Free brace algebras are free pre-Lie algebras

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ABSTRACT. Let $\mathfrak{g}$ be a free brace algebra. This structure implies that $\mathfrak{g}$ is also a pre-Lie algebra and a Lie algebra. It is already known that $\mathfrak{g}$ is a free Lie algebra. We prove here that $\mathfrak{g}$ is also a free pre-Lie algebra, using a description of $\mathfrak{g}$ with the help of planar rooted trees, a permutative product, and manipulations on the Poincaré-Hilbert series of $\mathfrak{g}$.

KEYWORDS. Pre-Lie algebras, brace algebras.

AMS CLASSIFICATION. 17A30, 05C05, 16W30.

Contents

Introduction

Let $\mathcal{D}$ be a set. The Connes-Kreimer Hopf algebra of rooted trees $\mathcal{H}_R^{\mathcal{D}}$ is introduced in [5] in the context of Quantum Field Theory and Renormalization. It is a graded, connected, commutative, non-cocommutative Hopf algebra. If the characteristic of the base field is zero, the Cartier-Quillen-Milnor-Moore theorem insures that its dual $(\mathcal{H}_R^{\mathcal{D}})^*$ is the enveloping algebra of a Lie algebra, based on rooted trees (note that $(\mathcal{H}_R^{\mathcal{D}})^*$ is isomorphic to the Grossman-Larson Hopf algebra [10, 11], as proved in [12, 16]). This Lie algebra admits an operadic interpretation: it is the free pre-Lie algebra $\mathcal{PL}(\mathcal{D})$ generated by $\mathcal{D}$, as shown in [4]; recall that a (left) pre-Lie algebra, also called a Vinberg algebra or a left-symmetric algebra, is a vector space V with a product $\circ$ satisfying:

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

A non-commutative version of these objects is introduced in [9, 13]. Replacing rooted trees by planar rooted trees, a Hopf algebra $\mathcal{H}_{PR}^{\mathcal{D}}$ is constructed. This self-dual Hopf algebra is isomorphic to the Loday-Ronco free dendriform algebra based on planar binary trees [15], so by the dendriform Milnor-Moore theorem [2, 18], the space of its primitive elements, or equivalently the space of the primitive elements of its dual, admits a structure of brace algebra, described in terms of trees in [8] by graftings of planar forests on planar trees, and is in fact the free brace algebra $\mathcal{B}r(\mathcal{D})$ generated by $\mathcal{D}$. This structure implies also a structure of pre-Lie algebra on $\mathcal{B}r(\mathcal{D})$.

As a summary, the brace structure of $\mathcal{B}r(\mathcal{D})$ implies a pre-Lie structure on $\mathcal{B}r(\mathcal{D})$, which implies a Lie structure on $\mathcal{B}r(\mathcal{D})$. It is already proved in several ways that $\mathcal{PL}(\mathcal{D})$ and $\mathcal{B}r(\mathcal{D})$ are free Lie algebras in characteristic zero [3, 8]. A remaining question was the structure of $\mathcal{B}r(\mathcal{D})$ as a pre-Lie algebra. The aim of the present text is to prove that $\mathcal{B}r(\mathcal{D})$ is a free pre-Lie algebra. We use for this the notion of non-associative permutative algebra [14] and a manipulation of formal series. More precisely, we introduce in the second section of this text a non-associative permutative product $\star$ on $\mathcal{B}r(\mathcal{D})$ and we show that $(\mathcal{B}r(\mathcal{D}), \star)$ is free. As a corollary, we prove that the abelianisation of $\mathcal{H}_{PR}^{\mathcal{D}}$ (which is not $\mathcal{H}_{R}^{\mathcal{D}}$), is isomorphic to a Hopf algebra $\mathcal{H}_{R}^{\mathcal{D}'}$ for a good choice of $\mathcal{D}'$. This implies that $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is a cofree coalgebra and we recover in a different way the result of freeness of $\mathcal{B}r(\mathcal{D})$ as a Lie algebra in characteristic zero. Note that a similar result for algebras with two compatible associative products is proved with the same pattern in [6].

Notations. We denote by K a commutative field of characteristic zero. All objects (vector spaces, algebras...) will be taken over K.

1 A description of free pre-Lie and brace algebras

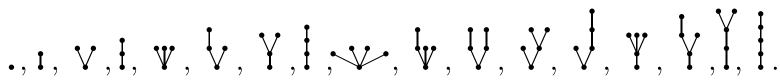
1.1 Rooted trees and planar rooted trees

Definition 1

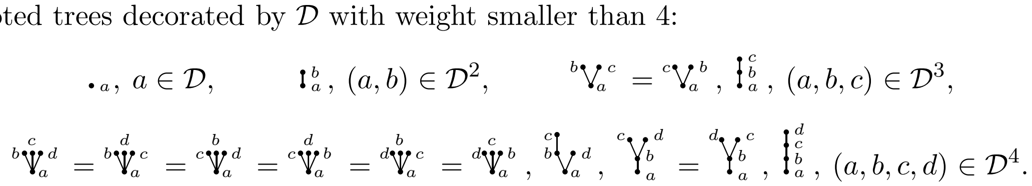
1. A *rooted tree* t is a finite graph, without loops, with a special vertex called the *root* of t. The *weight* of t is the number of its vertices. The set of rooted trees will be denoted by $\mathcal{T}$.
2. A *planar rooted tree* t is a rooted tree with an imbedding in the plane. the set of planar rooted trees will be denoted by $\mathcal{T}_P$.
3. Let $\mathcal{D}$ be a nonempty set. A rooted tree decorated by $\mathcal{D}$ is a rooted tree with an application from the set of its vertices into $\mathcal{D}$. The set of rooted trees decorated by $\mathcal{D}$ will be denoted by $\mathcal{T}^{\mathcal{D}}$.
4. Let $\mathcal{D}$ be a nonempty set. A planar rooted tree decorated by $\mathcal{D}$ is a planar tree with an application from the set of its vertices into $\mathcal{D}$. The set of planar rooted trees decorated by $\mathcal{D}$ will be denoted by $\mathcal{T}_P^{\mathcal{D}}$.

Examples.

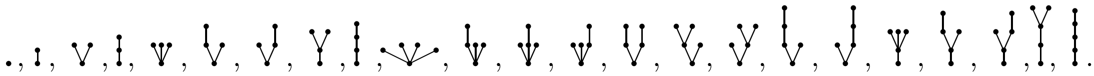
1. Rooted trees with weight smaller than 5:



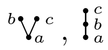
2. Rooted trees decorated by $\mathcal{D}$ with weight smaller than 4:

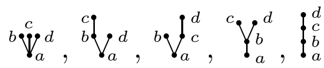


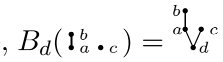
3. Planar rooted trees with weight smaller than 5:



4. Planar rooted trees decorated by $\mathcal{D}$ with weight smaller than 4:

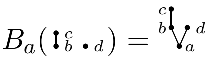
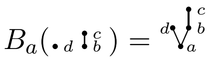
$\bullet_a$, $a \in \mathcal{D}$, , $(a,b) \in \mathcal{D}^2$, , $(a,b,c) \in \mathcal{D}^3$,

, $(a,b,c,d) \in \mathcal{D}^4$.

Let $t_1, \ldots, t_n$ be elements of $\mathcal{T}^{\mathcal{D}}$ and let $d \in \mathcal{D}$. We denote by $B_d(t_1 \ldots t_n)$ the rooted tree obtained by grafting $t_1, \ldots, t_n$ on a common root decorated by d. For example, . This application B_d can be extended in an operator:

$$B_d : \left\{ \begin{array}{rcl} K[\mathcal{T}^{\mathcal{D}}] & \longrightarrow & K\mathcal{T}^{\mathcal{D}} \\ t_1 \ldots t_n & \longrightarrow & B_d(t_1 \ldots t_n), \end{array} \right.$$

where $K[\mathcal{T}^{\mathcal{D}}]$ is the polynomial algebra generated by $\mathcal{T}^{\mathcal{D}}$ over K and $K\mathcal{T}^{\mathcal{D}}$ is the K-vector space generated by $\mathcal{T}^{\mathcal{D}}$. This operator is monic, and moreover $K\mathcal{T}^{\mathcal{D}}$ is the direct sum of the images of the B_d's, $d \in \mathcal{D}$.

Similarly, let $t_1, \ldots, t_n$ be elements of $\mathcal{T}_P^{\mathcal{D}}$ and let $d \in \mathcal{D}$. We denote by $B_d(t_1 \ldots t_n)$ the planar rooted tree obtained by grafting $t_1, \ldots, t_n$ in this order from left to right on a common root decorated by d. For example,  and . This application B_d can be extended in an operator:

$$B_d : \left\{ \begin{array}{rcl} K\langle\mathcal{T}_P^{\mathcal{D}}\rangle & \longrightarrow & K\mathcal{T}_P^{\mathcal{D}} \\ t_1 \ldots t_n & \longrightarrow & B_d(t_1 \ldots t_n), \end{array} \right.$$

where $K\langle\mathcal{T}_P^{\mathcal{D}}\rangle$ is the free associative algebra generated by $\mathcal{T}_P^{\mathcal{D}}$ over K and $K\mathcal{T}_P^{\mathcal{D}}$ is the K-vector space generated by $\mathcal{T}_P^{\mathcal{D}}$. This operator is monic, and moreover $K\mathcal{T}_P^{\mathcal{D}}$ is the direct sum of the images of the B_d's, $d \in \mathcal{D}$.

1.2 Free pre-Lie algebras

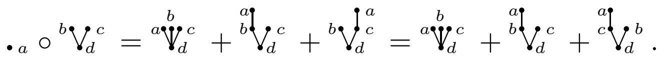
Definition 2 A (left) pre-Lie algebra is a couple $(A, \circ)$ where A is a vector space and $\circ : A \otimes A \longrightarrow A$ satisfying the following relation: for all $x, y, z \in A$,

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

Let $\mathcal{D}$ be a set. A description of the free pre-Lie algebra $\mathcal{PL}(\mathcal{D})$ generated by $\mathcal{D}$ is given in [4]. As a vector space, it has a basis given by $\mathcal{T}^{\mathcal{D}}$, and its pre-Lie product is given, for all $t_1, t_2 \in \mathcal{T}^{\mathcal{D}}$, by:

$$t_1 \circ t_2 = \sum_{s \text{ vertex of } t_2} \text{grafting of } t_1 \text{ on } s.$$

For example:



In other terms, the pre-Lie product can be inductively defined by:

$$\left\{ \begin{array}{rcl} t \circ \bullet_d & \longrightarrow & B_d(t), \\ t \circ B_d(t_1 \ldots t_n) & \longrightarrow & B_d(tt_1 \ldots t_n) + \displaystyle\sum_{i=1}^{n} B_d(t_1 \ldots (t \circ t_i) \ldots t_n). \end{array} \right.$$

Lemma 3 *Let $\mathcal{D}$ a set. We suppose that $\mathcal{D}$ has a gradation $(\mathcal{D}(n))_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $\mathcal{D}(n)$ is finite set of cardinality denoted by d_n, and $\mathcal{D}(0)$ is empty. We denote by $F_\mathcal{D}(x)$ the Poincaré-Hilbert series of this set:*

$$F_\mathcal{D}(x) = \sum_{n=1}^{\infty} d_n x^n.$$

This gradation induces a gradation $(\mathcal{PL}(\mathcal{D})(n))_{n\in\mathbb{N}}$ of $\mathcal{PL}(\mathcal{D})$. Moreover, for all $n \geq 0$, $\mathcal{PL}(\mathcal{D})(n)$ is finite-dimensional. We denote by $t_n^\mathcal{D}$ its dimension. Then the Poincaré-Hilbert series of $\mathcal{PL}(\mathcal{D})$ satisfies:

$$F_{\mathcal{PL}(\mathcal{D})}(x) = \sum_{n=1}^{\infty} t_n^\mathcal{D} x^n = \frac{F_\mathcal{D}(x)}{\prod_{i=1}^{\infty}(1-x^i)^{t_i^\mathcal{D}}}.$$

Proof. The formal series of the space $K[\mathcal{T}^\mathcal{D}]$ is given by:

$$F(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{t_i^\mathcal{D}}}.$$

Moreover, for all $d \in \mathcal{D}(n)$, B_d is homogeneous of degree n, so the Poincaré-Hilbert series of $Im(B_d)$ is $x^n F(x)$. As $\mathcal{PL}(\mathcal{D}) = K\mathcal{T}^\mathcal{D} = \bigoplus Im(B_d)$ as a graded vector space, its Poincaré-Hilbert formal series is:

$$F_{\mathcal{PL}(\mathcal{D})}(x) = F(x) \sum_{n=1}^{\infty} d_n x^n = F(x) F_\mathcal{D}(x),$$

which gives the announced result. $\Box$

1.3 Free brace algebras

Definition 4 [1, 2, 18] A brace algebra is a couple $(A, \langle\rangle)$ where A is a vector space and $\langle\rangle$ is a family of operators $A^{\otimes n} \longrightarrow A$ defined for all $n \geq 2$:

$$\left\{ \begin{array}{rcl} A^{\otimes n} & \longrightarrow & A \\ a_1 \otimes \ldots \otimes a_n & \longrightarrow & \langle a_1, \ldots, a_{n-1}; a_n \rangle, \end{array} \right.$$

with the following compatibilities: for all $a_1, \ldots, a_m$, $b_1, \ldots, b_n$, $c \in A$,

$$\langle a_1, \ldots, a_m; \langle b_1, \ldots, b_n; c \rangle \rangle = \sum \langle \langle A_0, \langle A_1; b_1 \rangle, A_2, \langle A_3; b_2 \rangle, A_4, \ldots, A_{2n-2}, \langle A_{2n-1}; b_n \rangle, A_{2n}; c \rangle,$$

where this sum runs over partitions of the ordered set $\{a_1, \ldots, a_n\}$ into (possibly empty) consecutive intervals $A_0 \sqcup \ldots \sqcup A_{2n}$. We use the convention $\langle a \rangle = a$ for all $a \in A$.

For example, if A is a brace algebra and $a, b, c \in A$:

$$\langle a; \langle b; c \rangle \rangle = \langle a, b; c \rangle + \langle b, a; c \rangle + \langle \langle a; b \rangle; c \rangle.$$

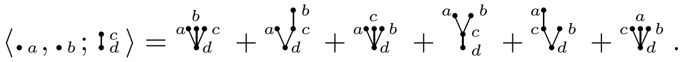
As an immediate corollary, $(A, \langle -; - \rangle)$ is a pre-Lie algebra. Here is another example of relation in a brace algebra: for all $a, b, c, d \in A$,

$$\langle a, b; \langle c; d \rangle \rangle = \langle a, b, c; d \rangle + \langle a, \langle b; c \rangle; d \rangle + \langle \langle a, b; c \rangle; d \rangle + \langle a, c, b; d \rangle + \langle \langle a; c \rangle, b; d \rangle + \langle c, a, b; d \rangle.$$

Let $\mathcal{D}$ be a set. A description of the free brace algebra $\mathcal{B}r(\mathcal{D})$ generated by $\mathcal{D}$ is given in [2, 9]. As a vector space, it has a basis given by $\mathcal{T}_P^\mathcal{D}$ and the brace structure is given, for all $t_1, \ldots, t_n \in \mathcal{T}_P^\mathcal{D}$, by:

$$\langle t_1, \ldots; t_n \rangle = \sum \text{graftings of } t_1 \ldots t_{n-1} \text{ over } t_n.$$

Note that for any vertex s of t_n, there are several ways of grafting a planar tree on s. For example:



As a consequence, the pre-Lie product of $\mathcal{B}r(\mathcal{D})$ can be inductively defined in this way:

$$\left\{\begin{array}{rcl}\langle t; \bullet_d\rangle & \longrightarrow & B_d(t),\\ \langle t; B_d(t_1\ldots t_n)\rangle & \longrightarrow & \displaystyle\sum_{i=0}^{n} B_d(t_1\ldots t_i t t_{i+1}\ldots t_n)+\sum_{i=1}^{n} B_d(t_1\ldots t_{i-1}\langle t;t_i\rangle t_{i+1}\ldots t_n).\end{array}\right.$$

Proposition 5 *$\mathcal{B}r(\mathcal{D})$ is the free brace algebra generated by $\mathcal{D}$.*

Proof. From [2, 9]. □

Lemma 6 *Let $\mathcal{D}$ a set, with the hypotheses and notations of lemma 3. The gradation of $\mathcal{D}$ induces a gradation $(\mathcal{B}r(\mathcal{D})(n))_{n\in\mathbb{N}}$ of $\mathcal{B}r(\mathcal{D})$. Moreover, for all $n \geq 0$, $\mathcal{B}r(\mathcal{D})(n)$ is finite-dimensional. Then the Poincaré-Hilbert series of $\mathcal{B}r(\mathcal{D})$ is:*

$$F_{\mathcal{B}r(\mathcal{D})}(x)=\sum_{n=1}^{\infty} t_n'^{\mathcal{D}} x^n=\frac{1-\sqrt{1-4F_{\mathcal{D}}(x)}}{2}.$$

Proof. The Poincaré-Hilbert formal series of $K\langle \mathcal{T}_P^{\mathcal{D}}\rangle$ is given by:

$$F(x)=\frac{1}{1-F_{\mathcal{B}r(\mathcal{D})}(x)}.$$

Moreover, for all $d \in \mathcal{D}(n)$, B_d is homogeneous of degree n, so the Poincaré-Hilbert series of $Im(B_d)$ is $x^n F(x)$. As $\mathcal{B}r(\mathcal{D}) = K\mathcal{T}_P^{\mathcal{D}} = \bigoplus Im(B_d)$ as a graded vector space, its Poincaré-Hilbert formal series is:

$$F_{\mathcal{B}r(\mathcal{D})}(x)=F(x)\sum_{n=1}^{\infty} d_n x^n = F(x)F_{\mathcal{D}}(x).$$

As a consequence, $F_{\mathcal{B}r(\mathcal{D})}(x) - F_{\mathcal{B}r(\mathcal{D})}(x)^2 = F_{\mathcal{D}}(x)$, which implies the announced result. □

2 A non-associative permutative product on $\mathcal{B}r(\mathcal{D})$

2.1 Definition and recalls

The following definition is introduced in [14]:

Definition 7 A (left) non-associative permutative algebra is a couple $(A, \star)$, where A is a vector space and $\star : A \otimes A \longrightarrow A$ satisfies the following property: for all $x, y, z \in A$,

$$x \star (y \star z) = y \star (x \star z).$$

Let $\mathcal{D}$ be a set. A description of the free non-associative permutative algebra $\mathcal{NAP}erm(\mathcal{D})$ generated by $\mathcal{D}$ is given in [14]. As a vector space, $\mathcal{NAP}erm(\mathcal{D})$ is equal to $K\mathcal{T}^{\mathcal{D}}$. The non-associative permutative product is given in this way: for all $t_1 \in \mathcal{T}^D$, $t_2 = B_d(F_2) \in \mathcal{T}^D$,

$$t_1 \star t_2 = B_d(t_1 F_2).$$

In other terms, $t_1 \star t_2$ is the tree obtained by grafting t_1 on the root of t_2. As $\mathcal{NAP}erm(\mathcal{D}) = \mathcal{PL}(\mathcal{D})$ as a vector space, lemma 3 is still true when one replaces $\mathcal{PL}(\mathcal{D})$ by $\mathcal{NAP}erm(\mathcal{D})$.

2.2 Permutative structures on planar rooted trees

Let us fix now a non-empty set $\mathcal{D}$. We define the following product on $\mathcal{B}r(\mathcal{D}) = K\mathcal{T}_P^{\mathcal{D}}$: for all $t \in \mathcal{T}_P^{\mathcal{D}}$, $t' = B_d(t_1 \dots t_n) \in \mathcal{T}_P^{\mathcal{D}}$,

$$t \star t' = \sum_{i=0}^{n} B_d(t_1 \dots t_i t t_{i+1} \dots t_n).$$

Proposition 8 *$(\mathcal{B}r(\mathcal{D}), \star)$ is a non-associative permutative algebra.*

Proof. Let us give $K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$ its shuffle product: for all $t_1, \dots, t_{m+n} \in \mathcal{T}_P^{\mathcal{D}}$,

$$(t_1 \dots t_m) * (t_{m+1} \dots t_{m+n}) = \sum_{\sigma \in Sh(m,n)} t_{\sigma^{-1}(1)} \dots t_{\sigma^{-1}(m+n)},$$

where $Sh(m,n)$ is the set of permutations of S_{m+n} which are increasing on $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$. It is well known that $*$ is an associative, commutative product. For example, for all $t, t_1, \dots, t_n \in \mathcal{T}_P^{\mathcal{D}}$:

$$t * (t_1 \dots t_n) = \sum_{i=0}^{n} t_1 \dots t_i t t_{i+1} \dots t_n.$$

As a consequence, for all $x \in K\mathcal{T}_P^{\mathcal{D}}$, $y \in K\langle \mathcal{T}_P^{\mathcal{D}} \rangle$, $d \in \mathcal{D}$:

$$x \star B_d(y) = B_d(x * y). \tag{1}$$

Let $t_1, t_2, t_3 = B_d(F_3) \in \mathcal{T}_P^{\mathcal{D}}$. Then, using (1):

$$\begin{array}{rcl} t_1 \star (t_2 \star t_3) & = & t_1 \star B_d(t_2 * F_3) \\ & = & B_d(t_1 * (t_2 * F_3)) \\ & = & B_d((t_1 * t_2) * F_3) \\ & = & B_d((t_2 * t_1) * F_3) \\ & = & B_d(t_2 * (t_1 * F_3)) \\ & = & t_2 \star (t_1 \star t_3). \end{array}$$

So $\star$ is a non-associative permutative product on $\mathcal{B}r(\mathcal{D})$. □

2.3 Freeness of $\mathcal{B}r(\mathcal{D})$ as a non-associative permutative algebra

We now assume that $\mathcal{D}$ is finite, of cardinality D. We can then assume that $\mathcal{D} = \{1, \dots, D\}$.

Theorem 9 *$(\mathcal{B}r(\mathcal{D}), \star)$ is a free non-associative permutative algebra.*

Proof. We graduate $\mathcal{D}$ by putting $\mathcal{D}(1) = \mathcal{D}$. Then $\mathcal{B}r(\mathcal{D})$ is graded, the degree of a tree $t \in \mathcal{T}_P^{\mathcal{D}}$ being the number of its vertices. By lemma 6, as the Poincaré-Hilbert series of $\mathcal{D}$ is $F_{\mathcal{D}}(x) = Dx$, the Poincaré-Hilbert series of $\mathcal{B}r(\mathcal{D})$ is:

$$F_{\mathcal{B}r(\mathcal{D})}(x) = \sum_{i=1}^{\infty} t_i'^{\mathcal{D}} x^i = \frac{1 - \sqrt{1 - 4Dx}}{2}. \tag{2}$$

We consider the following isomorphism of vector spaces:

$$B : \left\{ \begin{array}{rcl} (K\langle \mathcal{T}_P^{\mathcal{D}} \rangle)^d & \longrightarrow & \mathcal{B}r(\mathcal{D}) \\ (F_1, \dots, F_D) & \longrightarrow & \displaystyle\sum_{i=1}^{d} B_i(F_i). \end{array} \right.$$

Let us fix a graded complement V of the graded subspace $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$ in $\mathcal{B}r(\mathcal{D})$. Because $\mathcal{B}r(\mathcal{D})$ is a graded and connected (that is to say $\mathcal{B}r(\mathcal{D})(0) = (0)$), V generates $\mathcal{B}r(\mathcal{D})$ as a non-associative permutative algebra. By (1), $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D}) = B((\mathcal{T}_P^\mathcal{D} * K\langle \mathcal{T}_P^\mathcal{D}\rangle)^D)$.

Let us then consider $\mathcal{T}_P^\mathcal{D} * K\langle \mathcal{T}_P^\mathcal{D}\rangle$, that is to say the ideal of $(K\langle \mathcal{T}_P^\mathcal{D}\rangle, *)$ generated by $\mathcal{T}_P^\mathcal{D}$. It is known that $(K\langle \mathcal{T}_P^\mathcal{D}\rangle, *)$ is isomorphic to a symmetric algebra (see [17]). Hence, there exists a graded subspace W of $K\langle \mathcal{T}_P^\mathcal{D}\rangle$, such that $(K\langle \mathcal{T}_P^\mathcal{D}\rangle, *) \approx S(W)$ as a graded algebra. We can assume that W contains $K\mathcal{T}_P^\mathcal{D}$. As a consequence:

$$\frac{K\langle \mathcal{T}_P^\mathcal{D}\rangle}{\mathcal{T}_P^\mathcal{D} * K\langle \mathcal{T}_P^\mathcal{D}\rangle} \approx \frac{S(W)}{S(W)\mathcal{T}_P^\mathcal{D}} \approx S\left(\frac{W}{K\mathcal{T}_P^\mathcal{D}}\right). \tag{3}$$

We denote by w_i the dimension of $W(i)$ for all $i \in \mathbb{N}$. Then, the Poincaré-Hilbert formal series of $S\left(\frac{W}{K\mathcal{T}_P^\mathcal{D}}\right)$ is:

$$F_{S\left(\frac{W}{K\mathcal{T}_P^\mathcal{D}}\right)}(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i - t_i'^\mathcal{D}}}. \tag{4}$$

Moreover, the Poincaré-Hilbert formal series of $K\langle \mathcal{T}_P^\mathcal{D}\rangle \approx S(W)$ is, by (2):

$$F_{S(W)}(x) = \frac{1}{1 - F_{\mathcal{B}r(\mathcal{D})}(x)} = \frac{1-\sqrt{1-4Dx}}{2Dx} = \frac{F_{\mathcal{B}r(\mathcal{D})}(x)}{Dx} = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i}}. \tag{5}$$

So, from (3), using (4) and (5), the Poincaré-Hilbert series of $\mathcal{T}_P^\mathcal{D} * K\langle \mathcal{T}_P^\mathcal{D}\rangle$ is:

$$\begin{aligned} F_{\mathcal{T}_P^\mathcal{D} * K\langle \mathcal{T}_P^\mathcal{D}\rangle}(x) &= F_{S(W)}(x) - F_{S\left(\frac{W}{K\mathcal{T}_P^\mathcal{D}}\right)}(x) \\ &= \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{w_i}} \left(1 - \prod_{i=1}^{\infty}(1-x^i)^{t_i'^\mathcal{D}}\right) \\ &= \frac{F_{\mathcal{B}r(\mathcal{D})}(x)}{Dx}\left(1 - \prod_{i=1}^{\infty}(1-x^i)^{t_i'^\mathcal{D}}\right). \end{aligned}$$

As B is homogeneous of degree 1, the Poincaré-Hilbert formal series of $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$ is:

$$F_{\mathcal{B}r(\mathcal{D})\star\mathcal{B}r(\mathcal{D})}(x) = Dx F_{\mathcal{T}_P^\mathcal{D} * K\langle \mathcal{T}_P^\mathcal{D}\rangle}(x) = F_{\mathcal{B}r(\mathcal{D})}(x)\left(1 - \prod_{i=1}^{\infty}(1-x^i)^{t_i'^\mathcal{D}}\right).$$

Finally, the Poincaré-Hilbert formal series of V is:

$$F_V(x) = F_{\mathcal{B}r(\mathcal{D})}(x) - F_{\mathcal{B}r(\mathcal{D})\star\mathcal{B}r(\mathcal{D})}(x) = F_{\mathcal{B}r(\mathcal{D})}(x)\prod_{i=1}^{\infty}(1-x^i)^{t_i'^\mathcal{D}}.$$

Let us now fix a basis $(v_i)_{i\in I}$ of V, formed of homogeneous elements. There is a unique epimorphism of non-associative permutative algebras:

$$\Theta : \left\{ \begin{array}{rcl} \mathcal{NAP}erm(I) & \longrightarrow & \mathcal{B}r(\mathcal{D}) \\ \bullet_i & \longrightarrow & v_i. \end{array} \right.$$

We give to $i \in I$ the degree of $v_i \in \mathcal{B}r(\mathcal{D})$. With the induced gradation of $\mathcal{NAP}erm(I)$, Θ is a graded epimorphism. In order to prove that it is an isomorphism, it is enough to prove that the Poincaré-Hilbert series of $\mathcal{NAP}erm(I)$ and $\mathcal{B}r(\mathcal{D})$ are equal. By lemma 3, the formal series of $\mathcal{NAP}erm(I)$, or, equivalently, of $\mathcal{PL}(I)$, is:

$$F_{\mathcal{NAP}erm(I)}(x) = \sum_{n=1}^{\infty} t_i^\mathcal{D} x^i = \frac{F_V(x)}{\prod_{i=1}^{\infty}(1-x^i)^{t_i^\mathcal{D}}} = F_{\mathcal{B}r(\mathcal{D})}(x)\prod_{i=1}^{\infty}(1-x^i)^{t_i'^\mathcal{D} - t_i^\mathcal{D}}. \tag{6}$$

Let us prove inductively that $t_n = t'_n$ for all $n \in \mathbb{N}$. It is immediate if $n = 0$, as $t_0 = t'_0 = 0$. Let us assume that $t_i^{\mathcal{D}} = t_i'^{\mathcal{D}}$ for all $i < n$. Then:

$$\prod_{i=1}^{\infty} (1 - x^i)^{t_i^{\mathcal{D}} - t_i'^{\mathcal{D}}} = 1 + \mathcal{O}(x^n).$$

As $t'_0 = 0$, the coefficient of x^n in (6) is $t_n = t'_n$. So $F_{\mathcal{NAP}erm(I)}(x) = F_{S(W)}(x)$, and Θ is an isomorphism. □

3 Freeness of $\mathcal{B}r(\mathcal{D})$ as a pre-Lie algebra

3.1 Main theorem

Theorem 10 *Let $\mathcal{D}$ be a finite set. Then $\mathcal{B}r(\mathcal{D})$ is a free pre-Lie algebra.*

Proof. We give a $\mathbb{N}^2$-gradation on $\mathcal{B}r(\mathcal{D})$ in the following way:

$$\mathcal{B}r(\mathcal{D})(k,l) = Vect(t \in \mathcal{T}_P^{\mathcal{D}} \;/\; t \text{ has } k \text{ vertices and the fertility of its root is } l).$$

The following points are easy:

1. For all $i, j, k, l \in \mathbb{N}$, $\mathcal{B}r(\mathcal{D})(i,j) \star \mathcal{B}r(\mathcal{D})(k,l) \subseteq \mathcal{B}r(\mathcal{D})(i+k, l+1)$.
2. For all $i, j, k, l \in \mathbb{N}$, $t_1 \in \mathcal{B}r(\mathcal{D})(i,j)$, $t_2 \in \mathcal{B}r(\mathcal{D})(k,l)$, $\langle t_1; t_2\rangle - t_1 \star t_2 \in \mathcal{B}r(\mathcal{D})(i+k, l)$.

Let us fix a complement V of $\mathcal{B}r(\mathcal{D}) \star \mathcal{B}r(\mathcal{D})$ in $\mathcal{B}r(\mathcal{D})$ which is $\mathbb{N}^2$-graded. Then $\mathcal{B}r(\mathcal{D})$ is isomorphic as a $\mathbb{N}$-graded non-associative permutative algebra to $\mathcal{NAP}erm(V)$, the free non-associative permutative algebra generated by V.

Let us prove that V also generates $\mathcal{B}r(\mathcal{D})$ as a pre-Lie algebra. As $\mathcal{B}r(\mathcal{D})$ is $\mathbb{N}$-graded, with $\mathcal{B}r(\mathcal{D})(0)$, it is enough to prove that $\mathcal{B}r(\mathcal{D}) = V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D})\rangle$. Let $x \in \mathcal{B}r(\mathcal{D})(k,l)$, let us show that $x \in V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D})\rangle$ by induction on l. If $l = 0$, then $t \in \mathcal{B}r(\mathcal{D})(1) = V(1)$. If $l = 1$, we can suppose that $x = B_d(t)$, where $t \in \mathcal{T}_P^{\mathcal{D}}$. Then $x = \langle t; \bullet_d\rangle \in \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D})\rangle$. Let us assume the result for all $l' < l$. As V generates $(\mathcal{B}r(\mathcal{D}), \star)$, we can write x as:

$$x = x' + \sum_i x_i \star y_i,$$

where $x' \in V$ and $x_i, y_i \in \mathcal{B}r(\mathcal{D})$. By the first point, we can assume that:

$$\sum_i x_i \otimes y_i \in \bigoplus_{i+j=k} \mathcal{B}r(\mathcal{D})(i) \otimes \mathcal{B}r(\mathcal{D})(j, l-1).$$

So, by the second point:

$$\begin{aligned} x - x' - \sum_i \langle x_i; y_i\rangle &= \sum_i x_i \star y_i - \langle x_i; y_i\rangle \\ &\in \sum_{i+j=k} \mathcal{B}r(\mathcal{D})(i+j, l-1) \\ &\in V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D})\rangle, \end{aligned}$$

by the induction hypothesis. So $x \in V + \langle \mathcal{B}r(\mathcal{D}); \mathcal{B}r(\mathcal{D})\rangle$.

Hence, there is an homogeneous epimorphism:

$$\left\{ \begin{array}{rcl} \mathcal{PL}(V) & \longrightarrow & \mathcal{B}r(\mathcal{D}) \\ v \in V & \longrightarrow & v. \end{array} \right.$$

As $\mathcal{PL}(V)$, $\mathcal{NAP}erm(V)$ and $\mathcal{B}r(\mathcal{D})$ have the same Poincaré-Hilbert formal series, this is an isomorphism. □

We now give the number of generators of $\mathcal{B}r(\mathcal{D})$ in degree n when $card(\mathcal{D}) = D$ for small values of n, computed using lemmas 3 and 6:

1. For $n=1$, D.
2. For $n=2$, 0.
3. For $n=3$, $\frac{D^2(D-1)}{2}$.
4. For $n=4$, $\frac{D^2(2D-1)(2D+1)}{3}$.
5. For $n=5$, $\frac{D^2(31D^3-2D^2-3D-2)}{8}$.
6. For $n=6$, $\frac{D^2(356D^4-20D^3-5D^2+5D-6)}{30}$.
7. For $n=7$, $\frac{D^2(5441D^5-279D^4-91D^3-129D^2-22D-24)}{144}$.

3.2 Corollaries

Corollary 11 *Let $\mathcal{D}$ be any set. Then $\mathcal{B}r(\mathcal{D})$ is a free pre-Lie algebra.*

Proof. We graduate $\mathcal{B}r(\mathcal{D})$ by putting all the $\bullet_d$'s homogeneous of degree 1. Let V be a graded complement of $\langle \mathcal{B}r(\mathcal{D}), \mathcal{B}r(\mathcal{D})\rangle$. There exists an epimorphism of graded pre-Lie algebras:

$$\Theta : \left\{ \begin{array}{rcl} \mathcal{PL}(V) & \longrightarrow & \mathcal{B}r(\mathcal{D}) \\ \bullet_v & \longrightarrow & v. \end{array} \right.$$

Let x be in the kernel of Θ. There exists a finite subset $\mathcal{D}'$ of $\mathcal{D}$, such that all the decorations of the vertices of the trees appearing in x belong to $\mathcal{B}r(\mathcal{D}')$. By the preceding theorem, as $\mathcal{B}r(\mathcal{D}')$ is a free pre-Lie algebra, $x=0$. So Θ is an isomorphism. $\Box$

Corollary 12 *Let $\mathcal{D}$ be a graded set, satisfying the conditions of lemma 3. There exists a graded set $\mathcal{D}'$, such that $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is isomorphic, as a graded Hopf algebra, to $\mathcal{H}_R^{\mathcal{D}'}$.*

Proof. $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is isomorphic, as a graded Hopf algebra, to $\mathcal{U}(\mathcal{B}r(\mathcal{D}))^*$. For a good choice of $\mathcal{D}'$, $\mathcal{B}r(\mathcal{D})$ is isomorphic to $\mathcal{PL}(\mathcal{D}')$ as a pre-Lie algebra, so also as a Lie algebra. So $\mathcal{U}(\mathcal{B}r(\mathcal{D}))$ is isomorphic to $\mathcal{U}(\mathcal{PL}(\mathcal{D}'))$. Dually, $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is isomorphic to $\mathcal{H}_R^{\mathcal{D}'}$. $\Box$

Corollary 13 *Let $\mathcal{D}$ be graded set, satisfying the conditions of lemma 3. Then $(\mathcal{H}_{PR}^{\mathcal{D}})_{ab}$ is a cofree coalgebra. Moreover, $\mathcal{B}r(\mathcal{D})$ is free as a Lie algebra.*

Proof. It is proved in [7] that $(\mathcal{H}_R^{\mathcal{D}'})^*$ is a free algebra, so $Prim((\mathcal{H}_R^{\mathcal{D}'})^*) = \mathcal{PL}(\mathcal{D}')$ is a free Lie algebra and $\mathcal{H}_R^{\mathcal{D}'}$ is a cofree coalgebra. So $Prim((\mathcal{H}_{PR}^{\mathcal{D}})^*) = \mathcal{B}r(\mathcal{D})$ is a free Lie algebra and $\mathcal{H}_{PR}^{\mathcal{D}}$ is a cofree coalgebra. $\Box$

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