$, and let us introduce the pre-default relative price $\widetilde{Y}^*$ of the synthetic asset $\bar{Y}^2$ by setting

$$d\widetilde{Y}_t^* := d\widetilde{Y}_t^{2,3} - \frac{\widetilde{Y}_t^{2,3}}{\widetilde{Y}_t^{1,3}}\, d\widetilde{Y}_t^{1,3} = \widetilde{Y}_t^{2,3}\Big(\big(\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})\big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t\Big),$$

and let us assume that $\sigma_{1,t} - \sigma_{2,t} \neq 0$. It is also useful to note that the process $\widehat{Y}$, defined in Proposition 4.2.3, satisfies

$$d\widehat{Y}_t = \widehat{Y}_t\Big(\big(\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})\big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t\Big).$$

We shall show that in the case, where α given by (4.19) is deterministic, the process $\widehat{Y}$ has a nice financial interpretation as a credit-risk adjusted forward price of Y^2 relative to Y^1. Therefore, it is more convenient to work with the process $\widetilde{Y}^*$ when dealing with the general case, but to use the process $\widehat{Y}$ when analyzing a model with deterministic volatilities.

Consider an $\mathbb{F}$-predictable self-financing strategy ϕ satisfying the balance condition $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0$, and the corresponding wealth process

$$V_t(\phi) := \sum_{i=1}^{3} \phi_t^i Y_t^i = \phi_t^3 Y_t^3.$$

Let $\widetilde{V}_t(\phi) := \phi_t^3 \widetilde{Y}_t^3$. Since the process $\widetilde{V}(\phi)$ is $\mathbb{F}$-adapted, we see that this is the *pre-default price* process of the portfolio ϕ, that is, we have $\mathbb{1}_{\{\tau>t\}} V_t(\phi) = \mathbb{1}_{\{\tau>t\}} \widetilde{V}_t(\phi)$; we shall call this process the *pre-default wealth* of ϕ. Consequently, the process $\widetilde{V}_t^3(\phi) := \widetilde{V}_t(\phi)(\widetilde{Y}_t^3)^{-1} = \phi_t^3$ is termed the relative pre-default wealth.

Using Proposition 4.2.1, with suitably modified notation, we find that the $\mathbb{F}$-adapted process $\widetilde{V}^3(\phi)$ satisfies, for every $t \in [0,T]$,

$$\widetilde{V}_t^3(\phi) = \widetilde{V}_0^3(\phi) + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^*.$$

Define a new probability $\mathbb{Q}^*$ on $(\Omega, \mathcal{F}_T)$ by setting

$$d\mathbb{Q}^* = \eta_T^* \, d\mathbb{P},$$

where $d\eta_t^* = \eta_t^* \theta_t^* \, dW_t$, and

$$\theta_t^* = \frac{\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}}. \tag{4.30}$$

The process $\widetilde{Y}_t^*$, $t \in [0,T]$, is a (local) martingale under $\mathbb{Q}^*$ driven by a Brownian motion. We shall require that this process is in fact a true martingale; a sufficient condition for this is that

$$\int_0^T \mathbb{E}_{\mathbb{Q}^*} \left(\widetilde{Y}_t^{2,3}(\sigma_{2,t} - \sigma_{1,t}) \right)^2 dt < \infty.$$

From the predictable representation theorem, it follows that for any $X \in \mathcal{F}_T$, such that $X(\widetilde{Y}_T^3)^{-1}$ is square-integrable under $\mathbb{Q}$, there exists a constant x and an $\mathbb{F}$-predictable process ϕ^2 such that

$$X = \widetilde{Y}_T^3 \left(x + \int_0^T \phi_u^2 \, d\widetilde{Y}_u^* \right). \tag{4.31}$$

We now deduce from Proposition 4.2.2 that there exists a self-financing strategy ϕ with the pre-default wealth $\widetilde{V}_t(\phi) = \widetilde{Y}_t^3 \widetilde{V}_t^3$ for every $t \in [0,T]$, where we set

$$\widetilde{V}_t^3 = x + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^*. \tag{4.32}$$

Moreover, it satisfies the balance condition $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0$ for every $t \in [0,T]$. Since clearly $\widetilde{V}_T(\phi) = X$, we have that

$$V_T(\phi) = \phi_T^3 Y_T^3 = \mathbb{1}_{\{T<\tau\}} \phi_T^3 \widetilde{Y}_T^3 = \mathbb{1}_{\{T<\tau\}} \widetilde{V}_T(\phi) = \mathbb{1}_{\{T<\tau\}} X,$$

and thus this strategy replicates the survival claim $(X, 0, \tau)$. In fact, we have that $V_t(\phi) = 0$ on the random interval $[\![\tau, T]\!]$.

Definition 4.3.1 *We say that a survival claim $(X, 0, \tau)$ is attainable if the process $\widetilde{V}^3$ given by (4.32) is a martingale under $\mathbb{Q}^*$.*

The following result is an immediate consequence of (4.31) and (4.32).

Corollary 4.3.1 *Let $X \in \mathcal{F}_T$ be such that $X(\widetilde{Y}_T^3)^{-1}$ is square-integrable under $\mathbb{Q}^*$. Then the survival claim $(X, 0, \tau)$ is attainable. Moreover, the pre-default price $\widetilde{\pi}_t(X, 0, \tau)$ of the claim $(X, 0, \tau)$ is given by the conditional expectation*

$$\widetilde{\pi}_t(X, 0, \tau) = \widetilde{Y}_t^3 \, \mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}_T^3)^{-1} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{4.33}$$

The process $\widetilde{\pi}(X, 0, \tau)(\widetilde{Y}^3)^{-1}$ is an $\mathbb{F}$-martingale under $\mathbb{Q}$.

Proof. Since $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}$, we know from the predictable representation theorem that ϕ^2 in (4.31) is such that $\mathbb{E}_{\mathbb{Q}^*}\left(\int_0^T (\phi^2_t)^2\, d\langle \widetilde{Y}^*\rangle_t\right) < \infty$, so that the process $\widetilde{V}^3$ given by (4.32) is a true martingale under $\mathbb{Q}$. We conclude that $(X, 0, \tau)$ is attainable.

Now, let us denote by $\pi_t(X, 0, \tau)$ the time-t price of the claim $(X, 0, \tau)$. Since ϕ is a hedging portfolio for $(X, 0, \tau)$ we thus have $V_t(\phi) = \pi_t(X, 0, \tau)$ for each $t \in [0, T]$. Consequently,

$$\begin{aligned} \mathbb{1}_{\{\tau>t\}}\widetilde{\pi}_t(X,0,\tau) &= \mathbb{1}_{\{\tau>t\}}\widetilde{V}_t(\phi) = \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^3_t\, \mathbb{E}_{\mathbb{Q}^*}(\widetilde{V}^3_T \,|\, \mathcal{F}_t) \\ &= \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^3_t\, \mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}^3_T)^{-1} \,|\, \mathcal{F}_t) \end{aligned}$$

for each $t \in [0, T]$. This proves equality (4.33). □

In view of the last result, it is justified to refer to $\mathbb{Q}$ as the *pricing measure relative to* Y^3 for attainable survival claims.

Remark 4.3.3 It can be proved that there exists a unique absolutely continuous probability measure $\bar{\mathbb{Q}}$ on $(\Omega, \mathcal{G}_T)$ such that we have

$$Y^3_t\, \mathbb{E}_{\bar{\mathbb{Q}}}\left(\frac{\mathbb{1}_{\{\tau>T\}}X}{Y^3_T}\,\Big|\,\mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^3_t\, \mathbb{E}_{\mathbb{Q}^*}\left(\frac{X}{\widetilde{Y}^3_T}\,\Big|\,\mathcal{F}_t\right).$$

However, this probability measure is not equivalent to $\mathbb{Q}$, since its Radon-Nikodým density vanishes after τ (for a related result, see Collin-Dufresne *et al.* [32]).

Example 4.3.2 We provide here an explicit calculation of the pre-default price of a survival claim. For simplicity, we assume that $X = 1$, so that the claim represents a defaultable zero-coupon bond. Also, we set $\gamma_t = \gamma = const$, $\mu_{i,t} = 0$, and $\sigma_{i,t} = \sigma_i$, $i = 1, 2, 3$. Straightforward calculations yield the following pricing formula

$$\widetilde{\pi}_0(1, 0, \tau) = Y^3_0 e^{-(\gamma + \frac{1}{2}\sigma_3^2)T}.$$

We see that here the pre-default price $\widetilde{\pi}_0(1, 0, \tau)$ depends explicitly on the intensity γ, or rather, on the drift term in dynamics of pre-default value of defaultable asset. Indeed, from the practical viewpoint, the interpretation of the drift coefficient in dynamics of Y^2 as the real-world default intensity is questionable, since within our set-up the default intensity never appears as an independent variable, but is merely a component of the drift term in dynamics of pre-default value of Y^3.

Note also that we deal here with a model with three tradeable assets driven by a one-dimensional Brownian motion. No wonder that the model enjoys completeness, but as a downside, it has an undesirable property that the pre-default values of all three assets are perfectly correlated. Consequently, the drift terms in dynamics of traded assets are closely linked to each other, in the sense, that their behavior under an equivalent change of a probability measure is quite specific.

As we shall see later, if traded primary assets are judiciously chosen then, typically, the pre-default price (and hence the price) of a survival claim will not explicitly depend on the intensity process.

Remark 4.3.4 Generally speaking, we believe that one can classify a financial model as ‘realistic’ if its implementation does not require estimation of drift parameters in (pre-default) prices, at least for the purpose of hedging and valuation of a sufficiently large class of (defaultable) contingent claims of interest. It is worth recalling that the drift coefficients are not assumed to be market observables. Since the default intensity can formally interpreted as a component of the drift term in dynamics of pre-default prices, in a realistic model there is no need to estimate this quantity. From this perspective, the model considered in Example 4.3.2 may serve as an example of an ‘unrealistic’ model, since its implementation requires the knowledge of the drift parameter in the dynamics of Y^3. We do not pretend here that it is always possible to hedge derivative assets without using the drift coefficients in dynamics of tradeable assets, but it seems to us that a good idea is to develop models in which this knowledge is not essential.

Of course, a generic semimartingale model considered until now provides only a framework for a construction of realistic models for hedging of default risk. A choice of tradeable assets and specification of their dynamics should be examined on a case-by-case basis, rather than in a general semimartingale set-up. We shall address this important issue in the foregoing sections, in which we shall deal with particular examples of practically interesting defaultable claims.

Hedging a Recovery Process

Let us now briefly study the situation where the promised payoff equals zero, and the recovery payoff is paid at time τ and equals Z_τ for some $\mathbb{F}$-adapted process Z. Put another way, we consider a defaultable claim of the form $(0, Z, \tau)$. Once again, we make use of Propositions 4.2.1 and 4.2.2. In view of (4.15), we need to find a constant x and an $\mathbb{F}$-predictable process ϕ^2 such that

$$\psi_T := -\int_0^T \frac{Z_t}{Y_t^1}\, d\widetilde{Y}_t^{1,3} = x + \int_0^T \phi_t^2\, d\widetilde{Y}_t^*. \tag{4.34}$$

Similarly as before, we conclude that, under suitable integrability conditions on ψ_T, there exists ϕ^2 such that $d\psi_t = \phi_t^2\, dY_t^*$, where $\psi_t = \mathbb{E}_{\mathbb{Q}^*}(\psi_T \,|\, \mathcal{F}_t)$. We now set

$$\widetilde{V}_t^3 = x + \int_0^t \phi_u^2\, dY_u^* + \int_0^T \frac{\widetilde{Z}_u^3}{\widetilde{Y}_u^{1,3}}\, d\widetilde{Y}_u^{1,3},$$

so that, in particular, $\widetilde{V}_T^3 = 0$. Then it is possible to find processes ϕ^1 and ϕ^3 such that the strategy ϕ is self-financing and it satisfies: $\widetilde{V}_t(\phi) = \widetilde{V}_t^3 \widetilde{Y}_t^3$ and $V_t(\phi) = Z_t + \phi_t^3 Y_t^3$ for every $t \in [0, T]$. It is thus clear that $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \leq T\}$ and $V_T(\phi) = 0$ on the set $\{\tau > T\}$.

Bond Market

For the sake of concreteness, we assume that $Y_t^1 = B(t, T)$ is the price of a default-free ZCB with maturity T, and $Y_t^3 = D(t, T)$ is the price of a defaultable ZCB with zero recovery, that is, an asset with the terminal payoff $Y_T^3 = \mathbb{1}_{\{T<\tau\}}$. We postulate that the dynamics under $\mathbb{P}$ of the default-free ZCB are

$$dB(t, T) = B(t, T)\big(\mu(t, T)\, dt + b(t, T)\, dW_t\big) \tag{4.35}$$

for some $\mathbb{F}$-predictable processes $\mu(t, T)$ and $b(t, T)$. We choose the process $Y_t^1 = B(t, T)$ as a numéraire. Since the prices of the other two assets are not given a priori, we may choose any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ to play the role of $\mathbb{Q}^1$.

In such a case, an EMM $\mathbb{Q}^1$ is referred to as the *forward martingale measure* for the date T, and is denoted by $\mathbb{Q}_T$. Hence, the Radon-Nikodým density of $\mathbb{Q}_T$ with respect to $\mathbb{P}$ is given by (4.26) for some $\mathbb{F}$-predictable processes θ and ζ, and the process

$$W_t^T = W_t - \int_0^t \theta_u\, du, \quad \forall\, t \in [0, T],$$

is a Brownian motion under $\mathbb{Q}_T$. Under $\mathbb{Q}_T$ the default-free ZCB is governed by

$$dB(t, T) = B(t, T)\big(\widehat{\mu}(t, T)\, dt + b(t, T)\, dW_t^T\big)$$

where $\widehat{\mu}(t, T) = \mu(t, T) + \theta_t b(t, T)$. Let $\widehat{\Gamma}$ stand for the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_T$, so that $\widehat{\Gamma}_t = -\ln(1 - \widehat{F}_t)$, where $\widehat{F}_t = \mathbb{Q}_T(\tau \leq t \,|\, \mathcal{F}_t)$. Assume that the hypothesis (H) holds under $\mathbb{Q}_T$ so that, in particular, the process $\widehat{\Gamma}$ is increasing. We define the price process of a defaultable ZCB with zero recovery by the formula

$$D(t, T) := B(t, T)\mathbb{E}_{\mathbb{Q}_T}(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} B(t, T)\, \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,\big|\, \mathcal{F}_t\big),$$

It is then clear that $Y_t^{3,1} = D(t,T)(B(t,T))^{-1}$ is a $\mathbb{Q}_T$-martingale, and the pre-default price $\widetilde{D}(t,T)$ equals

$$\widetilde{D}(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}\,\big|\,\mathcal{F}_t\big).$$

The next result examines the basic properties of the auxiliary process $\widehat{\Gamma}(t,T)$ given as, for every $t \in [0,T]$,

$$\widehat{\Gamma}(t,T) = \widetilde{Y}_t^{3,1} = \widetilde{D}(t,T)(B(t,T))^{-1} = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}\,\big|\,\mathcal{F}_t\big).$$

The quantity $\widehat{\Gamma}(t,T)$ can be interpreted as the conditional probability (under $\mathbb{Q}_T$) that default will not occur prior to the maturity date T, given that we observe $\mathcal{F}_t$ and we know that the default has not yet happened. We will be more interested, however, in its volatility process $\beta(t,T)$ as defined in the following result.

Lemma 4.3.1 *Assume that the $\mathbb{F}$-hazard process $\widehat{\Gamma}$ of τ under $\mathbb{Q}_T$ is continuous. Then the process $\widehat{\Gamma}(t,T)$, $t \in [0,T]$, is a continuous $\mathbb{F}$-submartingale and*

$$d\widehat{\Gamma}(t,T) = \widehat{\Gamma}(t,T)\big(d\widehat{\Gamma}_t + \beta(t,T)\,dW_t^T\big) \tag{4.36}$$

for some $\mathbb{F}$-predictable process $\beta(t,T)$. The process $\widehat{\Gamma}(t,T)$ is of finite variation if and only if the hazard process $\widehat{\Gamma}$ is deterministic. In this case, we have $\widehat{\Gamma}(t,T) = e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}$.

Proof. We have

$$\widehat{\Gamma}(t,T) = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}\,|\,\mathcal{F}_t\big) = e^{\widehat{\Gamma}_t} L_t,$$

where we set $L_t = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T}\,|\,\mathcal{F}_t\big)$. Hence, $\widehat{\Gamma}(t,T)$ is equal to the product of a strictly positive, increasing, right-continuous, $\mathbb{F}$-adapted process $e^{\widehat{\Gamma}_t}$, and a strictly positive, continuous $\mathbb{F}$-martingale L. Furthermore, there exists an $\mathbb{F}$-predictable process $\widehat{\beta}(t,T)$ such that L satisfies

$$dL_t = L_t\widehat{\beta}(t,T)\,dW_t^T$$

with the initial condition $L_0 = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T}\big)$. Formula (4.36) now follows by an application of Itô's formula, by setting $\beta(t,T) = e^{-\widehat{\Gamma}_t}\widehat{\beta}(t,T)$. To complete the proof, it suffices to recall that a continuous martingale is never of finite variation, unless it is a constant process. □

Remark 4.3.5 It can be checked that $\beta(t,T)$ is also the volatility of the process

$$\Gamma(t,T) = \mathbb{E}_{\mathbb{P}}\big(e^{\Gamma_t - \Gamma_T}\,\big|\,\mathcal{F}_t\big).$$

Assume that $\widehat{\Gamma}_t = \int_0^t \widehat{\gamma}_u\,du$ for some $\mathbb{F}$-predictable, nonnegative process $\widehat{\gamma}$. Then we have the following auxiliary result, which gives, in particular, the volatility of the defaultable ZCB.

Corollary 4.3.2 *The dynamics under $\mathbb{Q}_T$ of the pre-default price $\widetilde{D}(t,T)$ equals*

$$d\widetilde{D}(t,T) = \widetilde{D}(t,T)\Big(\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T) + \widehat{\gamma}_t\big)\,dt + \big(b(t,T) + \beta(t,T)\big)\widetilde{d}(t,T)\,dW_t^T\Big).$$

Equivalently, the price $D(t,T)$ of the defaultable ZCB satisfies under $\mathbb{Q}_T$

$$dD(t,T) = D(t,T)\Big(\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T)\big)\,dt + \widetilde{d}(t,T)\,dW_t^T - dM_t\Big).$$

where we set $\widetilde{d}(t,T) = b(t,T) + \beta(t,T)$.

Note that the process $\beta(t,T)$ can be expressed in terms of market observables, since it is simply the difference of volatilities $\widetilde{d}(t,T)$ and $b(t,T)$ of pre-default prices of tradeable assets.

Credit-Risk-Adjusted Forward Price

Assume that the price Y^2 satisfies under the statistical probability $\mathbb{P}$

$$dY_t^2 = Y_t^2\big(\mu_{2,t}\,dt + \sigma_t\,dW_t\big) \tag{4.37}$$

with $\mathbb{F}$-predictable coefficients μ and σ. Let $F_{Y^2}(t,T) = Y_t^2(B(t,T))^{-1}$ be the forward price of Y_T^2. For an appropriate choice of θ (see 4.30), we shall have that

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\,dW_t^T.$$

Therefore, the dynamics of the pre-default synthetic asset $\widetilde{Y}_t^*$ under $\mathbb{Q}^T$ are

$$d\widetilde{Y}_t^* = \widetilde{Y}_t^{2,3}\big(\sigma_t - b(t,T)\big)\,\big(dW_t^T - \beta(t,T)\,dt\big),$$

and the process $\widehat{Y}_t = Y_t^{2,1}e^{-\alpha_t}$ (see Proposition 4.2.3 for the definition of α) satisfies

$$d\widehat{Y}_t = \widehat{Y}_t\big(\sigma_t - b(t,T)\big)\,\big(dW_t^T - \beta(t,T)\,dt\big).$$

Let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega,\mathcal{G}_T)$ such that $\widehat{Y}$ (or, equivalently, $\widetilde{Y}^*$) is a $\widehat{\mathbb{Q}}$-martingale. By virtue of Girsanov's theorem, the process $\widehat{W}$ given by the formula

$$\widehat{W}_t = W_t^T - \int_0^t \beta(u,T)\,du, \quad \forall\, t \in [0,T],$$

is a Brownian motion under $\widehat{\mathbb{Q}}$. Thus, the forward price $F_{Y^2}(t,T)$ satisfies under $\widehat{\mathbb{Q}}$

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\big(d\widehat{W}_t + \beta(t,T)\,dt\big). \tag{4.38}$$

It appears that the valuation results are easier to interpret when they are expressed in terms of forward prices associated with vulnerable forward contracts, rather than in terms of spot prices of primary assets. For this reason, we shall now examine credit-risk-adjusted forward prices of default-free and defaultable assets.

Definition 4.3.2 *Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the credit-risk-adjusted forward price of Y if the pre-default value at time t of the vulnerable forward contract represented by the claim $\mathbb{1}_{\{T<\tau\}}(Y-K)$ equals 0.*

Lemma 4.3.2 *The credit-risk-adjusted forward price $\widehat{F}_Y(t,T)$ of an attainable survival claim $(X,0,\tau)$, represented by a $\mathcal{G}_T$-measurable claim $Y = X\mathbb{1}_{\{T<\tau\}}$, equals $\widetilde{\pi}_t(X,0,\tau)(\widetilde{D}(t,T))^{-1}$, where $\widetilde{\pi}_t(X,0,\tau)$ is the pre-default price of $(X,0,\tau)$. The process $\widehat{F}_Y(t,T)$, $t \in [0,T]$, is an $\mathbb{F}$-martingale under $\widehat{\mathbb{Q}}$.*

Proof. The forward price is defined as an $\mathcal{F}_t$-measurable random variable K such that the claim

$$\mathbb{1}_{\{T<\tau\}}(X\mathbb{1}_{\{T<\tau\}} - K) = X\mathbb{1}_{\{T<\tau\}} - KD(T,T)$$

is worthless at time t on the set $\{t<\tau\}$. It is clear that the pre-default value at time t of this claim equals $\widetilde{\pi}_t(X,0,\tau) - K\widetilde{D}(t,T)$. Consequently, we obtain $\widetilde{F}_Y(t,T) = \widetilde{\pi}_t(X,0,\tau)(\widetilde{D}(t,T))^{-1}$. □

Let us now focus on default-free assets. Manifestly, the credit-risk-adjusted forward price of the bond $B(t,T)$ equals 1. To find the credit-risk-adjusted forward price of Y^2, let us write

$$\widehat{F}_{Y^2}(t,T) := F_{Y^2}(t,T)\,e^{\alpha_T-\alpha_t} = Y_t^{2,1}e^{\alpha_T-\alpha_t}, \tag{4.39}$$

where α is given by (see (4.19))

$$\alpha_t = \int_0^t \big(\sigma_u - b(u,T)\big)\beta(u,T)\,du = \int_0^t \big(\sigma_u - b(u,T)\big)\big(\widetilde{d}(u,T) - b(u,T)\big)\,du. \tag{4.40}$$

Lemma 4.3.3 *Assume that α given by (4.40) is a deterministic function. Then the credit-risk-adjusted forward price of Y^2 equals $\widehat{F}_{Y^2}(t,T)$ (defined in 4.39) for every $t \in [0,T]$.*

Proof. According to Definition 4.3.2, the price $\widehat{F}_{Y^2}(t,T)$ is an $\mathcal{F}_t$-measurable random variable K, which makes the forward contract represented by the claim $D(T,T)(Y_T^2 - K)$ worthless on the set $\{t < \tau\}$. Assume that the claim $Y_T^2 - K$ is attainable. Since $\widetilde{D}(T,T) = 1$, from equation (4.33) it follows that the pre-default value of this claim is given by the conditional expectation

$$\widetilde{D}(t,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2 - K \,\big|\, \mathcal{F}_t\big).$$

Consequently,

$$\widehat{F}_{Y^2}(t,T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2 \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(F_{Y^2}(T,T) \,\big|\, \mathcal{F}_t\big) = F_{Y^2}(t,T)\, e^{\alpha_T - \alpha_t},$$

as was claimed. □

It is worth noting that the process $\widehat{F}_{Y^2}(t,T)$ is a (local) martingale under the pricing measure $\widehat{\mathbb{Q}}$, since it satisfies

$$d\widehat{F}_{Y^2}(t,T) = \widehat{F}_{Y^2}(t,T)(\sigma_t - b(t,T))\, d\widehat{W}_t. \tag{4.41}$$

Under the present assumptions, the auxiliary process $\widehat{Y}$ introduced in Proposition 4.2.3 and the credit-risk-adjusted forward price $\widehat{F}_{Y^2}(t,T)$ are closely related to each other. Indeed, we have $\widehat{F}_{Y^2}(t,T) = \widehat{Y}_t e^{\alpha_T}$, so that the two processes are proportional.

Vulnerable Option on a Default-Free Asset

We shall now analyze a vulnerable call option with the payoff

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - K)^+.$$

Here K is a constant. Our goal is to find a replicating strategy for this claim, interpreted as a survival claim $(X, 0, \tau)$ with the promised payoff $X = C_T = (Y_T^2 - K)^+$, where C_T is the payoff of an equivalent non-vulnerable option. The method presented below is quite general, however, so that it can be applied to any survival claim with the promised payoff $X = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$ satisfying the usual integrability assumptions.

We assume that $Y_t^1 = B(t,T)$, $Y_t^3 = D(t,T)$ and the price of a default-free asset Y^2 is governed by (4.37). Then

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - KY_T^1)^+.$$

We are going to apply Proposition 4.2.3. In the present set-up, we have $Y_t^{2,1} = F_{Y^2}(t,T)$ and $\widehat{Y}_t = F_{Y^2}(t,T)e^{-\alpha_t}$. Since a vulnerable option is an example of a survival claim, in view of Lemma 4.3.2, its credit-risk-adjusted forward price satisfies $\widehat{F}_{C^d}(t,T) = \widetilde{C}_t^d(\widetilde{D}(t,T))^{-1}$.

Proposition 4.3.2 *Suppose that the volatilities σ, b and β are deterministic functions. Then the credit-risk-adjusted forward price of a vulnerable call option written on a default-free asset Y^2 equals*

$$\widehat{F}_{C^d}(t,T) = \widehat{F}_{Y^2}(t,T)N(d_+(\widehat{F}_{Y^2}(t,T),t,T)) - KN(d_-(\widehat{F}_{Y^2}(t,T),t,T)) \tag{4.42}$$

where

$$d_\pm(z,t,T) = \frac{\ln z - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_u - b(u,T))^2\, du.$$

The replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 B(t,T) = -\phi_t^2 Y_t^2, \;\; \phi_t^2 = \widetilde{D}(t,T)(B(t,T))^{-1}N(d_+(t,T))e^{\alpha_T - \alpha_t}, \;\; \phi_t^3 \widetilde{D}(t,T) = \widetilde{C}_t^d,$$

where $d_+(t,T) = d_+(\widehat{F}_{Y^2}(t,T),t,T)$.

Proof. In the first step, we establish the valuation formula. Assume for the moment that the option is attainable. Then the pre-default value of the option equals, for every $t \in [0,T]$,

$$\widetilde{C}_t^d = \widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big((F_{Y^2}(T,T)-K)^+ \,\big|\, \mathcal{F}_t\big) = \widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big((\widehat{F}_{Y^2}(T,T)-K)^+ \,\big|\, \mathcal{F}_t\big). \tag{4.43}$$

In view of (4.41), the conditional expectation above can be computed explicitly, yielding the valuation formula (4.42).

To find the replicating strategy, and establish attainability of the option, we consider the Itô differential $d\widehat{F}_{C^d}(t,T)$ and we identify terms in (4.32). It appears that

$$\begin{aligned} d\widehat{F}_{C^d}(t,T) &= N(d_+(t,T))\,d\widehat{F}_{Y^2}(t,T) = N(d_+(t,T))e^{\alpha_T}\,d\widehat{Y}_t \\ &= N(d_+(t,T))\widetilde{Y}_t^{3,1}e^{\alpha_T-\alpha_t}\,d\widetilde{Y}_t^*, \end{aligned} \tag{4.44}$$

so that the process ϕ^2 in (4.31) equals

$$\phi_t^2 = \widetilde{Y}_t^{3,1}N(d_+(t,T))e^{\alpha_T-\alpha_t}.$$

Moreover, ϕ^1 is such that $\phi_t^1 B(t,T) + \phi_t^2 Y_t^2 = 0$ and $\phi_t^3 = \widetilde{C}_t^d(\widetilde{D}(t,T))^{-1}$. It is easily seen that this proves also the attainability of the option. □

Let us examine the financial interpretation of the last result.

First, equality (4.44) shows that it is easy to replicate the option using vulnerable forward contracts. Indeed, we have

$$\widehat{F}_{C^d}(T,T) = X = \frac{\widetilde{C}_0^d}{\widetilde{D}(0,T)} + \int_0^T N(d_+(t,T))\,d\widehat{F}_{Y^2}(t,T)$$

and thus it is enough to invest the premium $\widetilde{C}_0^d = C_0^d$ in defaultable ZCBs of maturity T, and take at any instant t prior to default $N(d_+(t,T))$ positions in vulnerable forward contracts. It is understood that if default occurs prior to T, all outstanding vulnerable forward contracts become void.

Second, it is worth stressing that neither the arbitrage price, nor the replicating strategy for a vulnerable option, depend explicitly on the default intensity. This remarkable feature is due to the fact that the default risk of the writer of the option can be completely eliminated by trading in defaultable zero-coupon bond with the same exposure to credit risk as a vulnerable option.

In fact, since the volatility β is invariant with respect to an equivalent change of a probability measure, and so are the volatilities σ and $b(t,T)$, the formulae of Proposition 4.3.2 are valid for any choice of a forward measure $\mathbb{Q}_T$ equivalent to $\mathbb{P}$ (and, of course, they are valid under $\mathbb{P}$ as well). The only way in which the choice of a forward measure $\mathbb{Q}_T$ impacts these results is through the pre-default value of a defaultable ZCB.

We conclude that we deal here with the volatility based relative pricing a defaultable claim. This should be contrasted with more popular intensity-based risk-neutral pricing, which is commonly used to produce an arbitrage-free model of tradeable defaultable assets. Recall, however, that if tradeable assets are not chosen carefully for a given class of survival claims, then both hedging strategy and pre-default price may depend explicitly on values of drift parameters, which can be linked in our set-up to the default intensity (see Example 4.3.2).

Remark 4.3.6 Assume that $X = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$. Then the credit-risk-adjusted forward price of a survival claim satisfies $\widehat{F}_X(t,T) = v(t, \widehat{F}_{Y^2}(t,T))$, where the pricing function v solves the PDE

$$\partial_t v(t,z) + \frac{1}{2}(\sigma_t - b(t,T))^2 z^2 \partial_{zz} v(t,z) = 0$$

with the terminal condition $v(T,z) = G(z)$. The PDE approach is studied in Section 4.4 below.

Remark 4.3.7 Proposition 4.3.2 is still valid if the driving Brownian motion is two-dimensional, rather than one-dimensional. In an extended model, the volatilities $\sigma_t, b(t,T)$ and $\beta(t,T)$ take values in $\mathbb{R}^2$ and the respective products are interpreted as inner products in $\mathbb{R}^3$. Equivalently, one may prefer to deal with real-valued volatilities, but with correlated one-dimensional Brownian motions.

Vulnerable Swaption

In this section, we relax the assumption that Y^1 is the price of a default-free bond. We now let Y^1 and Y^2 to be arbitrary default-free assets, with dynamics

$$dY_t^i = Y_t^i\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t\big), \quad i = 1,2.$$

We still take $D(t,T)$ to be the third asset, and we maintain the assumption that the model is arbitrage-free, but we no longer postulate its completeness. In other words, we postulate the existence an EMM $\mathbb{Q}^1$, as defined in subsection on arbitrage free property, but not the uniqueness of $\mathbb{Q}^1$.

We take the first asset as a numéraire, so that all prices are expressed in units of Y^1. In particular, $Y_t^{1,1} = 1$ for every $t \in \mathbb{R}_+$, and the relative prices $Y^{2,1}$ and $Y^{3,1}$ satisfy under $\mathbb{Q}^1$ (cf. Proposition 4.3.1)

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\,d\widehat{W}_t, \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\,d\widehat{W}_t - d\widehat{M}_t\big). \end{aligned}$$

It is natural to postulate that the driving Brownian noise is two-dimensional. In such a case, we may represent the joint dynamics of $Y^{2,1}$ and $Y^{3,1}$ under $\mathbb{Q}^1$ as follows

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\,dW_t^1, \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\,dW_t^2 - d\widehat{M}_t\big), \end{aligned}$$

where W^1, W^2 are one-dimensional Brownian motions under $\mathbb{Q}^1$, such that $d\langle W^1, W^2\rangle_t = \rho_t\,dt$ for a deterministic instantaneous correlation coefficient ρ taking values in $[-1,1]$.

We assume from now on that the volatilities σ_i, $i = 1,2,3$ are deterministic. Let us set

$$\alpha_t = \langle \ln \widetilde{Y}^{2,1}, \ln \widetilde{Y}^{3,1}\rangle_t = \int_0^t \rho_u(\sigma_{2,u} - \sigma_{1,u})(\sigma_{3,u} - \sigma_{1,u})\,du, \tag{4.45}$$

and let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega, \mathcal{G}_T)$ such that the process $\widehat{Y}_t = Y_t^{2,1}e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale. To clarify the financial interpretation of the auxiliary process $\widehat{Y}$ in the present context, we introduce the concept of credit-risk-adjusted forward price relative to the numéraire Y^1.

Definition 4.3.3 *Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the time-t credit-risk-adjusted Y^1-forward price of Y if the pre-default value at time t of a vulnerable forward contract, represented by the claim*

$$\mathbb{1}_{\{T<\tau\}}(Y_T^1)^{-1}(Y - KY_T^1) = \mathbb{1}_{\{T<\tau\}}(Y(Y_T^1)^{-1} - K),$$

equals 0.

The credit-risk-adjusted Y^1-forward price of Y is denoted by $\widehat{F}_{Y|Y^1}(t,T)$, and it is also interpreted as an abstract defaultable swap rate. The following auxiliary results are easy to establish, along the same lines as Lemmas 4.3.2 and 4.3.3.

Lemma 4.3.4 *The credit-risk-adjusted Y^1-forward price of a survival claim $Y = (X, 0, \tau)$ equals*

$$\widehat{F}_{Y|Y^1}(t,T) = \widetilde{\pi}_t(X^1, 0, \tau)(\widetilde{D}(t,T))^{-1}$$

where $X^1 = X(Y_T^1)^{-1}$ is the price of X in the numéraire Y^1, and $\widetilde{\pi}_t(X^1, 0, \tau)$ is the pre-default value of a survival claim with the promised payoff X^1.

Proof. It suffices to note that for $Y = \mathbb{1}_{\{T<\tau\}}X$, we have

$$\mathbb{1}_{\{T<\tau\}}(Y(Y^1_T)^{-1} - K) = \mathbb{1}_{\{T<\tau\}}X^1 - KD(T,T),$$

where $X^1 = X(Y^1_T)^{-1}$, and to consider the pre-default values. □

Lemma 4.3.5 *The credit-risk-adjusted Y^1-forward price of the asset Y^2 equals*

$$\widehat{F}_{Y^2|Y^1}(t,T) = Y^{2,1}_t\, e^{\alpha_T-\alpha_t} = \widehat{Y}_t e^{\alpha_T}, \tag{4.46}$$

where α, assumed to be deterministic, is given by (4.45).

Proof. It suffices to find an $\mathcal{F}_t$-measurable random variable K for which

$$\widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y^2_T(Y^1_T)^{-1} - K \,\big|\, \mathcal{F}_t\big) = 0.$$

Consequently, $K = \widehat{F}_{Y^2|Y^1}(t,T)$, where

$$\widehat{F}_{Y^2|Y^1}(t,T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y^{2,1}_T \,\big|\, \mathcal{F}_t\big) = Y^{2,1}_t\, e^{\alpha_T-\alpha_t} = \widehat{Y}_t\, e^{\alpha_T},$$

where we have used the facts that $\widehat{Y}_t = Y^{2,1}_t e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale, and α is deterministic. □

We are in a position to examine a vulnerable option to exchange default-free assets with the payoff

$$C^d_T = \mathbb{1}_{\{T<\tau\}}(Y^1_T)^{-1}(Y^2_T - KY^1_T)^+ = \mathbb{1}_{\{T<\tau\}}(Y^{2,1}_T - K)^+. \tag{4.47}$$

The last expression shows that the option can be interpreted as a vulnerable swaption associated with the assets Y^1 and Y^2. It is useful to observe that

$$\frac{C^d_T}{Y^1_T} = \frac{\mathbb{1}_{\{T<\tau\}}}{Y^1_T}\left(\frac{Y^2_T}{Y^1_T} - K\right)^+,$$

so that, when expressed in the numéraire Y^1, the payoff becomes

$$C^{1,d}_T = D^1(T,T)(Y^{2,1}_T - K)^+,$$

where $C^{1,d}_t = C^d_t(Y^1_t)^{-1}$ and $D^1(t,T) = D(t,T)(Y^1_t)^{-1}$ stand for the prices relative to Y^1.

It is clear that we deal here with a model analogous to the model examined in previous subsections in which, however, all prices are now relative to the numéraire Y^1. This observation allows us to directly derive the valuation formula from Proposition 4.3.2.

Proposition 4.3.3 *Assume that the volatilities are deterministic. The credit-risk-adjusted Y^1-forward price of a vulnerable call option written with the payoff given by* (4.47) *equals*

$$\widehat{F}_{C^d|Y^1}(t,T) = \widehat{F}_{Y^2|Y^1}(t,T)N\big(d_+(\widehat{F}_{Y^2|Y^1}(t,T),t,T)\big) - KN\big(d_-(\widehat{F}_{Y^2|Y^1}(t,T),t,T)\big)$$

where

$$d_\pm(z,t,T) = \frac{\ln z - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{2,u} - \sigma_{1,u})^2\,du.$$

The replicating strategy ϕ in the spot market satisfies for every $t\in[0,T]$, on the set $\{t<\tau\}$,

$$\phi^1_t Y^1_t = -\phi^2_t Y^2_t, \quad \phi^2_t = \widetilde{D}(t,T)(Y^1_t)^{-1}N(d_+(t,T))e^{\alpha_T-\alpha_t}, \quad \phi^3_t\widetilde{D}(t,T) = \widetilde{C}^d_t,$$

where $d_+(t,T) = d_+\big(\widehat{F}_{Y^2}(t,T),t,T\big)$.

Proof. The proof is analogous to that of Proposition 4.3.2, and thus it is omitted. □

It is worth noting that the payoff (4.47) was judiciously chosen. Suppose instead that the option payoff is not defined by (4.47), but it is given by an apparently simpler expression

$$C^d_T = \mathbb{1}_{\{T<\tau\}}(Y^2_T - KY^1_T)^+. \tag{4.48}$$

Since the payoff C^d_T can be represented as follows

$$C^d_T = \widehat{G}(Y^1_T, Y^2_T, Y^3_T) = Y^3_T(Y^2_T - KY^1_T)^+,$$

where $\widehat{G}(y_1, y_2, y_3) = y_3(y_2 - Ky_1)^+$, the option can be seen an option to exchange the second asset for K units of the first asset, but with the payoff expressed in units of the defaultable asset. When expressed in relative prices, the payoff becomes

$$C^{1,d}_T = \mathbb{1}_{\{T<\tau\}}(Y^{2,1}_T - K)^+.$$

where $\mathbb{1}_{\{T<\tau\}} = D^1(T,T)Y^1_T$. It is thus rather clear that it is not longer possible to apply the same method as in the proof of Proposition 4.3.2.

4.3.2 Defaultable Asset with Non-Zero Recovery

We now assume that

$$dY^3_t = Y^3_{t-}(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y^3_0 > 0$, so that $Y^3_t > 0$ for every $t \in \mathbb{R}_+$. We shall briefly describe the same steps as in the case of a defaultable asset with total default.

Arbitrage-Free Property

As usual, we need first to impose specific constraints on model coefficients, so that the model is arbitrage-free. Indeed, an EMM $\mathbb{Q}^1$ exists if there exists a pair (θ, ζ) such that

$$\theta_t(\sigma_i - \sigma_1) + \zeta_t\xi_t\frac{\kappa_i - \kappa_1}{1+\kappa_1} = \mu_1 - \mu_i + \sigma_1(\sigma_i - \sigma_1) + \xi_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1+\kappa_1}, \quad i = 2, 3.$$

To ensure the existence of a solution (θ, ζ) on the set $\tau < t$, we impose the condition

$$\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},$$

that is,

$$\mu_1(\sigma_3 - \sigma_2) + \mu_2(\sigma_1 - \sigma_3) + \mu_3(\sigma_2 - \sigma_1) = 0.$$

Now, on the set $\tau \geq t$, we have to solve the two equations

$$\begin{aligned}\theta_t(\sigma_2 - \sigma_1) &= \mu_1 - \mu_2 + \sigma_1(\sigma_2 - \sigma_1),\\ \theta_t(\sigma_3 - \sigma_1) + \zeta_t\gamma\kappa_3 &= \mu_1 - \mu_3 + \sigma_1(\sigma_3 - \sigma_1).\end{aligned}$$

If, in addition, $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$, we obtain the unique solution

$$\begin{aligned}&\theta = \sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},\\ &\zeta = 0 > -1,\end{aligned}$$

so that the martingale measure $\mathbb{Q}^1$ exists and is unique.

4.3.3 Two Defaultable Assets with Total Default

We shall now assume that we have only two assets, and both are defaultable assets with total default. This case is also examined by Carr [27], who studies some imperfect hedging of digital options. Note that here we present results for perfect hedging.

We shall briefly outline the analysis of hedging of a survival claim. Under the present assumptions, we have, for $i = 1, 2$,

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t - dM_t\big), \tag{4.49}$$

where W is a one-dimensional Brownian motion, so that

$$Y_t^1 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_{i,t} + \gamma_t)\, dt + \sigma_{i,t}\, dW_t\big). \tag{4.50}$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies, for every $t \in [0, T]$,

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2\, d\widetilde{Y}_u^{2,1} \right),$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2 / \widetilde{Y}_t^1$. Since both primary traded assets are subject to total default, it is clear that the present model is incomplete, in the sense, that not all defaultable claims can be replicated. We shall check in the following subsection that, under the assumption that the driving Brownian motion W is one-dimensional, all survival claims satisfying natural technical conditions are hedgeable, however. In the more realistic case of a two-dimensional noise, we will still be able to hedge a large class of survival claims, including options on a defaultable asset and options to exchange defaultable assets.

Hedging a Survival Claim

For the sake of expositional simplicity, we assume in this section that the driving Brownian motion W is one-dimensional. This is definitely not the right choice, since we deal here with two risky assets, and thus they will be perfectly correlated. However, this assumption is convenient for the expositional purposes, since it will ensure the model completeness with respect to survival claims, and it will be later relaxed anyway.

We shall argue that in a model with two defaultable assets governed by (4.49), replication of a survival claim $(X, 0, \tau)$ is in fact equivalent to replication of the promised payoff X using the pre-default processes.

Lemma 4.3.6 *If a strategy ϕ^i, $i = 1, 2$, based on pre-default values $\widetilde{Y}^i$, $i = 1, 2$, is a replicating strategy for an $\mathcal{F}_T$-measurable claim X, that is, if ϕ is such that the process $\widetilde{V}_t(\phi) = \phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2$ satisfies, for every $t \in [0, T]$,*

$$\begin{aligned} d\widetilde{V}_t(\phi) &= \phi_t^1\, d\widetilde{Y}_t^1 + \phi_t^2\, d\widetilde{Y}_t^2, \\ \widetilde{V}_T(\phi) &= X, \end{aligned}$$

then for the process $V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2$ we have, for every $t \in [0, T]$,

$$\begin{aligned} dV_t(\phi) &= \phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2, \\ V_T(\phi) &= X \mathbb{1}_{\{T<\tau\}}. \end{aligned}$$

This means that the strategy ϕ replicates the survival claim $(X, 0, \tau)$.

Proof. It is clear that $V_t(\phi) = \mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)$. From

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = -(\phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2) \, dH_t + (1 - H_{t-})(\phi_t^1 \, d\widetilde{Y}_t^1 + \phi_t^2 \, d\widetilde{Y}_t^2),$$

it follows that

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = -\widetilde{V}_t(\phi) \, dH_t + (1 - H_{t-}) d\widetilde{V}_t(\phi),$$

that is,

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = d(\mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)) = dV_t(\phi).$$

It is also obvious that $V_T(\phi) = X \mathbb{1}_{\{T<\tau\}}$. □

Combining the last result with Lemma 4.2.1, we see that a strategy (ϕ^1, ϕ^2) replicates a survival claim $(X, 0, \tau)$ whenever we have

$$\widetilde{Y}_T^1 \Big(x + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1} \Big) = X$$

for some constant x and some $\mathbb{F}$-predictable process ϕ^2, where, in view of (4.50),

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1} \Big(\big(\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}) \big) \, dt + (\sigma_{2,t} - \sigma_{1,t}) \, dW_t \Big).$$

We introduce a probability measure $\widetilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and such that $\widetilde{Y}^{2,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. It is easily seen that the Radon-Nikodým density η satisfies, for $t \in [0, T]$,

$$d\widetilde{\mathbb{Q}} \,|\, _{\mathcal{G}_t} = \eta_t \, d\mathbb{P} \,|\, _{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^{\cdot} \theta_s \, dW_s \right) d\mathbb{P} \,|\, _{\mathcal{G}_t} \tag{4.51}$$

with

$$\theta_t = \frac{\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}},$$

provided, of course, that the process θ is well defined and satisfies suitable integrability conditions. We shall show that a survival claim is attainable if the random variable $X (\widetilde{Y}_T^1)^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable. Indeed, the pre-default value $\widetilde{V}_t$ at time t of a survival claim equals

$$\widetilde{V}_t = \widetilde{Y}_t^1 \, \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X (\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big),$$

and from the predictable representation theorem, we deduce that there exists a process ϕ^2 such that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X (\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big) = \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X (\widetilde{Y}_T^1)^{-1} \big) + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1}.$$

The component ϕ^1 of the self-financing trading strategy $\phi = (\phi^1, \phi^2)$ is then chosen in such a way that

$$\phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2 = \widetilde{V}_t, \quad \forall \, t \in [0, T].$$

To conclude, by focusing on pre-default values, we have shown that the replication of survival claims can be reduced here to classic results on replication of (non-defaultable) contingent claims in a default-free market model.

Option on a Defaultable Asset

In order to get a complete model with respect to survival claims, we postulated in the previous section that the driving Brownian motion in dynamics (4.49) is one-dimensional. This assumption is questionable, since it implies the perfect correlation of risky assets. However, we may relax this restriction, and work instead with the two correlated one-dimensional Brownian motions. The model will no longer be complete, but options on a defaultable assets will be still attainable. The payoff of a (non-vulnerable) call option written on the defaultable asset Y^2 equals

$$C_T = (Y^2_T - K)^+ = \mathbb{1}_{\{T<\tau\}}(\widetilde{Y}^2_T - K)^+,$$

so that it is natural to interpret this contract as a survival claim with the promised payoff $X = (\widetilde{Y}^2_T - K)^+$.

To deal with this option in an efficient way, we consider a model in which

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW^i_t - dM_t\big), \tag{4.52}$$

where W^1 and W^2 are two one-dimensional correlated Brownian motions with the instantaneous correlation coefficient ρ_t. More specifically, we assume that $Y^1_t = D(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}(t,T)$ represents a defaultable ZCB with zero recovery, and $Y^2_t = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}^2_t$ is a generic defaultable asset with total default. Within the present set-up, the payoff can also be represented as follows

$$C_T = G(Y^1_T, Y^2_T) = (Y^2_T - KY^1_T)^+,$$

where $g(y_1,y_2) = (y_2 - Ky_1)^+$, and thus it can also be seen as an option to exchange the second asset for K units of the first asset.

The requirement that the process $\widetilde{Y}^{2,1}_t = \widetilde{Y}^2_t(\widetilde{Y}^1_t)^{-1}$ follows an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$ implies that

$$d\widetilde{Y}^{2,1}_t = \widetilde{Y}^{2,1}_t\big((\sigma_{2,t}\rho_t - \sigma_{1,t})\,d\widetilde{W}^1_t + \sigma_{2,t}\sqrt{1-\rho^2_t}\,d\widetilde{W}^2_t\big), \tag{4.53}$$

where $\widetilde{W} = (\widetilde{W}^1, \widetilde{W}^2)$ follows a two-dimensional Brownian motion under $\widetilde{\mathbb{Q}}$. Since $\widetilde{Y}^1_T = 1$, replication of the option reduces to finding a constant x and an $\mathbb{F}$-predictable process ϕ^2 satisfying

$$x + \int_0^T \phi^2_t\,d\widetilde{Y}^{2,1}_t = (\widetilde{Y}^2_T - K)^+.$$

To obtain closed-form expressions for the option price and replicating strategy, we postulate that the volatilities $\sigma_{1,t}, \sigma_{2,t}$ and the correlation coefficient ρ_t are deterministic. Let $\widehat{F}_{Y^2}(t,T) = \widetilde{Y}^2_t(\widetilde{D}(t,T))^{-1}$ ($\widehat{F}_C(t,T) = \widetilde{C}_t(\widetilde{D}(t,T))^{-1}$, respectively) stand for the credit-risk-adjusted forward price of the second asset (the option, respectively). The proof of the following valuation result is fairly standard, and thus it is omitted.

Proposition 4.3.4 *Assume that the volatilities are deterministic and that Y^1 is a DZC. The credit-risk-adjusted forward price of the option written on Y^2 equals*

$$\widehat{F}_C(t,T) = \widehat{F}_{Y^2}(t,T)N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

Equivalently, the pre-default price of the option equals

$$\widetilde{C}_t = \widetilde{Y}^2_t N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - K\widetilde{D}(t,T)N\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big),$$

where

$$d_\pm(z,t,T) = \frac{\ln zf - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma^2_{1,u} + \sigma^2_{2,u} - 2\rho_u\sigma_{1,u}\sigma_{2,u})\,du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi^1_t = -KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big), \quad \phi^2_t = N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

4.4 PDE Approach to Valuation and Hedging

In the remaining part of this chapter, in which we follow Bielecki et al. [7] (see also Rutkowski and Yousiph [80]), we shall take a different perspective. We assume that trading occurs on the time interval $[0,T]$ and our goal is to replicate a contingent claim of the form

$$Y = \mathbb{1}_{\{T\geq\tau\}} g_1(Y_T^1, Y_T^2, Y_T^3) + \mathbb{1}_{\{T<\tau\}} g_0(Y_T^1, Y_T^2, Y_T^3) = G(Y_T^1, Y_T^2, Y_T^3, H_T),$$

which settles at time T. We do not need to assume here that the coefficients in dynamics of primary assets are $\mathbb{F}$-predictable. Since our goal is to develop the PDE approach, it will be essential, however, to postulate a Markovian character of a model. For the sake of simplicity, we assume that the coefficients are constant, so that

$$dY_t^i = Y_{t-}^i \big(\mu_i\, dt + \sigma_i\, dW_t + \kappa_i\, dM_t\big), \quad i = 1, 2, 3.$$

The assumption of constancy of coefficients is rarely, if ever, satisfied in practically relevant models of credit risk. It is thus important to note that it was postulated here mainly for the sake of notational convenience, and the general results established in this section can be easily extended to a non-homogeneous Markov case in which $\mu_{i,t} = \mu_i(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, H_{t-}), \sigma_{i,t} = \sigma_i(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, H_{t-})$, etc.

4.4.1 Defaultable Asset with Total Default

We first assume that Y^1 and Y^2 are default-free, so that $\kappa_1 = \kappa_2 = 0$, and the third asset is subject to total default, i.e. $\kappa_3 = -1$,

$$dY_t^3 = Y_{t-}^3 \big(\mu_3\, dt + \sigma_3\, dW_t - dM_t\big).$$

We work throughout under the assumptions of Proposition 4.3.1. This means that any $\mathbb{Q}^1$-integrable contingent claim $Y = G(Y_T^1, Y_T^2, Y_T^3; H_T)$ is attainable, and its arbitrage price equals

$$\pi_t(Y) = Y_t^1\, \mathbb{E}_{\mathbb{Q}^1}(Y (Y_T^1)^{-1} \,|\, \mathcal{G}_t), \quad \forall\, t \in [0,T]. \tag{4.54}$$

The following auxiliary result is thus rather obvious.

Lemma 4.4.1 *The process (Y^1, Y^2, Y^3, H) has the Markov property with respect to the filtration $\mathbb{G}$ under the martingale measure $\mathbb{Q}^1$. For any attainable claim $Y = G(Y_T^1, Y_T^2, Y_T^3; H_T)$ there exists a function $v : [0,T] \times \mathbb{R}^3 \times \{0,1\} \to \mathbb{R}$ such that $\pi_t(Y) = v(t, Y_t^1, Y_t^2, Y_t^3; H_t)$.*

We find it convenient to introduce the *pre-default* pricing function $v(\cdot\,;0) = v(t, y_1, y_2, y_3; 0)$ and the *post-default* pricing function $v(\cdot\,;1) = v(t, y_1, y_2, y_3; 1)$. In fact, since $Y_t^3 = 0$ if $H_t = 1$, it suffices to study the post-default function $v(t, y_1, y_2; 1) = v(t, y_1, y_2, 0; 1)$. Also, we write

$$\alpha_i = \mu_i - \sigma_i \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}, \quad b = (\mu_3 - \mu_1)(\sigma_1 - \sigma_2) - (\mu_1 - \mu_3)(\sigma_1 - \sigma_3).$$

Let $\gamma > 0$ be the constant default intensity under $\mathbb{P}$, and let $\zeta > -1$ be given by formula (4.28).

Proposition 4.4.1 *Assume that the functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ belong to the class $\mathrm{C}^{1,2}([0,T] \times \mathbb{R}_+^3, \mathbb{R})$. Then $v(t, y_1, y_2, y_3; 0)$ satisfies the PDE*

$$\begin{aligned}\partial_t v(\cdot\,;0) &+ \sum_{i=1}^{2} \alpha_i y_i \partial_i v(\cdot\,;0) + (\alpha_3 + \zeta) y_3 \partial_3 v(\cdot\,;0) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot\,;0) \\ &- \alpha_1 v(\cdot\,;0) + \left(\gamma - \frac{b}{\sigma_1 - \sigma_2}\right) \big[v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0)\big] = 0\end{aligned}$$

subject to the terminal condition $v(T,y_1,y_2,y_3;0)=G(y_1,y_2,y_3;0)$, *and* $v(t,y_1,y_2;1)$ *satisfies the PDE*

$$\partial_t v(\cdot\,;1)+\sum_{i=1}^{2}\alpha_i y_i\partial_i v(\cdot\,;1)+\frac{1}{2}\sum_{i,j=1}^{2}\sigma_i\sigma_j y_i y_j\partial_{ij}v(\cdot\,;1)-\alpha_1 v(\cdot\,;1)=0$$

subject to the terminal condition $v(T,y_1,y_2;1)=G(y_1,y_2,0;1)$.

Proof. For simplicity, we write $C_t=\pi_t(Y)$. Let us define

$$\Delta v(t,y_1,y_2,y_3)=v(t,y_1,y_2;1)-v(t,y_1,y_2,y_3;0).$$

Then the jump $\Delta C_t=C_t-C_{t-}$ can be represented as follows:

$$\Delta C_t=\mathbb{1}_{\{\tau=t\}}\big(v(t,Y_t^1,Y_t^2;1)-v(t,Y_t^1,Y_t^2,Y_{t-}^3;0)\big)=\mathbb{1}_{\{\tau=t\}}\,\Delta v(t,Y_t^1,Y_t^2,Y_{t-}^3).$$

We write ∂_i to denote the partial derivative with respect to the variable y_i, and we typically omit the variables $(t,Y_{t-}^1,Y_{t-}^2,Y_{t-}^3,H_{t-})$ in expressions $\partial_t v$, $\partial_i v$, Δv, etc. We shall also make use of the fact that for any Borel measurable function g we have

$$\int_0^t g(u,Y_u^2,Y_{u-}^3)\,du=\int_0^t g(u,Y_u^2,Y_u^3)\,du$$

since Y_u^3 and Y_{u-}^3 differ only for at most one value of u (for each ω). Let $\xi_t=\mathbb{1}_{\{t<\tau\}}\gamma$. An application of Itô's formula yields

$$\begin{aligned}
dC_t &= \partial_t v\,dt+\sum_{i=1}^{3}\partial_i v\,dY_t^i+\frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y_{t-}^i Y_{t-}^j\partial_{ij}v\,dt\\
&\quad+\Big(\Delta v+Y_{t-}^3\partial_3 v\Big)\,dH_t\\
&= \partial_t v\,dt+\sum_{i=1}^{3}\partial_i v\,dY_t^i+\frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y_{t-}^i Y_{t-}^j\partial_{ij}v\,dt\\
&\quad+\Big(\Delta v+Y_{t-}^3\partial_3 v\Big)\big(dM_t+\xi_t\,dt\big),
\end{aligned}$$

and this in turn implies that

$$\begin{aligned}
dC_t &= \partial_t v\,dt+\sum_{i=1}^{3}Y_{t-}^i\partial_i v\big(\mu_i\,dt+\sigma_i\,dW_t\big)+\frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y_{t-}^i Y_{t-}^j\partial_{ij}v\,dt\\
&\quad+\Delta v\,dM_t+\Big(\Delta v+Y_{t-}^3\partial_3 v\Big)\xi_t\,dt\\
&= \left\{\partial_t v+\sum_{i=1}^{3}\mu_i Y_{t-}^i\partial_i v+\frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y_{t-}^i Y_{t-}^j\partial_{ij}v+\Big(\Delta v+Y_{t-}^3\partial_3 v\Big)\xi_t\right\}dt\\
&\quad+\Big(\sum_{i=1}^{3}\sigma_i Y_{t-}^i\partial_i v\Big)\,dW_t+\Delta v\,dM_t.
\end{aligned}$$

We now use the integration by parts formula together with (4.22) to derive dynamics of the relative price $\widehat{C}_t=C_t(Y_t^1)^{-1}$. We find that

$$\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\Big((-\mu_1+\sigma_1^2)\,dt-\sigma_1\,dW_t\Big)\\
&\quad+(Y_{t-}^1)^{-1}\left\{\partial_t v+\sum_{i=1}^{3}\mu_i Y_{t-}^i\partial_i v+\frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y_{t-}^i Y_{t-}^j\partial_{ij}v+\Big(\Delta v+Y_{t-}^3\partial_3 v\Big)\xi_t\right\}dt\\
&\quad+(Y_{t-}^1)^{-1}\sum_{i=1}^{3}\sigma_i Y_{t-}^i\partial_i v\,dW_t+(Y_{t-}^1)^{-1}\Delta v\,dM_t-(Y_{t-}^1)^{-1}\sigma_1\sum_{i=1}^{3}\sigma_i Y_{t-}^i\partial_i v\,dt.
\end{aligned}$$

Hence, using (4.27), we obtain

$$\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\big(-\mu_1 + \sigma_1^2\big)\, dt + \widehat{C}_{t-}\Big(-\sigma_1\, d\widehat{W}_t - \sigma_1\theta\, dt\Big) \\
&+ (Y^1_{t-})^{-1}\left\{ \partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t \right\} dt \\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\, d\widehat{W}_t + (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v\, dt \\
&+ (Y^1_{t-})^{-1}\Delta v\, d\widehat{M}_t + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v\, dt - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma^i Y^i_{t-}\partial_i v\, dt.
\end{aligned}$$

This means that the process $\widehat{C}$ admits the following decomposition under $\mathbb{Q}^1$

$$\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\big(-\mu_1 + \sigma_1^2 - \sigma_1\theta\big)\, dt \\
&+ (Y^1_{t-})^{-1}\left\{ \partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t \right\} dt \\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v\, dt + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v\, dt \\
&- (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\, dt + \text{a } \mathbb{Q}^1\text{-martingale.}
\end{aligned}$$

From (4.54), it follows that the process $\widehat{C}$ is a martingale under $\mathbb{Q}^1$. Therefore, the continuous finite variation part in the above decomposition necessarily vanishes, and thus we get

$$\begin{aligned}
0 &= C_{t-}(Y^1_{t-})^{-1}\big(-\mu_1 + \sigma_1^2 - \sigma_1\theta\big) \\
&+ (Y^1_{t-})^{-1}\left\{ \partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \big(\Delta v + Y^3_{t-}\partial_3 v\big)\,\xi_t \right\} \\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v.
\end{aligned}$$

Consequently, we have that

$$\begin{aligned}
0 &= C_{t-}\big(-\mu_1 + \sigma_1^2 - \sigma_1\theta\big) \\
&+ \partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \big(\Delta v + Y^3_{t-}\partial_3 v\big)\,\xi_t \\
&+ \sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v + \zeta\xi_t\Delta v - \sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v.
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
&\partial_t v + \sum_{i=1}^2 \alpha_i Y^i_{t-}\partial_i v + (\alpha_3 + \xi_t)\, Y^3_{t-}\partial_3 v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v \\
&\quad - \alpha_1 C_{t-} + (1+\zeta)\xi_t\Delta v = 0.
\end{aligned}$$

Recall that $\xi_t = \mathbb{1}_{\{t<\tau\}}\gamma$. It is thus clear that the pricing functions $v(\cdot,0)$ and $v(\cdot;1)$ satisfy the PDEs given in the statement of the proposition. □

The next result deals with a replicating strategy for Y.

Proposition 4.4.2 *The replicating strategy ϕ for the claim Y is given by formulae*

$$\begin{aligned}
\phi^3_t Y^3_{t-} &= -\Delta v(t, Y^1_t, Y^2_t, Y^3_{t-}) = v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0) - v(t, Y^1_t, Y^2_t; 1),\\
\phi^2_t Y^2_t(\sigma_2 - \sigma_1) &= -(\sigma_1 - \sigma_3)\Delta v - \sigma_1 v + \sum_{i=1}^{3} Y^i_{t-}\sigma_i \partial_i v,\\
\phi^1_t Y^1_t &= v - \phi^2_t Y^2_t - \phi^3_t Y^3_t.
\end{aligned}$$

Proof. As a by-product of our computations, we obtain

$$d\widehat{C}_t = -(Y^1_t)^{-1}\sigma_1 v\, d\widehat{W}_t + (Y^1_t)^{-1}\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i v\, d\widehat{W}_t + (Y^1_t)^{-1}\Delta v\, d\widehat{M}_t.$$

The self-financing strategy that replicates Y is determined by two components ϕ^2, ϕ^3 and the following relationship:

$$d\widehat{C}_t = \phi^2_t\, dY^{2,1}_t + \phi^3_t\, dY^{3,1}_t = \phi^2_t Y^{2,1}_t(\sigma_2 - \sigma_1)\, d\widehat{W}_t + \phi^3_t Y^{3,1}_{t-}\left((\sigma_3 - \sigma_1)\, d\widehat{W}_t - d\widehat{M}_t\right).$$

By identification, we obtain $\phi^3_t Y^{3,1}_{t-} = (Y^1_t)^{-1}\Delta v$ and

$$\phi^2_t Y^2_t(\sigma_2 - \sigma_1) - (\sigma_3 - \sigma_1)\Delta v = -\sigma_1 C_t + \sum_{i=1}^{3} Y^i_{t-}\sigma_i\partial_i v.$$

This yields the claimed formulae. □

Corollary 4.4.1 *In the case of a total default claim, the hedging strategy satisfies the balance condition.*

Proof. A total default corresponds to the assumption that $G(y_1, y_2, y_3, 1) = 0$. We now have $v(t, y_1, y_2; 1) = 0$, and thus $\phi^3_t Y^3_{t-} = v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0)$ for every $t \in [0, T]$. Hence, the equality $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$ holds for every $t \in [0, T]$. The last equality is the balance condition for $Z = 0$. Recall that it ensures that the wealth of a replicating portfolio jumps to zero at default time. □

Hedging with the Savings Account

Let us now study the particular case where Y^1 is the savings account, i.e.,

$$dY^1_t = rY^1_t\, dt, \quad Y^1_0 = 1,$$

which corresponds to $\mu_1 = r$ and $\sigma_1 = 0$. Let us write $\widehat{r} = r + \widehat{\gamma}$, where

$$\widehat{\gamma} = \gamma(1 + \zeta) = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r - \mu_2)$$

stands for the intensity of default under $\mathbb{Q}^1$. The quantity $\widehat{r}$ has a natural interpretation as the risk-neutral *credit-risk adjusted* short-term interest rate. Straightforward calculations yield the following corollary to Proposition 4.4.1.

Corollary 4.4.2 *Assume that $\sigma_2 \neq 0$ and*

$$\begin{aligned}
dY^1_t &= rY^1_t\, dt,\\
dY^2_t &= Y^2_t\big(\mu_2\, dt + \sigma_2\, dW_t\big),\\
dY^3_t &= Y^3_{t-}\big(\mu_3\, dt + \sigma_3\, dW_t - dM_t\big).
\end{aligned}$$

Then the function $v(\cdot\,;0)$ satisfies

$$\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + \widehat{r}y_3\partial_3 v(t,y_2,y_3;0) - \widehat{r}v(t,y_2,y_3;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) + \widehat{\gamma}v(t,y_2;1) = 0$$

with $v(T,y_2,y_3;0) = G(y_2,y_3;0)$, and the function $v(\cdot\,;1)$ satisfies

$$\partial_t v(t,y_2;1) + ry_2\partial_2 v(t,y_2;1) + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} v(t,y_2;1) - rv(t,y_2;1) = 0$$

with $v(T,y_2;1) = G(y_2,0;1)$.

In the special case of a survival claim, the function $v(\cdot\,;1)$ vanishes identically, and thus the following result can be easily established.

Corollary 4.4.3 *The pre-default pricing function $v(\cdot\,;0)$ of a survival claim $Y = \mathbb{1}_{\{T<\tau\}}G(Y_T^2, Y_T^3)$ is a solution of the following PDE:*

$$\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + \widehat{r}y_3\partial_3 v(t,y_2,y_3;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) - \widehat{r}v(t,y_2,y_3;0) = 0$$

with the terminal condition $v(T,y_2,y_3;0) = G(y_2,y_3)$. The components ϕ^2 and ϕ^3 of the replicating strategy satisfy

$$\phi_t^2\sigma_2 Y_t^2 = \sum_{i=2}^{3}\sigma_i Y_{t-}^i \partial_i v(t, Y_t^2, Y_{t-}^3;0) + \sigma_3 v(t, Y_t^2, Y_{t-}^3;0),$$
$$\phi_t^3 Y_{t-}^3 = v(t, Y_t^2, Y_{t-}^3;0).$$

Example 4.4.1 Consider a survival claim $Y = \mathbb{1}_{\{T<\tau\}}g(Y_T^2)$, that is, a vulnerable claim with default-free underlying asset. Its pre-default pricing function $v(\cdot\,;0)$ does not depend on y_3, and satisfies the PDE (y stands here for y_2 and σ for σ_2)

$$\partial_t v(t,y;0) + ry\partial_2 v(t,y;0) + \frac{1}{2}\sigma^2 y^2 \partial_{22} v(t,y;0) - \widehat{r}v(t,y;0) = 0 \qquad (4.55)$$

with the terminal condition $v(T,y;0) = \mathbb{1}_{\{t<\tau\}}g(y)$. The solution to (4.55) is

$$v(t,y) = e^{(\widehat{r}-r)(t-T)}\, v^{r,g,2}(t,y) = e^{\widehat{\gamma}(t-T)}\, v^{r,g,2}(t,y),$$

where the function $v^{r,g,2}$ is the Black-Scholes price of $g(Y_T)$ in a Black-Scholes model for Y_t with interest rate r and volatility σ_2.

4.4.2 Defaultable Asset with Non-Zero Recovery

We now assume that

$$dY_t^3 = Y_{t-}^3(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y_0^3 > 0$, so that $Y_t^3 > 0$ for every $t \in \mathbb{R}_+$. We shall briefly describe the same steps as in the case of a defaultable asset with total default.

Pricing PDE and Replicating Strategy

We are in a position to derive the pricing PDEs. For the sake of simplicity, we assume that Y^1 is the savings account, so that Proposition 4.4.3 is a counterpart of Corollary 4.4.2. For the proof of Proposition 4.4.3, the interested reader is referred to Bielecki et al. [7].

Proposition 4.4.3 *Let $\sigma_2 \neq 0$ and let Y^1, Y^2, Y^3 satisfy*

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t\big). \end{aligned}$$

Assume, in addition, that $\sigma_2(r-\mu_3) = \sigma_3(r-\mu_2)$ and $\kappa_3 \neq 0$, $\kappa_3 > -1$. Then the price of a contingent claim $Y = G(Y_T^2, Y_T^3, H_T)$ can be represented as $\pi_t(Y) = v(t, Y_t^2, Y_t^3, H_t)$, where the pricing functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ satisfy the following PDEs

$$\begin{aligned} &\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v(t,y_2,y_3;0) - rv(t,y_2,y_3;0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) + \gamma\big(v(t,y_2,y_3(1+\kappa_3);1) - v(t,y_2,y_3;0)\big) = 0 \end{aligned}$$

and

$$\begin{aligned} &\partial_t v(t,y_2,y_3;1) + ry_2\partial_2 v(t,y_2,y_3;1) + ry_3\partial_3 v(t,y_2,y_3;1) - rv(t,y_2,y_3;1) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;1) = 0 \end{aligned}$$

subject to the terminal conditions

$$v(T,y_2,y_3;0) = G(y_2,y_3;0), \quad v(T,y_2,y_3;1) = G(y_2,y_3;1).$$

The replicating strategy ϕ equals

$$\begin{aligned} \phi_t^2 &= \frac{1}{\sigma_2 Y_t^2}\sum_{i=2}^{3}\sigma_i y_i \partial_i v(t, Y_t^2, Y_{t-}^3, H_{t-}) \\ &\quad - \frac{\sigma_3}{\sigma_2\kappa_3 Y_t^2}\big(v(t, Y_t^2, Y_{t-}^3(1+\kappa_3);1) - v(t, Y_t^2, Y_{t-}^3;0)\big), \\ \phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3}\big(v(t, Y_t^2, Y_{t-}^3(1+\kappa_3);1) - v(t, Y_t^2, Y_{t-}^3;0)\big), \end{aligned}$$

and ϕ_t^1 is given by $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = C_t$.

Hedging of a Survival Claim

We shall illustrate Proposition 4.4.3 by means of examples. First, consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}}\, g(Y_T^3).$$

Then the post-default pricing function $v^g(\cdot\,;1)$ vanishes identically, and the pre-default pricing function $v^g(\cdot\,;0)$ solves the PDE

$$\begin{aligned} &\partial_t v^g(\cdot\,;0) + ry_2\partial_2 v^g(\cdot\,;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v^g(\cdot\,;0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v^g(\cdot\,;0) - (r+\gamma)v^g(\cdot\,;0) = 0 \end{aligned}$$

with the terminal condition $v^g(T, y_2, y_3; 0) = g(y_3)$. Denote $\alpha = r - \kappa_3\gamma$ and $\beta = \gamma(1 + \kappa_3)$.

It is not difficult to check that $v^g(t, y_2, y_3; 0) = e^{\beta(T-t)} v^{\alpha,g,3}(t, y_3)$ is a solution of the above equation, where the function $w(t, y) = v^{\alpha,g,3}(t, y)$ is the solution of the standard Black-Scholes PDE equation

$$\partial_t w + y\alpha\partial_y w + \frac{1}{2}\sigma_3^2 y^2 \partial_{yy} w - \alpha w = 0$$

with the terminal condition $w(T, y) = g(y)$, that is, the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Let C_t be the current value of the contingent claim Y, so that

$$C_t = \mathbb{1}_{\{t<\tau\}} e^{\beta(T-t)} v^{\alpha,g,3}(t, Y_t^3).$$

The hedging strategy of the survival claim is, on the event $\{t < \tau\}$,

$$\begin{aligned} \phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} v^{\alpha,g,3}(t, Y_t^3) = -\frac{1}{\kappa_3} C_t, \\ \phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2}\left(Y_t^3 e^{-\beta(T-t)} \partial_y v^{\alpha,g,3}(t, Y_t^3) - \phi_t^3 Y_t^3\right). \end{aligned}$$

Hedging of a Recovery Payoff

As another illustration of Proposition 4.4.3, we shall now consider the contingent claim $G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T\geq\tau\}} g(Y_T^2)$, that is, we assume that recovery is paid at maturity and equals $g(Y_T^2)$. Let v^g be the pricing function of this claim. The post-default pricing function $v^g(\cdot\,; 1)$ does not depend on y_3. Indeed, the equation (we write here $y_2 = y$)

$$\partial_t v^g(\cdot\,; 1) + ry\partial_y v^g(\cdot\,; 1) + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy} v^g(\cdot\,; 1) - rv^g(\cdot\,; 1) = 0,$$

with $v^g(T, y; 1) = g(y)$, admits a unique solution $v^{r,g,2}$, which is the price of $g(Y_T)$ in the Black-Scholes model with interest rate r and volatility σ_2.

Prior to default, the price of the claim can be found by solving the following PDE

$$\begin{aligned} &\partial_t v^g(\cdot; 0) + ry_2\partial_2 v^g(\cdot; 0) + y_3\left(r - \kappa_3\gamma\right)\partial_3 v^g(\cdot; 0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} v^g(\cdot; 0) - (r + \gamma)v^g(\cdot; 0) = -\gamma v^g(t, y_2; 1) \end{aligned}$$

with $v^g(T, y_2, y_3; 0) = 0$. It is not difficult to check that

$$v^g(t, y_2, y_3; 0) = (1 - e^{\gamma(t-T)}) v^{r,g,2}(t, y_2).$$

The reader can compare this result with the one of Example 4.4.1. e now assume that

$$dY_t^3 = Y_{t-}^3(\mu_3\, dt + \sigma_3\, dW_t + \kappa_3\, dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y_0^3 > 0$, so that $Y_t^3 > 0$ for every $t \in \mathbb{R}_+$. We shall briefly describe the same steps as in the case of a defaultable asset with total default.

Arbitrage-Free Property

As usual, we need first to impose specific constraints on model coefficients, so that the model is arbitrage-free. Indeed, an EMM $\mathbb{Q}^1$ exists if there exists a pair (θ, ζ) such that

$$\theta_t(\sigma_i - \sigma_1) + \zeta_t\xi_t\frac{\kappa_i - \kappa_1}{1 + \kappa_1} = \mu_1 - \mu_i + \sigma_1(\sigma_i - \sigma_1) + \xi_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1 + \kappa_1}, \quad i = 2, 3.$$

To ensure the existence of a solution (θ, ζ) on the set $\tau < t$, we impose the condition

$$\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},$$

that is,

$$\mu_1(\sigma_3 - \sigma_2) + \mu_2(\sigma_1 - \sigma_3) + \mu_3(\sigma_2 - \sigma_1) = 0.$$

Now, on the set $\tau \geq t$, we have to solve the two equations

$$\begin{aligned} \theta_t(\sigma_2 - \sigma_1) &= \mu_1 - \mu_2 + \sigma_1(\sigma_2 - \sigma_1), \\ \theta_t(\sigma_3 - \sigma_1) + \zeta_t \gamma \kappa_3 &= \mu_1 - \mu_3 + \sigma_1(\sigma_3 - \sigma_1). \end{aligned}$$

If, in addition, $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$, we obtain the unique solution

$$\begin{aligned} &\theta = \sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3}, \\ &\zeta = 0 > -1, \end{aligned}$$

so that the martingale measure $\mathbb{Q}^1$ exists and is unique.

4.4.3 Two Defaultable Assets with Total Default

We shall now assume that we have only two assets, and both are defaultable assets with total default. We shall briefly outline the analysis of this case, leaving the details and the study of other relevant cases to the reader. We postulate that

$$dY^i_t = Y^i_{t-}\big(\mu_i\, dt + \sigma_i\, dW_t - dM_t\big), \ i = 1, 2, \tag{4.56}$$

so that

$$Y^1_t = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}^1_t, \quad Y^2_t = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}^2_t,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}^i_t = \widetilde{Y}^i_t\big((\mu_i + \gamma)\, dt + \sigma_i\, dW_t\big), \ i = 1, 2.$$

In the case where the promised payoff X is path-independent, so that

$$X\mathbb{1}_{\{T<\tau\}} = G(Y^1_T, Y^2_T)\mathbb{1}_{\{T<\tau\}} = G(\widetilde{Y}^1_T, \widetilde{Y}^2_T)\mathbb{1}_{\{T<\tau\}}$$

for some function G, it is possible to use the PDE approach in order to value and replicate survival claims prior to default (needless to say that the valuation and hedging after default are trivial here).

We know already from the martingale approach that hedging of a survival claim $X\mathbb{1}_{\{T<\tau\}}$ is formally equivalent to replicating the promised payoff X using the pre-default values of tradeable assets

$$d\widetilde{Y}^i_t = \widetilde{Y}^i_t\big((\mu_i + \gamma)\, dt + \sigma_i\, dW_t\big), \quad i = 1, 2.$$

We need not to worry here about the balance condition, since in case of default the wealth of the portfolio will drop to zero, as it should in view of the equality $Z = 0$.

We shall find the pre-default pricing function $v(t, y_1, y_2)$, which is required to satisfy the terminal condition $v(T, y_1, y_2) = G(y_1, y_2)$, as well as the hedging strategy (ϕ^1, ϕ^2). The replicating strategy ϕ is such that for the pre-default value $\widetilde{C}$ of our claim we have $\widetilde{C}_t := v(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t) = \phi^1_t \widetilde{Y}^1_t + \phi^2_t \widetilde{Y}^2_t$, and

$$d\widetilde{C}_t = \phi^1_t\, d\widetilde{Y}^1_t + \phi^2_t\, d\widetilde{Y}^2_t. \tag{4.57}$$

Proposition 4.4.4 *Assume that $\sigma_1 \neq \sigma_2$. Then the pre-default pricing function v satisfies the PDE*

$$\partial_t v + y_1\left(\mu_1 + \gamma - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right)\partial_1 v + y_2\left(\mu_2 + \gamma - \sigma_2 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right)\partial_2 v$$
$$+ \frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) = \left(\mu_1 + \gamma - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right) v$$

with the terminal condition $v(T, y_1, y_2) = G(y_1, y_2)$.

Proof. We shall merely sketch the proof. By applying Itô's formula to $v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)$, and comparing the diffusion terms in (4.57) and in the Itô differential $dv(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)$, we find that

$$y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v = \phi^1 y_1\sigma_1 + \phi^2 y_2\sigma_2, \tag{4.58}$$

where $\phi^i = \phi^i(t, y_1, y_2)$. Since $\phi^1 y_1 = v(t, y_1, y_2) - \phi^2 y_2$, we deduce from (4.58) that

$$y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v = v\sigma_1 + \phi^2 y_2(\sigma_2 - \sigma_1),$$

and thus

$$\phi^2 y_2 = \frac{y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v - v\sigma_1}{\sigma_2 - \sigma_1}.$$

On the other hand, by identification of drift terms in (4.58), we obtain

$$\partial_t v \quad + \quad y_1(\mu_1 + \gamma)\partial_1 v + y_2(\mu_2 + \gamma)\partial_2 v$$
$$+ \frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) = \phi^1 y_1(\mu_1 + \gamma) + \phi^2 y_2(\mu_2 + \gamma).$$

Upon elimination of ϕ^1 and ϕ^2, we arrive at the stated PDE. □

Recall that the historically observed drift terms are $\widehat{\mu}_i = \mu_i + \gamma$, rather than μ_i. The pricing PDE can thus be simplified as follows:

$$\partial_t v + y_1\left(\widehat{\mu}_1 - \sigma_1 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1}\right)\partial_1 v + y_2\left(\widehat{\mu}_2 - \sigma_2 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1}\right)\partial_2 v$$
$$+ \frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) = v\left(\widehat{\mu}_1 - \sigma_1 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1}\right).$$

The pre-default pricing function v depends on the market observables (drift coefficients, volatilities, and pre-default prices), but not on the (deterministic) default intensity.

To make one more simplifying step, we make an additional assumption about the payoff function. Suppose, in addition, that the payoff function is such that $G(y_1, y_2) = y_1 g(y_2/y_1)$ for some function $g : \mathbb{R}_+ \to \mathbb{R}$ (or equivalently, $G(y_1, y_2) = y_2 h(y_1/y_2)$ for some function $h : \mathbb{R}_+ \to \mathbb{R}$). Then we may focus on relative pre-default prices $\widehat{C}_t = \widetilde{C}_t(\widetilde{Y}_t^1)^{-1}$ and $\widetilde{Y}^{2,1} = \widetilde{Y}_t^2(\widetilde{Y}_t^1)^{-1}$. The corresponding pre-default pricing function $\widehat{v}(t, z)$, such that $\widehat{C}_t = \widehat{v}(t, Y_t^{2,1})$ will satisfy the PDE

$$\partial_t \widehat{v} + \frac{1}{2}(\sigma_2 - \sigma_1)^2 z^2 \partial_{zz}\widehat{v} = 0$$

with terminal condition $\widehat{v}(T, z) = g(z)$. If the price processes Y^1 and Y^2 in (4.49) are driven by the correlated Brownian motions W and $\widehat{W}$ with the constant instantaneous correlation coefficient ρ, then the PDE becomes

$$\partial_t \widehat{v} + \frac{1}{2}(\sigma_2^2 + \sigma_1^2 - 2\rho\sigma_1\sigma_2) z^2 \partial_{zz}\widehat{v} = 0.$$

Consequently, the pre-default price $\widetilde{C}_t = \widetilde{Y}_t^1 \widehat{v}(t, \widetilde{Y}_t^{2,1})$ will not depend directly on the drift coefficients $\widehat{\mu}_1$ and $\widehat{\mu}_2$, and thus, in principle, we should be able to derive an expression the price of the claim in terms of market observables: the prices of the underlying assets, their volatilities and the correlation coefficient. Put another way, neither the default intensity nor the drift coefficients of the underlying assets appear as independent parameters in the pre-default pricing function.

Chapter 5

Dependent Defaults and Credit Migrations

Modeling of dependent defaults is the most important and challenging research area with regard to credit risk and credit derivatives. We describe the case of conditionally independent default time, the industry standard copula-based approach, as well as the Jarrow and Yu [57] approach to the modeling of default times with dependent stochastic intensities. We conclude by summarizing one of the approaches that were recently developed for the purpose of modeling joint credit ratings migrations for several firms. It should be acknowledged that several other methods of modeling dependent defaults proposed in the literature are not covered by this text.

Let us start by providing a tentative classification of issues and techniques related to dependent defaults and credit ratings.

Valuation of basket credit derivatives covers, in particular:

- Default swaps of type F (Duffie [40], Kijima and Muromachi [62]) – they provide a protection against the first default in a basket of defaultable claims.
- Default swaps of type D (Kijima and Muromachi [62]) – a protection against the first two defaults in a basket of defaultable claims.
- The i^{th}-to-default claims (Bielecki and Rutkowski [14]) – a protection against the first i defaults in a basket of defaultable claims.

Technical issues arising in the context of dependent defaults include:

- Conditional independence of default times (Kijima and Muromachi [62]).
- Simulation of correlated defaults (Duffie and Singleton [42]).
- Modeling of infectious defaults (Davis and Lo [35]).
- Asymmetric default intensities (Jarrow and Yu [57]).
- Copulas (Laurent and Gregory [66], Schönbucher and Schubert [82]).
- Dependent credit ratings (Lando [64], Bielecki and Rutkowski [13]).
- Simulation of dependent credit migrations (Kijima et al. [61], Bielecki [3]).
- Simulation of correlated defaults via Marshall-Olkin copula (Elouerkhaoui [46]).

5.1 Basket Credit Derivatives

Basket credit derivatives are credit derivatives deriving their cash flows values (and thus their values) from credit risks of several reference entities (or prespecified credit events).

Standing assumptions. We assume that:

- We are given a collection of default times $\tau_1, \ldots, \tau_n$ defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q})$.
- $\mathbb{Q}\{\tau_i = 0\} = 0$ and $\mathbb{Q}\{\tau_i > t\} > 0$ for every i and t.
- $\mathbb{Q}\{\tau_i = \tau_j\} = 0$ for arbitrary $i \neq j$ (in a continuous time setup).

We associate with the collection $\tau_1, \ldots, \tau_n$ of default times the ordered sequence $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$, where $\tau_{(i)}$ stands for the random time of the i^{th} default. Formally,

$$\tau_{(1)} = \min\{\tau_1, \tau_2, \ldots, \tau_n\}$$

and for $i = 2, \ldots, n$

$$\tau_{(i)} = \min\big\{\tau_k : k = 1, \ldots, n,\, \tau_k > \tau_{(i-1)}\big\}.$$

In particular,

$$\tau_{(n)} = \max\{\tau_1, \tau_2, \ldots, \tau_n\}.$$

5.1.1 The i^{th}-to-Default Contingent Claims

We set $H_t^i = \mathbb{1}_{\{\tau_i \leq t\}}$ and we denote by $\mathbb{H}^i$ the filtration generated by the process H^i, that is, by the observations of the default time τ_i. In addition, we are given a reference filtration $\mathbb{F}$ on the space $(\Omega, \mathcal{G}, \mathbb{Q})$. The filtration $\mathbb{F}$ is related to some other market risks, for instance, to the interest rate risk. Finally, we introduce the enlarged filtration $\mathbb{G}$ by setting

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2 \vee \ldots \vee \mathbb{H}^n.$$

The σ-field $\mathcal{G}_t$ models the information available at time t.

A general i^{th}-to-default contingent claim which matures at time T is specified by the following covenants:

- If $\tau_{(i)} = \tau_k \leq T$ for some $k = 1, \ldots, n$ it pays at time $\tau_{(i)}$ the amount $Z^k_{\tau_{(i)}}$ where Z^k is an $\mathbb{F}$-predictable recovery process.
- If $\tau_{(i)} > T$ it pays at time T an $\mathcal{F}_T$-measurable promised amount X.

5.1.2 Case of Two Entities

For the sake of notational simplicity, we shall frequently consider the case of two reference credit risks.

Cash flows of the first-to-default contract (FDC):

- If $\tau_{(1)} = \min\{\tau_1, \tau_2\} = \tau_i \leq T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$.
- If $\min\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

Cash flows of the last-to-default contract (LDC):

- If $\tau_{(2)} = \max\{\tau_1, \tau_2\} = \tau_i \leq T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$.
- If $\max\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

We recall that throughout these lectures the savings account B equals

$$B_t = \exp\Big(\int_0^t r_u\,du\Big),$$

and $\mathbb{Q}$ stands for the martingale measure for our model of the financial market (including defaultable securities, such as: corporate bonds and credit derivatives). Consequently, the price $P(t,T)$ of a zero-coupon default-free bond equals

$$P(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}\,|\,\mathcal{G}_t\big) = B_t\,\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}\,|\,\mathcal{F}_t\big).$$

Values of FDC and LDC

In general, the value at time t of a defaultable claim (X,Z,τ) is given by the *risk-neutral valuation formula*

$$S_t = B_t\,\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1}\,dD_u\,\Big|\,\mathcal{G}_t\Big)$$

where D is the *dividend process,* which describes all the cash flows of the claim. Consequently, the value at time t of the FDC equals:

$$\begin{aligned} S_t^{(1)} &= B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_1}^{-1}Z_{\tau_1}^1\mathbb{1}_{\{\tau_1<\tau_2,\,t<\tau_1\le T\}}\,\Big|\,\mathcal{G}_t\Big) \\ &\quad + B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_2}^{-1}Z_{\tau_2}^2\mathbb{1}_{\{\tau_2<\tau_1,\,t<\tau_2\le T\}}\,\Big|\,\mathcal{G}_t\Big) \\ &\quad + B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1}X\mathbb{1}_{\{T<\tau_{(1)}\}}\,\Big|\,\mathcal{G}_t\Big). \end{aligned}$$

The value at time t of the LDC equals:

$$\begin{aligned} S_t^{(2)} &= B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_1}^{-1}Z_{\tau_1}^1\mathbb{1}_{\{\tau_2<\tau_1,\,t<\tau_1\le T\}}\,\Big|\,\mathcal{G}_t\Big) \\ &\quad + B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_2}^{-1}Z_{\tau_2}^2\mathbb{1}_{\{\tau_1<\tau_2,\,t<\tau_2\le T\}}\,\Big|\,\mathcal{G}_t\Big) \\ &\quad + B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1}X\mathbb{1}_{\{T<\tau_{(2)}\}}\,\Big|\,\mathcal{G}_t\Big). \end{aligned}$$

Both expressions above are merely special cases of a general formula. The goal is to derive more explicit representations under various assumptions about τ_1 and τ_2, or to provide ways of efficient calculation of involved expected values by means of simulation (using perhaps another probability measure).

5.1.3 Role of the Hypothesis (H)

If one assumes that (H) hypothesis holds between the filtrations $\mathbb{F}$ and $\mathbb{G}$, then, it holds between the filtrations $\mathbb{F}$ and $\mathbb{F}\vee\mathbb{H}^{i_1}\vee\cdots\vee\mathbb{H}^{i_k}$ for any $i_1,\dots,i_k$. However, there is no reason for the hypothesis (H) to hold between $\mathbb{F}\vee\mathbb{H}^{i_1}$ and $\mathbb{G}$. Note that, if (H) holds then one has, for $t_1\le\ldots\le t_n\le T$,

$$\mathbb{Q}(\tau_1>t_1,\dots,\tau_n>t_n\,|\,\mathcal{F}_T) = \mathbb{Q}(\tau_1>t_1,\dots,\tau_n>t_n\,|\,\mathcal{F}_\infty).$$

5.2 Conditionally Independent Defaults

Definition 5.2.1 *The random times τ_i, $i=1,\dots,n$ are said to be conditionally independent with respect to $\mathbb{F}$ under $\mathbb{Q}$ if we have, for any $T>0$ and any $t_1,\dots,t_n\in[0,T]$,*

$$\mathbb{Q}\{\tau_1>t_1,\dots,\tau_n>t_n\,|\,\mathcal{F}_T\} = \prod_{i=1}^n \mathbb{Q}\{\tau_i>t_i\,|\,\mathcal{F}_T\}.$$

Let us comment briefly on Definition 5.2.1.

- Conditional independence has the following intuitive interpretation: the reference credits (credit names) are subject to common risk factors that may trigger credit (default) events. In addition, each credit name is subject to idiosyncratic risks that are specific for this name.
- Conditional independence of default times means that once the common risk factors are fixed then the idiosyncratic risk factors are independent of each other.
- The property of conditional independence is not invariant with respect to an equivalent change of a probability measure.
- Conditional independence fits into static and dynamic theories of default times.
- A stronger condition would be a full conditionally independence, i.e., for any $T > 0$ and any intervals $I_1, \ldots, I_n$ we have:

$$\mathbb{Q}(\tau_1 \in I_1, \ldots, \tau_n \in I_n \,|\, \mathcal{F}_T) = \prod_{i=1}^n \mathbb{Q}(\tau_i \in I_i \,|\, \mathcal{F}_T).$$

5.2.1 Canonical Construction

Let Γ^i, $i = 1, \ldots, n$ be a given family of $\mathbb{F}$-adapted, increasing, continuous processes, defined on a probability space $(\tilde{\Omega}, \mathbb{F}, \mathbb{Q})$. We assume that $\Gamma^i_0 = 0$ and $\Gamma^i_\infty = \infty$. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ be an auxiliary probability space with a sequence ξ_i, $i = 1, \ldots, n$ of mutually independent random variables uniformly distributed on $[0, 1]$. We set

$$\tau_i(\tilde{\omega}, \hat{\omega}) = \inf \{ \, t \in \mathbb{R}_+ \, : \, \Gamma^i_t(\tilde{\omega}) \geq -\ln \xi_i(\hat{\omega}) \, \}$$

on the product probability space $(\Omega, \mathcal{G}, \mathbb{Q}) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F}_\infty \otimes \hat{\mathcal{F}}, \mathbb{Q} \otimes \hat{\mathbb{P}})$. We endow the space $(\Omega, \mathcal{G}, \mathbb{Q})$ with the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$.

Proposition 5.2.1 *If the random variables ξ_k are i.i.d., the process Γ^i is the $\mathbb{F}$-hazard process of τ_i:*

$$\mathbb{Q}\{\tau_i > s \,|\, \mathcal{F}_t \vee \mathcal{H}^i_t\} = \mathbb{1}_{\{\tau_i > t\}} \, \mathbb{E}_{\mathbb{Q}}\big(e^{\Gamma^i_t - \Gamma^i_s} \,|\, \mathcal{F}_t\big).$$

We have $\mathbb{Q}\{\tau_i = \tau_j\} = 0$ for every $i \neq j$. Moreover, default times $\tau_1, \ldots, \tau_n$ are conditionally independent with respect to $\mathbb{F}$ under $\mathbb{Q}$.

Proof. It suffices to note that, for $t_i < T$,

$$\begin{aligned} \mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n | \, \mathcal{F}_T) &= \mathbb{Q}(\Gamma^1_{t_1} \geq -\ln \xi_1, \ldots, \Gamma^n_{t_n} \geq -\ln \xi_n | \, \mathcal{F}_T) \\ &= \prod_{i=1}^n e^{\Gamma^i_{t_i}} \end{aligned}$$

□

Recall that if $\Gamma^i_t = \int_0^t \gamma^i_u \, du$ then γ^i is the $\mathbb{F}$-intensity of τ_i. Intuitively

$$\mathbb{Q}\{\tau_i \in [t, t + dt] \,|\, \mathcal{F}_t \vee \mathcal{H}^i_t\} \approx \mathbb{1}_{\{\tau_i > t\}} \gamma^i_t \, dt.$$

In the more general case where the random variables ξ_i are correlated, we introduce their joint cumulative distribution function

$$C(u_1, \ldots, u_n) = \mathbb{Q}(\xi_1 > u_1, \ldots, \xi_n > u_n).$$

Proposition 5.2.2 *If the random variables ξ_k have the joint cumulative distribution function C, the process Γ^i is the $\mathbb{F}$-hazard process of τ_i, that is,*

$$\mathbb{Q}(\tau_i > s \,|\, \mathcal{F}_t \vee \mathcal{H}^i_t) = \mathbb{1}_{\{\tau_i > t\}} \, \mathbb{E}_{\mathbb{Q}}\big(e^{\Gamma^i_t - \Gamma^i_s} \,|\, \mathcal{F}_t\big).$$

5.2.2 Independent Default Times

We shall first examine the case of default times $\tau_1, \ldots, \tau_n$ that are mutually independent under $\mathbb{Q}$. Suppose that for every $k = 1, \ldots, n$ we know the cumulative distribution function $F_k(t) = \mathbb{Q}\{\tau_k \le t\}$ of the default time of the k^{th} reference entity. The cumulative distribution functions of $\tau_{(1)}$ and $\tau_{(n)}$ are:

$$F_{(1)}(t) = \mathbb{Q}\{\tau_{(1)} \le t\} = 1 - \prod_{k=1}^{n}(1 - F_k(t))$$

and

$$F_{(n)}(t) = \mathbb{Q}\{\tau_{(n)} \le t\} = \prod_{k=1}^{n} F_k(t).$$

More generally, for any $i = 1, \ldots, n$ we have

$$F_{(i)}(t) = \mathbb{Q}\{\tau_{(i)} \le t\} = \sum_{m=i}^{n} \sum_{\pi \in \Pi^m} \prod_{j \in \pi} F_{k_j}(t) \prod_{l \notin \pi}(1 - F_{k_l}(t))$$

where Π^m denote the family of all subsets of $\{1, \ldots, n\}$ consisting of m elements.

Suppose, in addition, that the default times $\tau_1, \ldots, \tau_n$ admit deterministic intensity functions $\gamma_1(t), \ldots, \gamma_n(t)$, such that

$$H_t^i - \int_0^{t \wedge \tau_i} \gamma_i(s) ds$$

are $\mathbb{H}^i$-martingales. Recall that $\mathbb{Q}\{\tau_i > t\} = e^{-\int_0^t \gamma_i(v)\, dv}$. It is easily seen that, for any $t \in \mathbb{R}_+$,

$$\mathbb{Q}\{\tau_{(1)} > t\} = \prod \mathbb{Q}\{\tau_i > t\} = e^{-\int_0^t \gamma_{(1)}(v)\, dv}.$$

where

$$\gamma_{(1)}(t) = \gamma_1(t) + \ldots + \gamma_n(t)$$

hence

$$H_t^{(1)} - \int_0^{t \wedge \tau_{(1)}} \gamma_{(1)}(t) dt$$

is a $\mathbb{H}^{(1)}$-martingale, where $\mathcal{H}_t^{(1)} = \sigma(\tau_{(1)} \wedge t)$. By direct calculations, it is also possible to find the intensity function of the i^{th} default time.

Example 5.2.1 We shall consider a digital default put of basket type. To be more specific, we postulate that a contract pays a fixed amount (e.g., one unit of cash) at the i^{th} default time $\tau_{(i)}$ provided that $\tau_{(i)} \le T$. Assume that the interest rates are non-random. Then the value at time 0 of the contract equals

$$S_0 = \mathbb{E}_{\mathbb{Q}}\big(B_\tau^{-1} \mathbb{1}_{\{\tau_{(i)} \le T\}}\big) = \int_{]0,T]} B_u^{-1}\, dF_{(i)}(u).$$

If $\tau_1, \ldots, \tau_n$ admit intensities then

$$S_0 = \int_0^T B_u^{-1}\, dF_{(i)}(u) = \int_0^T B_u^{-1} \gamma_{(i)}(u) e^{-\int_0^u \gamma_{(i)}(v) dv}\, du.$$

5.2.3 Signed Intensities

Some authors (e.g., Kijima and Muromachi [62]) examine credit risk models in which the negative values of "intensities" are not precluded. In that case, the process chosen as the "intensity" does not play the role of a real intensity, in particular, it is not true that $H_t - \int_0^{t \wedge \tau} \gamma_t\, dt$ is a martingale and negative values of the "intensity" process clearly contradict the interpretation of the intensity as the conditional probability of survival over an infinitesimal time interval. More precisely, for a given collection Γ^i, $i = 1, \ldots, n$ of $\mathbb{F}$-adapted continuous stochastic processes, with $\Gamma^i_0 = 0$, defined on $(\widehat{\Omega}, \mathbb{F}, \widehat{\mathbb{P}})$. one can define τ_i, $i = 1, \ldots, n$, on the enlarged probability space $(\Omega, \mathcal{G}, \mathbb{Q})$:

$$\tau_i = \inf \{ \, t \in \mathbb{R}_+ : \Gamma^i_t(\hat{\omega}) \geq - \ln \xi_i(\hat{\omega}) \, \}.$$

Let us denote $\hat{\Gamma}^i_t = \max_{u \leq t} \Gamma^i_u$. Observe that if the process Γ^i is absolutely continuous, than so it the process $\hat{\Gamma}^i$; in this case the intensity of τ_i is obtained as the derivative of $\hat{\Gamma}^i$ with respect to the time variable.

The following result examines the case of signed intensities.

Lemma 5.2.1 *Random times τ_i, $i = 1, \ldots, n$ are conditionally independent with respect to $\mathbb{F}$ under $\mathbb{Q}$. In particular, for every $t_1, \ldots, t_n \leq T$,*

$$\mathbb{Q}\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \prod_{i=1}^{n} e^{-\hat{\Gamma}^i_{t_i}} = e^{-\sum_{i=1}^n \hat{\Gamma}^i_{t_i}}.$$

5.2.4 Valuation of FDC and LDC

Valuation of the first-to-default or last-to-default contingent claim in relatively straightforward under the assumption of conditional independence of default times. We have the following result in which, for notational simplicity, we consider only the case of two entities. As usual, we do not state explicitly integrability conditions that should be imposed on recovery processes Z^j and the terminal payoff X.

Proposition 5.2.3 *Let the default times τ_j, $j = 1, 2$ be $\mathbb{F}$-conditionally independent with $\mathbb{F}$-intensities γ^j, that is, $H^i_t - \int_0^{t \wedge \tau_i} \gamma^i_s \, ds$ are $\mathbb{G}^i$-martingales and γ^i is $\mathbb{F}$ adapted. Assume that the recovery Z is an $\mathbb{F}$-predictable process, and that the terminal payoff X is $\mathcal{F}_T$-measurable.*
(i) If the hypothesis (H) holds between $\mathbb{F}$ and $\mathbb{G}$, then the price at time $t = 0$ of the first-to-default claim equals

$$S^{(1)}_0 = \sum_{i,j=1,\, i \neq j}^{2} \mathbb{E}_{\mathbb{Q}} \Big(\int_0^T B_u^{-1} Z^j_u \, e^{-\Gamma^i_u} \gamma^j_u e^{-\Gamma^j_u} \, du \Big) + \mathbb{E}_{\mathbb{Q}} \big(B_T^{-1} X G \big),$$

where we denote

$$G = e^{-(\Gamma^1_T + \Gamma^2_T)} = \mathbb{Q}(\tau_1 > T, \, \tau_2 > T \,|\, \mathcal{F}_T).$$

(ii) In the general case, setting $F^i_t = \mathbb{Q}(\tau_i \leq t \,|\, \mathcal{F}_t) = Z^i_t + A^i_t$, where Z^i is an $\mathbb{F}$ martingale, we have that

$$S^{(1)}_0 = \mathbb{E}_{\mathbb{Q}} \int_0^T Z_u (e^{-(\Gamma^1_u + \Gamma^2_u)} (\gamma^1_u + \gamma^2_u) \, du + d\langle Z^1, Z^2 \rangle_u) + \mathbb{E}_{\mathbb{Q}} \big(B_T^{-1} X G \big).$$

Proof. We need to compute $\mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{\tau < T\}})$ for $\tau = \tau_1 \wedge \tau_2$. We know that, if Z is $\mathbb{F}$-predictable

$$\mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{\tau < T\}}) = \mathbb{E}_{\mathbb{Q}} \Big(\int_0^T Z_u \, dF_u \Big)$$

where $F_u = \mathbb{Q}(\tau \leq u \,|\, \mathcal{F}_u)$. For $\tau = \tau_1 \wedge \tau_2$, the conditional independence assumption yields

$$1 - F_u = \mathbb{Q}(\tau_1 > u, \tau_2 > u \,|\, \mathcal{F}_u) = \mathbb{Q}(\tau_1 > u \,|\, \mathcal{F}_u) \, \mathbb{Q}(\tau_2 > u \,|\, \mathcal{F}_u) = (1 - F^1_u)(1 - F^2_u).$$

• If we assume that the hypothesis (H) holds between $\mathbb{F}$ and $\mathbb{G}^i$, for $i = 1, 2$, the processes F^i are increasing, and thus

$$dF_u = e^{-\Gamma^1_u} \, dF^2_u + e^{-\Gamma^2_u} \, dF^1_u = e^{-\Gamma^1_u} e^{-\Gamma^2_u} (\gamma^1_u + \gamma^2_u) \, du$$

It follows that

$$\mathbb{E}_{\mathbb{Q}}(Z_{\tau_1\wedge\tau_2}\mathbb{1}_{\{\tau_1\wedge\tau_2<T\}}) = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T Z_u e^{-\Gamma^1_u}e^{-\Gamma^2_u}(\gamma^1_u+\gamma^2_u)\,du\Big)$$

• In the general case, the Doob-Meyer decomposition of F_i is $F_i = Z_i + A_i$ and

$$H^i_t - \int_0^{t\wedge\tau_i}\gamma^i_s\,ds$$

is a $\mathcal{G}^i$-martingale where $\gamma^i_s = \frac{a^i_s}{1-F^i_s}$. We now have

$$dF_u = e^{-\Gamma^1_u}\,dF^2_u + e^{-\Gamma^2_u}\,dF^1_u + d\langle Z^1,Z^2\rangle_u.$$

It follows that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(Z_{\tau_1\wedge\tau_2}\mathbb{1}_{\tau_1\wedge\tau_2<T}) &= \mathbb{E}_{\mathbb{Q}}\int_0^T Z_u(e^{-\Gamma^1_u}dA^2_u + e^{-\Gamma^2_u}dA^1_u + d\langle Z^1,Z^2\rangle_u)\\ &= \mathbb{E}_{\mathbb{Q}}\int_0^T Z_u(e^{-(\Gamma^1_u+\Gamma^2_u)}(\gamma^1_u+\gamma^2_u) + d\langle Z^1,Z^2\rangle_u)\end{aligned}$$

The bracket must be related with some correlation of default times. □

Exercise 5.2.1 Compute the conditional expectation $\mathbb{E}_{\mathbb{Q}}(Z_\tau\mathbb{1}_{\{\tau<T\}}\,|\,\mathcal{G}_t)$.

5.3 Copula-Based Approaches

5.3.1 Direct Application

In a direct application, we first postulate a (univariate marginal) probability distribution for each random variable τ_i. Let us denote the marginal distribution by F_i for $i=1,2,\dots,n$. Then, a suitable copula function C is chosen in order to introduce an appropriate dependence structure of the random vector $(\tau_1,\dots,\tau_n)$. Finally, the joint distribution of the random vector $(\tau_1,\dots,\tau_n)$ is postulated, specifically,

$$\mathbb{Q}\{\tau_1\le t_1,\dots,\tau_n\le t_n\} = C\big(F_1(t_1),\dots,F_n(t_n)\big).$$

In the finance industry, the most commonly used are elliptical copulas (such as the Gaussian copula and the t-copula). The direct approach has an apparent drawback. It is essentially a static approach; it makes no account of changes in credit ratings, and no conditioning on the flow of information is present. Let us mention, however, an interesting theoretical issue, namely, the study of the effect of a change of probability measures on the copula structure.

5.3.2 Indirect Application

A less straightforward application of copulas is based on an extension of the canonical construction of conditionally independent default times. This can be considered as the first step towards a dynamic theory, since the techniques of copulas is merged with the flow of available information, in particular, the information regarding the observations of defaults.

Assume that the cumulative distribution function of $(\xi_1,\dots,\xi_n)$ in the canonical construction (cf. Section 5.2.1) is given by an n-dimensional copula C, and that the univariate marginal laws are uniform on $[0,1]$. Similarly as in Section 5.2.1, we postulate that $(\xi_1,\dots,\xi_n)$ are independent of $\mathbb{F}$, and we set

$$\tau_i(\tilde\omega,\hat\omega) = \inf\{\,t\in\mathbb{R}_+\,:\,\Gamma^i_t(\tilde\omega)\ge -\ln\xi_i(\hat\omega)\,\}.$$

Then, $\{\tau_i > t_i\} = \{e^{-\Gamma^i_{t_i}} > \xi_i\}$. However, we do not assume that the ξ_k are i.i.d. and we denote by C their copula.

Then:

• The case of default times conditionally independent with respect to $\mathbb{F}$ corresponds to the choice of the product copula Π. In this case, for $t_1, \dots, t_n \leq T$ we have

$$\mathbb{Q}\{\tau_1 > t_1, \dots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \Pi(Z^1_{t_1}, \dots, Z^n_{t_n}),$$

where we set $Z^i_t = e^{-\Gamma^i_t}$.

• In general, for $t_1, \dots, t_n \leq T$ we obtain

$$\mathbb{Q}\{\tau_1 > t_1, \dots, \tau_n > t_n \,|\, \mathcal{F}_T\} = C(Z^1_{t_1}, \dots, Z^n_{t_n}),$$

where C is the copula used in the construction of $\xi_1, \dots, \xi_n$.

Survival Intensities

We follow here Schönbucher and Schubert [82].

Proposition 5.3.1 *For arbitrary $s \leq t$ on the set $\{\tau_1 > s, \dots, \tau_n > s\}$ we have*

$$\mathbb{Q}\{\tau_i > t \,|\, \mathcal{G}_s\} = \mathbb{E}_{\mathbb{Q}}\left(\frac{C(Z^1_s, \dots, Z^i_t, \dots, Z^n_s)}{C(Z^1_s, \dots, Z^n_s)} \,\Big|\, \mathcal{F}_s \right).$$

Proof. The proof is rather straightforward. We have

$$\mathbb{Q}\{\tau_i > t \,|\, \mathcal{G}_s\} \mathbb{1}_{\{\tau_1 > s, \dots, \tau_n > s\}} = \mathbb{1}_{\{\tau_1 > s, \dots, \tau_n > s\}} \frac{\mathbb{Q}(\tau_1 > s, \dots, \tau_i > t, \dots, \tau_n > s \,|\, \mathcal{F}_s)}{\mathbb{Q}(\tau_1 > s, \dots, \tau_i > s, \dots, \tau_n > s \,|\, \mathcal{F}_s)},$$

where we used the key lemma. □

Under the assumption that the derivatives $\gamma^i_t = \frac{d\Gamma^i_t}{dt}$ exist, the i^{th} intensity of survival equals, on the set $\{\tau_1 > t, \dots, \tau_n > t\}$,

$$\lambda^i_t = \gamma^i_t \, Z^i_t \, \frac{\frac{\partial}{\partial v_i} C(Z^1_t, \dots, Z^n_t)}{C(Z^1_t, \dots, Z^n_t)} = \gamma^i_t \, Z^i_t \, \frac{\partial}{\partial v_i} \ln C(Z^1_t, \dots, Z^n_t),$$

where λ^i_t is understood as the following limit

$$\lambda^i_t = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_1 > t, \dots, \tau_n > t\}.$$

It appears that, in general, the i^{th} intensity of survival jumps at time t, if the j^{th} entity defaults at time t for some $j \neq i$. In fact, it holds that

$$\lambda^{i,j}_t = \gamma^i_t \, Z^i_t \, \frac{\frac{\partial^2}{\partial v_i \partial v_j} C(Z^1_t, \dots, Z^n_t)}{\frac{\partial}{\partial v_j} C(Z^1_t, \dots, Z^n_t)},$$

where

$$\lambda^{i,j}_t = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_k > t, k \neq j, \tau_j = t\}.$$

Schönbucher and Schubert [82] examine also the intensities of survival after the default times of some entities. Let us fix s, and let $t_i \leq s$ for $i = 1, 2, \dots, k < n$, and $T_i \geq s$ for $i = k+1, k+2, \dots, n$. Then,

$$\begin{aligned} &\mathbb{Q}\big\{\tau_i > T_i, \, i = k+1, k+2, \dots, n \,|\, \mathcal{F}_s, \, \tau_j = t_j, \, j = 1, 2, \dots, k, \\ &\qquad \tau_i > s, \, i = k+1, k+2, \dots, n\big\} \\ &= \frac{\mathbb{E}_{\mathbb{Q}}\Big(\frac{\partial^k}{\partial v_1 \dots \partial v_k} C(Z^1_{t_1}, \dots, Z^k_{t_k}, Z^{k+1}_{T_{k+1}}, \dots, Z^n_{T_n}) \,\Big|\, \mathcal{F}_s \Big)}{\frac{\partial^k}{\partial v_1 \dots \partial v_k} C(Z^1_{t_1}, \dots, Z^k_{t_k}, Z^{k+1}_s, \dots, Z^n_s)}. \end{aligned} \tag{5.1}$$

Remark 5.3.1 The jumps of intensities cannot be efficiently controlled, except for the choice of C. In the approach described above, the dependence between the default times is implicitly introduced through Γ^is, and explicitly introduced by the choice of a copula C.

Laurent and Gregory Model

Laurent and Gregory [66] examine a simplified version of the framework of Schönbucher and Schubert [82]. Namely, they assume that the reference filtration is trivial – that is, $\mathcal{F}_t = \{\Omega, \emptyset\}$ for every $t \in \mathbb{R}_+$. This implies, in particular, that the default intensities γ^i are deterministic functions, and

$$\mathbb{Q}(\tau_i > t) = 1 - F_i(t) = e^{-\int_0^t \gamma^i(u)\,du}.$$

They obtain closed-form expressions for certain conditional intensities of default.

Example 5.3.1 This example describes the use of the one-factor Gaussian copula model, which is the BIS (Bank of International Settlements) standard. Let

$$X_i = \rho V + \sqrt{1-\rho^2}\, V_i,$$

where V, V_i, $i = 1, 2, \ldots, n$, are independent, standard Gaussian variables under $\mathbb{Q}$. Define

$$\tau_i = \inf\Big\{ t \in \mathbb{R}_+ : \int_0^t \gamma^i(u)\,du > -\ln U_i \Big\} = \inf\{ t \in \mathbb{R}_+ : 1 - F_i(t) < U_i \}$$

where the random barriers are defined as $U_i = 1 - N(X_i)$. As usual, N stands for the cumulative distribution function of a standard Gaussian random variable. Then the following equalities hold

$$\{\tau_i \le t\} = \{U_i \ge 1 - F_i(t)\} = \Big\{ X_i \le \frac{N^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}} \Big\}.$$

Define $q_t^{i|V} = \mathbb{Q}(\tau_i > t \,|\, V)$ and $p_t^{i|V} = 1 - q_t^{i|V}$. Then

$$\mathbb{Q}\{\tau_1 \le t_1, \ldots, \tau_n \le t_n\} = \int_{\mathbb{R}} \prod_{i=1}^n p_{t_i}^{i|v} f(v)\,dv$$

where f is the density of V. It is easy to check that

$$p_t^{i|V} = N\left(\frac{N^{-1}(F_i(t)) - \rho_i V}{\sqrt{1-\rho_i^2}} \right)$$

and thus

$$\mathbb{Q}\{\tau_1 \le t_1, \ldots, \tau_n \le t_n\} = \int_{\mathbb{R}} \prod_{i=1}^n N\left(\frac{N^{-1}(F_i(t_i)) - \rho_i V}{\sqrt{1-\rho_i^2}} \right) f(v)\,dv.$$

5.4 Jarrow and Yu Model

Jarrow and Yu [57] approach can be considered as another step towards a dynamic theory of dependence between default times. For a given finite family of reference credit names, Jarrow and Yu [57] propose to make a distinction between the *primary firms* and the *secondary firms*.

At the intuitive level:

- The class of primary firms encompasses these entities whose probabilities of default are influenced by macroeconomic conditions, but not by the credit risk of counterparties. The pricing of bonds issued by primary firms can be done through the standard intensity-based methodology.

- It suffices to focus on securities issued by secondary firms, that is, firms for which the intensity of default depends on the status of some other firms.

Formally, the construction is based on the assumption of asymmetric information. Unilateral dependence is not possible in the case of complete (i.e., symmetric) information.

5.4.1 Construction and Properties of the Model

Let $\{1, \dots, n\}$ represent the set of all firms, and let $\mathbb{F}$ be the reference filtration. We postulate that:

- For any firm from the set $\{1, \dots, k\}$ of primary firms, the 'default intensity' depends only on $\mathbb{F}$.
- The 'default intensity' of each firm belonging to the set $\{k+1, \dots, n\}$ of secondary firms may depend not only on the filtration $\mathbb{F}$, but also on the status (default or no-default) of the primary firms.

Construction of Default Times $\tau_1, \dots, \tau_n$

First step. We first model default times of primary firms. To this end, we assume that we are given a family of $\mathbb{F}$-adapted 'intensity processes' $\lambda^1, \dots, \lambda^k$ and we produce a collection $\tau_1, \dots, \tau_k$ of $\mathbb{F}$-conditionally independent random times through the canonical method:

$$\tau_i = \inf \Big\{ t \in \mathbb{R}_+ \, : \, \int_0^t \lambda^i_u \, du \geq - \ln \xi_i \Big\}$$

where ξ_i, $i = 1, \dots, k$ are mutually independent identically distributed random variables with uniform law on $[0, 1]$ under the martingale measure $\mathbb{Q}$.

Second step. We now construct default times of secondary firms. We assume that:

- The probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is large enough to support a family ξ_i, $i = k+1, \dots, n$ of mutually independent random variables, with uniform law on $[0, 1]$.
- These random variables are independent not only of the filtration $\mathbb{F}$, but also of the already constructed in the first step default times $\tau_1, \dots, \tau_k$ of primary firms.

The default times τ_i, $i = k+1, \dots, n$ are also defined by means of the standard formula:

$$\tau_i = \inf \Big\{ t \in \mathbb{R}_+ \, : \, \int_0^t \lambda^i_u \, du \geq - \ln \xi_i \Big\}.$$

However, the 'intensity processes' λ^i for $i = k+1, \dots, n$ are now given by the following expression:

$$\lambda^i_t = \mu^i_t + \sum_{l=1}^{k} \nu^{i,l}_t \mathbb{1}_{\{\tau_l \leq t\}},$$

where μ^i and $\nu^{i,l}$ are $\mathbb{F}$-adapted stochastic processes. If the default of the j^{th} primary firm does not affect the default intensity of the i^{th} secondary firm, we set $\nu^{i,j} \equiv 0$.

Main Features

Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \dots \vee \mathbb{H}^n$ stand for the enlarged filtration and let $\hat{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{k+1} \vee \dots \vee \mathbb{H}^n$ be the filtration generated by the reference filtration $\mathbb{F}$ and the observations of defaults of secondary firms. Then:

- The default times $\tau_1, \dots, \tau_k$ of primary firms are conditionally independent with respect to $\mathbb{F}$.

- The default times $\tau_1, \dots, \tau_k$ of primary firms are no longer conditionally independent when we replace the filtration $\mathbb{F}$ by $\hat{\mathbb{F}}$.
- In general, the default intensity of a primary firm with respect to the filtration $\hat{\mathbb{F}}$ differs from the intensity λ^i with respect to $\mathbb{F}$.

We conclude that defaults of primary firms are also 'dependent' of defaults of secondary firms.

Case of Two Firms

To illustrate the present model, we now consider only two firms, A and B say, and we postulate that A is a primary firm, and B is a secondary firm. Let the constant process $\lambda_t^1 \equiv \lambda_1$ represent the $\mathbb{F}$-intensity of default for firm A, so that

$$\tau_1 = \inf \Big\{ t \in \mathbb{R}_+ \, : \, \int_0^t \lambda_u^1 \, du = \lambda_1 t \geq -\ln \xi_1 \Big\},$$

where ξ_1 is a random variable independent of $\mathbb{F}$, with the uniform law on $[0,1]$. For the second firm, the 'intensity' of default is assumed to satisfy

$$\lambda_t^2 = \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \leq t\}}$$

for some positive constants λ_2 and α_2, and thus

$$\tau_2 = \inf \Big\{ t \in \mathbb{R}_+ \, : \, \int_0^t \lambda_u^2 \, du \geq -\ln \xi_2 \Big\}$$

where ξ_2 is a random variable with the uniform law, independent of $\mathbb{F}$, and such that ξ_1 and ξ_2 are mutually independent. Then the following properties hold:

- λ^1 is the intensity of τ_1 with respect to $\mathbb{F}$,
- λ^2 is the intensity of τ_2 with respect to $\mathbb{F} \vee \mathbb{H}^1$,
- λ^1 is not the intensity of τ_1 with respect to $\mathbb{F} \vee \mathbb{H}^2$.

Let $\tau_i = \inf\{t \, : \, \Lambda_i(t) \geq \Theta_i\}, i = 1, 2$ where $\Lambda_i(t) = \displaystyle\int_0^t \lambda_i(s) ds$ and Θ_i are independent random variables with exponential law of parameter 1. Jarrow and Yu [57] study the case where λ_1 is a constant and

$$\lambda_2(t) = \lambda_2 + (\alpha_2 - \lambda_2)\mathbb{1}_{\{\tau_1 \leq t\}} = \lambda_2 \mathbb{1}_{\{t < \tau_1\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \leq t\}} \, .$$

Assume for simplicity that $r = 0$ and compute the value of defaultable zero-coupon bonds with default time τ_i, with a rebate δ_i:

$$P_{i,d}(t,T) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau_i > T\}} + \delta_i \mathbb{1}_{\{\tau_i < T\}} \,|\, \mathcal{G}_t), \text{ for } \mathcal{G}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2.$$

We need to compute the joint law of the pair (τ_1, τ_2). Let $G(s,t) = \mathbb{Q}(\tau_1 > s, \tau_2 > t)$.

- **Case** $t \leq s$

For $t < s$, one has $\lambda_2(t) = \lambda_2 t$ on the set $s < \tau_1$. Hence, the following equality

$$\{\tau_1 > s\} \cap \{\tau_2 > t\} = \{\tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\} = \{\tau_1 > s\} \cap \{\lambda_2 t < \Theta_2\} = \{\lambda_1 s < \Theta_1\} \cap \{\lambda_2 t < \Theta_2\}$$

leads to

$$\text{for } \; t < s, \; \mathbb{Q}(\tau_1 > s, \tau_2 > t) = e^{-\lambda_1 s} e^{-\lambda_2 t} \, .$$

- **Case** $t > s$

$$\begin{aligned}\{\tau_1 > s\} \cap \{\tau_2 > t\} &= \{\{t > \tau_1 > s\} \cap \{\tau_2 > t\}\} \cup \{\{\tau_1 > t\} \cap \{\tau_2 > t\}\} \\ \{t > \tau_1 > s\} \cap \{\tau_2 > t\} &= \{t > \tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\} \\ &= \{t > \tau_1 > s\} \cap \{\lambda_2 \tau_1 + \alpha_2(t - \tau_1) < \Theta_2\}\end{aligned}$$

The independence between Θ_1 and Θ_2 implies that the r.v. τ_1 is independent from Θ_2 (use that $\tau_1 = \Theta_1(\lambda_1)^{-1}$), hence

$$\begin{aligned}\mathbb{Q}(t > \tau_1 > s, \tau_2 > t) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{t>\tau_1>s\}} e^{-(\lambda_2\tau_1 + \alpha_2(t-\tau_1))}\right) \\ &= \int du\, \mathbb{1}_{\{t>u>s\}} e^{-(\lambda_2 u + \alpha_2(t-u))} \lambda_1 e^{-\lambda_1 u} \\ &= \frac{1}{\lambda_1 + \lambda_2 - \alpha_2} \lambda_1 e^{-\alpha_2 t}\left(e^{-s(\lambda_1+\lambda_2-\alpha_2)} - e^{-t(\lambda_1+\lambda_2-\alpha_2)}\right).\end{aligned}$$

Setting $\Delta = \lambda_1 + \lambda_2 - \alpha_2$, it follows that

$$\mathbb{Q}(\tau_1 > s, \tau_2 > t) = \frac{1}{\Delta}\lambda_1 e^{-\alpha_2 t}\left(e^{-s\Delta} - e^{-t\Delta}\right) + e^{-\lambda_1 t} e^{-\lambda_2 t}. \tag{5.2}$$

In particular, for $s = 0$,

$$\mathbb{Q}(\tau_2 > t) = \frac{1}{\Delta}\left(\lambda_1\left(e^{-\alpha_2 t} - e^{-(\lambda_1+\lambda_2)t}\right) + \Delta e^{-(\lambda_1+\lambda_2)t}\right)$$

- The computation of $P_{1,d}$ reduces to that of

$$\mathbb{Q}(\tau_1 > T \,|\, \mathcal{G}_t) = \mathbb{Q}(\tau_1 > T \,|\, \mathcal{F}_t \vee \mathcal{H}^1_t)$$

where $\mathcal{F}_t = \mathcal{H}^2_t$.

We have

$$\mathbb{Q}(\tau_1 > T \,|\, \mathcal{G}_t) = 1 - DZC^1_t = \mathbb{1}_{\{\tau_1 > t\}}\left(\mathbb{1}_{\{\tau_2 \le t\}} \frac{\partial_2 G(T, \tau_2)}{\partial_2 G(t, \tau_2)} + \mathbb{1}_{\{\tau_2 > t\}} \frac{G(T,t)}{G(t,t)}\right)$$

Therefore,

$$P_{1,d}(t,T) = \delta_1 + \mathbb{1}_{\{\tau_1 > t\}}(1 - \delta_1) e^{-\lambda_1(T-t)}.$$

One can also use

- The computation of $P_{2,d}$ follows from the computation of

$$\mathbb{Q}(\tau_2 > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_2\}} \frac{\mathbb{Q}(\tau_2 > T \,|\, \mathcal{H}^1_t)}{\mathbb{Q}(\tau_2 > t \,|\, \mathcal{H}^1_t)} + \mathbb{1}_{\{\tau_2 < t\}} \mathbb{Q}(\tau_2 > T \,|\, \tau_2)$$

$$\begin{aligned}P_{2,d}(t,T) &= \delta_2 + (1-\delta_2)\mathbb{1}_{\{\tau_2 > t\}}\Big(\mathbb{1}_{\{\tau_1 \le t\}} e^{-\alpha_2(T-t)} \\ &\quad + \mathbb{1}_{\{\tau_1 > t\}} \frac{1}{\Delta}(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)(T-t)})\Big)\end{aligned}$$

Special Case: Zero Recovery

Assume that $\lambda_1 + \lambda_2 - \alpha_2 \neq 0$ and the bond is subject to the zero recovery scheme. For the sake of brevity, we set $r = 0$ so that $P(t,T) = 1$ for $t \le T$. Under the present assumptions:

$$P_{d,2}(t,T) = \mathbb{Q}\{\tau_2 > T \,|\, \mathcal{H}^1_t \vee \mathcal{H}^2_t\}$$

and the general formula yields

$$P_{d,2}(t,T) = \mathbb{1}_{\{\tau_2 > t\}} \frac{\mathbb{Q}\{\tau_2 > T \,|\, \mathcal{H}^1_t\}}{\mathbb{Q}\{\tau_2 > t \,|\, \mathcal{H}^1_t\}} \,.$$

If we set $\Lambda^2_t = \int_0^t \lambda^2_u \, du$ then

$$P_{d,2}(t,T) = \mathbb{1}_{\{\tau_2 > t\}} \, \mathbb{E}_{\mathbb{Q}}(e^{\Lambda^2_t - \Lambda^2_T} \,|\, \mathcal{H}^1_t).$$

Finally, we have the following explicit result.

Corollary 5.4.1 *If $\delta_2 = 0$ then $P_{d,2}(t,T) = 0$ on $\{\tau_2 \le t\}$. On the set $\{\tau_2 > t\}$ we have*

$$\begin{aligned} P_{d,2}(t,T) &= \mathbb{1}_{\{\tau_1 \le t\}} \, e^{-\alpha_2(T-t)} \\ &+ \mathbb{1}_{\{\tau_1 > t\}} \, \frac{1}{\lambda - \alpha_2} \Big(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2 - \alpha_2) e^{-\lambda(T-t)} \Big). \end{aligned}$$

5.5 Extension of the Jarrow and Yu Model

We shall now argue that the assumption that some firms are primary while other firms are secondary is not relevant. For simplicity of presentation, we assume that:

- We have $n = 2$, that is, we consider two firms only.
- The interest rate r is zero, so that $B(t,T) = 1$ for every $t \le T$.
- The reference filtration $\mathbb{F}$ is trivial.
- Corporate bonds are subject to the zero recovery scheme.

Since the situation is symmetric, it suffices to analyze a bond issued by the first firm. By definition, the price of this bond equals

$$P_{d,1}(t,T) = \mathbb{Q}\{\tau_1 > T \,|\, \mathcal{H}^1_t \vee \mathcal{H}^2_t\}.$$

For the sake of comparison, we shall also evaluate the following values, which are based on partial observations,

$$\tilde{P}_{d,1}(t,T) = \mathbb{Q}\{\tau_1 > T \,|\, \mathcal{H}^2_t\}$$

and

$$\widehat{P}_{d,1}(t,T) = \mathbb{Q}\{\tau_1 > T \,|\, \mathcal{H}^1_t\}.$$

5.5.1 Kusuoka's Construction

We follow here Kusuoka [63]. Under the original probability measure $\mathbb{Q}$ the random times τ_i, $i = 1, 2$ are assumed to be mutually independent random variables with exponential laws with parameters λ_1 and λ_2, respectively.

Girsanov's theorem. For a fixed $T > 0$, we define a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta_T, \quad \mathbb{P}\text{-a.s.}$$

where the Radon-Nikodým density process η_t, $t \in [0,T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-} \kappa^i_u \, dM^i_u$$

where in turn

$$M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \lambda_i \, du.$$

Here $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$ and processes κ^1 and κ^2 are given by

$$\kappa_t^1 = \mathbb{1}_{\{\tau_2 < t\}} \Big(\frac{\alpha_1}{\lambda_1} - 1 \Big), \quad \kappa_t^2 = \mathbb{1}_{\{\tau_1 < t\}} \Big(\frac{\alpha_2}{\lambda_2} - 1 \Big).$$

It can be checked that the martingale intensities of τ_1 and τ_2 under $\mathbb{Q}$ are

$$\begin{aligned} \lambda_t^1 &= \lambda_1 \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le t\}}, \\ \lambda_t^2 &= \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le t\}}. \end{aligned}$$

Main features. We focus on τ_1 and we denote $\Lambda_t^1 = \int_0^t \lambda_u^1 \, du$. Let us make few observations. First, the process λ^1 is $\mathbb{H}^2$-predictable, and the process

$$M_t^1 = H_t^1 - \int_0^{t\wedge\tau_1} \lambda_u^1 \, du = H_t^1 - \Lambda_{t\wedge\tau_1}^1$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}$. Next, the process λ^1 is not the 'true' intensity of the default time τ_1 with respect to $\mathbb{H}^2$ under $\mathbb{Q}$. Indeed, in general, we have

$$\mathbb{Q}\{\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\} \ne \mathbb{1}_{\{\tau_1 > t\}} \, \mathbb{E}_{\mathbb{Q}} \big(e^{\Lambda_t^1 - \Lambda_s^1} \,|\, \mathcal{H}_t^2 \big).$$

Finally, the process λ^1 represents the intensity of the default time τ_1 with respect to $\mathbb{H}^2$ under a probability measure $\mathbb{Q}^1$ equivalent to $\mathbb{P}$, where

$$\frac{d\mathbb{Q}^1}{d\mathbb{P}} = \tilde{\eta}_T, \quad \mathbb{P}\text{-a.s.}$$

and the Radon-Nikodým density process $\tilde{\eta}_t,\ t \in [0,T],$ satisfies

$$\tilde{\eta}_t = 1 + \int_{]0,t]} \tilde{\eta}_{u-} \kappa_u^2 \, dM_u^2.$$

For $s > t$ we have

$$\mathbb{Q}^1\{\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\} = \mathbb{1}_{\{\tau_1 > t\}} \, \mathbb{E}_{\mathbb{Q}^1} \big(e^{\Lambda_t^1 - \Lambda_s^1} \,|\, \mathcal{F}_t \big),$$

but also

$$\mathbb{Q}\{\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\} = \mathbb{Q}^1\{\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\}.$$

5.5.2 Interpretation of Intensities

Recall that the processes λ_1 and λ_2 have jumps if $\alpha_i \ne \lambda_i$. The following result shows that the intensities λ^1 and λ^2 are 'local intensities' of default with respect to the information available at time t. It shows also that the model can in fact be reformulated as a two-dimensional Markov chain (see Lando [64]).

Proposition 5.5.1 *For $i = 1, 2$ and every $t \in \mathbb{R}_+$ we have*

$$\lambda_i = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_i \le t + h \,|\, \tau_1 > t,\, \tau_2 > t\}. \tag{5.3}$$

Moreover:

$$\alpha_1 = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_1 \le t + h \,|\, \tau_1 > t,\, \tau_2 \le t\}.$$

and

$$\alpha_2 = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_2 \le t + h \,|\, \tau_2 > t,\, \tau_1 \le t\}.$$

5.5.3 Bond Valuation

Proposition 5.5.2 *The price $P_{d,1}(t,T)$ on $\{\tau_1 > t\}$ equals*

$$P_{d,1}(t,T) = \mathbb{1}_{\{\tau_2 \le t\}}\, e^{-\alpha_1(T-t)} + \mathbb{1}_{\{\tau_2 > t\}} \frac{1}{\lambda-\alpha_1}\Big(\lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1-\alpha_1)e^{-\lambda(T-t)}\Big).$$

Furthermore

$$\tilde{P}_{d,1}(t,T) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{(\lambda-\alpha_2)\lambda_2 e^{-\alpha_1(T-\tau_2)}}{\lambda_1\alpha_2 e^{(\lambda-\alpha_2)\tau_2} + \lambda(\lambda_2-\alpha_2)} + \mathbb{1}_{\{\tau_2 > t\}} \frac{\lambda-\alpha_2}{\lambda-\alpha_1}\, \frac{(\lambda_1-\alpha_1)e^{-\lambda(T-t)} + \lambda_2 e^{-\alpha_1(T-t)}}{\lambda_1 e^{-(\lambda-\alpha_2)t} + \lambda_2 - \alpha_2}$$

and

$$\widehat{P}_{d,1}(t,T) = \mathbb{1}_{\{\tau_1 > t\}} \frac{\lambda_2 e^{-\alpha_1 T} + (\lambda_1-\alpha_1)e^{-\lambda T}}{\lambda_2 e^{-\alpha_1 t} + (\lambda_1-\alpha_1)e^{-\lambda t}}.$$

Observe that:

- Formula for $P_{d,1}(t,T)$ coincides with the Jarrow and Yu formula for the bond issued by a secondary firm.
- Processes $P_{d,1}(t,T)$ and $\widehat{P}_{d,1}(t,T)$ represent ex-dividend values of the bond, and thus they vanish after default time τ_1.
- The latter remark does not apply to the process $\tilde{P}_{d,1}(t,T)$.

5.6 Markovian Models of Credit Migrations

In this section we give a brief description of a Markovian market model that can be efficiently used for evaluating and hedging basket credit instruments. This framework, is a special case of a more general model introduced in Bielecki et al. [4], which allows to incorporate information relative to the dynamic evolution of credit ratings and credit migration processes in the pricing of basket instruments. Empirical study of the model is carried in Bielecki et al. [15].

We start with some notation. Let the underlying probability space be denoted by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, where $\mathbb{Q}$ is a risk neutral measure inferred from the market (we shall discuss this in further detail when addressing the issue of model calibration), $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ is a filtration containing all information available to market agents. The filtration $\mathbb{H}$ carries information about evolution of credit events, such as changes in credit ratings or defaults of respective credit names. The filtration $\mathbb{F}$ is a reference filtration containing information pertaining to the evolution of relevant macroeconomic variables.

We consider L obligors (or credit names) and we assume that the current credit quality of each reference entity can be classified into $\mathcal{K} := \{1, 2, \ldots, K\}$ rating categories. By convention, the category K corresponds to default. Let X^ℓ, $\ell = 1, 2, \ldots, L$ be some processes on $(\Omega, \mathcal{G}, \mathbb{Q})$ taking values in the finite state space $\mathcal{K}$. The processes X^ℓ represent the evolution of credit ratings of the ℓ^{th} reference entity. We define the *default time* τ_l of the ℓ^{th} reference entity by setting

$$\tau_l = \inf\{\, t \in \mathbb{R}_+ : X^\ell_t = K\} \tag{5.4}$$

We assume that the default state K is absorbing, so that for each name the default event can only occur once.

We denote by $X = (X^1, X^2, \ldots, X^L)$ the joint credit rating process of the portfolio of L credit names. The state space of X is $\mathcal{X} := \mathcal{K}^L$ and the elements of $\mathcal{X}$ will be denoted by x. We postulate that the filtration $\mathbb{H}$ is the natural filtration of the process X and that the filtration $\mathbb{F}$ is generated by a $\mathbb{R}^n$ valued factor process, Y, representing the evolution of relevant economic variables, like short rate or equity price processes.

5.6.1 Infinitesimal Generator

We assume that the factor process Y takes values in $\mathbb{R}^n$ so that the state space for the process $M = (X, Y)$ is $\mathcal{X} \times \mathbb{R}^n$. At the intuitive level, we wish to model the process $M = (X, Y)$ as a combination of a Markov chain X modulated by the Lévy-like process Y and a Lévy-like process Y modulated by a Markov chain X. To be more specific, we postulate that the *infinitesimal generator* $\mathbf{A}$ of M is given as

$$\begin{aligned}\mathbf{A}f(x,y) &= (1/2)\sum_{i,j=1}^{n} a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(x,y)\partial_i f(x,y)\\ &+ \gamma(x,y)\int_{\mathbb{R}^n}\big(f(x,y+g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy') + \sum_{x'\in\mathcal{X}} \lambda(x,x';y)f(x',y),\end{aligned}$$

where $\lambda(x,x';y) \geq 0$ for every $x = (x^1, x^2, \ldots, x^L) \neq (x'^1, x'^2, \ldots, x'^L) = x'$, and

$$\lambda(x,x;y) = -\sum_{x'\in\mathcal{X},\, x'\neq x} \lambda(x,x';y).$$

Here ∂_i denotes the partial derivative with respect to the variable y^i. The existence and uniqueness of a Markov process M with the generator $\mathbf{A}$ will follow (under appropriate technical conditions) from the respective results regarding martingale problems.

We find it convenient to refer to X (Y, respectively) as the *Markov chain component* of M (the *jump-diffusion component* of M, respectively). At any time t, the intensity matrix of the Markov chain component is given as $\Lambda_t = [\lambda(x,x';Y_t)]_{x,x'\in\mathcal{X}}$. The jump-diffusion component satisfies the SDE:

$$dY_t = b(X_t, Y_t)\,dt + \sigma(X_t, Y_t)\,dW_t + \int_{\mathbb{R}^n} g(X_{t-}, Y_{t-}, y')\,\pi(X_{t-}, Y_{t-}; dy', dt),$$

where, for a fixed $(x, y) \in \mathcal{X} \times \mathbb{R}^n$, $\pi(x, y; dy', dt)$ is a Poisson measure with the intensity measure $\gamma(x, y)\Pi(x, y; dy')dt$, and where $\sigma(x, y)$ satisfies the equality $\sigma(x, y)\sigma(x, y)^{\mathsf{T}} = a(x, y)$.

Remarks 5.6.1 If we take $g(x, y, y') = y'$, and we suppose that the coefficients $\sigma = [\sigma_{ij}]$, $b = [b_i]$, γ, and the measure Π do not depend on x and y then the factor process Y is a Poisson-Lévy process with the characteristic triplet (a, b, ν), where the diffusion matrix is $a(x, y) = \sigma(x, y)\sigma(x, y)^{\mathsf{T}}$, the "drift" vector is $b(x, y)$, and the Lévy measure is $\nu(dy) = \gamma\Pi(dy)$. In this case, the migration process X is modulated by the factor process Y, but not vice versa. We shall not study here the "infinite activity" case, that is, the case when the jump measure π is not a Poisson measure, and the related Lévy measure is an infinite measure.

We shall provide with more structure the Markov chain part of the generator $\mathbf{A}$. Specifically, we make the following standing assumption.

Asumption (M). The infinitesimal generator of the process $M = (X, Y)$ takes the following form

$$\begin{aligned}\mathbf{A}f(x,y) &= (1/2)\sum_{i,j=1}^{n} a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(x,y)\partial_i f(x,y)\\ &+ \gamma(x,y)\int_{\mathbb{R}^n}\big(f(x,y+g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy') \qquad (5.5)\\ &+ \sum_{l=1}^{L}\sum_{x^{l'}\in\mathcal{K}} \lambda^l(x,x'_l;y)f(x'_l,y),\end{aligned}$$

where we write $x'_l = (x^1, x^2, \ldots, x^{l-1}, x'^l, x^{l+1}, \ldots, x^L)$.

Note that x'_l is the vector $x = (x^1, x^2, \ldots, \ldots, x^L)$ with the l^{th} coordinate x^l replaced by x'^l. In the case of two obligors (i.e., for $L = 2$), the generator becomes

$$\begin{aligned}\mathbf{A}f(x,y) &= (1/2)\sum_{i,j=1}^{n} a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(x,y)\partial_i f(x,y)\\ &+ \gamma(x,y)\int_{\mathbb{R}^n}\big(f(x,y+g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy')\\ &+ \sum_{x'^1\in\mathcal{K}} \lambda^1(x,x'_1;y)f(x'_1,y) + \sum_{x'^2\in\mathcal{K}} \lambda^2(x,x'_2;y)f(x'_2,y),\end{aligned}$$

where $x = (x^1, x^2)$, $x'_1 = (x'^1, x^2)$ and $x'_2 = (x^1, x'^2)$. In this case, coming back to the general form, we have for $x = (x^1, x^2)$ and $x' = (x'^1, x'^2)$

$$\lambda(x,x';y) = \begin{cases} \lambda^1(x,x'_1;y), & \text{if } x^2 = x'^2,\\ \lambda^2(x,x'_2;y), & \text{if } x^1 = x'^1,\\ 0, & \text{otherwise.}\end{cases}$$

Similar expressions can be derived in the case of a general value of L. Note that the model specified by (5.5) does not allow for simultaneous jumps of the components X^l and $X^{l'}$ for $l \neq l'$. In other words, the ratings of different credit names may not change simultaneously.

Nevertheless, this is not a serious lack of generality, as the ratings of both credit names may still change in an arbitrarily small time interval. The advantage is that, for the purpose of simulation of paths of process X, rather than dealing with $\mathcal{X}\times\mathcal{X}$ intensity matrix $[\lambda(x,x';y)]$, we shall deal with L intensity matrices $[\lambda^l(x,x'_l;y)]$, each of dimension $\mathcal{K}\times\mathcal{K}$ (for any fixed y). The structure (5.5) is assumed in the rest of the paper. Let us stress that within the present set-up the current credit rating of the credit name l directly impacts the intensity of transition of the rating of the credit name l', and vice versa. This property, known as *frailty*, may contribute to default contagion.

Remarks 5.6.2 (i) It is clear that we can incorporate in the model the case when some – possibly all – components of the factor process Y follow Markov chains themselves. This feature is important, as factors such as economic cycles may be modeled as Markov chains. It is known that default rates are strongly related to business cycles.

(ii) Some of the factors $Y^1, Y^2, \ldots, Y^d$ may represent cumulative duration of visits of rating processes X^l in respective rating states. For example, we may set $Y^1_t = \int_0^t \mathbb{1}_{\{X^1_s=1\}}\,ds$. In this case, we have $b_1(x,y) = \mathbb{1}_{\{x^1=1\}}(x)$, and the corresponding components of coefficients σ and g equal zero.

(iii) In the area of *structural arbitrage*, so called *credit–to–equity* (C2E) models and/or *equity–to–credit* (E2C) models are studied. Our market model nests both types of interactions, that is C2E and E2C. For example, if one of the factors is the price process of the equity issued by a credit name, and if credit migration intensities depend on this factor (implicitly or explicitly) then we have a E2C type interaction. On the other hand, if credit ratings of a given obligor impact the equity dynamics (of this obligor and/or some other obligors), then we deal with a C2E type interaction.

As already mentioned, $S = (H, X, Y)$ is a Markov process on the state space $\{0, 1, \ldots, L\}\times\mathcal{X}\times\mathbb{R}^d$ with respect to its natural filtration. Given the form of the generator of the process (X, Y), we can easily describe the generator of the process (H, X, Y). It is enough to observe that the transition intensity at time t of the component H from the state H_t to the state $H_t + 1$ is equal to $\sum_{l=1}^{L}\lambda^l(X_t, K; X^{(l)}_t, Y_t)$, provided that $H_t < L$ (otherwise, the transition intensity equals zero), where we write

$$X^{(l)}_t = (X^1_t, \ldots, X^{l-1}_t, X^{l+1}_t, \ldots, X^L_t)$$

and we set $\lambda^l(x^l, x'^l; x^{(l)}, y) = \lambda^l(x, x'_l; y)$.

5.6.2 Specification of Credit Ratings Transition Intensities

One always needs to find a compromise between realistic features of a financial model and its complexity. This issue frequently nests the issues of functional representation of a model, as well as its parameterization. We present here an example of a particular model for credit ratings transition rates, which is rather arbitrary, but is nevertheless relatively simple and should be easy to estimate/calibrate.

Let $\bar{X}_t$ be the average credit rating at time t, so that

$$\bar{X}_t = \frac{1}{L}\sum_{l=1}^{L} X_t^l.$$

Let $\mathcal{L} = \{l_1, l_2, \ldots, l_{\widehat{L}}\}$ be a subset of the set of all obligors, where $\widehat{L} < L$. We consider $\mathcal{L}$ to be a collection of "major players" whose economic situation, reflected by their credit ratings, effectively impacts all other credit names in the pool. The following exponential-linear "regression" model appears to be a plausible model for the rating transition intensities:

$$\begin{aligned} \ln \lambda^l(x, x_l'; y) &= \alpha_{l,0}(x^l, x'^l) + \sum_{j=1}^{n} \alpha_{l,j}(x^l, x'^l) y_j + \beta_{l,0}(x^l, x'^l) h \\ &+ \sum_{i=1}^{\bar{L}} \beta_{l,i}(x^l, x'^l) x^i + \tilde{\beta}_l(x^l, x'^l)\bar{x} + \widehat{\beta}_l(x^l, x'^l)(x^l - x'^l), \end{aligned} \tag{5.6}$$

where h represents a generic value of H_t, so that $h = \sum_{l=1}^{L} \mathbb{1}_{\{K\}}(x^l)$, and $\bar{x}$ represents a generic value of $\bar{X}_t$, that is, $\bar{x} = \frac{1}{L}\sum_{l=1}^{L} x^l$.

The number of parameters involved in (5.6) can easily be controlled by the number of model variables, in particular – the number of factors and the number of credit ratings, as well as structure of the transition matrix (see Section 5.7.3 below). In addition, the reduction of the number of parameters can be obtained if the pool of all L obligors is partitioned into a (small) number of homogeneous sub-pools. All of this is a matter of practical implementation of the model. Assume, for instance, that there are $\widetilde{L} \ll L$ homogeneous sub-pools of obligors, and the parameters $\alpha, \beta, \tilde{\beta}$ and $\widehat{\beta}$ in (5.6) do not depend on x^l, x'^l. Then the migration intensities (5.6) are parameterized by $\widetilde{L}(n + \widehat{L} + 4)$ parameters.

5.6.3 Conditionally Independent Migrations

Suppose that the intensities $\lambda^l(x, x_l'; y)$ do not depend on $x^{(l)} = (x^1, x^2, \ldots, x^{l-1}, x^{l+1}, \ldots, x^L)$ for every $l = 1, 2, \ldots, L$. In addition, assume that the dynamics of the factor process Y do not depend on the the migration process X. It turns out that in this case, given the structure of our generator as in (5.5), the migration processes X^l, $l = 1, 2, \ldots, L$, are conditionally independent given the sample path of the process Y.

We shall illustrate this point in the case of only two credit names in the pool (i.e., for $L = 2$) and assuming that there is no factor process, so that conditional independence really means independence between migration processes X^1 and X^2. For this, suppose that X^1 and X^2 are independent Markov chains, each taking values in the state space $\mathcal{K}$, with infinitesimal generator matrices Λ^1 and Λ^2, respectively. It is clear that the joint process $X = (X^1, X^2)$ is a Markov chain on $\mathcal{K} \times \mathcal{K}$. An easy calculation reveals that the infinitesimal generator of the process X is given as

$$\Lambda = \Lambda^1 \otimes \mathrm{Id}_K + \mathrm{Id}_K \otimes \Lambda^2,$$

where Id_K is the identity matrix of order K and $\otimes$ denotes the matrix tensor product. This agrees with the structure (5.5) in the present case.

5.6.4 Examples of Markov Market Models

We shall now present three pertinent examples of Markov market models. We assume here that a numaraire β is given; the choice of β depends on the problem at hand.

Markov Chain Migration Process

We assume here that there is no factor process Y. Thus, we only deal with a single migration process X. In this case, an attractive and efficient way to model credit migrations is to postulate that X is a *birth-and-death process* with absorption at state K. In this case, the intensity matrix Λ is tri-diagonal. To simplify the notation, we shall write $p_t(k,k') = \mathbb{Q}(X_{s+t} = k' \,|\, X_s = k)$. The transition probabilities $p_t(k,k')$ satisfy the following system of ODEs, for $t \geq 0$ and $k' \in \{1,2,\dots,K\}$,

$$\frac{dp_t(1,k')}{dt} = -\lambda(1,2)p_t(1,k') + \lambda(1,2)p_t(2,k'),$$

$$\frac{dp_t(k,k')}{dt} = \lambda(k,k-1)p_t(k-1,k') - (\lambda(k,k-1) + \lambda(k,k+1))p_t(k,k') + \lambda(k,k+1)p_t(k+1,k')$$

for $k = 2,3,\dots,K-1$, and

$$\frac{dp_t(K,k')}{dt} = 0,$$

with initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$. Once the transition intensities $\lambda(k,k')$ are specified, the above system can be easily solved. Note, in particular, that $p_t(K,k') = 0$ for every t if $k' \neq K$. The advantage of this representation is that the number of parameters can be kept small.

A slightly more flexible model is produced if we allow for jumps to the default state K from any other state. In this case, the master ODEs take the following form, for $t \geq 0$ and $k' \in \{1,2,\dots,K\}$,

$$\frac{dp_t(1,k')}{dt} = -(\lambda(1,2) + \lambda(1,K))p_t(1,k') + \lambda(1,2)p_t(2,k') + \lambda(1,K)p_t(K,k'),$$

$$\begin{aligned}\frac{dp_t(k,k')}{dt} &= \lambda(k,k-1)p_t(k-1,k') - (\lambda(k,k-1) + \lambda(k,k+1) + \lambda(k,K))p_t(k,k') \\ &\quad + \lambda(k,k+1)p_t(k+1,k') + \lambda(k,K)p_t(K,k')\end{aligned}$$

for $k = 2,3,\dots,K-1$, and

$$\frac{dp_t(K,k')}{dt} = 0,$$

with initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$. Some authors model migrations of credit ratings using a (proxy) diffusion, possibly with jumps to default. The birth-and-death process with jumps to default furnishes a Markov chain counterpart of such proxy diffusion models. The nice feature of the Markov chain model is that the credit ratings are (in principle) observable state variables – whereas in case of the proxy diffusion models they are not.

Diffusion-type Factor Process

We now add a factor process Y to the model. We postulate that the factor process is a diffusion process and that the generator of the process $M = (X,Y)$ takes the form

$$\begin{aligned}\mathbf{A}f(x,y) &= (1/2)\sum_{i,j=1}^{n} a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(x,y)\partial_i f(x,y) \\ &\quad + \sum_{x' \in \mathcal{K},\, x' \neq x} \lambda(x,x';y)(f(x',y) - f(x,y)).\end{aligned}$$

Let $\phi(t,x,y,x',y')$ be the transition probability of M. Formally,

$$\phi(t,x,y,x',y')\,dy' = \mathbb{Q}(X_{s+t}=x', Y_{s+t}\in dy' \,|\, X_s=x, Y_s=y).$$

In order to determine the function ϕ, we need to study the following Kolmogorov equation

$$\frac{dv(s,x,y)}{ds} + \mathbf{A}v(s,x,y) = 0. \tag{5.7}$$

For the generator $\mathbf{A}$ of the present form, equation (5.7) is commonly known as the *reaction-diffusion equation.* It is worth mentioning that a reaction-diffusion equation is a special case of a more general integro-partial-differential equation (IPDE). In a future work, we shall deal with issue of practical solving of equations of this kind.

CDS Spread Factor Model

Suppose now that the factor process $Y_t = \kappa^{(1)}(t,T^S,T^M)$ is the forward CDS spread (for the definition of $\kappa^{(1)}(t,T^S,T^M)$, see Section 5.6.5 below), and that the generator for (X,Y) is

$$\mathbf{A}f(x,y) = (1/2)y^2a(x)\frac{d^2f(x,y)}{dy^2} + \sum_{x'\in\mathcal{K},\,x'\neq x}\lambda(x,x')(f(x',y)-f(x,y)).$$

Thus, the credit spread satisfies the following SDE

$$d\kappa^{(1)}(t,T^S,T^M) = \kappa^{(1)}(t,T^S,T^M)\sigma(X_t)\,dW_t$$

for some Brownian motion process W, where $\sigma(x)=\sqrt{a(x)}$. Note that in this example $\kappa^{(1)}(t,T^S,T^M)$ is a conditionally log-Gaussian process given a sample path of the migration process X, so that we are in the position to make use of Proposition 5.6.1 below.

5.6.5 Forward CDS

As before, the reference claim is a defaultable bond maturing at time U. We now consider a *forward (start) CDS* with the maturity date $T^M<U$ and the start date $T^S<T^M$. If default occurs prior to or at time T^S the contract is terminated with no exchange of payments. Therefore, the two legs of this CDS are manifestly T^S-survival claims, and the valuation of a forward CDS is not much different from valuation a straight CDS discussed above.

Default Payment Leg

As before, we let $N=1$ be the notional amount of the bond, and we let δ be a deterministic recovery rate in case of default. The recovery is paid at default, so that the cash flow associated with the default payment leg of the forward CDS can be represented as follows

$$(1-\delta)\mathbb{1}_{\{T^S<\tau\le T^M\}}\mathbb{1}_\tau(t).$$

For any $t\le T^S$, the time-t value of the default payment leg is equal to

$$A_t^{(1),T^S} = (1-\delta)B_t\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T^S<\tau\le T^M\}}B_\tau^{-1}\,|\,M_t\big).$$

As explained above, we can compute this conditional expectation. If B is a deterministic function of time then simply

$$B_t\,\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{T^S<\tau\le T^M\}}B_\tau^{-1}\,|\,M_t\right) = B_t\int_{T^S}^{T^M}B_s^{-1}\,\mathbb{Q}(\tau\in ds\,|\,M_t).$$

Premium Payment Leg

Let $\mathcal{T} = \{T_1, T_2, \dots, T_J\}$ be the tenor of premium payment, where $T^S < T_1 < \dots < T_J < T^M$. As before, we assume that the premium accrual covenant is in force, so that the cash flows associated with the premium payment leg are

$$\kappa \left(\sum_{j=1}^{J} \mathbb{1}_{\{T_j < \tau\}} \mathbb{1}_{T_j}(t) + \sum_{j=1}^{J} \mathbb{1}_{\{T_{j-1} < \tau \le T_j\}} \mathbb{1}_{\tau}(t) \frac{t - T_{j-1}}{T_j - T_{j-1}} \right).$$

Thus, for any $t \le T^S$ the time-t value of the premium payment leg is $\kappa B_t^{(1),T^S}$, where

$$B_t^{(1),T^S} = \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\{T_S < \tau\}} \left[\sum_{j=1}^{J} \frac{B_t}{B_{T_j}} \mathbb{1}_{\{T_j < \tau\}} + \sum_{j=1}^{J} \frac{B_t}{B_\tau} \mathbb{1}_{\{T_{j-1} < \tau \le T_j\}} \frac{\tau - T_{j-1}}{T_j - T_{j-1}} \right] \middle| M_t \right).$$

Again, knowing the conditional density $\mathbb{Q}(\tau \in ds \,|\, M_t)$, we can evaluate this conditional expectation.

5.6.6 Credit Default Swaptions

We consider a forward CDS swap starting at T^S and maturing at $T^M > T^S$, as described in the previous section. We shall now value the corresponding *credit default swaption* with expiry date $T < T^S$. Let K be the strike CDS rate of the swaption. Then the swaption's cash flow at expiry date T equals

$$\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+,$$

so that the price of the swaption equals, for any $t \le T$,

$$B_t \, \mathbb{E}_{\mathbb{Q}} \Big(B_T^{-1} \big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,\Big|\, M_t \Big) = B_t \, \mathbb{E}_{\mathbb{Q}} \Big(B_T^{-1} B_T^{(1),T^S} \big(\kappa^{(1)}(t, T^S, T^M) - K\big)^+ \,\Big|\, M_t \Big),$$

where $\kappa^{(1)}(t, T^S, T^M) := A_t^{(1),T^S} / B_t^{(1),T^S}$ is the *forward CDS rate.* Note that the random variables $A_t^{(1),T^S}$ and $B_t^{(1),T^S}$ are strictly positive on the event $\{\tau > T\}$ for $t \le T < T^S$, and thus $\kappa^{(1)}(t, T^S, T^M)$ enjoys the same property.

Conditionally Gaussian Case

We shall now provide a more explicit representation for the value of a CDS swaption. To this end, we assume that the forward CDS swap rates $\kappa^{(1)}(t, T^S, T^M)$ are conditionally log-Gaussian under $\mathbb{Q}$ for $t \le T$ (for an example of such a model, see Section 5.6.4). Then we have the following result.

Proposition 5.6.1 *Suppose that, on the set $\{\tau > T\}$ and for arbitrary $t < t_1 < \dots < t_n \le T$, the conditional distribution*

$$\mathbb{Q}\Big(\kappa^{(1)}(t_1, T^S, T^M) \le k_1, \kappa^{(1)}(t_2, T^S, T^M) \le k_2, \dots, \kappa^{(1)}(t_n, T^S, T^M) \le k_n \,\Big|\, \sigma(M_t) \vee \mathcal{F}_T^X \Big)$$

is lognormal, $\mathbb{Q}$-a.s. Let $\sigma(s, T^S, T^M)$, $s \in [t, T]$, denote the conditional volatility of the process $\kappa^{(1)}(s, T^S, T^M)$, $s \in [t, T]$, given the σ-field $\sigma(M_t) \vee \mathcal{F}_T^X$. Then the price of a CDS swaption equals, for $t < T$,

$$\begin{aligned} &B_t \, \mathbb{E}_{\mathbb{Q}} \Big(B_T^{-1} \big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,\Big|\, M_t \Big) \\ &\quad = B_t \, \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\{\tau > T\}} B_T^{-1} B_T^{(1),T^S} \left[\kappa^{(1)}(t, T^S, T^M) N \left(\frac{\ln \frac{\kappa^{(1)}(t, T^S, T^M)}{K}}{\upsilon_{t,T}} + \frac{\upsilon_{t,T}}{2} \right) \right. \right. \\ &\quad \left. \left. - KN \left(\frac{\ln \frac{\kappa^{(1)}(t, T^S, T^M)}{K}}{\upsilon_{t,T}} - \frac{\upsilon_{t,T}}{2} \right) \right] \middle| M_t \right), \end{aligned}$$

where

$$\upsilon_{t,T}^2 = \upsilon(t,T,T^S,T^M)^2 := \int_t^T \sigma(s,T^S,T^M)^2\,ds.$$

Proof. We have

$$\begin{aligned} B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1}\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,\Big|\, M_t\Big) &= B_t\,\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau>T\}}B_T^{-1}\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,\Big|\, M_t\Big) \\ &= B_t\,\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau>T\}}B_T^{-1}\mathbb{E}_{\mathbb{Q}}\Big(\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,|\, \sigma(M_t)\vee\mathcal{F}_T^X\Big) \,\Big|\, M_t\Big) \\ &= B_t\,\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau>T\}}B_T^{-1}B_T^{(1),T^S}\mathbb{E}_{\mathbb{Q}}\Big(\big(\kappa^{(1)}(T,T^S,T^M) - K\big)^+ \,|\, \sigma(M_t)\vee\mathcal{F}_T^X\Big) \,\Big|\, M_t\Big). \end{aligned}$$

In view of our assumptions, we obtain

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\Big(\big(\kappa^{(1)}(T,T^S,T^M) - K\big)^+ \,\Big|\, \sigma(M_t)\vee\mathcal{F}_T^X\Big) \\ &\quad = \kappa^{(1)}(t,T^S,T^M)N\left(\frac{\ln\frac{\kappa^{(1)}(t,T^S,T^M)}{K}}{\upsilon_{t,T}} + \frac{\upsilon_{t,T}}{2}\right) - KN\left(\frac{\ln\frac{\kappa^{(1)}(t,T^S,T^M)}{K}}{\upsilon_{t,T}} - \frac{\upsilon_{t,T}}{2}\right). \end{aligned}$$

By combining the above equalities, we arrive at the stated formula. □

5.7 Basket Credit Derivatives

We shall now discuss the case of credit derivatives with several underlying credit names. Feasibility of closed-form calculations, such as analytic computation of relevant conditional expected values, depends to a great extent on the type and amount of information one wants to utilize. Typically, in order to efficiently deal with exact calculations of conditional expectations, one will need to amend specifications of the underlying model so that information used in calculations is given by a coarser filtration, or perhaps by some proxy filtration.

5.7.1 k^{th}-to-Default CDS

We shall now discuss the valuation of a generic k^{th}*-to-default credit default swap* relative to a portfolio of L reference defaultable bonds. The deterministic notional amount of the i^{th} bond is denoted as N_i, and the corresponding deterministic recovery rate equals δ_i. We suppose that the maturities of the bonds are $U_1, U_2, \ldots, U_L$, and the maturity of the swap is $T < \min\{U_1, U_2, \ldots, U_L\}$.

As before, we shall only discuss a vanilla basket CDS written on such a portfolio of corporate bonds under the fractional recovery of par covenant. Thus, in the event that $\tau_{(k)} < T$, the buyer of the protection is paid at time $\tau_{(k)}$ a cumulative compensation

$$\sum_{i\in\mathcal{L}_k}(1-\delta_i)N_i,$$

where $\mathcal{L}_k$ is the (random) set of all reference credit names that defaulted in the time interval $]0,\tau_{(k)}]$. This means that the protection buyer is protected against the cumulative effect of the first k defaults. Recall that, in view of our model assumptions, the possibility of simultaneous defaults is excluded.

Default Payment Leg

The cash flow associated with the default payment leg is given by the expression

$$\sum_{i\in\mathcal{L}_k}(1-\delta_i)N_i\mathbb{1}_{\{\tau_{(k)}\le T\}}\mathbb{1}_{\tau_{(k)}}(t),$$

so that the time-t value of the default payment leg is equal to

$$A_t^{(k)} = B_t \, \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\{t<\tau_{(k)} \le T\}} B_{\tau_{(k)}}^{-1} \sum_{i \in \mathcal{L}_k} (1-\delta_i) N_i \,\Big|\, M_t \right).$$

In general, this expectation will need to be evaluated numerically by means of simulations.

A special case of a k^{th}-to-default-swap is when the protection buyer is protected against losses associated with the last default only. In the case of a *last-to-default credit default swap*, the cash flow associated with the default payment leg is given by the expression

$$(1-\delta_{\iota^{(k)}}) N_{\iota^{(k)}} \mathbb{1}_{\{\tau_{(k)} \le T\}} \mathbb{1}_{\tau_{(k)}}(t) = \sum_{i=1}^{L} (1-\delta_i) N_i \mathbb{1}_{\{H_{\tau_i}=k\}} \mathbb{1}_{\{\tau_{(i)} \le T\}} \mathbb{1}_{\tau_{(i)}}(t),$$

where $\iota^{(k)}$ stands for the identity of the k^{th} defaulting credit name. Assuming that the numeraire process B is deterministic, we can represent the value at time t of the default payment leg as follows:

$$\begin{aligned} A_t^{(k)} &= \sum_{i=1}^{L} B_t \, \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\{t<\tau_i \le T\}} \mathbb{1}_{\{H_{\tau_i}=k\}} B_{\tau_i}^{-1} (1-\delta_i) N_i \,|\, M_t \right) \\ &= \sum_{i=1}^{L} B_t (1-\delta_i) N_i \int_t^T B_s^{-1} \, \mathbb{Q}(H_s = k \,|\, \tau_i = s, M_t) \, \mathbb{Q}(\tau_i \in ds \,|\, M_t). \end{aligned}$$

Note that the conditional probability $\mathbb{Q}(H_s = k \,|\, \tau_i = s, M_t)$ can be approximated as

$$\mathbb{Q}(H_s = k \,|\, \tau_i = s, M_t) \approx \frac{\mathbb{Q}(H_s = k, X_{s-\epsilon}^i \neq K, X_s^i = K \,|\, M_t)}{\mathbb{Q}(X_{s-\epsilon}^i \neq K, X_s^i = K \,|\, M_t)}.$$

Hence, if the number L of credit names is small, so that the Kolmogorov equations for the conditional distribution of the process (H, X, Y) can be solved, the value of $A_t^{(k)}$ can be approximated analytically.

Premium Payment Leg

Let $\mathcal{T} = \{T_1, T_2, \dots, T_J\}$ denote the tenor of the premium payment, where $0 = T_0 < T_1 < \dots < T_J < T$. If the premium accrual covenant is in force, then the cash flows associated with the premium payment leg admit the following representation:

$$\kappa^{(k)} \left(\sum_{j=1}^{J} \mathbb{1}_{\{T_j < \tau_{(k)}\}} \mathbb{1}_{T_j}(t) + \sum_{j=1}^{J} \mathbb{1}_{\{T_{j-1} < \tau_{(k)} \le T_j\}} \mathbb{1}_{\tau_{(k)}}(t) \frac{t - T_{j-1}}{T_j - T_{j-1}} \right),$$

where $\kappa^{(k)}$ is the CDS premium. Thus, the time-t value of the premium payment leg is $\kappa^{(k)} B_t^{(k)}$, where

$$\begin{aligned} B_t^{(k)} &= \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\{t<\tau_{(k)}\}} \sum_{j=j(t)}^{N} \frac{B_t}{B_{T_j}} \mathbb{1}_{\{T_j < \tau_{(k)}\}} \,\Big|\, M_t \right) \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\{t<\tau_{(k)}\}} \sum_{j=j(t)}^{J} \frac{B_t}{B_{\tau_{(k)}}} \mathbb{1}_{\{T_{j-1} < \tau_{(k)} \le T_j\}} \frac{\tau_{(k)} - T_{j-1}}{T_j - T_{j-1}} \,\Big|\, M_t \right), \end{aligned}$$

where $j(t)$ is the smallest integer such that $T_{j(t)} > t$. Again, in general, the above conditional expectation will need to be approximated by simulation. And again, for a small portfolio size L, if either exact or numerical solution of relevant Kolmogorov equations can be derived, then an analytical computation of the expectation can be done (at least in principle).

5.7.2 Forward k^{th}-to-Default CDS

Forward k^{th}-to-default CDS has an analogous structure to the forward CDS. The notation used here is consistent with the notation used previously in Sections 5.6.5 and 5.7.1.

Default Payment Leg

The cash flow associated with the default payment leg can be expressed as follows

$$\sum_{i\in\mathcal{L}_k}(1-\delta_i)N_i\mathbb{1}_{\{T^S<\tau_{(k)}\le T^M\}}\mathbb{1}_{\tau_{(k)}}(t).$$

Consequently, the time-t value of the default payment leg equals, for every $t\le T^S$,

$$A_t^{(k),T^S}=B_t\,\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T^S<\tau_{(k)}\le T^M\}}B^{-1}_{\tau_{(k)}}\sum_{i\in\mathcal{L}_k}(1-\delta_i)N_i\,\Big|\,M_t\Big).$$

Premium Payment Leg

As before, let $\mathcal{T}=\{T_1,T_2,\ldots,T_J\}$ be the tenor of a generic premium payment leg, where $T^S<T_1<\cdots<T_J<T^M$. Under the premium accrual covenant, the cash flows associated with the premium payment leg are

$$\kappa^{(k)}\left(\sum_{j=1}^J\mathbb{1}_{\{T_j<\tau_{(k)}\}}\mathbb{1}_{T_j}(t)+\sum_{j=1}^J\mathbb{1}_{\{T_{j-1}<\tau_{(k)}\le T_j\}}\mathbb{1}_{\tau_{(k)}}(t)\frac{t-T_{j-1}}{T_j-T_{j-1}}\right),$$

where $\kappa^{(k)}$ is the CDS premium. Thus, the time-t value of the premium payment leg is $\kappa^{(k)}B_t^{(k),T^S}$, where

$$B_t^{(k),T^S}=\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{t<\tau_{(k)}\}}\left[\sum_{j=1}^N\frac{B_t}{B_{T_j}}\,\mathbb{1}_{\{T_j<\tau\}}+\sum_{j=1}^J\frac{B_t}{B_\tau}\,\mathbb{1}_{\{T_{j-1}<\tau_{(k)}\le T_j\}}\frac{\tau-T_{j-1}}{T_j-T_{j-1}}\right]\,\middle|\,M_t\right).$$

5.7.3 Model Implementation

The last section is devoted to a brief discussion of issues related to the model implementation.

Curse of Dimensionality

When one deals with basket products involving multiple credit names, direct computations may not be feasible. The cardinality of the state space $\mathbf{K}$ for the migration process X is equal to K^L. Thus, for example, in case of $K=18$ rating categories, as in Moody's ratings,[1] and in case of a portfolio of $L=100$ credit names, the state space $\mathbf{K}$ has 18^{100} elements.[2] If one aims at closed-form expressions for conditional expectations, but K is large, then it will typically be infeasible to work directly with information provided by the state vector $(X,Y)=(X^1,X^2,\ldots,X^L,Y)$ and with the corresponding generator $\mathbf{A}$. A reduction in the amount of information that can be effectively used for analytical computations will be needed. Such reduction may be achieved by reducing the number of distinguished rating categories – this is typically done by considering only two categories: pre-default and default.

[1] We think here of the following Moody's rating categories: Aaa, Aa1, Aa2, Aa3, A1, A2, A3, Baa1, Baa2, Baa3, Ba1, Ba2, Ba3, B1, B2, B3, Caa, D(efault).

[2] The number known as *Googol* is equal to 10^{100}. It is believed that this number is greater than the number of atoms in the entire observed Universe.

However, this reduction may still not be sufficient enough, and further simplifying structural modifications to the model may need to be called for. Some types of additional modifications, such as *homogeneous grouping* of credit names and the *mean-field interactions* between credit names.

Recursive Simulation Procedure

When closed-form computations are not feasible, but one does not want to give up on potentially available information, an alternative may be to carry approximate calculations by means of either approximating some involved formulae and/or by simulating sample paths of underlying random processes. This is the approach that we opt for.

In general, a simulation of the evolution of the process X will be infeasible, due to the curse of dimensionality. However, the structure of the generator $\mathbf{A}$ that we postulate (cf. (5.5)) makes it so that simulation of the evolution of process X reduces to recursive simulation of the evolution of processes X^l whose state spaces are only of size K each. In order to facilitate simulations even further, we also postulate that each migration process X^l behaves like a birth-and-death process with absorption at default, and with possible jumps to default from every intermediate state (cf. Section 5.6.4). Recall that

$$X_t^{(l)} = (X_t^1, \ldots, X_t^{l-1}, X_t^{l+1}, \ldots, X_t^L).$$

Given the state $(x^{(l)}, y)$ of the process $(X^{(l)}, Y)$, the intensity matrix of the l^{th} migration process is sub-stochastic and is given as:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \vdots \\ K-1 \\ K \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & K-1 & K \end{array} \\
\begin{pmatrix}
\lambda^l(1,1) & \lambda^l(1,2) & 0 & \cdots & 0 & \lambda^l(1,K) \\
\lambda^l(2,1) & \lambda^l(2,2) & \lambda^l(2,3) & \cdots & 0 & \lambda^l(2,K) \\
0 & \lambda^l(3,2) & \lambda^l(3,3) & \cdots & 0 & \lambda^l(3,K) \\
 & & & & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda^l(K-1,K-1) & \lambda^l(K-1,K) \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{array},$$

where we set $\lambda^l(x^l, x'^l) = \lambda^l(x, x'_l; y)$. Also, we find it convenient to write

$$\lambda^l(x^l, x'^l; x^{(l)}, y) = \lambda^l(x, x'_l; y)$$

in what follows.

Then the diagonal elements are specified as follows, for $x^l \neq K$,

$$\begin{aligned}
\lambda^l(x, x; y) = & -\lambda^l(x^l, x^l - 1; x^{(l)}, y) - \lambda^l(x^l, x^l + 1; x^{(l)}, y) - \lambda^l(x^l, K; x^{(l)}, y) \\
& - \sum_{i \neq l} \Big(\lambda^i(x^i, x^i - 1; x^{(i)}, y) + \lambda^i(x^i, x^i + 1; x^{(i)}, y) + \lambda^i(x^i, K; x^{(i)}, y) \Big)
\end{aligned}$$

with the convention that $\lambda^l(1, 0; x^{(l)}, y) = 0$ for every $l = 1, 2, \ldots, L$.

It is implicit in the above description that $\lambda^l(K, x^l; x^{(l)}, y) = 0$ for any $l = 1, 2, \ldots, L$ and $x^l = 1, 2, \ldots, K$. Suppose now that the current state of the process (X, Y) is (x, y). Then the intensity of a jump of the process X equals

$$\lambda(x, y) := -\sum_{l=1}^{L} \lambda^l(x, x; y).$$

Conditional on the occurrence of a jump of X, the probability distribution of a jump for the component X^l, $l = 1, 2, \ldots, L$, is given as follows:

- probability of a jump from x^l to $x^l - 1$ equals $p^l(x^l, x^l - 1; x^{(l)}, y) := \frac{\lambda^l(x^l, x^l-1; x^{(l)}, y)}{\lambda(x,y)}$,
- probability of a jump from x^l to $x^l + 1$ equals $p^l(x^l, x^l + 1; x^{(l)}, y) := \frac{\lambda^l(x^l, x^l+1; x^{(l)}, y)}{\lambda(x,y)}$,
- probability of a jump from x^l to K equals $p^l(x^l, K; x^{(l)}, y) := \frac{\lambda^l(x^l, K; x^{(l)}, y)}{\lambda(x,y)}$.

As expected, we have that

$$\sum_{l=1}^{L} \big(p^l(x^l, x^l - 1; x^{(l)}, y) + p^l(x^l, x^l + 1; x^{(l)}, y) + p^l(x^l, K; x^{(l)}, y) \big) = 1.$$

For a generic state $x = (x^1, x^2, \dots, x^L)$ of the migration process X, we define the *jump space*

$$\mathcal{J}(x) = \bigcup_{l=1}^{L} \{(x^l - 1, l), (x^l + 1, l), (K, l)\}$$

with the convention that $(K+1, l) = (K, l)$. The notation (a, l) refers to the l^{th} component of X. Given that the process (X, Y) is in the state (x, y), and conditional on the occurrence of a jump of X, the process X jumps to a point in the jump space $\mathcal{J}(x)$ according to the probability distribution denoted by $p(x, y)$ and determined by the probabilities p^l described above. Thus, if a random variable J has the distribution given by $p(x, y)$ then, for any $(x'^l, l) \in \mathcal{J}(x)$, we have that

$$\mathbb{Q}(J = (x'^l, l)) = p^l(x^l, x'^l; x^{(l)}, y).$$

Simulation Algorithm: Special Case

We shall now present in detail the case when the dynamics of the factor process Y do not depend on the credit migrations process X. The general case appears to be much harder.

Under the assumption that the dynamics of the factor process Y do not depend on the process X, the simulation procedure splits into two steps. In Step 1, a sample path of the process Y is simulated; then, in Step 2, for a given sample path Y, a sample path of the process X is simulated. We consider here simulations of sample paths over some generic time interval, say $[t_1, t_2]$, where $0 \le t_1 < t_2$. We assume that the number of defaulted names at time t_1 is less than k, that is $H_{t_1} < k$. We conduct the simulation until the k^{th} default occurs or until time t_2, whichever occurs first.

Step 1: The dynamics of the factor process are now given by the SDE

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dW_t + \int_{\mathbb{R}^n} g(Y_{t-}, y)\pi(Y_{t-}; dy, dt), \quad t \in [t_1, t_2].$$

Any standard procedure can be used to simulate a sample path of Y. Let us denote by $\widehat{Y}$ the simulated sample path of Y.

Step 2: Once a sample path of Y has been simulated, simulate a sample path of X on the interval $[t_1, t_2]$ until the k^{th} default time.

We exploit the fact that, according to our assumptions about the infinitesimal generator $\mathbf{A}$, the components of the process X do not jump simultaneously. Thus, the following algorithm for simulating the evolution of X appears to be feasible:

Step 2.1: Set the counter $n = 1$ and simulate the first jump time of the process X in the time interval $[t_1, t_2]$. Towards this end, simulate first a value, say $\widehat{\eta}_1$, of a unit exponential random variable η_1. The simulated value of the first jump time, τ_1^X, is then given as

$$\widehat{\tau}_1^X = \inf\Big\{ t \in [t_1, t_2] \,:\, \int_{t_1}^{t} \lambda(X_{t_1}, \widehat{Y}_u)\, du \geq \widehat{\eta}_1 \Big\},$$

where by convention the infimum over an empty set is $+\infty$. If $\widehat{\tau}_1^X = +\infty$, set the simulated value of the k^{th} default time to be $\widehat{\tau}_{(k)} = +\infty$, stop the current run of the simulation procedure and go to Step 3. Otherwise, go to Step 2.2.

Step 2.2: Simulate the jump of X at time $\widehat{\tau}_1^X$ by drawing from the distribution $p(X_{t_1}, \widehat{Y}_{\widehat{\tau}_1^X -})$ (cf. discussion in Section 5.7.3). In this way, one obtains a simulated value $\widehat{X}_{\widehat{\tau}_1^X}$, as well as the simulated value of the number of defaults $\widehat{H}_{\widehat{\tau}_1^X}$. If $\widehat{H}_{\widehat{\tau}_1^X} < k$ then let $n := n + 1$ and go to Step 2.3; otherwise, set $\widehat{\tau}_{(k)} = \widehat{\tau}_1^X$ and go to Step 3.

Step 2.3: Simulate the n^{th} jump of process X. Towards this end, simulate a value, say $\widehat{\eta}_n$, of a unit exponential random variable η_n. The simulated value of the n^{th} jump time τ_n^X is obtained from the formula

$$\widehat{\tau}_n^X = \inf\Big\{ t \in [\widehat{\tau}_{n-1}^X, t_2] \,:\, \int_{\widehat{\tau}_{n-1}^X}^{t} \lambda(X_{\widehat{\tau}_{n-1}^X}, \widehat{Y}_u)\, du \geq \widehat{\eta}_n \Big\}.$$

In case $\widehat{\tau}_n^X = +\infty$, let the simulated value of the k^{th} default time to be $\widehat{\tau}_{(k)} = +\infty$; stop the current run of the simulation procedure, and go to Step 3. Otherwise, go to Step 2.4.

Step 2.4: Simulate the jump of X at time $\widehat{\tau}_n^X$ by drawing from the distribution $p(X_{\widehat{\tau}_{n-1}^X}, \widehat{Y}_{\widehat{\tau}_n^X -})$. In this way, produce a simulated value $\widehat{X}_{\widehat{\tau}_n^X}$, as well as the simulated value of the number of defaults $\widehat{H}_{\widehat{\tau}_n^X}$. If $\widehat{H}_{\widehat{\tau}_n^X} < k$, let $n := n + 1$ and go to Step 2.3; otherwise, set $\widehat{\tau}_{(k)} = \widehat{\tau}_n^X$ and go to Step 3.

Step 3: Calculate a simulated value of a relevant functional. For example, in case of the k^{th}-to-default CDS, compute

$$\widehat{A}_{t_1}^{(k)} = \mathbb{1}_{\{t_1 < \widehat{\tau}_{(k)} \leq T\}} \widehat{B}_{t_1} \widehat{B}_{\widehat{\tau}_{(k)}}^{-1} \sum_{i \in \widehat{\mathcal{L}}_k} (1 - \delta_i) N_i \tag{5.8}$$

and

$$\widehat{B}_{t_1}^{(k)} = \sum_{j=j(t_1)}^{N} \frac{\widehat{B}_{t_1}}{\widehat{B}_{T_j}} \mathbb{1}_{\{T_j < \widehat{\tau}_{(k)}\}} + \sum_{j=j(t_1)}^{J} \frac{\widehat{B}_{t_1}}{\widehat{B}_{\widehat{\tau}_{(k)}}} \mathbb{1}_{\{T_{j-1} < \widehat{\tau}_{(k)} \leq T_j\}} \frac{\widehat{\tau}_{(k)} - T_{j-1}}{T_j - T_{j-1}}, \tag{5.9}$$

where, as usual, the 'hat' indicates that we deal with simulated values.

5.7.4 Standard Credit Basket Products

In this section, we describe the cash-flows associated to the mainstream basket credit products, focusing in particular on the recently developed standardized instruments like the Dow Jones Credit Default Swap indices (iTraxx and CDX), and the relative derivative contracts (Collateralized Debt Obligations and First-to-Default Swaps).

CDS Indices

CDS indices are static portfolios of L equally weighted credit default swaps (CDSs) with standard maturities, typically five or ten years. Typically, the index matures few months before the underlying CDSs. For instance, the five years iTraxx S3 (series three) and its underlying CDSs mature on June 2010 and December 2010 respectively. The debt obligations underlying the CDSs in the pool are selected from among those with highest CDS trading volume in the respective industry sector.

We shall refer to the underlying debt obligations as reference entities. We shall denote by $T > 0$ the maturity of any given CDS index.

CDS indices are typically issued by a pool of licensed financial institutions, which we shall call the market maker. At time of issuance of a CDS index, say at time $t = 0$, the market maker determines an annual rate known as *index spread*, to be paid out to investors on a periodic basis. We shall denote this rate by η_0.

In what follows, we shall assume that, at some time $t \in [0, T]$, an investor purchases one unit of CDS index issued at time zero. By purchasing the index, an investor enters into a binding contract whose main provisions are summarized below:

- The time of issuance of the contract 0. The inception time of the contract is time t; the maturity time of the contract is T.
- By purchasing the index, the investor sells protection to the market makers. Thus, the investor assumes the role of a protection seller and the market makers assume the role of protection buyers. In practice, the investors agrees to absorb all losses due to defaults in the reference portfolio, occurring between the time of inception t and the maturity T. In case of default of a reference entity, the protection seller pays to the market makers the protection payment in the amount of $(1 - \delta)$, where $\delta \in [0, 1]$ is the agreed recovery rate (typically 40%). We assume that the face value of each reference entity is one. Thus the total notional of the index is L. The notional on which the market maker pays the spread, henceforth referred to as *residual protection* is then reduced by some amount. For instance, after the first default, the residual protection is updated as follows (we adopt hereafter the former convention)

$$L \to L - (1 - \delta) \quad \text{or} \quad L \to L - 1.$$

- In exchange, the protection seller receives from the market maker a periodic fixed premium on the residual protection at the annual rate of η_t, that represents the fair index spread. (Whenever a reference entity defaults, its weight in the index is set to zero. By purchasing one unit of index the protection seller owes protection only on those names that have not yet defaulted at time of inception.) If, at inception of the contract, the market index spread is different from the issuance spread, i.e. $\eta_t \neq \eta_0$, the present value of the difference is settled through an upfront payment.

We denote by τ_i the random default time of the i^{th} name in the index and by H_t^i the right continuous process defined as $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$, $i = 1, 2, \ldots, L$. Also, let $\{t_j, j = 0, 1, \ldots, J\}$ with $t = t_0$ and $t_J \le T$ denote the tenor of the premium leg payments dates. The discounted cumulative cash flows associated with a CDS index are as follows:

$$\textit{Premium Leg} = \sum_{j=0}^{J} \frac{B_t}{B_{t_j}} \Big(\sum_{i=1}^{L} 1 - H_{t_j}^i (1 - \delta) \Big) \eta_t$$

and

$$\textit{Protection Leg} = \sum_{i=1}^{L} \frac{B_t}{B_{\tau_i}} \Big((1 - \delta)(H_T^i - H_t^i) \Big).$$

Collateralized Debt Obligations

Collateralized Debt Obligations (CDO) are credit derivatives backed by portfolios of assets. If the underlying portfolio is made up of bonds, loans or other securitized receivables, such products are known as *cash* CDOs. Alternatively, the underlying portfolio may consist of credit derivatives referencing a pool of debt obligations. In the latter case, CDOs are said to be *synthetic*. Because of their recently acquired popularity, we focus our discussion on standardized (synthetic) CDO contracts backed by CDS indices.

We begin with an overview of the product:

- The time of issuance of the contract is 0. The time of inception of the contract is $t \geq 0$, the maturity is T. The notional of the CDO contract is the residual protection of the underlying CDS index at the time of inception.

- The credit risk (the potential loss due to credit events) borne by the reference pool is layered into different risk levels. The range in between two adjacent risk levels is called a *tranche*. The lower bound of a tranche is usually referred to as *attachment point* and the upper bound as *detachment point*. The credit risk is sold in these tranches to protection sellers. For instance, in a typical CDO contract on iTraxx, the credit risk is split into equity, mezzanine, and senior tranches corresponding to $0-3\%, 3-6\%, 6-9\%, 9-12\%$, and $12-22\%$ of the losses, respectively. At inception, the notional value of each tranche is the CDO residual notional weighted by the respective *tranche width*.

- The tranche buyer sells partial protection to the pool owner, by agreeing to absorb the pool's losses comprised in between the tranche attachment and detachment point. This is better understood by an example. Assume that, at time t, the protection seller purchases one currency unit worth of the $6-9\%$ tranche. One year later, consequently to a default event, the cumulative loss breaks through the attachment point, reaching 8%. The protection seller then fulfills his obligation by disbursing two thirds ($= \frac{8\%-6\%}{9\%-6\%}$) of a currency unit. The tranche notional is then reduced to one third of its pre-default event value. We refer to the remaining tranche notional as *residual tranche protection*.

- In exchange, as of time t and up to time T, the CDO issuer (protection buyer) makes periodic payments to the tranche buyer according to a predetermined rate (termed tranche spread) on the residual tranche protection. We denote the time t spread of the l^{th} tranche by κ^l_t. Returning to our example, after the loss reaches 8%, premium payments are made on $\frac{1}{3}$ ($= \frac{9\%-8\%}{9\%-6\%}$) of the tranche notional, until the next credit event occurs or the contract matures.

We denote by L_l and U_l the lower and upper attachment points for the l^{th} tranche, κ^l_t its time t spread. It is also convenient to introduce the percentage loss process,

$$\Gamma^t_s = \frac{\sum_{i=1}^L (H^i_s - H^i_t)(1-\delta)}{\sum_{i=1}^L (1 - H^i_t)}$$

where L is the number of reference names in the basket. (Note that the loss is calculated only on the names which are not defaulted at the time of inception t.) Finally define by $C^l = U_l - L_l$ the portion of credit risk assigned to the l^{th} tranche.

Purchasing one unit of the l^{th} tranche at time t generates the following discounted cash flows:

$$\textit{Premium Leg} = \sum_{j=0}^J \frac{B_t}{B_{t_j}} \kappa^l_t \sum_{i=1}^L (1 - H^i_t)\Big(C^l - \min(C^l, \max(\Gamma^t_{t_j} - L_l, 0))\Big)$$

and

$$\textit{Protection Leg} = \sum_{i=1}^L \frac{B_t}{B_{t_j}} (H^i_T - H^i_t)(1-\delta)\mathbb{1}_{\{L_k \leq \Gamma^t_{\tau_i} \leq U_k\}}.$$

We remark here that the equity tranche of the CDO on iTraxx or CDX is quoted as an upfront rate, say κ^0_t, on the total tranche notional, in addition to 500 basis points (5% rate) paid annually on the residual tranche protection. The premium leg payment, in this case, is as follows:

$$\kappa^0_t C^0 \sum_{i=1}^L (1 - H^i_t) + \sum_{j=0}^J \frac{B_t}{B_{t_j}} (.05) \sum_{i=1}^L (1 - H^i_t)\Big(C^0 - \min\left(C^0, \max(\Gamma^t_{t_j} - L_0, 0)\right)\Big)$$

First-to-Default Swaps

The k$^{\text{th}}$-to-default swaps (NTDS) are basket credit instruments backed by portfolios of single name CDSs. Since the growth in popularity of CDS indices and the associated derivatives, NTDS have become rather illiquid. Currently, such products are typically customized bank to client contracts, and hence relatively bespoke to the client's credit portfolio. For this reason, we focus our attention on First-to-Default Swaps issued on the iTraxx index, which are the only ones with a certain degree of liquidity. Standardized FTDS are now issued on each of the iTraxx sector sub-indices. Each FTDS is backed by an equally weighted portfolio of five single name CDSs in the relative sub-index, chosen according to some liquidity criteria.

The main provisions contained in a FTDS contract are the following:

- The time of issuance of the contract is 0. The time of inception of the contract is t, the maturity is T.
- By investing in a FTDS, the protection seller agrees to absorb the loss produced by the first default in the reference portfolio
- In exchange, the protection seller is paid a periodic premium, known as FTDS spread, computed on the residual protection. We denote the time-t spread by φ_t.

Recall that $\{t_j, j = 0, 1, \ldots, J\}$ with $t = t_0$ and $t_J \leq T$ denotes the tenor of the premium leg payments dates. Also, denote by $\tau_{(1)}$ the (random) time of the first default in the pool. The discounted cumulative cash flows associated with a FTDS on an iTraxx sub-index containing N names are as follows (again we assume that each name in the basket has notional equal to one):

$$\textit{Premium Leg} = \sum_{j=0}^{J} \varphi_t \frac{B_t}{B_{t_j}} \mathbb{1}_{\{\tau_{(1)} \geq t_j\}}$$

and

$$\textit{Protection Leg} = \frac{B_t}{B_{\tau_{(1)}}} (1-\delta) \mathbb{1}_{\{\tau_{(1)} \leq T\}}.$$

Step-up Corporate Bonds

As of now, these products are not traded in baskets, however they are of interest because they offer protection against credit events other than defaults. In particular, step up bonds are corporate coupon issues for which the coupon payment depends on the issuer's credit quality: the coupon payment increases when the credit quality of the issuer declines. In practice, for such bonds, credit quality is reflected in credit ratings assigned to the issuer by at least one credit ratings agency (Moody's-KMV or Standard&Poor's). The provisions linking the cash flows of the step-up bonds to the credit rating of the issuer have different step amounts and different rating event triggers. In some cases, a step-up of the coupon requires a downgrade to the trigger level by both rating agencies. In other cases, there are step-up triggers for actions of each rating agency. Here, a downgrade by one agency will trigger an increase in the coupon regardless of the rating from the other agency. Provisions also vary with respect to step-down features which, as the name suggests, trigger a lowering of the coupon if the company regains its original rating after a downgrade. In general, there is no step-down below the initial coupon for ratings exceeding the initial rating.

Let X_t stand for some indicator of credit quality at time t. Assume that t_i, $i = 1, 2, \ldots, n$ are coupon payment dates and let $c_n = c(X_{t_{n-1}})$ be the coupons ($t_0 = 0$). The time t cumulative cash flow process associated to the step-up bond equals

$$D_t = (1 - H_T) \frac{B_t}{B_T} + \int_{(t,T]} (1 - H_u) \frac{B_t}{B_u} \, dC_u + \text{ possible recovery payment}$$

where $C_t = \sum_{t_i \leq t} c_i$.

5.7.5 Valuation of Standard Basket Credit Derivatives

We now discuss the pricing of the basket instruments introduced in previous sub-section. In particular, computing the fair spreads of such products involves evaluating the conditional expectation under the martingale measure $\mathbb{Q}$ of some quantities related to the cash flows associated to each instrument. In the case of CDS indexes, CDOs and FTDS, the fair spread is such that, at inception, the value of the contract is exactly zero, i.e the risk neutral expectations of the fixed leg and protection leg payments are identical.

The following expressions can be easily derived from the discounted cumulative cash flows given in the previous subsection.

- the time t fair spread of a single name CDS:

$$\eta_t^\ell = \frac{\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\left(\frac{B_t}{B_{\tau_\ell}} H_T^\ell\right)(1-\delta)}{\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\left(\sum_{j=0}^J \frac{B_t}{B_{t_j}}(1-H_{t_j}^\ell)\right)}$$

- the time t fair spread of a CDS index is:

$$\eta_t = \frac{\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\left(\sum_{i=1}^L \frac{B_t}{B_{\tau_i}}(1-\delta)(H_T^i - H_t^i)\right)}{\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\left(\sum_{j=0}^J \frac{B_t}{B_{t_j}}\left(\sum_{i=1}^L 1 - H_{t_j}^i(1-\delta)\right)\right)}$$

- the time t fair spread of the CDO equity tranche is:

$$\kappa_t^0 = \frac{1}{C^0\sum_{i=1}^L(1-H_t^i)}\Bigg(\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\sum_{i=1}^L \frac{B_t}{B_{\tau_i}}(H_T^i - H_t^i)(1-\delta)\mathbb{1}_{\{L_0\le\Gamma_{\tau_i}^t\le U_0\}}$$
$$-\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\sum_{j=0}^J \frac{B_t}{B_{t_j}}(.05)\sum_{i=1}^L(1-H_t^i)\Big(C^0 - \min(C^0, \max(\Gamma_{t_j}^t - L_0, 0))\Big)\Bigg)$$

- the time t fair spread of the ℓ^{th} CDO tranche is:

$$\kappa_t^\ell = \frac{\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\left(\sum_{i=1}^L \frac{B_t}{B_{\tau_i}}(H_T^i - H_t^i)(1-\delta)\mathbb{1}_{\{L_\ell\le\Gamma_{\tau_i}^t\le U_\ell\}}\right)}{\mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\left(\sum_{j=0}^J \frac{B_t}{B_{t_j}}\sum_{i=1}^L(1-H_t^i)\Big(C^l - \min(C^l, \max(\Gamma_{t_j}^t - L_l, 0))\Big)\right)}$$

- the time t fair spread of a first-to-default swap is:

$$\varphi_t = \frac{\frac{B_t}{B_{\tau_{(1)}}}(1-\delta)(\mathbb{1}_{\{\tau_{(1)}\le T\}})}{\sum_{j=0}^J \frac{B_t}{B_{t_j}}(\mathbb{1}_{\{\tau_{(1)}\ge t_j\}})}$$

- the time t fair value of the step up bond is:

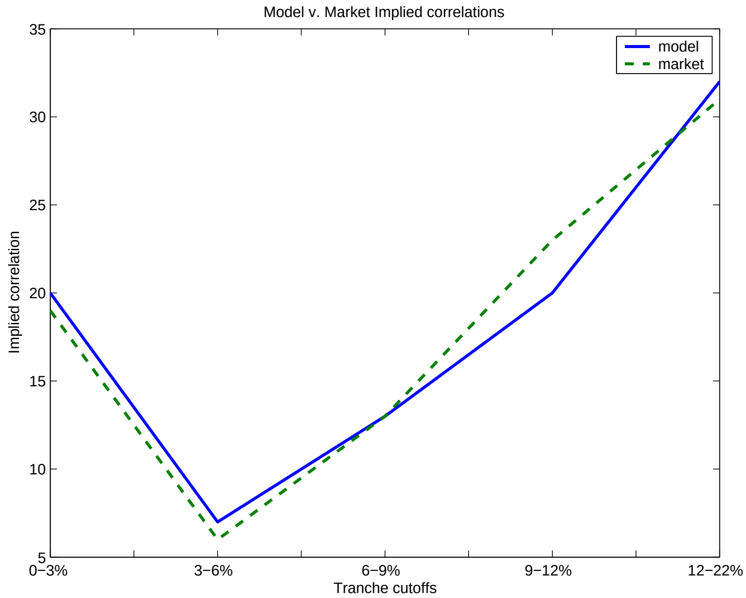
$$B^{su} = \mathbf{E}_{\mathbb{Q}}^{X_t,Y_t}\left((1-H_T)\frac{B_t}{B_T} + \int_{(t,T]}(1-H_u)\frac{B_t}{B_u}\,dC_u + \text{ possible recovery payment}\right)$$

Depending on the dimensionality of the problem, the above conditional expectations will be evaluated either by means of Monte Carlo simulation, or by means of some other numerical method and, in the low-dimensional case, even analytically.

It is perhaps worth mentioning that we have already done some numerical tests of our model so to see whether the model can reproduce so called market correlation skews. The picture below shows that the model performs very well in this regard.[3] For further examples of model's implementations, the interested reader is referred to Bielecki et al. [15].

[3]We thank Andrea and Luca Vidozzi from Applied Mathematics Department at the Illinois Institute of Technology for numerical implementation of the model and, in particular, for generating the picture.

Implied correlation skews for CDO tranches



5.7.6 Portfolio Credit Risk

The issue of evaluating functionals associated with multiple credit migrations, defaults in particular, is also prominent with regard to portfolio credit risk. In some segments of the credit markets, only the deterioration of the value of a portfolio of debts (bonds or loans) due to defaults is typically considered. In fact, such is the situation regarding various tranches of (either cash or synthetic) collateralized debt obligations, as well as with various tranches of recently introduced CDS indices, such as, DJ CDX NA IG or DJ iTraxx Europe.[4] Nevertheless, it is rather apparent that a valuation model reflecting the possibility of intermediate credit migrations, and not only defaults, is called for in order to better account for changes in creditworthiness of the reference credit names. Likewise, for the purpose of managing risks of a debt portfolio, it is necessary to account for changes in value of the portfolio due to changes in credit ratings of the components of the portfolio.

[4]See http://www.creditflux.com/public/publications/0409CFindexGuide.pdf.

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