Chebyshev diagrams for rational knots

P. -V. Koseleff, D. Pecker

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Abstract

We show that every rational knot K of crossing number N admits a polynomial parametrization $x = T_a(t)$, $y = T_b(t), z = C(t)$ where $T_k(t)$ are the Chebyshev polynomials, $a = 3$ and $b + \deg C = 3N$. We show that every rational knot also admits a polynomial parametrization with $a = 4$. If $C(t) = T_c(t)$ is a Chebyshev polynomial, we call such a knot a harmonic knot. We give the classification of harmonic knots for $a \le 4$.

keywords: Polynomial curves, Chebyshev curves, rational knots, continued fractions
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Contents

1 Introduction

We study the polynomial parametrization of knots, viewed as non singular space curves. Vassiliev proved that any knot can be represented by a polynomial embedding $\mathbf{R} \to \mathbf{R}^3 \subset \mathbf{S}_3$ ([Va]). Shastri ([Sh]) gave another proof of this theorem, he also found explicit parametrizations of the trefoil and of the figure-eight knot (see also [Mi]).

We shall study polynomial embeddings of the form $x = T_a(t)$, $y = T_b(t)$, $z = C(t)$ where a and b are coprime integers and $T_n(t)$ are the classical Chebyshev polynomials defined by $T_n(\cos t) = \cos nt$. The projection of such a curve on the xy-plane is the Chebyshev curve $\mathcal{C}(a, b) : T_b(x) = T_a(y)$ which has exactly $\frac{1}{2}(a-1)(b-1)$ crossing points ([Fi, P1, P2]). We will say that such a knot has the Chebyshev diagram $\mathcal{C}(a, b)$.

We observed in [KP1] that the trefoil can be parametrized by Chebyshev polynomials: $x = T_3(t)$; $y = T_4(t)$; $z = T_5(t)$. This led us to study Chebyshev knots in [KP3].

Definition 1.1. *A knot in $\mathbf{R}^3 \subset \mathbf{S}^3$ is the Chebyshev knot $\mathcal{C}(a, b, c, \varphi)$ if it admits the one-to-one parametrization*

$$x = T_a(t);\ \ y = T_b(t);\ \ z = T_c(t + \varphi)$$

where $t \in \mathbf{R}$, a and b are coprime integers, c is an integer and φ is a real constant.

When $\varphi = 0$ and a, b, c are coprime, it is denoted by $\mathrm{H}(a, b, c)$ and is called a harmonic knot.

We proved that any knot is a Chebyshev knot. Our proof uses theorems on braids by Hoste, Zirbel and Lamm ([HZ]), and a density argument. In a joint work with F. Rouillier ([KPR]), we developed an effective method to enumerate all the knots $\mathcal{C}(a, b, c, \varphi), \varphi \in \mathbf{R}$ where $a = 3$ or $a = 4$, a and b coprime.

Chebyshev knots are polynomial analogues of Lissajous knots that admit a parametrization of the form

$$x = \cos(at);\ \ y = \cos(bt + \varphi);\ \ z = \cos(ct + \psi)$$

where $0 \le t \le 2\pi$ and where a, b, c are pairwise coprime integers. These knots, introduced in [BHJS], have been studied by V. F. R. Jones, J. Przytycki, C. Lamm, J. Hoste and L. Zirbel. Most known properties of Lissajous knots are deduced from their symmetries (see [BDHZ, Cr, HZ, JP, La1]).

The symmetries of harmonic knots, obvious from the parity of Chebyshev polynomials, are different from those of Lissajous. For example, the figure-eight knot which is amphicheiral but not a Lissajous knot, is the harmonic knot $\mathrm{H}(3, 5, 7)$.

We proved that the harmonic knot $\mathrm{H}(a, b, ab - a - b)$ is alternate, and deduced that there are infinitely many amphicheiral harmonic knots and infinitely many strongly invertible harmonic knots. We also proved ([KP3]) that the torus knot $T(2, 2n + 1)$ is the harmonic knot $\mathrm{H}(3, 3n + 2, 3n + 1)$.

In this article, we give the classification of the harmonic knots $\mathrm{H}(a, b, c)$ for $a \le 4$. We also give explicit polynomial parametrizations of all rational knots. The diagrams of our knots are Chebyshev curves of minimal degrees with a small number of crossing points. The degrees of the height polynomials are small.

In section **2.** we recall the Conway notation for rational knots, and the computation of their Schubert fractions with continued fractions. We observe that Chebyshev diagrams correspond to continued fractions of the form $[\pm 1, \ldots, \pm 1]$ when $a = 3$ and of the form $[\pm 1, \pm 2, \ldots, \pm 1, \pm 2]$ when $a = 4$. We show results on our continued fraction expansion:

Theorem 2.6.
Every rational number r has a unique continued fraction expansion $r = [e_1, e_2, \ldots, e_n]$, $e_i = \pm 1$, where there are no two consecutive sign changes in the sequence $(e_1, \ldots, e_n)$.

We have a similar theorem for continued fractions of the form $r = [\pm 1, \pm 2, \ldots, \pm 1, \pm 2]$. We provide a formula for the crossing number of the corresponding knots. Then we study the matrix interpretation of these continued fraction expansions. As an application, we give optimal Chebyshev diagrams for the torus knots $T(2, N)$, the twist knots $\mathcal{T}_n$, the generalized stevedore knots and some others.

In section **3.** we describe the harmonic knots $\mathrm{H}(a, b, c)$ where $a \le 4$. We begin with a careful analysis of the nature of the crossing points, giving the Schubert fractions of $\mathrm{H}(3, b, c)$ and $\mathrm{H}(4, b, c)$. Being rather long, the proofs of these results will be given in the last paragraph. We deduce the following algorithmic classification theorems.

Theorem 3.7.
Let $K = \mathrm{H}(3, b, c)$. There exists a unique pair (b', c') such that (up to mirror symmetry)

$$K = \mathrm{H}(3, b', c'),\ b' < c' < 2b',\ b' \not\equiv c' \pmod 3.$$

The crossing number of K is $\frac{1}{3}(b' + c')$.
The Schubert fractions $\frac{\alpha}{\beta}$ of K are such that $\beta^2 \equiv \pm 1 \pmod \alpha$.

Theorem 3.13.
Let $K = \mathrm{H}(4, b, c)$. There exists a unique pair (b', c') such that (up to mirror symmetry)

$$K = \mathrm{H}(4, b', c'),\ b' < c' < 3b',\ b' \not\equiv c' \pmod 4.$$

The crossing number of K is $\frac{1}{4}(3b' + c' - 2)$.
K has a Schubert fraction $\frac{\alpha}{\beta}$ such that $\beta^2 \equiv \pm 2 \pmod \alpha$.

We notice that the trefoil is the only knot which is both of form $\mathrm{H}(3, b, c)$ and $\mathrm{H}(4, b, c)$. We remark that the 6_1 knot (the stevedore knot) is not a harmonic knot $\mathrm{H}(a, b, c),\ a \le 4$.

In section **4.** we find explicit polynomial parametrizations of all rational knots. We first compute the optimal Chebyshev diagrams for $a = 3$ and $a = 4$. Then we define a height polynomial of small degree. More precisely:

Theorem 4.4.
Every rational knot of crossing number N can be parametrized by $x = T_3(t), y = T_b(t), z = C(t)$ where $b + \deg C = 3N$. Furthermore, when the knot is amphicheiral, b is odd and we can choose C to be an odd polynomial.

In the same way we show: *Every rational knot of crossing number N can be parametrized by $x = T_4(t), y = T_b(t), z = C(t)$* where b is odd and C is an odd polynomial.

As a consequence, we see that any rational knot has a representation $K \subset \mathbf{R}^3$ such that K is symmetrical about the y-axis (with reversed orientation). It clearly implies the classical result: *every rational knot is strongly invertible.*

We give polynomial parametrizations of the torus knots $T(2, 2n+1)$. We also give the first polynomial parametrizations of the twist knots and the generalized stevedore knots. We conjecture that the lexicographic degrees of our polynomials are minimal (among odd or even polynomials).

2 Continued fractions and rational Chebyshev knots

A two-bridge knot (or link) admits a diagram in Conway's normal form. This form, denoted by $C(a_1, a_2, \ldots, a_n)$ where a_i are integers, is explained by the following picture (see [Con], [Mu] p. 187). The number of twists is denoted by the integer $|a_i|$, and the sign of a_i is

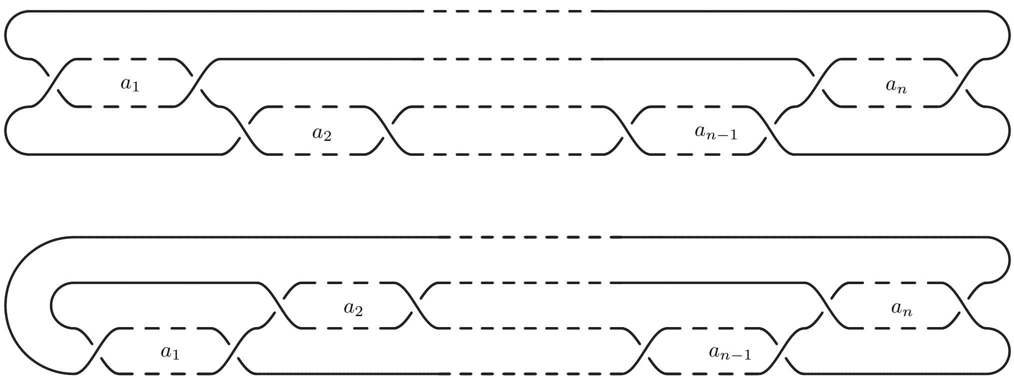
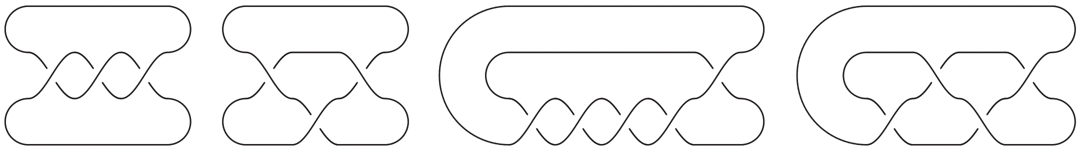


Figure 1: Conway's normal forms

defined as follows: if i is odd, then the right twist is positive, if i is even, then the right twist is negative. On Fig. 1 the a_i are positive (the a_1 first twists are right twists).

Examples 2.1. The trefoil has the following Conway's normal forms $C(3)$, $C(-1, -1, -1)$, $C(4, -1)$ and $C(1, 1, -1, -1)$. The diagrams in Figure 2 clearly represent the same trefoil.



$C(3)$ $C(-1,-1,-1)$ $C(4,-1)$ $C(1,1,-1,-1)$

Figure 2: Diagrams of the standard trefoil

The two-bridge links are classified by their Schubert fractions

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}} = [a_1, \ldots, a_n], \quad \alpha > 0.$$

We shall denote $S(\frac{\alpha}{\beta})$ a two-bridge link with Schubert fraction $\frac{\alpha}{\beta}$. The two-bridge links $S(\frac{\alpha}{\beta})$ and $S(\frac{\alpha'}{\beta'})$ are equivalent if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} (\mathrm{mod}\ \alpha)$. The integer α is odd for a knot, and even for a two-component link. If $K = S(\frac{\alpha}{\beta})$, its mirror image is $\overline{K} = S(\frac{\alpha}{-\beta})$.

We shall study knots with a Chebyshev diagram $\mathcal{C}(3, b)$: $x = T_3(t), y = T_b(t)$. It is remarkable that such a diagram is already in Conway normal form (see Figure 1). Consequently, the Schubert fraction of such a knot is given by a continued fraction of the form $[\pm 1, \pm 1, \ldots, \pm 1]$. For example the only diagrams of Figure 2 which may be Chebyshev are the second and the last (in fact they are Chebyshev).

Figure 3 shows a typical example of a knot with a Chebyshev diagram.

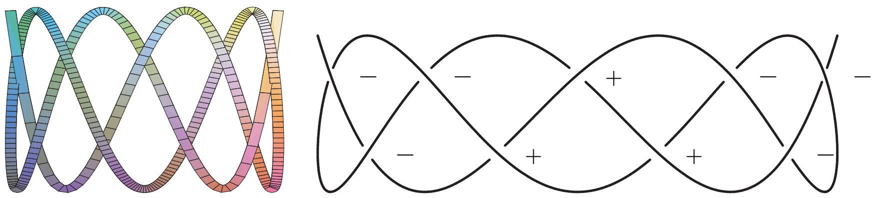


Figure 3: A Chebyshev diagram of the torus knot $T(2, 7)$

This knot is defined by $x = T_3(t)$, $y = T_{10}(t)$, $z = -T_{11}(t)$. Its xy-projection is in the Conway normal form $C(-1,-1,-1,1,1,1,-1,-1,-1)$. Its Schubert fraction is then $\frac{7}{-6}$ and this knot is the torus knot $T(2, 7) = S(\frac{7}{-6}) = S(7)$.

We shall also need to study knots with a diagram illustrated by the following picture. In

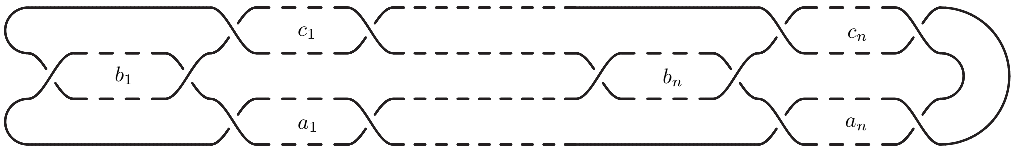
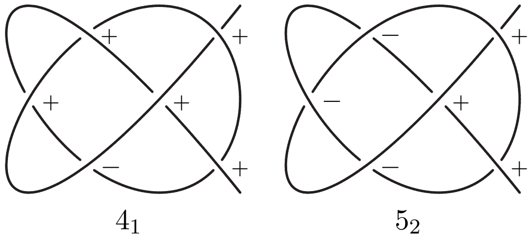


Figure 4: A knot isotopic to $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \ldots, b_n, a_n + c_n)$

this case, the a_i and the c_i are positive if they are left twists, the b_i are positive if they are right twists (on our figure a_i, b_i, c_i are positive). Such a knot is equivalent to a knot with Conway's normal form $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \ldots, b_n, a_n + c_n)$ (see [Mu] p. 183-184).

We shall study the knots with a Chebyshev diagram $\mathcal{C}(4, k) : x = T_4(t),\ y = T_k(t)$. In this case we get diagrams of the form illustrated by Figure 4. Consequently, such a knot has a Schubert fraction of the form $[b_1, d_1, b_2, d_2, \ldots, b_n, d_n]$ with $b_i = \pm 1$, $d_i = \pm 2$ or $d_i = 0$.

Once again, the situation is best explained by typical examples. Figure 5 represents two knots with the same Chebyshev diagram $\mathcal{C}(4, 5) : x = T_4(t),\ y = T_5(t)$. A Schubert fraction of the first knot is $\dfrac{5}{2} = [1, 0, 1, 2]$, it is the figure-eight knot. A Schubert fraction of the second knot is $\dfrac{7}{-4} = [-1, -2, 1, 2]$, it is the twist knot 5_2.



4_1
$C(1, 0, 1, 2)$

5_2
$C(-1, -2, 1, 2)$

Figure 5: Knots with the Chebyshev diagram $\mathcal{C}(4, 5)$

2.1 Continued fractions

Let α, β be relatively prime integers. Then $\dfrac{\alpha}{\beta}$ admits the continued fraction expansion $\dfrac{\alpha}{\beta} = [q_1, q_2, \ldots, q_n]$ if and only if there exist integers r_i such that

$$\left\{\begin{array}{rcl} \alpha &=& q_1\beta + r_2, \\ \beta &=& q_2 r_2 + r_3, \\ &\vdots& \\ r_{n-2} &=& q_{n-1}r_{n-1} + r_n, \\ r_{n-1} &=& q_n r_n. \end{array}\right.$$

The integers q_i are called the quotients of the continued fraction. Euclidean algorithms provide various continued fraction expansions which are useful to the study of two-bridge knots (see [BZ, St]).

Definition 2.2. *Let $r > 0$ be a rational number, and $r = [q_1, \ldots, q_n]$ be a continued fraction with $q_i > 0$. The* crossing number *of r is defined by $\mathrm{cn}\,(r) = q_1 + \cdots + q_n$.*

Remark 2.3. When q_i are positive integers, the continued fraction expansion $[q_1, q_2, \ldots, q_n]$ is unique up to $[q_n] = [q_n - 1, 1]$. $\mathrm{cn}\,(\dfrac{\alpha}{\beta})$ is the crossing number of the knot $K = S(\dfrac{\alpha}{\beta})$. It means that it is the minimum number of crossing points for all diagrams of K ([Mu]).

We shall be interested by algorithms where the sequence of remainders is not necessarily decreasing anymore. In this case, if $\dfrac{\alpha}{\beta} = [a_1, \ldots, a_n]$, we have $\mathrm{cn}\,(\dfrac{\alpha}{\beta}) \le \sum_{k=1}^n |a_i|$.

Definition 2.4. *A continued fraction $[a_1, a_2, \ldots, a_n]$ is regular if it has the following properties:*

$$a_i \neq 0,\ a_{n-1}a_n > 0,\ \textit{and } a_i a_{i+1} < 0 \Rightarrow a_{i+1}a_{i+2} > 0,\ \ i = 1, \ldots, n-2.$$

If $a_1 a_2 > 0$ she shall say that the continued fraction is biregular.

Proposition 2.5. *Let $\dfrac{\alpha}{\beta} = [a_1, \ldots, a_n]$ be a biregular continued fraction. Its crossing number is*

$$\mathrm{cn}\,(\frac{\alpha}{\beta}) = \sum_{k=1}^{n} |a_i| - \sharp\{i, a_i a_{i+1} < 0\}. \tag{1}$$

Proof. We prove this result by induction on the number of sign changes $k = \sharp\{i, a_i a_{i+1} < 0\}$. If k is 0, then K is alternate and the result is true. If $k > 0$ let us consider the first change of sign. The Conway normal form of K is $[x, a, b, -c, -d, -y]$ where a, b, c, d are positive integers and x is a sequence (possibly empty) of positive integers and y is a sequence of integers. We have $[x, a, b, -c, -d, -y] = [x, a, b-1, 1, c-1, d, y]$.

- Suppose $(b-1)(c-1) > 0$, then the sum of absolute values has decreased by 1 and the number of changes of sign has also decreased by 1.
- Suppose $b = 1, c \neq 1$ (resp. $c = 1, b \neq 1$). Then $[x, a, b, -c, -d, -y] = [x, a, 0, 1, c-1, d, y] = [x, a+1, c-1, d, y]$. (resp. $[x, a-1, c+1, d, y]$). The sum of absolute values has decreased by 1 and the number of changes of sign has also decreased by 1.
- Suppose $b = c = 1$. Then $[x, a, b, -c, -d, -y] = [x, a, 0, 1, 0, d, y] = [x, a+d+1, y]$. The sum of absolute values has decreased by 1 and the number of changes of sign has also decreased by 1. □

Note that Formula (1) is also true in other cases. For instance, Formula (1) still holds when $a_1, \ldots, a_n$ are non zero even integers (see [St]).

We shall now use the basic (subtractive) Euclidean algorithm to get continued fractions of the form $[\pm 1, \pm 1, \ldots, \pm 1]$.

2.2 Continued fractions $[\pm 1, \pm 1, \ldots, \pm 1]$

We will consider the following homographies:

$$P : x \mapsto [1, x] = 1 + \frac{1}{x},\ M : x \mapsto [1, -1, -x] = \frac{1}{1+x}. \tag{2}$$

Let E be the set of positive real numbers. We have $P(E) =]1, \infty[$ and $M(E) =]0, 1[$. $P(E)$ and $M(E)$ are disjoint subsets of E.

Theorem 2.6. *Let $\frac{\alpha}{\beta} > 0$ be a rational number. There is a unique regular continued fraction such that*

$$\frac{\alpha}{\beta} = [1, e_2, \ldots, e_n],\ e_i = \pm 1.$$

Furthermore, $\alpha > \beta$ if and only if $[e_1, e_2, \ldots, e_n]$ is biregular.

Proof. Let us prove the existence by induction on the height $h(\frac{\alpha}{\beta}) = \alpha + \beta$.

- If $h = 2$ then $\frac{\alpha}{\beta} = 1 = [1]$ and the result is true.
- If $\alpha > \beta$, we have $\frac{\alpha}{\beta} = P(\frac{\beta}{\alpha - \beta}) = [1, \frac{\beta}{\alpha - \beta}]$. Since $h(\frac{\beta}{\alpha - \beta}) < h(\frac{\alpha}{\beta})$, we get our regular continued fraction for r by induction.
- If $\beta > \alpha$ we have $\frac{\alpha}{\beta} = M(\frac{\beta - \alpha}{\alpha}) = [1, -1, -\frac{\beta - \alpha}{\alpha}]$. And we also get a regular continued fraction for r.

Conversely, let r be defined by the regular continued fraction $r = [1, r_2, \ldots, r_n]$, $r_i = \pm 1$, $n \geq 2$. Let us prove, by induction on the length n of the continued fraction, that $r > 0$ and that $r > 1$ if and only if $r_2 = 1$.

- If $r_2 = 1$ we have $r = P([1, r_3, \dots, r_n])$, and by induction $r \in P(E)$ and then $r > 1$.
- If $r_2 = -1$, we have $r_3 = -1$ and $r = M([1, -r_4, \dots, -r_n])$. By induction, $r \in M(E) =]0, 1[$.

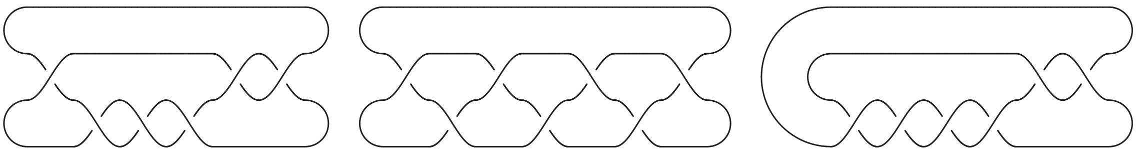
The uniqueness is now easy to prove. Let $r = [1, r_2, \dots, r_n] = [1, r'_2, \dots, r'_{n'}]$.

- If $r > 1$ then $r_2 = r'_2 = 1$ and $[1, 1, r_3, \dots, r_n] = [1, 1, r'_3, \dots r'_{n'}]$. Consequently, $[1, r_3, \dots, r_n] = [1, r'_3, \dots, r'_{n'}]$, and by induction $r_i = r'_i$ for all i.
- If $r < 1$, then $r_2 = r_3 = r'_2 = r'_3 = -1$ and $[1, -1, -1, r_4, \dots, r_n] = [1, -1, -1, r'_4, \dots, r'_{n'}]$. Then, $[1, -r_4, \dots, -r_n] = [1, -r'_4, \dots, -r'_{n'}]$ and by induction $r_i = r'_i$ for all i. □

Definition 2.7. *Let $\dfrac{\alpha}{\beta} > 0$ be the regular continued fraction $[e_1, \dots, e_n]$, $e_i = \pm 1$. We will denote its length n by $\ell(\dfrac{\alpha}{\beta})$. Note that $\ell(-\dfrac{\alpha}{\beta}) = \ell(\dfrac{\alpha}{\beta})$.*

Examples 2.8. Using our algorithm we obtain

$$\begin{aligned}\frac{9}{7} &= [1, \frac{7}{2}] = [1, 1, \frac{2}{5}] = [1, 1, 1, -1, -\frac{3}{2}] = [1, 1, 1, -1, -1, -\frac{2}{1}] = [1, 1, 1, -1, -1, -1, -1],\\ \frac{9}{2} &= [1, \frac{2}{7}] = [1, 1, -1, -\frac{5}{2}] = [1, 1, -1, -1, -\frac{2}{3}] = [1, 1, -1, -1, -1, 1, \frac{1}{2}]\\ &= [1, 1, -1, -1, -1, 1, 1, -1, -1] = [4, 2].\end{aligned}$$



$C(1,3,2)$ $C(1,1,1,-1,-1,-1,-1)$ $C(4,2)$

Figure 6: Diagrams of the knot $6_1 = S(\frac{9}{7})$ and its mirror image $S(\frac{9}{2})$

We will rather use the notation

$$\frac{9}{7} = P^2 M P^3(\infty),\ \frac{9}{2} = PMPM^2P(\infty).$$

We get $\ell(\frac{9}{7}) = 7$, $\ell(\frac{9}{2}) = 9$. The crossing numbers of these fractions are $\mathrm{cn}\,(\frac{9}{7}) = \mathrm{cn}\,([1, 3, 2]) = 6 = 7 - 1$ and $\mathrm{cn}\,(\frac{9}{2}) = \mathrm{cn}\,([4, 2]) = 6 = 9 - 3$. If the fractions $\frac{9}{7}$ and $\frac{9}{2}$ have the same crossing number, it is because the knot $S(\frac{9}{7})$ is the mirror image of $S(\frac{9}{2})$.

In order to get a full description of two-bridge knots we shall need a more detailed study of the homographies P and M.

Proposition 2.9. *The multiplicative monoid* $\mathbf{G} = \langle P, M\rangle$ *is free. The mapping* $g : G \mapsto G(\infty)$ *is a bijection from* $\mathbf{G} \cdot P$ *to* $\mathbf{Q}_{>0}$ *and* $g(P \cdot \mathbf{G} \cdot P) = \mathbf{Q}_{>1}$*, the set of rational numbers greater than* 1.

Proof. Suppose that $PX = MX'$ for some X, X' in $\mathbf{G}$. Then we would have $PX(1) = MX'(1) \in P(E) \bigcap M(E) = \emptyset$. Clearly, this means that $\mathbf{G}$ is free. Similarly, from $P(\infty) = 1$, we deduce that the mapping $G \mapsto G \cdot P(\infty)$ is injective. From Theorem 2.6 and $P(\infty) = 1$, we deduce that g is surjective. □

Remark 2.10. Let $r = G(\infty) = [e_1, \ldots, e_n]$, $e_i = \pm 1$, be a regular continued fraction. It is easy to find the unique homography $G \in \mathbf{G} \cdot P$ such that $r = G(\infty)$. Consider the sequence $(e_1, \ldots, e_n)$. For any i such that $e_i e_{i+1} < 0$, replace the couple (e_i, e_{i+1}) by M, and then replace each remaining e_i by P.

Let $G = P^{p_1} M^{m_1} \cdots M^{m_k} P^{p_{k+1}}$. Let $p = p_1 + \cdots + p_{k+1}$ be the degree of G in P and $m = m_1 + \cdots + m_k$ its degree in M. Then we have $n = \ell(r) = p + 2m$ and $\mathrm{cn}\,(r) = p + m$.

We shall consider matrix notations for many proofs. We will consider

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = P^{p_1} M^{m_1} \cdots M^{m_k} P^{p_{k+1}} \begin{bmatrix}1\\ 0\end{bmatrix}, \quad P = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}, \quad M = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}.$$

Lemma 2.11. *Let* $\dfrac{\alpha}{\beta} = [e_1, \ldots, e_n]$ *be a regular continued fraction* ($e_i = \pm 1$)*. We have*

- $n \equiv 2 \pmod 3$ *if and only if* α *is even and* β *is odd.*
- $n \equiv 0 \pmod 3$ *if and only if* α *is odd and* β *is even.*
- $n \equiv 1 \pmod 3$ *if and only if* α *and* β *are odd.*

Proof. Let us write $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = P^{p_1} M^{m_1} \cdots M^{m_k} P^{p_{k+1}} \begin{bmatrix}1\\ 0\end{bmatrix}$. We have (from remark 2.10) $n = p + 2m$. Since $P^3 \equiv \mathbf{Id}$ and $M \equiv P^2 \pmod 2$, we get $P^{p_1} M^{m_1} \cdots M^{m_k} P^{p_{k+1}} \equiv P^n \pmod 2$

If $n \equiv 2 \pmod 3$, then $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \equiv M \begin{bmatrix}1\\ 0\end{bmatrix} \equiv \begin{bmatrix}0\\ 1\end{bmatrix} \pmod 2$.

If $n \equiv 1 \pmod 3$, then $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \equiv P \begin{bmatrix}1\\ 0\end{bmatrix} \equiv \begin{bmatrix}1\\ 1\end{bmatrix} \pmod 2$.

If $n \equiv 0 \pmod 3$, then $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \equiv \begin{bmatrix}1\\ 0\end{bmatrix} \pmod 2$. □

Definition 2.12.
We define on $\mathbf{G}$ *the anti-homomorphism* $G \mapsto \overline{G}$ *with* $\overline{M} = M$, $\overline{P} = P$.
We define on $\mathbf{G}$ *the homomorphism* $G \mapsto \widehat{G}$ *with* $\widehat{M} = P$, $\widehat{P} = M$.

Proposition 2.13. *Let* $\alpha > \beta$ *and consider* $\dfrac{\alpha}{\beta} = PGP(\infty)$ *and* $N = \mathrm{cn}\,(\dfrac{\alpha}{\beta})$. *Let* β' *such that* $0 < \beta' < \alpha$ *and* $\beta\beta' \equiv (-1)^{N-1} \pmod{\alpha}$. *Then we have*

$$\frac{\beta}{\alpha} = M\widehat{G}P(\infty), \quad \frac{\alpha}{\alpha-\beta} = P\widehat{G}P(\infty), \quad \frac{\alpha}{\beta'} = P\overline{G}P(\infty).$$

We also have

$$\ell(\frac{\beta}{\alpha}) + \ell(\frac{\alpha}{\beta}) = 3N-1, \quad \ell(\frac{\alpha}{\alpha-\beta}) + \ell(\frac{\alpha}{\beta}) = 3N-2, \quad \ell(\frac{\alpha}{\beta'}) = \ell(\frac{\alpha}{\beta}).$$

Proof. We use matrix notations for this proof. Let us consider

$$PGP = \begin{bmatrix} \alpha & \beta' \\ \beta & \alpha' \end{bmatrix} = P^{p_1}M^{m_1}\cdots M^{m_k}P^{p_{k+1}}.$$

From $\det P = \det M = -1$, we obtain $\alpha\alpha' - \beta\beta' = (-1)^N$. Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ be a matrix such that $0 \le c \le a$ and $0 \le d \le b$. From $PA = \begin{bmatrix} a+c & b+d \\ a & c \end{bmatrix}$ and $MA = \begin{bmatrix} b & d \\ a+b & c+d \end{bmatrix}$ we deduce that PGP satisfies $0 < \alpha' < \beta$ and $0 < \beta' < \alpha$. We therefore conclude that, β' is the integer defined by $0 < \beta' < \alpha$, $\beta\beta' \equiv (-1)^{N-1} \pmod{\alpha}$. By transposition we deduce that

$$\begin{bmatrix} \alpha & \beta \\ \beta' & \alpha' \end{bmatrix} = P^{p_{k+1}}M^{m_k}\cdots M^{m_1}P^{p_1} = P\overline{G}P,$$

which implies $\dfrac{\alpha}{\beta'} = P\overline{G}P(\infty)$.

Let us introduce $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We have $J^2 = \mathbf{Id}, M = JPJ$ and $P = JMJ$. Therefore

$$\begin{bmatrix} \beta \\ \alpha \end{bmatrix} = J\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = M^{p_1}P^{m_1}\cdots P^{m_k}M^{p_{k+1}-1}JP\begin{bmatrix} 1 \\ 0 \end{bmatrix} = M^{p_1}P^{m_1}\cdots P^{m_k}M^{p_{k+1}-1}P\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

that is $\dfrac{\beta}{\alpha} = M\widehat{G}P(\infty)$.

Then, $\begin{bmatrix} \alpha \\ \alpha-\beta \end{bmatrix} = PM^{-1}\begin{bmatrix} \beta \\ \alpha \end{bmatrix} = P\widehat{G}P\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. That is $\dfrac{\alpha}{\alpha-\beta} = P\widehat{G}P(\infty)$.

Relations on lengths are derived from the previous relations and remark 2.10. □

We deduce

Proposition 2.14. *Let* $G \in \mathbf{G}$ *and* $\dfrac{\alpha}{\beta} = [e_1,\ldots,e_n] = PGP(\infty)$. *Let* $K = S(\dfrac{\alpha}{\beta})$ *and* $N = \mathrm{cn}\,(K)$. *The following properties are equivalent:*

1. *G is palindromic (i.e. $\overline{G} = G$).*
2. *the sequence of sign changes in $[e_1, \ldots, e_n]$ is palindromic (i.e. $e_i e_{i+1} = e_{n-i} e_{n-i+1}$).*
3. *$\beta^2 \equiv (-1)^{N-1} \pmod{\alpha}$.*

Moreover we have

- *$\beta^2 \equiv -1 \pmod{\alpha}$ (i.e. $K = \overline{K}$ is amphicheiral) if and only if N is even and $G = \overline{G}$. Furthermore, the length $n = \ell(\frac{\alpha}{\beta})$ is even and the sequence $[e_1, \ldots, e_n]$ is palindromic (i.e. $e_i = e_{n-i+1}$).*
- *$\beta^2 \equiv 1 \pmod{\alpha}$ if and only if N is odd and $G = \overline{G}$ or N is even and $\widehat{G} = \overline{G}$ (in this case K is a two-component link).*

Proof. From Remark 2.10, it is straightforward that if $[\varepsilon_1, \ldots, \varepsilon_n] = PGP(\infty)$ then $P\overline{G}P(\infty) = \varepsilon_n[\varepsilon_n, \ldots, \varepsilon_1]$. We deduce that $G = \overline{G}$ is palindromic if and and only if the sequence of sign changes in $[e_1, \ldots, e_n]$ is palindromic.

Let $0 < \beta' < \alpha$ such that $\beta'\beta \equiv (-1)^{N-1} \pmod{\alpha}$. We have from the previous proposition: $\dfrac{\alpha}{\beta'} = P\overline{G}P(\infty)$ We thus deduce that $G = \overline{G}$ is equivalent to $\beta = \beta'$, that is $\beta^2 \equiv (-1)^{N-1} \pmod{\alpha}$.

Suppose now that $\beta^2 \equiv 1 \pmod{\alpha}$. If N is even then $\beta' = \alpha - \beta$, that is $P\overline{G}P = P\widehat{G}P$ and $\overline{G} = \widehat{G}$. We have $p + 2m = m + 2p - 2$ and then $2n = 2(p + 2m) = 3N - 2$. This implies $n \equiv 2 \pmod 3$. By Lemma 2.11, α is even and K is a two-component link. If N is odd then $\beta' = \beta$ and $G = \overline{G}$ by the first part of our proof.

Suppose now that $\beta^2 \equiv -1 \pmod{\alpha}$. If N is odd then $\beta' = \alpha - \beta$ and by the same argument we should have $n = 3N - n - 2$, which would imply that N is even. We deduce that amphicheiral rational links have even crossing numbers and from $\beta' = \beta$ we get $G = \overline{G}$. The crossing number $N = m + p$ is even and G is palindromic so m and p are both even. Consequently $n = p + 2m$ is even and the number of sign changes is even. We thus have $e_n = 1$ and $(e_n, \ldots, e_1) = (e_1, \ldots, e_n)$. □

We deduce also a method to find a minimal Chebyshev diagram for any rational knot.

Proposition 2.15. *Let K be a two-bridge knot with crossing number N.*

1. *There exists $\dfrac{\alpha}{\beta} > 1$ such that $K = S(\pm\dfrac{\alpha}{\beta})$ and $n = \ell(\dfrac{\alpha}{\beta}) < \frac{3}{2}N - 1$.*
2. *There exists a biregular sequence $(e_1, \ldots, e_n)$, $e_i = \pm 1$, such that $K = C(e_1, \ldots, e_n)$.*
3. *If $K = C(\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_i = \pm 1$, then $m \geq n \geq N$.*

Let K be a two-bridge knot. Let $r = \dfrac{\alpha}{\beta} > 1$ such that $K = S(r)$. We have $\overline{K} = S(r')$ where $r' = \dfrac{\alpha}{\alpha - \beta}$. From Proposition 2.13 and Proposition 2.5, we have $\ell(r) + \ell(r') = 3N - 2$ and therefore $N \leq \min(\ell(r), \ell(r')) < \frac{3}{2}N$. From Lemma 2.11, we have $\ell(r) \not\equiv 2 \pmod 3$ so $\ell(r) \neq \ell(r')$ and $\min(\ell(r), \ell(r')) < \frac{3}{2}N - 1$. We may suppose that $n = \ell(r)$. Let

$r = \pm[e_1, \ldots, e_n]$, then $C(e_1, \ldots, e_n)$ is a Conway normal form for K with $n < \frac{3}{2}N - 1$. This Conway normal form is a Chebyshev diagram $\mathcal{C}(3, n+1) : x = T_3(t),\ y = T_{n+1}(t)$.

Let us consider γ such that $\beta\gamma \equiv 1 \,(\mathrm{mod}\, \alpha)$ and $0 < \gamma < \alpha$. Let $\rho = \dfrac{\alpha}{\gamma}$ and $\rho' = \dfrac{\alpha}{\alpha - \gamma}$. We have $K = S(\rho)$ and $\overline{K} = S(\rho')$ and from Proposition 2.13: $\min(\ell(\rho), \ell(\rho')) = \min(\ell(r), \ell(r'))$. Suppose that $K = C(\varepsilon_1, \ldots, \varepsilon_m)$ then $K = S(x)$ where $x = [\varepsilon_1, \ldots, \varepsilon_m]$. We thus deduce that $x = \dfrac{\alpha}{\beta + k\alpha}$ or $x' = \dfrac{\alpha}{\gamma + k\alpha}$ where $k \in \mathbf{Z}$.

- If $k = 2p > 0$ then we have $x = (MP)^p r$ so $m = \ell(x) = \ell(r) + 3p > \ell(r)$.
- If $k = 2p+1 > 0$ then $x = (MP)^p M(\dfrac{1}{r})$ so $\ell(x) = \ell(1/r)+3p+2 = \ell(r')+3p+3 > \ell(r')$.
- If $k = -(2p+1) < 0$ then $-x = (MP)^p(r')$ so $\ell(x) = \ell(-x) = \ell(r') + 3p > \ell(r')$.
- If $k = -2p > 0$ then $-x = (MP)^{p-1} M(\dfrac{1}{r'})$ so $\ell(x) = \ell(1/r')+3p-1 = \ell(r)+3p > \ell(r)$.

If $x' = \dfrac{\alpha}{\gamma + k\alpha}$, we obtain the same relations. We deduce that $m \geq \min(\ell(r), \ell(r'))$. □

Remark 2.16 (Computing the minimal Chebyshev diagram $\mathcal{C}(3, b)$).
Let $K = S(\dfrac{\alpha}{\beta})$ with $\dfrac{\alpha}{\beta} = PGP(\infty)$. The condition $\ell(\dfrac{\alpha}{\beta})$ is minimal, that is $\ell(\dfrac{\alpha}{\beta}) < \frac{3}{2}N - 1$, is equivalent to $p \geq m + 3$ where $p = \deg_P(PGP)$ and $m = \deg_M(PGP)$. In this case $b = \ell(\dfrac{\alpha}{\beta}) + 1$ is the smallest integer such that $K = S(\dfrac{\alpha}{\beta})$ has a Chebyshev diagram $x = T_3(t),\ y = T_b(t)$. If $p < m + 3$, using Proposition 2.13, K has a Chebyshev diagram $\mathcal{C}(3, b')$ with $b' = 3N - b < \frac{3}{2}N$. This last diagram is minimal.

Example 2.17 (Torus knots). The Schubert fraction of the torus knot $T(2, 2k+1)$ is $2k+1$. We have $PM(x) = x + 2$, and then $(PM)^k(x) = x + 2k, (PM)^k P(x) = 2k + 1 + \dfrac{1}{x}$. So we get the continued fraction of length $3k+1$: $2k+1 = (PM)^k P(\infty)$. This shows that the torus knot $T(2, 2k+1)$ has a Chebyshev diagram $\mathcal{C}(3, 3k+2)$. This is not a minimal diagram.

On the other hand, we get $(PM)^{k-1}P^2(\infty) = 2k$ so $\dfrac{2k+1}{2k} = P(PM)^{k-1}P^2(\infty) > 1$. This shows that the torus knot $T(2, 2k+1)$ has a Chebyshev diagram $\mathcal{C}(3, 3k+1)$. This diagram is minimal by Remark 2.16. We will see that $T(2, 2k+1)$ is in fact a harmonic knot.

Example 2.18 (Twist knots). The twist knot $\mathcal{T}_n$ is defined by $\mathcal{T}_n = S(n + \dfrac{1}{2})$.

From $P^3(x) = \dfrac{3x+2}{2x+1}$, we get the continued fraction of length $3k+3$: $\dfrac{4k+3}{2} = (PM)^k P^3(\infty)$. This shows that the twist knot $\mathcal{T}_{2k+1}$ has a Chebyshev diagram $\mathcal{C}(3, 3k+4)$, which is minimal by Remark 2.16.

We also deduce that $P(PM)^{k-1}P^3(\infty) = P\left(\frac{1}{2}(4k-1)\right) = \dfrac{4k+1}{4k-1}$. This shows that the twist knot $\mathcal{T}_{2k}$ has a minimal Chebyshev diagram $\mathcal{C}(3, 3k+2)$.

We shall see that these knots are not harmonic knots for $a = 3$ and we will give explicit bounds for their polynomial parametrizations.

Example 2.19 (Generalized stevedore knots). The generalized stevedore knot $\mathcal{S}_k$ is defined by $\mathcal{S}_k = S(2k+2+\dfrac{1}{2k})$. We have

$$(MP)^k = \begin{bmatrix} 1 & 0 \\ 2k & 1 \end{bmatrix},\ (PM)^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$$

so $2k+2+\dfrac{1}{2k} = (PM)^{k+1}(MP)^k(\infty)$. This shows that the stevedore knots have a Chebyshev diagram $\mathcal{C}(3, 6k+4)$. It is not minimal and we see, using Remark 2.16, that the stevedore knot $\mathcal{S}_k$ also has a minimal Chebyshev diagram $\mathcal{C}(3, 6k+2)$. Moreover, using Proposition 2.13, we get

$$\frac{(k+1)^2}{(k+1)^2-2k} = P^2(MP)^k(PM)^{k-1}P^2(\infty).$$

2.3 Continued fractions $[\pm\mathbf{1}, \pm\mathbf{2}, \ldots, \pm\mathbf{1}, \pm\mathbf{2}]$

Let us consider the homographies $A(x) = [1, 2, x] = \dfrac{3x+1}{2x+1}$, $B(x) = [1, -2, -x] = \dfrac{x+1}{2x+1}$, $S(x) = -x$. We shall also use the classical matrix notation for these homographies

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},\ S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Lemma 2.20. *The monoid* $\mathbf{\Gamma} = \langle (AS)^kA, (AS)^{k+1}B, B(SA)^k,\ B(SA)^{k+1}SB,\ k \geq 0\rangle$ *is free. Let* $\mathbf{\Gamma}^*$ *be the subset of elements of* $\mathbf{\Gamma}$ *that are not of the form* $M \cdot (AS)^{k+1}B$, *or* $M \cdot B(SA)^{k+1}SB$. *There is an injection* $h : \mathbf{\Gamma}^* \to \mathbf{Q}_{>0}$ *such that* $h(G) = G(\infty)$.

Proof. Let us denote $E = \mathbf{R}_+^* \bigcup\{\infty\} =]0, \infty]$. We will describe $G(E)$ for any generator G of $\mathbf{\Gamma}$. Let $C = AS = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$. We get $C^2 = 2C + \mathbf{Id}$, and then $C^{k+2}A = 2C^{k+1}A + C^kA$. From $CA = ASA = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$ we deduce by induction that $(AS)^kA(E) \subset]1, \infty[$. Similarly, we get $(AS)^{k+1}B(E) \subset]1, \infty]$, $B(SA)^k(E) \subset [0, 1[$, and $B(SA)^{k+1}SB(E) \subset]0, 1[$.

Now, we shall prove that if G and G' are distinct generators of $\mathbf{\Gamma}$, the relation $G\cdot M = G'\cdot M'$ is impossible. This is clear in cases where $G(E)$ and $G'(E)$ are disjoint. Suppose that $(AS)^kAM = (AS)^{k'}BM'$. If $k < k'$, we would get $M = S\Big((AS)^{k''}BM'\Big) = SM''$, which is impossible because SM'' has some negative coefficients. If $k \geq k'$, we get $(AS)^{k''}AM = BM'$, which is also impossible because $B(E)$ and $(AS)^{k''}A(E)$ are disjoint. The proof of the impossibility of $B(SA)^k M = B(SA)^{k'+1} SBM'$ is analogous to the preceding one. This completes the proof that $\mathbf{\Gamma}$ is free.

Now, let us prove that $M \neq M',\ M, M' \in \mathbf{\Gamma}^*$ implies $M(\infty) \neq M'(\infty)$. There are two cases that are not obvious.

If $(AS)^k AM(\infty) = (AS)^{k'} BM'(\infty),\ M, M' \in \mathbf{\Gamma}^*$. If $k < k'$, we get $M(\infty) = S(M''(\infty))$, which is impossible since $M(\infty) \in]0, \infty[$, and $SM''(\infty) \in] - \infty, 0[$.
If $k \geq k'$, we get $BM'(\infty) = (AS)^{k''} AM(\infty)$, which is also impossible because $BM'(\infty) < 1$ and $(AS)^{k''} AM(\infty) > 1$.

The proof of the impossibility of $B(SA)^k M(\infty) = B(SA)^{k'+1} SBM'(\infty)$ is analogous to the preceding one. This completes the proof of the injectivity of h. □

Theorem 2.21. *Let $r = \dfrac{\alpha}{\beta} > 0$ be a rational number. Then r has a continued fraction expansion $\dfrac{\alpha}{\beta} = [1, 2e_2, \ldots, e_{2n-1}, 2e_{2n}]$, $e_i = \pm 1$, if and only if β is even. Furthermore, we can suppose that there are no three consecutive sign changes. In this case the continued fraction is unique, and $\alpha > \beta$ if and only if $e_1 = e_2 = 1$.*

Proof. Let us suppose that $\dfrac{\alpha}{\beta} = [1, \pm 2, \ldots, \pm 1, \pm 2]$. It means that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = G \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $G \in \langle A, B, S \rangle$. But we have $A \equiv B \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \pmod 2$, $S \equiv \mathbf{Id} \pmod 2$. Consequently, $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pmod 2$ and β is even.

Let us suppose that β is even. We shall use induction on the height $h(\dfrac{\alpha}{\beta}) = \max(|\alpha|, |\beta|)$.

- If $h(\dfrac{\alpha}{\beta}) = 2$, then $\alpha = 1$ and $\beta = 2$ and we have $r = [1, -2] = B(\infty)$.
- We have two cases to consider
  - If $\alpha > \beta > 0$ then we write $\dfrac{\alpha}{\beta} = [1, 2, -1, 2, \dfrac{\alpha - 2\beta}{\beta}] = ASB\left(\dfrac{\alpha - 2\beta}{\beta}\right)$. We have $h(\dfrac{\alpha - 2\beta}{\beta}) < h(\dfrac{\alpha}{\beta})$ and we conclude by induction.
  - If $\beta > \alpha > 0$ we write $\dfrac{\alpha}{\beta} = [1, -2, \dfrac{\alpha - \beta}{2\alpha - \beta}] = B\left(\dfrac{\beta - \alpha}{2\alpha - \beta}\right)$. From $|2\beta - \alpha| \leq \beta$ we have $h(\dfrac{\alpha - \beta}{2\alpha - \beta}) < h(\dfrac{\alpha}{\beta})$ and we conclude by induction.

The existence of a continued fraction $[1, \pm 2, \ldots, \pm 1, \pm 2]$ is proved.

Using the identities $BSBS(x) = [1, -2, 1, -2, x] = x$ and $[2, -1, 2, -1, x] = x$, we can delete all subsequences with three consecutive sign changes in our sequence. We deduce also that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = G \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $G \in \langle A, S, B \rangle$. Furthermore, from $BSB(x) = [1, -2, 1, -2, -x]$ we see that G contains no BSB. We also see from $ASBSA(x) = [1, 2, -1, 2, -1, -2, -x]$ that G

contains no SBS. Consequently G is an element of $\mathbf{\Gamma}^*$ and, by Lemma 2.20, the continued fraction $\frac{\alpha}{\beta} = [1, \pm 2, \dots, \pm 1, \pm 2]$ is unique. □

Example 2.22 (Torus knots).
Since $BSA = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$, we get by induction

$$A(BSA)^k = \begin{bmatrix} 4k+3 & 1 \\ 4k+2 & 1 \end{bmatrix}, \ A(BSA)^k B = \begin{bmatrix} 4k+5 & 4k+4 \\ 4k+4 & 4k+3 \end{bmatrix}.$$

We deduce the following continued fractions

$$\frac{4k+3}{4k+2} = [1, 2, \underline{1, -2, 1, 2}, \dots, \underline{1, -2, 1, 2}], \qquad \frac{4k+5}{4k+4} = [1, 2, \underline{1, -2, 1, 2}, \dots, \underline{1, -2, 1, 2}, 1, -2]$$

of length $4k+2$ and $4k+4$. Since the knot with Schubert fraction $\frac{N}{N-1}$ is the torus knot $\overline{T}(2, N)$, we see that this knot admits a Chebyshev diagram $x = T_4(t), y = T_N(t)$.

Example 2.23 (Twist knots).
The twist knot is the knot $\mathcal{T}_m = S(\frac{2m+1}{2}) = S(\frac{2m+1}{m+1}) = \overline{S}(\frac{2m+1}{m})$. We have the continued fractions

$$\frac{8k+1}{4k} = (ASB)(BSA)^k(\infty), \qquad \frac{8k+5}{4k+2} = (ASB)(BSA)^k B(\infty).$$

We deduce that $\mathcal{T}_{2n}$ has a Chebyshev diagram $x = T_4(t)$, $y = T_{2n+5}(t)$. We get similarly

$$\frac{8k+7}{4k+4} = ASA(BSA)^k(\infty), \ \frac{8k+3}{4k+2} = ASA(BSA)^{k-1}B(\infty),$$

and we deduce that $\mathcal{T}_{2n+1}$ has a Chebyshev diagram $x = T_4(t)$, $y = T_{2n+3}(t)$.

Example 2.24 (Generalized stevedore knots).
The generalized stevedore knot $\mathcal{S}_m$ is defined by $\mathcal{S}_m = S(2m+2+\frac{1}{2m})$, $m \geq 1$. The stevedore knot is $\mathcal{S}_1 = \overline{6}_1$. We have $\overline{\mathcal{S}}_m = S\Big(\frac{(2m+1)^2}{2m+2}\Big)$ and the continued fractions

$$\frac{(4k+1)^2}{4k+2} = (ASB)^{2k}(BSA)^k B(\infty), \ \frac{(4k-1)^2}{4k} = (ASB)^{2k-1}(BSA)^k(\infty).$$

Consequently, the stevedore knot $\mathcal{S}_m$ has a Chebyshev diagram $\mathcal{C}(4, 6m+3)$.

These examples show that our continued fractions are not necessarily regular. In fact, the subsequences $\pm(2, -1, 2)$ of the continued fraction correspond bijectively to factors (ASB).

There is a formula to compute the crossing number of such knots.

Proposition 2.25. *Let* $\dfrac{\alpha}{\beta} = [e_1, 2e_2, \ldots, e_{2n-1}, 2e_{2n}]$, $e_i = \pm 1$, *where there are no three consecutive sign changes and* $e_1 = e_2$. *We say that* i *is an islet in* $[a_1, a_2, \ldots, a_n]$ *when*

$$|a_i| = 1,\ a_{i-1} = a_{i+1} = -2a_i.$$

We denote by σ *the number of islets. We have*

$$\mathrm{cn}\,(\frac{\alpha}{\beta}) = \sum_{k=1}^{n} |a_i| - \sharp\{i, a_i a_{i+1} < 0\} - 2\sigma. \tag{3}$$

Proof. By induction on the number of double sign changes $k = \sharp\{i, e_{i-1}e_i < 0,\ e_i e_{i+1} < 0\}$. If $k = 0$, the sequence is biregular and Formula (3) is true by Proposition 2.5. Suppose $k \geq 1$. First, we have $a_1 a_2 > 0$. Then $\dfrac{\alpha}{\beta} = \pm[x, a, b, -c, d, e, y]$ where $[x, a, b]$ is a biregular sequence and $a, b, c, d, e > 0$. We have $[x, a, b, -c, d, e, y] = [x, a, b-1, 1, c-1, -d, -e, -y]$.

- If $c = 2$ (there is no islet at c), then we have $b = d = 1$ and $[x, a, b, -c, d, e, y] = [x, a+1, c-1, -d, -e, -y]$. The sum of the absolute values has decreased by 1, as has the number of sign changes, σ is unchanged.
- If $c = 1$ (there is an islet at c) then $[x, a, b, -c, d, e, y] = [x, a, b-1, -(d-1), -e, -y]$. The sum of absolute values has decreased by 3, the number of sign changes by 1, σ by 1.

In both cases, $\sum_{k=1}^{n} |a_i| - \sharp\{i, a_i a_{i+1} < 0\} - 2\sharp\{i, e_{2i}e_{2i+1} < 0,\ e_{2i+1}e_{2i+2} < 0\}$ remains unchanged while k has decreased by 1. ☐

We shall need a specific result for biregular fractions $[\pm 1, \pm 2, \ldots, \pm 1, \pm 2]$.

Proposition 2.26. *Let* $r = \dfrac{\alpha}{\beta}$ *be a rational number given by a biregular continued fraction of the form* $r = [e_1, 2e_2, e_3, 2e_4, \ldots e_{2m-1}, 2e_{2m}]$, $e_1 = 1$, $e_i = \pm 1$. *If the sequence of sign changes is palindromic, i.e. if* $e_k e_{k+1} = e_{2m-k}e_{2m-k+1}$, *we have* $\beta^2 \equiv \pm 2 \pmod{\alpha}$.

Proof. From Theorem 2.21, and because $\dfrac{\alpha}{\beta} = [1, 2e_2, e_3, 2e_4, \ldots e_{2m-1}, 2e_{2m}]$ is regular, we have $\dfrac{\alpha}{\beta} = G(\infty)$ where $G \in \langle B, (AS)^k A,\ k \geq 0\rangle \subset \mathbf{\Gamma}$.

We shall consider the mapping (analogous to matrix transposition)

$$\varphi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & \frac{c}{2} \\ 2b & d \end{bmatrix}.$$

We have $\varphi(XY) = \varphi(Y)\varphi(X)$, $\varphi(A) = A$, $\varphi(B) = B$ and $\varphi((AS)^k A) = (AS)^k A$.

Let us show that G is a palindromic product of terms $A_k = (AS)^k A$ and B.

By induction on $s = \sharp\{i, e_i e_{i+1} < 0\}$. If $s = 0$ then $G = A^m$. Let $k = \min\{i, e_i = -e_{i+1}\}$. If $k = 2p$ then $G = A^p G'$ and $G' \in S \cdot \mathbf{\Gamma}$. We have $e_1 = \ldots = e_{2p}$ and $e_{2(m-p)+1} = \ldots = e_{2m}$ that is $G = A^p S G' S A^p$. The subsequence $(-e_{2p+1}, \ldots, -e_{2m-2p})$ is still palindromic and corresponds to $G'(\infty)$. By regularity $e_{2p+1} = e_{2p+2}$, and we conclude by induction. If $k = 2p+1$ we have $G = A^p B G' B A^p$ and we conclude also by induction.

Hence $\varphi(G) = G$, and since $G \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, we see that G has the form $G = \begin{bmatrix} \alpha & \gamma \\ \beta & \lambda \end{bmatrix}$, with $\beta = 2\gamma$. Using the fact that $\det(G) = \pm 1$, we get $\beta^2 \equiv \pm 2 \pmod{\alpha}$. □

3 The harmonic knots $\mathrm{H}(\mathbf{a}, \mathbf{b}, \mathbf{c})$

In this paragraph we shall study Chebyshev knots with $\varphi = 0$. Comstock (1897) found the number of crossing points of the harmonic curve parametrized by $x = T_a(t), y = T_b(t), z = T_c(t)$. In particular, he proved that this curve is non-singular if and only if a, b, c are pairwise coprime integers ([Com]). Such curves will be named harmonic knots $\mathrm{H}(a, b, c)$.

We shall need the following result proved in [JP, KP3]

Proposition 3.1. *Let a and b be coprime integers. The $\frac{1}{2}(a-1)(b-1)$ double points of the Chebyshev curve $x = T_a(t), y = T_b(t)$ are obtained for the parameter pairs*

$$t = \cos(\frac{k}{a} + \frac{h}{b})\pi, \ \ s = \cos(\frac{k}{a} - \frac{h}{b})\pi,$$

where h, k are positive integers such that $\dfrac{k}{a} + \dfrac{h}{b} < 1$.

Using the symmetries of Chebyshev polynomials, we see that this set of parameters is symmetrical about the origin. We will write $x \sim y$ when $\mathrm{sign}\,\big(x\big) = \mathrm{sign}\,\big(y\big)$. We shall need the following result proved in [KP3].

Lemma 3.2. *Let $\mathrm{H}(a, b, c)$ be the harmonic knot: $x = T_a(t)$, $y = T_b(t)$, $z = T_c(t)$. A crossing point of parameter $t = \cos\left(\dfrac{k}{a} + \dfrac{h}{b}\right)\pi$, is a right twist if and only if*

$$D = \Big(z(t) - z(s)\Big) x'(t) y'(t) > 0$$

where

$$z(t) - z(s) = T_c(t) - T_c(s) = -2 \sin\Big(\frac{ch}{b}\pi\Big) \sin\Big(\frac{ck}{a}\pi\Big).$$

and

$$x'(t) y'(t) \sim (-1)^{h+k} \sin\Big(\frac{ah}{b}\pi\Big) \sin\Big(\frac{bk}{a}\pi\Big).$$

From this lemma we immediately deduce

Corollary 3.3. *Let a, b, c be coprime integers. Suppose that the integer c' verifies $c' \equiv c \pmod{2a}$ and $c' \equiv -c \pmod{2b}$. Then the knot $\mathrm{H}(a, b, c')$ is the mirror image of $\mathrm{H}(a, b, c)$.*

Proof. At each crossing point we have $T_{c'}(t) - T_{c'}(s) = -\Big(T_c(t) - T_c(s)\Big)$. □

Corollary 3.4. *Let a, b, c be coprime integers. Suppose that the integer c is of the form $c = \lambda a + \mu b$ with $\lambda, \mu > 0$. Then there exists $c' < c$ such that $\mathrm{H}(a, b, c) = \overline{\mathrm{H}}(a, b, c')$*

Proof. Let $c' = |\lambda a - \mu b|$. The result follows immediately from corollary 3.3 □

In a recent paper, G. and J. Freudenburg have proved the following stronger result. *There is a polynomial automorphism Φ of $\mathbf{R}^3$ such that $\Phi(\mathrm{H}(a, b, c)) = \mathrm{H}(a, b, c')$.* They also conjectured that the knots $\mathrm{H}(a, b, c)$, $a < b < c$, $c \neq \lambda a + \mu b$, $\lambda, \mu > 0$ are different knots ([FF], Conjecture 6.2).

3.1 The harmonic knots $\mathrm{H}(\mathbf{3}, \mathbf{b}, \mathbf{c})$

The following result is the main step in the classification of the harmonic knots $\mathrm{H}(3, b, c)$.

Theorem 3.5. *Let $b = 3n + 1$, $c = 2b - 3\lambda$, $(\lambda, b) = 1$. The Schubert fraction of the knot $\mathrm{H}(3, b, c)$ is*

$$\frac{\alpha}{\beta} = [e_1, e_2, \ldots, e_{3n}], \text{ where } e_k = \mathrm{sign}\left(\sin k\theta\right) \text{ and } \theta = \frac{\lambda}{b}\pi.$$

If $0 < \lambda < \frac{b}{2}$, its crossing number is $N = b - \lambda = \frac{b + c}{3}$, and we have $\beta^2 \equiv \pm 1 \pmod{\alpha}$.

Proof. Will be given in section 5, p. 31.

Corollary 3.6. *The knots $\mathrm{H}(3, b, c)$ where $\frac{c}{2} < b < 2c$, $b \equiv 1 \pmod 3$, $c \equiv 2 \pmod 3$ are different knots (even up to mirroring). Their crossing number is given by $b + c = 3N$.*

Proof. Let $K = \mathrm{H}(3, b, c)$ and $\frac{\alpha}{\beta} > 1$ be its biregular Schubert fraction given by Theorem 3.5. From Prop 2.15, $\min(b, c)$ is the minimum length of any Chebyshev diagram of K and $\max(b, c) = 3N - \min(b, c)$. The pair (b, c) is uniquely determined. □

The following result gives the classification of harmonic knots $\mathrm{H}(3, b, c)$.

Theorem 3.7.
Let $K = \mathrm{H}(3, b, c)$. There exists a unique pair (b', c') such that (up to mirror symmetry)

$$K = \mathrm{H}(3, b', c'),\ b' < c' < 2b',\ b' + c' \equiv 0 \pmod 3).$$

The crossing number of K is $\frac{1}{3}(b'+c')$, its fractions $\dfrac{\alpha}{\beta}$ are such that $\beta^2 \equiv \pm 1 \pmod{\alpha}$. Furthermore, there is an algorithm to find the pair (b', c').

Proof. Let $K = \mathrm{H}(a, b, c)$ We will show that if the pair (b, c) does not satisfy the condition of the theorem, then it is possible to reduce it.

If $c < b$ we consider $\mathrm{H}(3, c, b) = \overline{\mathrm{H}}(3, b, c)$.

If $b \equiv c \pmod 3$, we have $c = b + 3\mu,\ \mu > 0$. Let $c' = |b - 3\mu|$. We have $c' \equiv \pm c \pmod{2b}$ and $c' \equiv \mp c \pmod 6$. By Lemma 3.3, we see that $K = \overline{\mathrm{H}}(3, b, c')$ and we get a smaller pair.

If $b \not\equiv c \pmod 3$ and $c > 2b$, we have $c = 2b + 3\mu,\ \mu > 0$. Let $c' = |2b - 3\mu|$. Similarly, we get $K = \overline{\mathrm{H}}(3, b, c')$. This completes the proof of existence. This uniqueness is a direct consequence of Corollary 3.6. □

Remark 3.8. Our theorem gives a positive answer to the Freudenburg conjecture for $a = 3$.

Examples

As applications of Proposition 2.5, let us deduce the following results (already in [KP3])

Corollary 3.9. *The harmonic knot $\mathrm{H}(3, 3n+2, 3n+1)$ is the torus knot $T(2, 2n+1)$.*

Proof. The harmonic knot $K = \mathrm{H}(3, 3n+1, 3n+2)$ is obtained for $b = 3n+1$, $c = 2b - 3\lambda,\ \lambda = n,\ \theta = \dfrac{n}{3n+1}\pi$. If $j = 1, 2$, or 3, and $k = 0, \ldots, n-1$ we have $(3k+j)\theta = k\pi + \dfrac{jk-n}{3n+1}$, hence $\mathrm{sign}\bigl(\sin(3k+j)\theta\bigr) = (-1)^k$, so that the Schubert fraction of K is

$$[1, 1, 1, -1, -1, -1, \ldots, (-1)^{n+1}, (-1)^{n+1}, (-1)^{n+1}] = \frac{2n+1}{2n} \approx -(2n+1).$$

We see that K is the mirror image of $T(2, 2n+1)$, which completes the proof. □

It is possible to parameterize the knot $T(2, 2n+1)$ by polynomials of the same degrees and a diagram with only $2n+1$ crossings ([KP2]). However, our Chebyshev parametrizations are easier to visualize. We conjecture that these degrees are minimal (see also [RS, FF, KP1]).

Corollary 3.10. *The harmonic knot $\mathrm{H}(3, b, 2b-3)$ ($b \not\equiv 0 \pmod 3$) is alternate and has crossing number $b - 1$.*

Proof. For this knot we have $\lambda = 1,\ \theta = \dfrac{\pi}{b}$. The Schubert fraction is given by the continued fraction of length $b-1$: $[1,1, \ldots ,1] = \dfrac{F_b}{F_{b-1}}$ where F_n are the Fibonacci numbers ($F_0 = 0, F_1 = 1, \ldots$). J. C. Turner named these knots Fibonacci knots ([Tu]). □

The two previous examples describe infinite families of harmonic knots. They have a Schubert fraction $\dfrac{\alpha}{\beta}$ with $\beta^2 \equiv 1 \pmod{\alpha}$ (torus knots) or with $\beta^2 \equiv -1 \pmod{\alpha}$ (Fibonacci knots with odd b). There is also an infinite number of two-bridge knots with $\beta^2 = \pm 1 \pmod{\alpha}$ that are not harmonic.

Proposition 3.11. *The knots (or links)* $K_n = S(\dfrac{5F_{n+1}}{F_{n+1}+F_{n-1}})$, $n > 1$ *are not harmonic knots* $\mathrm{H}(3,b,c)$. *Their crossing number is* $n+4$ *and we have* $(F_{n+1}+F_{n-1})^2 \equiv (-1)^{n+1} \pmod{5F_{n+1}}$.

Proof. Using the fact that $P^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ we deduce that

$$PMP^nMP = \begin{bmatrix} 5F_{n+1} & F_{n+1}+F_{n-1} \\ F_{n+1}+F_{n-1} & F_{n-1} \end{bmatrix}.$$

Taking determinants, we get $(F_{n+1}+F_{n-1})^2 \equiv (-1)^{n+1} \pmod{5F_{n+1}}$. We also have

$$\frac{5F_{n+1}}{F_{n+1}+F_{n-1}} = PMP^nMP(\infty) = [1, 1, \underbrace{-1, \ldots, -1}_{n+2}, 1, 1].$$

Since $n+2 \geq 4$, it cannot be of the form $[\mathrm{sign}\left(\sin\theta\right), \mathrm{sign}\left(\sin 2\theta\right), \ldots, \mathrm{sign}\left(\sin k\theta\right)]$.

If $n \equiv 2 \pmod 3$, K_n is a two-component link.

If $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$, the Schubert fraction $\dfrac{\alpha}{\beta}$ satisfies $\beta^2 \equiv 1 \pmod{\alpha}$.

If $n \equiv 0 \pmod 6$ or $n \equiv 4 \pmod 6$, K_n is amphicheiral. □

3.2 The harmonic knots $\mathrm{H}(\mathbf{4}, \mathbf{b}, \mathbf{c})$.

The following result will allow us to classify the harmonic knots of the form $\mathrm{H}(4, b, c)$.

Theorem 3.12. *Let* b, c *be odd integers such that* $b \not\equiv c \pmod 4$. *The Schubert fraction of the knot* $K = \mathrm{H}(4, b, c)$ *is given by the continued fraction*

$$\frac{\alpha}{\beta} = [e_1, 2e_2, e_3, 2e_4, \ldots, e_{b-2}, 2e_{b-1}],\ e_j = -\mathrm{sign}\left(\sin(b-j)\theta\right),\ \theta = \frac{3b-c}{4b}\pi.$$

If $b < c < 3b$, *this fraction is biregular, the crossing number of* K *is* $N = \dfrac{3b+c-2}{4}$, *and* $\beta^2 \equiv \pm 2 \pmod 3$.

Proof. Will be given in section 5, p. 34. □

We are now able to classify the harmonic knots of the form $\mathrm{H}(4, b, c)$.

Theorem 3.13.
Let $K = \mathrm{H}(4, b, c)$. *There is a unique pair* (b', c') *such that (up to mirroring)*

$$K = \mathrm{H}(4, b', c'),\ b' < c' < 3b',\ b' \not\equiv c' \pmod 4.$$

The crossing number of K is $\frac{1}{4}(3b'+c'-2)$. K has a Schubert fraction $\dfrac{\alpha}{\beta}$ such that $\beta^2 \equiv \pm 2 \pmod{\alpha}$. Furthermore, there is an algorithm to find (b', c').

Proof. Let us first prove the uniqueness of this pair. Let $K = \mathrm{H}(4,b,c)$ with $b<c<3b$, $c \not\equiv b \pmod 4$. By theorem 3.12, K has a Schubert fraction $\dfrac{\alpha}{\beta} = \pm[1,\pm 2,\ldots,\pm 1,\pm 2]$ of length $b-1$ with β even, $-\alpha<\beta<\alpha$, and $\beta^2 \equiv \pm 2 \pmod{\alpha}$. The other fraction of K is $\dfrac{\alpha}{\beta'}$, where β' is even and $|\beta'|<\alpha$ and $\beta\beta' \equiv 1 \pmod{\alpha}$.

If $\alpha>3$, we cannot have ${\beta'}^2 \equiv \pm 2 \pmod{\alpha}$. We deduce that b is uniquely determined by K: $b = \ell(\dfrac{\alpha}{\beta})+1$ where $\dfrac{\alpha}{\beta}$ is a Schubert fraction of K such that $\beta^2 \equiv \pm 2 \pmod{\alpha}$. Since we also have $3b+c-2 = 4\mathrm{cn}\,(K)$, we conclude that (b,c) is uniquely determined by K.

Now, let us prove the existence of the pair (b',c'). Let $K = \mathrm{H}(4,b,c)$, $b<c$. We have only to show that if the pair (b,c) does not satisfy the condition of the theorem, then it is possible to reduce it.

If $c \equiv b \pmod 4$, then $c = b+4\mu$, $\mu>0$. Let $c' = |b-4\mu|$. Then $K = \overline{\mathrm{H}}(4,b,c')$, and the pair (b,c') is smaller than (b,c).

If $c \not\equiv b \pmod 4$ and $c>3b$, we have $c = 3b+4\mu$, $\mu>0$. Let $c' = |3b-4\mu|$. We have, $K = \overline{\mathrm{H}}(4,b,c')$ with (b,c') smaller than (b,c). This completes the proof. ☐

Remark 3.14. Our theorem gives a positive answer to the Freudenburg conjecture for $a=4$.

We also see that the only knot belonging to the two families of knots $\mathrm{H}(3,b,c)$ and $\mathrm{H}(4,b,c)$ is the trefoil $\mathrm{H}(3,4,5) = \overline{\mathrm{H}}(4,3,5)$.

Example 3.15 ($\mathrm{H}(\mathbf{4},\mathbf{2k-1},\mathbf{2k+1})$).
From Theorem 3.12, we know that $\mathrm{H}(4,2k-1,2k+1)$ has crossing number $2k-1$. Using this theorem, the Conway sequence of this knot is $[e_1, 2e_2, \ldots, e_{2k-3}, 2e_{2k-2}]$, where

$$e_j = -\mathrm{sign}\left(\sin((2k-1-j)\frac{(k-1)\pi}{2k-1})\right) = (-1)^{k+1}\mathrm{sign}\left(\sin(\frac{j(k-1)\pi}{2k-1})\right) = (-1)^{k+\lfloor\frac{j+1}{2}\rfloor}.$$

We deduce that the Schubert fraction of $\mathrm{H}(4,2k-1,2k+1)$ is

$$\frac{\alpha_k}{\beta_k} = (-1)^{k+1}[1,2,-1,-2,1,2,\ldots,(-1)^k,2(-1)^k] = (-1)^{k+1}(AS)^{k-2}A(\infty).$$

Using recurrence formula in Lemma 2.20, we deduce that

$$\begin{array}{l} \alpha_2 = 3, \quad \alpha_3 = 7, \quad \alpha_{k+2} = 2\alpha_{k+1}+\alpha_k \\ \beta_2 = -2, \quad \beta_3 = 4, \quad |\beta_{k+2}| = 2\,|\beta_{k+1}| + |\beta_k|\,. \end{array}$$

Let us consider the homography $G(x) = [2, x]$, and its matrix $G = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Let the sequence u_k be defined by

$$G^k = \begin{bmatrix} u_{k+1} & u_k \\ u_k & u_{k-1} \end{bmatrix}.$$

The sequence u_k verifies the same recurrence formula $u_{k+2} = 2u_{k+1} + u_k$. We deduce $\alpha_k = u_k + u_{k-1}, |\beta_k| = 2u_{k-1}$. We also have

$$P^2 G^{k-2} P(\infty) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_{k-1} & u_{k-2} \\ u_{k-2} & u_{k-3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_k + u_{k-1} \\ 2u_k \end{bmatrix} = \frac{\alpha_k}{|\beta_k|}.$$

Finally, we get $r_k = (-1)^{k+1}[1, 1, \underbrace{2, \ldots 2}_{k-2}, 1]$.

Example 3.16 (The twist knots). The knots $\mathcal{T}_n$ are not harmonic knots $\mathrm{H}(4, b, c)$ for $n > 3$.

Proof. The Schubert fractions of $\mathcal{T}_n = S(n + \dfrac{1}{2})$ with an even denominator are $\dfrac{2n+1}{2}$, and $\dfrac{2n+1}{-n}$ or $\dfrac{2n+1}{n+1}$ according to the parity of n. The only such fractions verifying $\beta^2 \equiv \pm 2 \pmod{\alpha}$ are $\dfrac{3}{2}, \dfrac{7}{4}, \dfrac{9}{4}$. The first two are the Schubert fractions of the trefoil and the $\overline{5}_2$ knot, which are harmonic for $a = 4$. We have only to study the case of $6_1 = S(\dfrac{9}{4})$. We have $\dfrac{9}{4} = [1, 2, -1, 2, 1, -2, 1, 2]$. Since this fraction is not biregular, we see that 6_1 is not of the form $\mathrm{H}(4, b, c)$. □

But there also exist infinitely many rational knots whose Schubert fractions $\dfrac{\alpha}{\beta}$ satisfy $\beta^2 \equiv -2 \pmod{\alpha}$ that are not harmonic for $a = 4$.

Proposition 3.17. *The knots $S(n+\dfrac{1}{2n})$ are not harmonic knots $\mathrm{H}(4, b, c)$ for $n > 1$. Their crossing number is $3n$ and their Schubert fractions $\dfrac{\alpha}{\beta} = \dfrac{2n^2+1}{2n}$ satisfy $\beta^2 \equiv -2 \pmod{\alpha}$.*

Proof. If $n = 2k$, we deduce from $(ASB)^k(x) = 2k + x$ and $(BSA)^k(\infty) = \dfrac{1}{4k}$, that

$$n + \frac{1}{2n} = (ASB)^k (BSA)^k(\infty).$$

If $n = 2k + 1$, we use (see the torus knots, example 2.22) $\dfrac{2n+1}{2n} = A(BSA)^k(\infty)$, so

$$n + \frac{1}{2n} = n - 1 + \frac{2n+1}{2n} = (ASB)^k A(BSA)^k(\infty).$$

For $n > 1$ these continued fractions are not biregular, and since $\beta^2 \equiv -2 \pmod{\alpha}$, they do not correspond to harmonic knots $\mathrm{H}(4, b, c)$. □

4 Chebyshev diagrams of rational knots

Definition 4.1. *We say that a knot in* $\mathbf{R}^3 \subset \mathbf{S}^3$ *has a Chebyshev diagram* $\mathcal{C}(a,b)$*, if* a *and* b *are coprime and the Chebyshev curve*

$$\mathcal{C}(a,b) : x = T_a(t);\ \ y = T_b(t)$$

is the projection of some knot which is isotopic to K*.*

4.1 Chebyshev diagrams with $a = 3$

Using the previous results of our paper (Proposition 2.15) we have

Theorem 4.2. *Let* K *be a two-bridge knot with crossing number* N*. There is an algorithm to determine the smallest* b *such that* K *has a Chebyshev diagram* $\mathcal{C}(3,b)$ *with* $N < b < \frac{3}{2}N$*.*

Proof. Let $\frac{\alpha}{\beta} > 1$ and $\frac{\alpha}{\alpha-\beta} > 1$ be Schubert fractions of K and $\overline{K}$. By Proposition 2.15, $b = \min\Bigl(\ell(\frac{\alpha}{\beta}), \ell(\frac{\alpha}{\alpha-\beta})\Bigr) + 1$ has the required property. □

Definition 4.3. *Let* $\mathcal{D}(K)$ *be a diagram of a knot having crossing points corresponding to the parameters* $t_1, \ldots, t_{2m}$*. The Gauss sequence of* $\mathcal{D}(K)$ *is defined by* $g_k = 1$ *if* t_k *corresponds to an overpass, and* $g_k = -1$ *if* t_k *corresponds to an underpass.*

Theorem 4.4. *Let* K *be a two-bridge knot of crossing number* N*. Let* $x = T_3(t),\ y = T_b(t)$ *be the minimal Chebyshev diagram of* K*. Let* c *denote the number of sign changes in the corresponding Gauss sequence. Then we have*

$$b + c = 3N.$$

Proof. Let s be the number of sign changes in the Conway normal form of K. By Proposition 2.5 we have $N = b - 1 - s$. From this we deduce that our condition is equivalent to $3s + c = 2b - 3$. Let us prove this assertion by induction on s. If $s = 0$ then the diagram of K is alternate, and we deduce $c = 2(b-1) - 1 = 2b - 3$.

Let $C(e_1, e_2, \ldots, e_{b-1})$ be the Conway normal form of K. We may suppose $e_1 = 1$. We shall denote by $M_1, \ldots, M_{b-1}$ the crossing points of the diagram, and by $x_1 < x_2 < \cdots < x_{b-1}$ their abscissae. Let e_k be the first negative coefficient in this form. By the regularity of the sequence we get $e_{k+1} < 0$, and $3 \le k \le b-1$.

Let us consider the knot K' defined by its Conway normal form

$$K' = C(e_1, e_2, \ldots, e_{k-1}, -e_k, -e_{k+1}, \ldots, -e_{b-1}).$$

We see that the number of sign changes in the Conway sequence of K' is $s' = s - 1$. By induction, we get for the knot K': $3s' + c' = 2b - 3$.

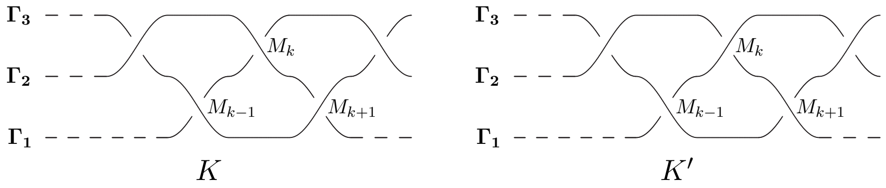


Figure 7: The modification of Gauss sequences

The plane curve $x = T_3(t),\ y = T_b(t)$ is the union of three arcs where $x(t)$ is monotonic. Let Γ be one of these arcs. Γ contains (at least) one point M_k or M_{k+1}. Let j be the first integer in $\{k, k+1\}$ such that M_j is on Γ, and let $j_- < j$ be the greatest integer such that $M_{j_-} \in \Gamma$. In figure 7, we have for Γ_1: $j = k, j^- = k-1$, for Γ_2: $j = k, j^- = k-2$, for Γ_3: $j = k+1, j^- = k-1$.

On each arc Γ, there is a sign change in the Gauss sequence iff the corresponding Conway signs are equal. Then, since the Conway signs $s(M_{j_-})$ and $s(M_j)$ are different, we see that the corresponding Gauss signs are equal. Now, consider the modifications in the Gauss sequences when we transform K into K'. Since the the Conway signs $s(M_h)$, $h \geq k$ are changed, we see that we get one more sign change on every arc Γ. Thus the number of sign changes in the Gauss sequence of K' is $c' = c + 3$. We get $3s + c = 3(s'+1) + c' - 3 = 3s' + c' = 2b - 3$, which completes our induction proof. □

Corollary 4.5. *Let K be a two-bridge knot with crossing number N. Then there exist b, c, $b+c = 3N$, and an polynomial C of degree c such that the knot $x = T_3(t)$, $y = T_b(t)$, $z = C(t)$ is isotopic to K.*

If K is amphicheiral, then b is odd, and the polynomial $C(t)$ can be chosen odd.

Proof. Let $b = n + 1$ be the smallest integer such that K has a Chebyshev diagram $x = T_3(t)$, $y = T_b(t)$. By our theorem 4.4, the Gauss sequence $(g(t_1), \ldots, g(t_{2n}))$ of this diagram has $c = 3N - b$ sign changes. We choose C such that $C(t_i)g(t_i) > 0$ and we can realize it by choosing the roots of C to be $\frac{1}{2}(t_i + t_{i+1})$ where $g(t_i)g(t_{i+1}) < 0$.

If K is amphicheiral, then b is odd and the Conway form is palindromic by Proposition 2.14. Then our Chebyshev diagram is symmetrical about the origin. We see that the Gauss sequence is odd: $g(t_h) = -g(-t_h)$. This implies that the polynomial $C(t)$ is odd. □

Remark 4.6. This corollary gives a simple proof of a famous theorem of Hartley and Kawauchi: *every amphicheiral rational knot is strongly negative amphicheiral* ([HK, Kaw]).

Example 4.7 (The knot 6_1).
The knot $\overline{6}_1 = S(\dfrac{9}{2})$ is not harmonic with $a = 4$. It is not even harmonic with $a = 3$ because $2^2 \not\equiv \pm 1 \pmod 9$. Its crossing number is 6. In the example 2.8, we get $\ell(\dfrac{9}{2}) = 9$, $\ell(\dfrac{9}{7}) = 7$.

$b = 8$ is the minimal value for which $x = T_3(t)$, $y = T_8(t)$ is a Chebyshev diagram for $\overline{6}_1$. The Gauss sequence associated to the Conway form $\overline{6}_1 = C(-1, -1, -1, 1, 1, 1, 1)$ has exactly 10 sign changes. It is precisely

$$[1, -1, -1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1].$$

We can check that

$$x = T_3(t),\ y = T_8(t),\ z = (8\,t + 7)\,(5\,t - 4)\,\left(15\,t^2 - 14\right)\,\left(2\,t^2 - 1\right)\,\left(3\,t^2 - 1\right)\,\left(15\,t^2 - 1\right)$$

is a parametrization of $\overline{6}_1$ of degree $(3, 8, 10)$. In [KP3] we gave the Chebyshev parametrization $6_1 = \mathcal{C}(3, 8, 10, \frac{1}{100})$. We will give another parametrization in example 4.11.

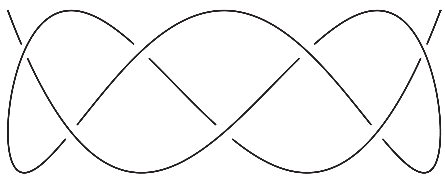


Figure 8: The knot 6_1

4.2 Chebyshev diagrams with $a = 4$

It is also possible to get Chebyshev diagrams of the form $\mathcal{C}(4, b)$. The following result is analogous to the Theorem 4.4.

Theorem 4.8. *Let K be a two-bridge knot of crossing number N and Schubert fraction $\dfrac{\alpha}{\beta}$, β even. Let $\dfrac{\alpha}{\beta} = \pm[1, \pm 2, \ldots, \pm 1, \pm 2]$ be a continued fraction expansion of minimal length $b-1$, and σ be the number of islets (subsequences of the form $\pm(2, -1, 2)$) in this expansion. Let $x = T_4(t)$, $y = T_b(t)$ be the corresponding Chebyshev diagram of K and c be the number of sign changes in the corresponding Gauss sequence of K. Then we have*

$$3b + c - 2 = 4N + 12\sigma.$$

Proof. Let s be the number of sign changes in the given continued fraction. Since $N = \frac{3}{2}(b-1) - s - 2\sigma$ the formula is equivalent to $3b + c - 2 = 6(b-1) - 4s - 8\sigma + 12\sigma$, that is

$$4s + c - 4\sigma = 3b - 4.$$

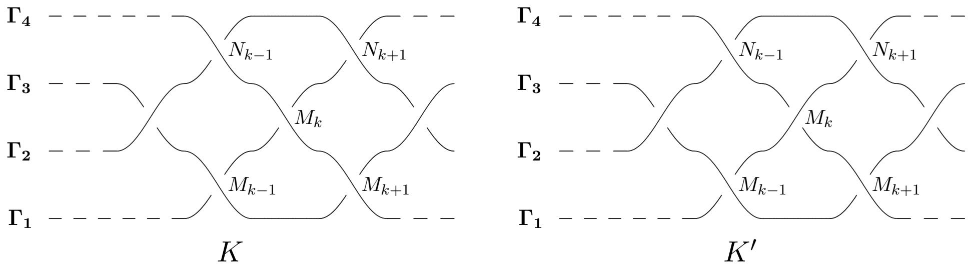
We shall prove this formula by induction on s.

If $s = 0$, the knot is alternate. We have $c = 3b - 4$, $\sigma = 0$, and the formula is true.

We shall need precise notations. Let $M_1, M_2, N_2, M_3, M_4, N_4, \ldots, M_{b-1}, N_{b-1}$ be the crossing points where $x(M_k) = x_k$, $x(N_{2k}) = x_{2k}$, and $x_1 < x_2 < \cdots < x_{b-1}$.

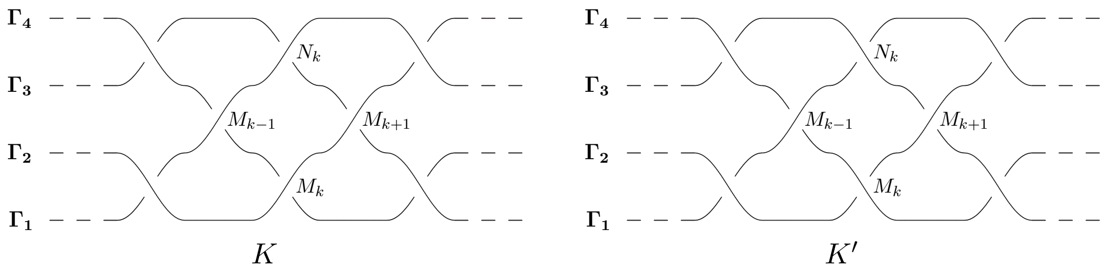
The plane curve $x = T_4(t),\ y = T_b(t)$ is the union of four arcs where $x(t)$ is monotonic (see the following figures). On each arc there is a sign change in the Gauss sequences iff the corresponding Conway signs are equal. Let $C(e_1, 2e_2, \ldots, e_{b-2}, 2e_{b-1}),\ e_i = \pm 1$ be the Conway form of K. Let k be the first integer such that $e_{k-1}e_k < 0$. We have three cases to consider.

k is odd, and $\mathbf{e_k e_{k+1}} < 0$.
In this case, $(2e_{k-1}, e_k, 2e_{k+1}) = \pm(2, -1, 2)$ is an islet. Let us consider the knot K' obtained by changing only the sign of e_k. The number of sign changes in the Conway sequence of K' is $s' = s - 2$. By induction we get for $K' : 4s' + c' - 4\sigma' = 3b - 4$. The number of islets of K' is $\sigma' = \sigma - 1$. Let us look the modification of Gauss sequences. There are only two arcs containing M_k. On each of these arcs there is no sign change in the Gauss sequence of K, and then there are two sign changes in the Gauss sequence of K'. Consequently, $c' = c + 4$. Finally, we get $4s + c - 4\sigma = 4(s' + 2) + (c' - 4) - 4(\sigma' + 1) = 4s' + c' - 4\sigma' = 3b - 4$, which



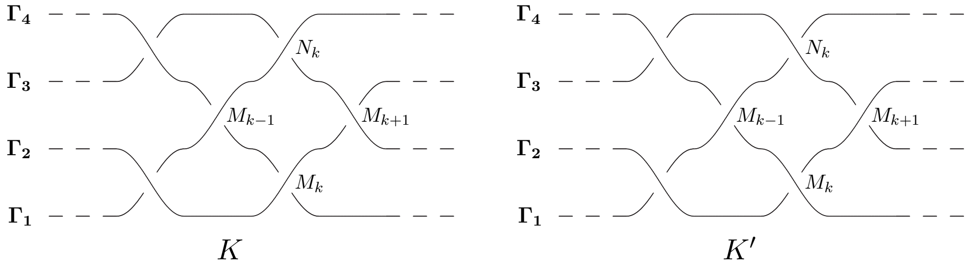
completes the proof in this case.

k is even, and $\mathbf{e_k e_{k+1}} < 0$.
In this case there are two crossing points M_k and N_k with $x(M_k) = x(N_k) = x_k$. Each arc contains one of these points. Let us consider the knot K' obtained by changing only the sign of e_k. The number of sign changes in the Conway sequence of K' is $s' = s - 2$. By induction the formula is true for K'. On each arc the Gauss sequence gains two sign changes, so that



$c' = c + 8$. Since we have $\sigma' = \sigma$, we get $4s + c - 4\sigma = 4(s' + 2) + (c' - 8) - 4\sigma' = 3b - 4$.

The case $\mathbf{e_k e_{k+1}} > 0$.
In this case we consider the knot K' obtained by changing the signs of $e_j,\ j \geq k$. For K'



we have $s' = s - 1$, and by induction $4s' + c' - 4\sigma' = 3b - 4$.
On each of the four arcs the Gauss sequence gains one sign change, and then $c' = c + 4$. Since $\sigma' = \sigma$, we conclude

$$4s + c - 4\sigma = 4(s' + 1) + (c' - 4) - 4\sigma' = 4s' + c' - 4\sigma' = 3b - 4.$$

This completes the proof of the last case. □

Corollary 4.9. *Let K be a two-bridge knot of crossing number N and Schubert fraction $\dfrac{\alpha}{\beta}$, β even. Let $\dfrac{\alpha}{\beta} = \pm[1, \pm 2, \ldots, \pm 1, \pm 2]$ be a continued fraction expansion of minimal length $b-1$ and σ be the number of islets (subsequences of the form $\pm(2, -1, 2)$) in this expansion. There exists an odd polynomial $C(t)$ of degree c such that $3b + c - 2 = 4N + 12\sigma$ and such that the knot defined by $x = T_4(t)$, $y = T_b(t)$, $z = C(t)$ is isotopic to K.*

Proof. This proof is similar to the proof of Corollary 4.5. We must bear in mind that in this case, the Gauss sequence is odd: $g(t_i) = -g(-t_i)$ and $t_{3(b-1)+1-i} = -t_i$. □

Corollary 4.9 gives an algorithm to represent any rational knot as a polynomial knot which is rotationally symmetric around the y-axis. This gives a strong evidence for the following classical result.

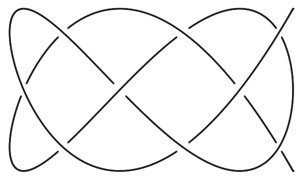
Corollary 4.10. *Every rational knot is strongly invertible.*

Example 4.11 (The stevedore knot $\overline{6}_1$). The stevedore knot 6_1 is $S(\dfrac{9}{2}) = S(-\dfrac{9}{4})$. We get $\dfrac{9}{4} = (ASB)(BSA)(\infty) = [1, 2, -1, 2, 1, -2, 1, 2]$. We deduce that it can be parametrized by $x = T_4(t),\ y = T_9(t),\ z = C(t)$ where $\deg C = 11$. We find

$$C(t) = t\left(34\,t^2 - 33\right)\left(2\,t^2 - 1\right)\left(3\,t^2 - 1\right)\left(4\,t^2 - 1\right)\left(6\,t^2 - 1\right).$$

On the other hand, we find $S(\dfrac{9}{14}) = [1, -2, -1, -2, -1, 2] = BASB(\infty)$. We find that 6_1 can also be represented by polynomials of degrees $(4, 7, 9)$:

$$x = T_4(t),\ y = T_7(t),\ z = t\left(10\,t^2 - 9\right)\left(4\,t^2 - 3\right)\left(4\,t^2 - 1\right)\left(6\,t^2 - 1\right).$$



$\overline{6}_1 = S(\dfrac{9}{14})$

4.3 Examples

In this section, we give several examples of polynomial parametrizations of rational knots with Chebyshev diagrams $\mathcal{C}(3, b)$ and $\mathcal{C}(4, b')$.

Parametrizations of the torus knots

The torus knot $T(2, N),\ N = 2n + 1$ is the harmonic knot $\overline{\mathrm{H}}(3, 3n + 1, 3n + 2)$.

The torus knot $T(2, N)$ can be parametrized by $x = T_4(t)$, $y = T_N(t)$, $z = C(t)$, where $C(t)$ is an odd polynomial of degree $\deg(C) = N + 2 = 2n + 3$.

Proof. We have already seen (example 2.22) that $T(2, N)$ has a Chebyshev diagram $x = T_4(t), y = T_N(t)$. Since there is no islet in the corresponding continued fractions, we see that the number of sign changes in the Gauss sequence is $c = N + 2$. By symmetry of the diagram, we can find an odd polynomial of degree c giving this diagram. □

In both cases, the diagrams have the same number of crossing points : $\frac{3}{2}(N - 1) = 3n$.

As an example, we obtain for $\overline{T}(2, 5)$:

$$x = T_4(t),\ y = T_5(t),\ z = t\left(2\,t^2 - 1\right)\left(3\,t^2 - 1\right)\left(5\,t^2 - 4\right).$$

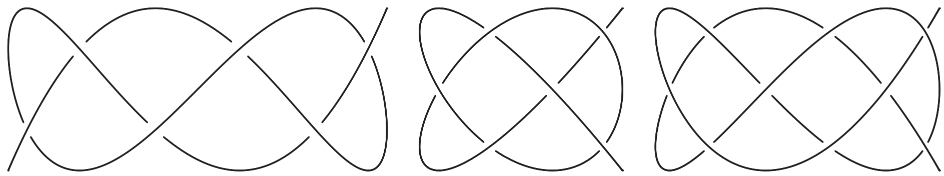
In this case the Chebyshev diagram has exactly 6 crossing points as it is for $\mathrm{H}(3, 7, 8)$. Note that we also obtain $T(2, 5) = S(\dfrac{5}{6})$:

$$x = T_4(t),\ y = T_7(t),\ z = t\left(21\,t^2 - 20\right)\left(4\,t^2 - 1\right).$$

We therefore obtain parametrizations of degrees $(4, 5, 7)$ or $(4, 7, 5)$.

Parametrizations of the twist knots

The twist knot $\mathcal{T}_m = S(m + \dfrac{1}{2})$ has crossing number $m + 2$. We have seen (example 3.16) that the only twist knots that are harmonic for $a = 4$ are the trefoil and the $\overline{5}_2$ knot. The knot $\mathcal{T}_m$ is not harmonic for $a = 3$ because $2^2 \not\equiv \pm 1 \pmod{2m + 1}$ except when $m = 2$ (the figure-eight knot) or $m = 1$ (trefoil). From example 2.18, we know that:



$\mathrm{H}(3,7,8)$ $\quad S(\frac{5}{4})$ $\quad S(\frac{5}{6})$

Figure 9: Diagrams of the torus knot $\overline{T}(2,5)$ and its mirror image

- $\mathcal{T}_{2k+1}$ can be parametrized by $x = T_3(t)$, $y = T_{3k+4}$, $z = C(t)$ where $\deg(C) = 3k+5$.
- $\mathcal{T}_{2k}$ can be parametrized by $x = T_3(t)$, $y = T_{3k+2}$, $z = C(t)$ where $\deg(C) = 3k+4$.

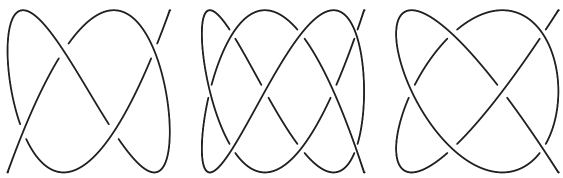
Using results of 2.23, we deduce other parametrizations

- $\mathcal{T}_{2k+1}$ can be represented by $x = T_4(t),\ y = T_{2k+3}(t),\ z = C(t)$ where $C(t)$ is an odd polynomial of degree $2k+5$.
- $\mathcal{T}_{2k}$ can be represented by $x = T_4(t), y = T_{2k+5}(t), z = C(t)$ where $C(t)$ is an odd polynomial of degree $2k+7$.

Proof. The proof is very similar to the preceding one, except that there is an islet in the continued fractions for $2k$ even. □

Note that Chebyshev diagrams we obtain ($a = 3$ or $a = 4$) for $\mathcal{T}_{2k+1}$ have the same number of crossing points: $3k+3$.

Example 4.12 (The figure-eight knot). $\mathcal{T}_2$ is the figure-eight knot. Note that we obtain



$\overline{\mathrm{H}}(3,5,7)$ $\quad S(\frac{5}{2})$ $\quad S(-\frac{5}{8})$

Figure 10: The figure-eight knot

the figure-eight knot as the harmonic knot $\mathrm{H}(3,5,7)$ or as a Chebyshev knot

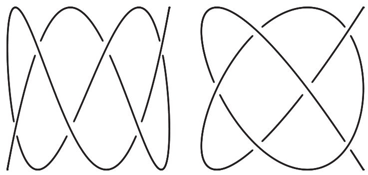
$$x = T_4(t),\ y = T_7(t), z = t\left(10\,t^2 - 9\right)\left(4\,t^2 - 3\right)\left(3\,t^2 - 2\right)\left(2\,t^2 - 1\right).$$

But, considering $S(-5/8)$, we obtain a better parametrization

$$x = T_4(t),\ y = T_5(t),\ z = t\left(11\,t^2 - 10\right)\left(5\,t^2 - 4\right)\left(5\,t^2 - 1\right).$$

Example 4.13 (The 3-twist knot). $\mathcal{T}_3$ is the 3-twist knot $\overline{5}_2$. It is the harmonic knot H(4, 5, 7). It can also be parametrized by

$$x = T_3(t),\ y = T_7(t),\ z = t\,(4\,t + 3)\,(3\,t + 1)\,(6\,t - 5)\,\left(12\,t^2 - 11\right)\,\left(2\,t^2 - 1\right)$$



$S(\frac{7}{2})$ $S(\frac{7}{4})$

Figure 11: Diagrams of the 3-twist knot $\overline{5}_2$

Parametrizations of the generalized stevedore knots

The stevedore knot $\mathcal{S}_m = S(2m + 2 + \dfrac{1}{2m})$ can be represented by $x = T_3(t),\ y = T_{6m+2}(t),\ z = C(t)$ where $C(t)$ is a polynomial of degree $6m + 4$.

The stevedore knot $\mathcal{S}_m = S(2m + 2 + \dfrac{1}{2m})$ can be represented by $x = T_4(t),\ y = T_{6m+3}(t),\ z = C(t)$ where $C(t)$ is an odd polynomial of degree $c = 10m + 1$.

Proof. The case $a = 3$ is deduced from 2.19 and Corollary 4.5. The case $a = 4$ is a consequence of Theorem 4.8. In this case $b = 6m+3$, and the crossing number is $N = 4m+2$. For $m = 2k-1$ the number of islets in $\dfrac{(4k-1)^2}{4k} = (ASB)^{2k-1}(BSA)^k(\infty)$ is $\sigma = 2k-1 = m$. For $m = 2k$, we also find $\sigma = m$. Consequently we get $3(6m+3)+c-2 = 4(4m+2)+12m$, that is, $c = 10m + 1$. The rest of the proof is analogous to the preceding ones. □

Remark 4.14. There is an algorithm to determine minimal Chebyshev diagrams for $a = 3$ (Remark 2.16 and Prop. 2.15). When $a = 4$, we can determine Chebyshev diagrams using Theorem 2.21 but we do not know yet if they are minimal (consider for example $S(\frac{9}{14})$ and $S(-\frac{5}{8})$).

5 Proofs of theorems 3.5 and 3.12

Proof of Theorem 3.5

We study here the diagram of H(3, b, c) where $b = 3n + 1$ and $c = 2b - 3\lambda$. The crossing points of the plane projection of H(3, b, c) are obtained for pairs of values (t, s) where

$t = \cos(\frac{m}{3b}\pi),\ s = \cos(\frac{m'}{3b}\pi)$. For $k = 0, \ldots, n-1$, let us consider

- A_k obtained for $m = 3k+1,\ m' = 2b - m$.
- B_k obtained for $m = 3k+2,\ m' = 2b + m$.
- C_k obtained for $m = 2b - 3k - 3,\ m' = 4b - m$.

Then we have

- $x(A_k) = \cos(\frac{3k+1}{b}\pi),\ \ y(A_k) = \frac{1}{2}(-1)^k$.
- $x(B_k) = \cos(\frac{3k+2}{b}\pi),\ \ y(B_k) = \frac{1}{2}(-1)^{k+1}$.
- $x(C_k) = \cos(\frac{3k+3}{b}\pi),\ \ y(C_k) = \frac{1}{2}(-1)^k$.

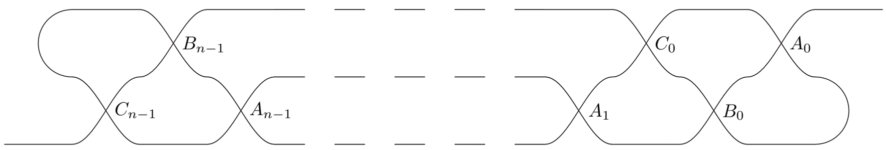


Figure 12: $\mathrm{H}(3, 3n+1, c)$, n even

Hence our $3n$ points satisfy

$$x(A_{k-1}) > x(B_{k-1}) > x(C_{k-1}) > x(A_k) > x(B_k) > x(C_k),\ \ k = 1, \ldots, n-1.$$

Using the identity $T'_a(\cos\tau) = a\dfrac{\sin a\tau}{\sin\tau}$, we get $x'(t)y'(t) \sim \sin\bigl(\frac{m}{b}\pi\bigr)\sin\bigl(\frac{m}{3}\pi\bigr)$. We obtain

$$\begin{aligned}
\text{for } A_k\text{:}\quad x'(t)y'(t) &\sim \sin(\frac{3k+1}{b}\pi)\sin(\frac{3k+1}{3}\pi) \sim (-1)^k.\\
\text{for } B_k\text{:}\quad x'(t)y'(t) &\sim \sin\bigl(\frac{3k+2}{b}\pi\bigr)\sin\bigl(\frac{3k+2}{3}\pi\bigr) \sim (-1)^k.\\
\text{for } C_k\text{:}\quad x'(t)y'(t) &\sim \sin(\frac{2b-3k-3}{b}\pi)\sin(\frac{2b-3k-3}{3}\pi)\\
&\sim -\sin(\frac{3k+3}{b}\pi)\sin(-\frac{3k+1}{3}\pi) \sim (-1)^k.
\end{aligned}$$

The following identity will be useful in computing the sign of $z(t) - z(s)$.

$$T_c(t) - T_c(s) = 2\sin\Bigl(\frac{c}{6b}(m'-m)\pi\Bigr)\sin\Bigl(\frac{c}{6b}(m+m')\pi\Bigr).$$

We have, with $c = 2b - 3\lambda$, $\theta = \dfrac{\lambda}{b}\pi$, (and $b = 3n+1$), for A_k: $z(t) - z(s) = -2\sin c\frac{\pi}{3}\sin\Bigl(c\frac{m-b}{3b}\pi\Bigr)$. But

$$\sin c\frac{\pi}{3} = \sin\Bigl(\frac{6n+2-3\lambda}{3}\pi\Bigr) = (-1)^{\lambda}\sin\frac{2\pi}{3}, \tag{4}$$

and

$$\sin\Bigl(c\frac{b-m}{3b}\pi\Bigr) = \sin\Bigl((2-\frac{3\lambda}{b})\,\frac{b-m}{3}\pi\Bigr) = \sin\bigl(\frac{\lambda}{b}(m-b)\pi\bigr) = (-1)^{\lambda}\sin(3k+1)\theta.$$

We deduce that $z(t) - z(s) \sim \sin(3k+1)\theta$. Finally, we obtain

$$\operatorname{sign}\bigl(D(A_k)\bigr) = (-1)^k \operatorname{sign}\bigl(\sin(3k+1)\theta\bigr).$$

- for B_k: $z(t) - z(s) = 2\sin c\frac{\pi}{3}\sin\bigl(\frac{c}{b}\cdot\frac{b+m}{3}\pi\bigr)$. We have

$$\begin{aligned}\sin\bigl(\frac{c}{b}\cdot\frac{b+m}{3}\pi\bigr) &= \sin\bigl((2-\frac{3\lambda}{b})\frac{b+m}{3}\pi\bigr)\\ &= -\sin\bigl(\frac{\lambda}{b}(b+m)\pi\bigr) = (-1)^{\lambda+1}\sin(3k+2)\theta.\end{aligned}$$

Then, using Equation 4, we get $z(t) - z(s) \sim -\sin(3k+2)\theta$, and finally

$$\operatorname{sign}\bigl(D(B_k)\bigr) = (-1)^{k+1}\operatorname{sign}\bigl(\sin(3k+2)\theta\bigr).$$

- for C_k:
$$\begin{aligned} z(t)-z(s) &\sim \sin\frac{2c}{3}\pi\sin\bigl(\frac{c}{b}(k+1)\pi\bigr)\\ &\sim \sin\frac{4\pi}{3}\sin\bigl((2-\frac{3\lambda}{b})(k+1)\pi\bigr) \sim \sin(3k+3)\theta.\end{aligned}$$

We obtain

$$\operatorname{sign}\bigl(D(C_k)\bigr) = (-1)^k\operatorname{sign}\bigl(\sin(3k+3)\theta\bigr).$$

These results give the Conway normal form. If n is odd, the Conway's signs of our points are

$$\begin{array}{lcccl} s(A_k) & \sim & (-1)^k D(A_k) & \sim & \sin(3k+1)\theta,\\ s(B_k) & \sim & (-1)^{k+1} D(B_k) & \sim & \sin(3k+2)\theta,\\ s(C_k) & \sim & (-1)^k D(C_k) & \sim & \sin(3k+3)\theta.\end{array}$$

In this case our result follows, since the fractions $[e_1, e_2, \ldots, a_{3n}]$ and $(-1)^{3n+1}[a_{3n}, \ldots, a_1]$ define the same knot. If n is even, the Conway's signs are the opposite signs, and we also get the Schubert fraction of our knot.

Since $0 < \theta < \frac{\pi}{2}$, we see that there are not two consecutive sign changes in our sequence. We also see that the first two terms are of the same sign, and so are the last two terms. The Conway normal form is biregular and the total number of sign changes in this sequence is $\lambda - 1$: the crossing number of our knot is then $b - \lambda$. Finally, we get $\beta^2 \equiv \pm 1$ by Proposition 2.14. □

Proof of Theorem 3.12

The crossing points of the plane projection of $\mathrm{H} = \mathrm{H}(4, b, c)$ are obtained for parameter pairs (t, s) where $t = \cos(\frac{m}{4b}\pi)$, $s = \cos(\frac{m'}{4b}\pi)$. We shall denote $\lambda = \frac{3b-c}{4}$, (or $c = 3b - 4\lambda$) and $\theta = \frac{\lambda}{b}\pi$. We will consider the two following cases.

The case $\mathbf{b = 4n + 1}$.

For $k = 0, \ldots, n-1$, let us consider the following crossing points

- A_k corresponding to $m = 4k + 1$, $m' = 2b - m$,
- B_k corresponding to $m = 4k + 2$, $m' = 4b - m$,
- C_k corresponding to $m = 4k + 3$, $m' = 2b + m$,
- D_k corresponding to $m = 2b - 4(k + 1)$, $m' = 4b - m$.

Then we have

- $x(A_k) = \cos(\frac{4k+1}{b}\pi)$, $y(A_k) = (-1)^k \cos\frac{\pi}{4} \neq 0$,
- $x(B_k) = \cos(\frac{4k+2}{b}\pi)$, $y(B_k) = 0$,
- $x(C_k) = \cos(\frac{4k+3}{b}\pi)$, $y(C_k) = (-1)^k \cos\frac{3\pi}{4} \neq 0$,
- $x(D_k) = \cos(\frac{4k+4}{b}\pi)$, $y(D_k) = 0$.

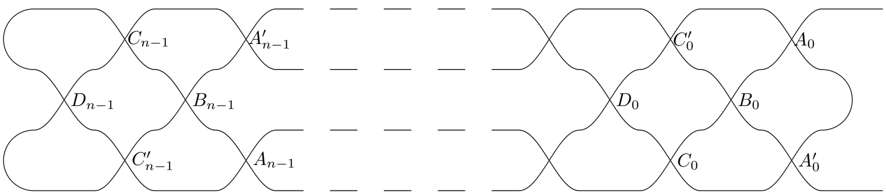


Figure 13: $\mathrm{H}(4, 4n + 1, c)$, n even

Hence our $4n$ points satisfy

$$x(A_{k-1}) > x(B_{k-1}) > x(C_{k-1}) > x(D_{k-1}) >$$
$$x(A_k) > x(B_k) > x(C_k) > x(D_k), \qquad k = 1, \ldots, n-1.$$

We remark that these points together with the symmetric points A'_k (resp. C'_k) of A_k (resp. C_k) with respect to the $y-$axis form the totality of the crossing points.

The Conway sign of a crossing point M is $s(M) = \mathrm{sign}\big(D(M)\big)$ if $y(M) = 0$, and $s(M) = -\mathrm{sign}\big(D(M)\big)$ if $y(M) \neq 0$.

By symmetry, we have $s(A'_k) = s(A_k)$ and $s(C'_k) = s(C_k)$ because symmetric points correspond to opposite parameters. The Conway form of H is then (see paragraph 2) :

$$C\Big(s(D_{n-1}), 2s(C_{n-1}), s(B_{n-1}), 2s(A_{n-1}), \dots, s(B_0), 2s(A_0)\Big).$$

Using the identity $T'_a(\cos\tau) = a\dfrac{\sin a\tau}{\sin\tau}$, we get $x'(t)y'(t) \sim \sin(\dfrac{m}{b}\pi)\sin(\dfrac{m}{4}\pi)$. Consequently,

- For A_k we have $x'(t)y'(t) \sim \sin(\dfrac{4k+1}{b}\pi)\sin(\dfrac{4k+1}{4}\pi) \sim (-1)^k$.
- Similarly, for B_k and C_k we get $x'(t)y'(t) \sim (-1)^k$.
- For D_k we get $x'(t)y'(t) \sim \sin(\dfrac{2b-4k-4}{b}\pi)\sin(\dfrac{2b-4k-4}{4}\pi) \sim (-1)^{k+1}$.

On the other hand, at the crossing points we have

$$z(t) - z(s) = 2\sin\Big(\frac{c}{8b}(m'-m)\pi\Big)\,\sin\Big(\frac{c}{8b}(m+m')\pi\Big).$$

We obtain the signs of our crossing points, with $c = 3b - 4\lambda,\ \theta = \dfrac{\lambda}{b}, t.$

- For A_k we get: $z(t) - z(s) = 2\sin\dfrac{c}{b}(n-k)\pi\,\sin c\dfrac{\pi}{4}$.

  We have $\sin c\dfrac{\pi}{4} = \sin\dfrac{12n+3-4\lambda}{4}\pi = (-1)^{n+\lambda}\sin\dfrac{3\pi}{4} \sim (-1)^{n+\lambda}$

  and also
  $$\begin{aligned}\sin(\frac{c}{b}(n-k)\pi) &= \sin\Big(\big(3-\frac{4\lambda}{b}\big)(n-k)\pi\Big)\\ &= (-1)^{n+k}\sin\Big(\frac{4k-4n}{b}\lambda\pi\Big) = (-1)^{n+k+\lambda}\sin(4k+1)\theta\end{aligned}.$$

  Consequently, the sign of A_k is

  $$s(A_k) = -\mathrm{sign}\,\big(\sin(4k+1)\theta\big).$$

- For B_k, we have: $z(t) - z(s) = 2\sin(\dfrac{c}{b}(2n-k)\pi)\sin c\dfrac{\pi}{2} = -2\sin(\dfrac{c}{b}(2n-k)\pi)$.

  But
  $$\begin{aligned}\sin(\frac{c}{b}(2n-k)\pi) &= \sin\Big(\big(3-\frac{4\lambda}{b}\big)(2n-k)\pi\Big)\\ &= (-1)^k\sin\Big(\frac{\lambda}{b}(4k-8n)\pi\Big) = (-1)^k\sin(4k+2)\theta.\end{aligned}$$

  Therefore the sign of B_k is

  $$s(B_k) = -\mathrm{sign}\,\big(\sin(4k+2)\theta\big).$$

- For C_k: $z(t) - z(s) = 2\sin(\dfrac{c}{4}\pi)\sin(\dfrac{c}{b}(n+k+1)\pi)$.

  We know that $\sin\dfrac{c\pi}{4} \sim (-1)^{n+\lambda}$. Let us compute the second factor

  $$\begin{aligned}\sin\Big(\big(3-\frac{4\lambda}{b}\big)(n+k+1)\pi\Big) &= (-1)^{n+k}\sin\Big(\frac{\lambda}{b}(4n+4k+4)\pi\Big)\\ &= (-1)^{n+k}\sin\Big(\frac{\lambda}{b}(b+4k+3)\pi\Big)\\ &= (-1)^{n+k+\lambda}\sin(4k+3)\theta.\end{aligned}$$

Hence

$$s(C_k) = -\text{sign}\left(\sin(4k+3)\theta\right).$$

- For D_k: $$\begin{aligned} z(t)-z(s) &= 2\sin\bigl(\frac{c}{b}(k+1)\pi\bigr)\sin\bigl(c\frac{\pi}{2}\bigr) \\ &= 2\sin\Bigl(\bigl(3-\frac{4\lambda}{b}\bigr)\bigl(k+1\bigr)\pi\Bigr) = (-1)^k\sin(4k+4)\theta. \end{aligned}$$

  We conclude

$$s(D_k) = -\text{sign}\left(\sin(4k+1)\theta\right).$$

This completes the computation of our Conway normal form of H in this first case.

The case $\mathbf{b = 4n+3}$.
Here, the diagram is different. Let us consider the following $4n+2$ crossing points.
For $k = 0, \ldots, n$

- A_k corresponding to $m = 4k+1,\ m' = 2b+m$,
- B_k corresponding to $m = 4k+2,\ m' = 4b-m$.

For $k = 0, \ldots, n-1$

- C_k corresponding to $m = 4k+3,\ m' = 2b-m$,
- D_k corresponding to $m = 2b+4(k+1),\ m' = 4b-m$.

These points are chosen so that

$$x(A_0) > x(B_0) > x(C_0) > x(D_0) > \cdots > x(D_{n-1}) > x(A_n) > x(B_n),$$

and we have $\text{sign}\left(x'(t)y'(t)\right) = (-1)^k$.

- For A_k we get

$$z(t)-z(s) = 2\sin\bigl(c\frac{\pi}{4}\bigr)\sin\bigl(\frac{c}{b}(n+k+1)\pi\bigr).$$

  We easily get $\text{sign}\left(\sin c\frac{\pi}{4}\right) = (-1)^{n+\lambda}$. We also get

$$\begin{aligned} \sin\bigl(\frac{c}{b}(n+k+1)\pi\bigr) &= \sin\Bigl(\bigl(3-\frac{4\lambda}{b}\bigr)\bigl(n+k+1\bigr)\pi\Bigr) \\ &= (-1)^{n+k}\sin\bigl(\frac{\lambda}{b}(b+4k+1)\pi\bigr) = (-1)^{n+k+\lambda}\sin(4k+1)\theta. \end{aligned}$$

  Hence the sign of A_k is

$$s(A_k) = -\text{sign}\left(\sin(4k+1)\theta\right).$$

- For B_k we get

$$z(t)-z(s) = 2\sin\bigl(\frac{c}{b}(2n+1-k)\pi\bigr)\sin c\frac{\pi}{2}.$$

We have $\sin\bigl(c\frac{\pi}{2}\bigr) = 1 > 0$, and

$$\begin{aligned}\sin\bigl(\frac{c}{b}(2n+1-k)\pi\bigr) &= \sin\Bigl(\bigl(3-\frac{4\lambda}{b}\bigr)\bigl(2n+1-k\bigr)\pi\Bigr)\\ &= (-1)^{k+1}\sin\bigl(\frac{\lambda}{b}(4k-8n-4)\pi\bigr) = (-1)^{k+1}\sin(4k+2)\theta.\end{aligned}$$

Then, the sign of B_k is

$$s(B_k) = -\mathrm{sign}\,\bigl(\sin(4k+2)\theta\bigr).$$

- For C_k we have

$$z(t)-z(s) = 2\sin\bigl(\frac{c}{b}(n-k)\pi\bigr)\sin c\frac{\pi}{4}.$$

We get

$$\begin{aligned}\sin\bigl(\frac{c}{b}(n-k)\pi\bigr) &= \sin\Bigl(\bigl(3-\frac{4\lambda}{b}\bigr)\bigl(n-k\bigr)\pi\Bigr)\\ &= (-1)^{n+k}\sin\bigl(\frac{4k-4n}{b}\lambda\pi\bigr) = (-1)^{n+k+\lambda}\sin(4k+3)\theta.\end{aligned}$$

The sign of C_k is then

$$s(C_k) = -\mathrm{sign}\,\bigl(\sin(4k+3)\theta\bigr).$$

- For D_k we get

$$z(t)-z(s) = 2\sin\bigl(-\frac{c}{b}(k+1)\pi\bigr)\sin c\frac{\pi}{2}.$$

We have $\sin c\frac{\pi}{2} > 0$. We also have

$$\sin\bigl(-\frac{c}{b}(k+1)\pi\bigr) = \sin\Bigl(\bigl(\frac{4\lambda}{b}-3\bigr)\bigl(k+1\bigr)\pi\Bigr)(-1)^{k+1}\sin(4k+4)\theta.$$

Consequently, the sign of D_k is

$$s(D_k) = -\mathrm{sign}\,\bigl(\sin(4k+4)\theta\bigr).$$

This concludes the computation of the Conway normal form of $\mathrm{H}(4,b,c)$.

If $b < c < 3b$, we get $\lambda < \frac{b}{2}$, and then $\theta < \frac{\pi}{2}$. Consequently, our sequence is biregular. Furthermore, the total number of sign changes is $\lambda - 1$. We conclude that the crossing number is $N = \frac{3(b-1)}{2} - (\lambda-1) = \frac{3b+c-2}{4}$. The fact that $\beta^2 \equiv \pm 2 \pmod{\alpha}$ is a consequence of Proposition 2.26. □

References

[BZ] G. Burde, H. Zieschang, *Knots*, Walter de Gruyter, 2003

[BDHZ] A. Boocher, J. Daigle, J. Hoste, W. Zheng, *Sampling Lissajous and Fourier knots*, arXiv:0707.4210, (2007).

[BHJS] M. G. V. Bogle, J. E. Hearst, V. F. R. Jones, L. Stoilov, *Lissajous knots*, Journal of Knot Theory and its Ramifications, 3(2): 121-140, (1994).

[Com] E. H. Comstock, *The Real Singularities of Harmonic Curves of three Frequencies*, Trans. of the Wisconsin Academy of Sciences, Vol XI : 452-464, (1897).

[Con] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), 329–358 Pergamon, Oxford (1970)

[Cr] P. R. Cromwell, *Knots and links*, Cambridge University Press, Cambridge, 2004. xviii+328 pp.

[Fi] G. Fischer, *Plane Algebraic Curves*, A.M.S. Student Mathematical Library Vol 15, 2001.

[FF] G. Freudenburg, J. Freudenburg, *Curves defined by Chebyshev polynomials*, 19 p., (2009), `arXiv:0902.3440`

[HK] R. Hartley, A. Kawauchi, *Polynomials of amphicheiral knots*, Math. Ann. **243 (1)**: 63-70 (1979)

[HZ] J. Hoste, L. Zirbel, *Lissajous knots and knots with Lissajous projections,* `arXiv:math.GT/0605632v1`, (2006). To appear in Kobe Journal of mathematics, vol 24, n°2

[JP] V. F. R. Jones, J. Przytycki, *Lissajous knots and billiard knots,* Banach Center Publications, 42:145-163, (1998).

[Kaw] A. Kawauchi, editor, *A Survey of Knot Theory*, Birhäuser, 1996.

[KP1] P. -V. Koseleff, D. Pecker, *On polynomial Torus Knots*, Journal of Knot Theory and its Ramifications, Vol. **17 (12)** (2008), 1525-1537

[KP2] P. -V. Koseleff, D. Pecker, *A construction of polynomial torus knots*, to appear in Journal of AAECC
`http://arxiv.org/abs/0712.2408`

[KP3] P. -V. Koseleff, D. Pecker, *Chebyshev knots*, `arXiv:0812.1089`, (2008).

[KPR] P. -V. Koseleff, D. Pecker, F. Rouillier, *The first rational Chebyshev knots*, Conference MEGA 2009, Barcelona.

[La1] C. Lamm, *There are infinitely many Lissajous knots,* Manuscripta Math., 93: 29-37, (1997).

[La2] C. Lamm, *Fourier Knots,* preprint

[Mi] R. Mishra, *Polynomial representations of strongly-invertible knots and strongly-negative amphicheiral knots,* Osaka J. Math., 43 625-639 (2006).

[Mu] K. Murasugi, *Knot Theory and its Applications*, Boston, Birkhäuser, 341p., 1996.

[P1] D. Pecker, *Simple constructions of algebraic curves with nodes*, Compositio Math. 87 (1993), no. 1, 1–4.

[P2] D. Pecker, *Sur le genre arithmétique des courbes rationnelles. (French) [On the arithmetic genus of rational curves]*, Ann. Inst. Fourier (Grenoble) 46 (1996), no. 2, 293–306.

[RS] A. Ranjan and R. Shukla, *On polynomial representation of torus knots,* Journal of knot theory and its ramifications, Vol. 5 (2) (1996) 279-294.

[Ro] D. Rolfsen, *Knots and Links*, Math. Lecture Series 7, Publish or Perish, 1976.

[Sh] A.R. Shastri, *Polynomial representation of knots*, Tôhoku Math. J. **44** (1992), 11-17.

[St] A. Stoimenow, *Generating functions, Fibonacci numbers and rational knots*, J. Algebra **310(2)** (2007), 491–525.

[Tu] J.C. Turner, *On a class of knots with Fibonacci invariant numbers*, Fibonacci Quart. **24** (1986), n°1, 61-66.

[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its Applications, Advances Soviet Maths Vol 1 (1990)

Pierre-Vincent Koseleff,
Équipe-project INRIA Salsa & Université Pierre et Marie Curie (UPMC-Paris 6)
e-mail: `koseleff@math.jussieu.fr`

Daniel Pecker,
Université Pierre et Marie Curie (UPMC-Paris 6)
e-mail: `pecker@math.jussieu.fr`