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Local existence and uniqueness of the mild solution to the 1D Vlasov–Poisson system with an initial condition of bounded variation

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SUMMARY

We propose a result of local existence and uniqueness of a mild solution to the one-dimensional Vlasov–Poisson system. We establish the result for an initial condition lying in the space $W^{1,1}(\mathbb{R}^2)$, then we extend it to initial conditions lying in the space $BV(\mathbb{R}^2)$, without any assumption of continuity, boundedness or compact support. Copyright © 2009 John Wiley & Sons, Ltd.

KEY WORDS: Vlasov-Poisson system, bounded variation functions, Banach contraction mapping principle

1. Introduction

1.1. Position of the problem

In this paper we study the one-dimensional Vlasov–Poisson system:

$$\forall (t,x,v) \in [0,T]\times\mathbb{R}^2, \quad \frac{\partial f}{\partial t}(t,x,v) + v\frac{\partial f}{\partial x}(t,x,v) + E(t,x)\frac{\partial f}{\partial v}(t,x,v) = 0, \tag{1}$$

$$\forall (t,x) \in [0,T]\times\mathbb{R}, \quad \frac{\partial E}{\partial x}(t,x) = \int_{\mathbb{R}} f(t,x,v)\,dv, \tag{2}$$

$$\forall (x,v) \in \mathbb{R}^2, \quad f(0,x,v) = f_0(x,v). \tag{3}$$

This system models the behaviour of a gas of protons in its self-consistent electrostatic field when the collisions between particles are neglected. In [6], Cooper and Klimas show the existence and uniqueness of a global mild solution to this system, i.e. a solution defined by characteristics, for a continuous and bounded initial condition which has its first two moments in v uniformly bounded in x. This was extended by Bostan [4] to the initial-boundary value problem, with slightly more general hypotheses on the initial and boundary conditions, namely, that they are bounded but not necessarily continuous, and have one moment in v uniformly bounded in x. In [10], Guo showed that there exists a unique local weak solution to (1–3) in the space $L^\infty([0,T], BV(\mathbb{R}^2))$ for initial and boundary conditions with compact support and in the space $L^\infty(\mathbb{R}^2)\cap BV(\mathbb{R}^2)$.

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In this article we extend the results of Guo to the initial value problem with intial data in the space $BV(\mathbb{R}^2)$, thus not necessarily compactly supported, bounded, or continuous. Our proof is based on the contraction mapping principle of Banach, and consists of two steps: first we establish the local existence and uniqueness of a mild solution for an initial data f_0 in $W^{1,1}(\mathbb{R}^2)$, then we extend the result to $f_0 \in BV(\mathbb{R}^2)$.

1.2. Notations and main results

We introduce the following notations (see [10]). Given $T > 0$, we denote

$$U_T = (0,T) \times \mathbb{R} \ \text{ and } \ V_T = (0,T) \times \mathbb{R}^2.$$

For $s \in [0,T]$, we denote $\Pi_s = \{s\} \times \mathbb{R}^2$ the slice $t = s$ of $\overline{V_T}$. Then we introduce the following functional spaces:

$$L(T) = L^\infty(0,T; W^{1,1}(\mathbb{R}^2)), \quad X(T) = L^\infty(0,T; W^{1,\infty}(\mathbb{R})).$$

The space $L(T)$ will be that of the solutions f to the Vlasov equation with initial data $f_0 \in W^{1,1}(\mathbb{R}^2)$; we equip it with its natural norm. As for $X(T)$, it is a space of electrostatic fields E for which the characteristic curves are globally well defined and Lipschitz-continuous in all their variables [4]. This can be shown by adapting the proof of the Cauchy–Lipschitz theorem: the only difference is that we integrate L^∞ functions instead of C^0 functions and so we get continuous solutions differentiable almost everywhere in the time variable and with bounded derivative. We equip it with the following norm:

$$\forall E \in X(T), \quad \|E\|_{X(T)} = \max(\|E\|_{L^\infty(U_T)}, \|\partial_x E\|_{L^\infty(U_T)}).$$

Morover, for any $E \in X(T)$, we set

$$C(E) = \max(\|\partial_x E\|_{L^\infty(U_T)}, 1), \tag{4}$$

and we denote by Y_E the Vlasov differential operator:

$$Y_E = \frac{\partial}{\partial t} + v\frac{\partial}{\partial t} + E(t,x)\frac{\partial}{\partial v}. \tag{5}$$

We recall the definition of the total variation of a function $f \in L^1(\mathbb{R}^2)$ (see for example [7, p. 39]):

$$\forall f \in L^1(\mathbb{R}^2),\ TV[f] = TV_x[f] + TV_v[f], \tag{6}$$

where:

$$TV_x[f] = \limsup_{\epsilon \longrightarrow 0} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\frac{f(x+\epsilon,v) - f(x,v)}{\epsilon}\right| dxdv \tag{7}$$

$$TV_v[f] = \limsup_{\epsilon \longrightarrow 0} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\frac{f(x,v+\epsilon) - f(x,v)}{\epsilon}\right| dxdv \tag{8}$$

The space of functions of bounded variation is defined as:

$$BV(\mathbb{R}^2) = \{f \in L^1(\mathbb{R}^2) : TV[f] < +\infty\}, \tag{9}$$

and equipped with the norm $\|f\|_{BV(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)} + TV[f]$.

Finally, we denote by $L^{bv}(T)$ the space $L^\infty(0,T; BV(\mathbb{R}^2))$ equipped with its natural norm. We shall establish the following two theorems:

Theorem 1 (Local existence and uniqueness in $W^{1,1}$)
Let $f_0 \in W^{1,1}(\mathbb{R}^2)$, and let

$$R \geq \max(\|f_0\|_1, |f_0|_{W^{1,1}(\mathbb{R}^2)}, 1) \quad \text{and} \quad T \in \left[0, \frac{1}{R}\ln\left(\frac{R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\right)\right].$$

Then there exists a unique mild solution $(f,E) \in L(T) \times X(T)$ to (1–3).

Moreover, we have a lower bound on the existence time $T_{\rm ex}$ of the maximal solution to (1–3) with initial condition f_0. Setting $R_0 := \max(1, \|f_0\|_1)$, there holds:

$$T_{\rm ex} \geq \frac{1}{(\mathrm{e}-1)\,|f_0|_{W^{1,1}}} \quad \text{if} \quad |f_0|_{W^{1,1}} \geq \frac{R_0}{\mathrm{e}},$$
$$T_{\rm ex} \geq \frac{1}{R_0}\left[\ln\frac{R_0}{|f_0|_{W^{1,1}}} + \frac{1}{(\mathrm{e}-1)}\right] \quad \text{if} \quad |f_0|_{W^{1,1}} \leq \frac{R_0}{\mathrm{e}}.$$

Theorem 2 (Local existence and uniqueness in BV)
Let $f_0 \in BV(\mathbb{R}^2)$, and let

$$R \geq \max(\|f_0\|_1, TV[f_0], 1) \quad \text{and} \quad T \in \left[0, \frac{1}{R}\ln\left(\frac{R}{TV[f_0]}\right)\right].$$

Then there exists a unique mild solution $(f,E) \in L^{bv}(T) \times X(T)$ to (1–3). The existence time of the maximal solution is bounded as in the $W^{1,1}$ case, with $|f_0|_{W^{1,1}}$ replaced with $TV[f_0]$.

The proof is organised as follows. In §2, we recall the definitions of weak and mild solutions to the linear Vlasov equation (i.e. (1) and (3) with E a known function of (t,x)) and to the Vlasov–Poisson system (1–3). Then, in §3, we estimate the mild solutions to the linear Vlasov equation with initial data in $W^{1,1}(\mathbb{R}^2)$, and use these results to construct a contraction mapping on a suitable set, whose fixed point gives a mild solution to the Vlasov–Poisson system. Finally, we extend these results to initial conditions lying in $BV(\mathbb{R}^2)$ in §4.

2. Weak and mild solutions

2.1. *Definition of a weak solution*

We recall the definition of a weak solution to (1–3) by using the spaces of test functions and the functionals introduced by Guo in [9]. We define two spaces of test functions, one for the Vlasov equation and the other for the Poisson equation:

$$\mathcal{V} = C_c^\infty([0,T) \times \mathbb{R}^2), \quad \mathcal{M} = C_c^\infty([0,T) \times \mathbb{R}).$$

We define for $(E, f, f_0) \in L^\infty_{\rm loc}(U_T) \times L^1_{\rm loc}(V_T) \times L^1_{\rm loc}(\mathbb{R}^2)$ and $\alpha \in \mathcal{V}$ (still like in [9]) the following functional:

$$A(f,E,f_0,\alpha) = \int_{\mathbb{R}^2} f_0(x,v)\,\alpha(0,x,v)\,dxdv + \int_0^T \int_{\mathbb{R}^2} [(Y_E\alpha)\,f](t,x,v)\,dxdv\,dt.$$

We define for $(E,f) \in L^\infty_{\rm loc}(U_T) \times L^1_{\rm loc}((0,T)\times\mathbb{R}_x; L^1(\mathbb{R}_v))$ and $\psi \in \mathcal{M}$ the following functional:

$$C(f,E,\psi) = \int_0^T \int_{\mathbb{R}} E(t,x)\,\partial_x\psi(t,x)dx\,dt \;+ \int_0^T \int_{\mathbb{R}} \psi(t,x) \int_{\mathbb{R}} f(t,x,v)\,dv\,dx\,dt.$$

These functionals are well-defined.

A weak solution to the linear Vlasov equation associated to $E \in L^\infty_{\rm loc}(U_T)$ with initial condition $f_0 \in L^1_{\rm loc}(\mathbb{R}^2)$ is a function $f \in L^1_{\rm loc}(V_T)$ which satisfies:

$$\forall \alpha \in \mathcal{V}, \quad A(f,E,f_0,\alpha) = 0.$$

A weak solution to the one-dimensional Vlasov–Poisson system with intial condition $f_0 \in L^1_{\rm loc}(\mathbb{R}^2)$ is a pair $(E,f) \in L^\infty_{\rm loc}(U_T) \times L^1_{\rm loc}((0,T)\times\mathbb{R}_x; L^1(\mathbb{R}_v))$ which verifies:

$$\forall (\alpha,\psi) \in \mathcal{V}\times\mathcal{M}, \quad A(f,E,f_0,\alpha) = 0 \quad \text{and} \quad C(f,E,\psi) = 0.$$

2.2. Characteristic curves associated to $E \in X(T)$

We recall the following results on the characteristic curves of a transport equation, see for example [6] or [11]. Given $E \in X(T)$ and $(t, x, v) \in \overline{V_T}$, we consider the differential system :

$$\begin{aligned} &\frac{dX}{ds}(s) = V(s), \\ &\frac{dV}{ds}(s) = E(s, X(s)), \\ &(X(t), V(t)) = (x, v). \end{aligned} \tag{10}$$

As remarked above, this system admits a unique solution for all $(t, x, v) \in \overline{V_T}$, which we denote $\Gamma(s; t, x, v) = (X(s; t, x, v), V(s; t, x, v))$ and is called the *characteristic curve* passing by (t, x, v).

As E is bounded on $[0, T] \times \mathbb{R}$, every characteristic curve is defined from $s = 0$ to $s = T$; moreover, the characteristic curves form a partition of V_T. Thus for every characteristic $\Gamma(s; t, x, v)$, we can define an origin on Π_0: $\Gamma(0; t, x, v) = (X(0; t, x, v), V(0; t, x, v))$.

Let $(t, s) \in [0, T]$. We denote by $\phi_{t,s}$ the *characteristic flow* of E, namely the function:

$$\begin{aligned} \phi_{t,s} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, v) &\longmapsto \Gamma(s; t, x, v). \end{aligned} \tag{11}$$

$\phi_{t,s}$ transports a point (t, x, v) of the slice Π_t to a point (s, x', v') of the slice Π_s by following the characteristic curve passing by (t, x, v). It is well-known that $\phi_{t,s}$ is a bijection (one-to-one and onto mapping) of $\mathbb{R}^2$, which admits bounded partial derivatives and whose Jacobian is identically equal to 1.

2.3. Definition of a mild solution

Let $E \in X(T)$ and (X, V) be the associated characteristic curves. A mild solution to the linear Vlasov equation associated to E with initial condition $f_0 \in L^1_{\text{loc}}(\mathbb{R}^2)$ is a function $f \in L^1_{\text{loc}}(V_T)$ which satisfies:

$$f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)) \quad \text{for a.e. } (t, x, v) \in \overline{V_T}.$$

We recall the following result (see for example [2]):

Proposition 3
Let $E \in X(T)$ and $f_0 \in L^1(\mathbb{R}^2)$. Then $f \in L^1(V_T)$ is a weak solution to the linear Vlasov equation associated to E with initial condition f_0 if and only if it is a mild solution.

This can be shown by using the characteristic change of variables: $(t, x, v) \mapsto (t, x_0, v_0) = (t, \phi_{t,0}(x, v))$, as e.g. in Guo [10]. We deduce the existence and uniqueness of a solution $f \in L^1(V_T)$ to the linear Vlasov equation associated to a field $E \in X(T)$:

Corollary 4
Let $E \in X(T)$ and $f_0 \in L^1(\mathbb{R}^2)$. The linear Vlasov equation associated to E with initial condition f_0 admits a unique weak solution in $L^1(V_T)$ defined as: $\forall (t, x, v) \in V_T,\ f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v))$.

Finally, a mild solution to the Vlasov–Poisson system with intial condition $f_0 \in L^1(\mathbb{R}^2)$ is defined as a weak solution (E, f), which belongs to $X(T) \times L^1(V_T)$, and such that f coincides a.e. with the mild solution to the linear Vlasov equation associated to E with initial condition f_0.

3. Proof of Theorem 1

3.1. A priori *estimates*

The proof of Theorem 1 relies on the following two theorems whose version for a half space is given by Guo in [10].

Theorem 5
Let $E \in X(T)$ and $p \in [1, +\infty)$. We suppose that $u \in L^p(V_T)$ and $Y_E u \in L^p(V_T)$. Then:

1. There exists $u_0 \in L^1_{\text{loc}}(\Pi_0) \simeq L^1_{\text{loc}}(\mathbb{R}^2)$, called the trace of u on Π_0, such that $\forall \alpha \in C_c^\infty([0,T[\times\mathbb{R}^2)$,
$$\int_{V_T} (Y_E u\, \alpha + u\, Y_E \alpha)(t,x,v)\, dxdv\, dt = -\int_{\mathbb{R}^2} u_0(x,v)\, \alpha(0,x,v)\, dxdv.$$
2. If $u_0 \in L^p(\mathbb{R}^2)$, then $\forall s \in [0,T],\ u(s) \in L^p(\mathbb{R}^2)$ and
$$\int_{\mathbb{R}^2} |u(s)|^p\, dxdv = \int_{\mathbb{R}^2} |u_0|^p\, dxdv + p\int_0^s \int_{\mathbb{R}^2} (\operatorname{sgn} u\, |u|^{p-1}\, Y_E u)(\tau)\, dxdv\, d\tau.$$

Theorem 6
Let $E \in X(T)$, and $\phi_{t,s}$ be its characteristic flow. We suppose that $u \in L^1(V_T)$ and $Y_E u \in L^1(V_T)$. Let u_0 be the trace of u on Π_0 defined in Theorem 5. If K is a measurable set of $\mathbb{R}^2$ with non-vanishing Lebesgue measure, then:
$$\int_{\phi_{0,s}(K)} |u(s)|\, dxdv = \int_K |u_0|\, dxdv + \int_0^s \int_{\phi_{0,\tau}(K)} (\operatorname{sgn} u\, Y_E u)(\tau)\, dxdv\, d\tau.$$
The proofs rely on the characteristic change of variables and are entirely similar to those of [10].

With these results, we can prove the fundamental estimate on the solutions to the linear Vlasov equation. We introduce the semi-norm $|\cdot|_{W^{1,1}}$ defined by
$$\forall f \in W^{1,1}(\mathbb{R}^2),\ |f|_{W^{1,1}} = \|\partial_x f\|_1 + \|\partial_v f\|_1.$$

Theorem 7
Let $E \in X(T)$ and $f_0 \in W^{1,1}(\mathbb{R}^2)$. Let f be the unique mild solution in $L^1(V_T)$ of the linear Vlasov equation associated to E with initial condition f_0. Then $\forall s \in [0,T],\ f(s) \in W^{1,1}(\mathbb{R}^2)$ and
$$\|f(s)\|_{L^1(\mathbb{R}^2)} = \|f_0\|_{L^1(\mathbb{R}^2)}\,; \qquad (12)$$
$$|f(s)|_{W^{1,1}(\mathbb{R}^2)} \le |f_0|_{W^{1,1}(\mathbb{R}^2)}\ \exp(C(E)s). \qquad (13)$$
Thus, integrating from 0 to T:
$$\int_0^T |f(\tau)|_{W^{1,1}(\mathbb{R}^2)}\, d\tau \le |f_0|_{W^{1,1}(\mathbb{R}^2)}\, \frac{\exp(C(E)T) - 1}{C(E)}. \qquad (14)$$

Proof
Equation (12) is an immediate consequence of point 2 of Theorem 5 (with $p = 1$), or of Theorem 6 (with $K = \mathbb{R}^2$), given that $Y_E f = 0$.

We now establish the estimate (13) on derivatives. The set of the indefinitely differentiable functions with compact support on $\mathbb{R}^2$ is dense in $W^{1,1}(\Pi_0)$ [1, p. 54]. Thus there exists a sequence $(f_0^n)_n$ of elements of $C_c^\infty(\mathbb{R}^2)$, such that $\|f_0^n - f_0\|_{W^{1,1}(\mathbb{R}^2)} \longrightarrow 0$ when $n \longrightarrow +\infty$.

Similarly, we regularise $E \in L^\infty(0,T;,W^{1,\infty}(\mathbb{R}))$ in the following way. We define for all $t \in [0,T]$, $E_n(t,.) = E(t,.) * \rho_n$, where $(\rho_n) \in C_c^\infty(\mathbb{R}_x)$ is a mollifying sequence. The sequence $(E_n)_n$ satisfies: $E_n \in L^\infty(0,T;W^{1,\infty} \cap C^1(\mathbb{R}_x))$; $\|E_n\|_{L^\infty(U_T)} \leq \|E\|_{L^\infty(U_T)}$, $\|\partial_x E_n\|_{L^\infty(U_T)} \leq \|\partial_x E\|_{L^\infty(U_T)}$, and $\|E - E_n\|_{L^\infty(U_T)} \longrightarrow 0$ when $n \longrightarrow +\infty$. We denote by (X^n, V^n) and $\phi^n_{t,s}$ the characteristic curves and flow associated to E_n.

Let f_n be the solution to the linear problem associated to E_n with initial condition f_0^n; we recall that this solution is given for a.e. $(t,x,v) \in \overline{V_T}$ by $f_n(t,x,v) = f_0^n(X^n(0;t,x,v), V^n(0;t,x,v))$. As f_0^n is compactly supported, so is f_n, as $\mathrm{Supp} f_n$ is contained in the image of the compact $[0,T] \times \mathrm{Supp} f_0^n$ by the continuous mapping $(s,x_0,v_0) \mapsto (s, X(s;0,x_0,v_0), V(s;0,x_0,v_0))$. Moreover, the characteristics associated to E_n are Lipschitz-continuous in all their variables (s,t,x,v), therefore $f_n \in W^{1,\infty}(V_T)$.

All together, we have $\partial_x f_n$ and $\partial_v f_n \in L_c^\infty(V_T)$, thus $\partial_x f_n$ and $\partial_v f_n$ lie in $L^1(V_T)$. Moreover $Y_{E_n} \partial_x f_n = -\partial_x E_n \; \partial_v f_n$ in $\mathcal{D}'(V_T)$, thus $Y_{E_n} \partial_x f_n$ lie in $L^1(V_T)$. By an integration by parts, it can be shown that the trace of $\partial_x f_n$ on Π_0 is $\partial_x f_0^n$. If K is a measurable subset of $\mathbb{R}^2$ of non-vanishing Lebesgue measure, we get by Theorem 6:

$$\int_{\phi^n_{0,s}(K)} |\partial_x f_n(s)| = \int_K |\partial_x f_0^n| - \int_0^s \int_{\phi^n_{0,\tau}(K)} (\mathrm{sgn}(\partial_x f_n)\, \partial_x E_n \; \partial_v f_n)(\tau)\, d\tau\,;$$

for the sake of brevity we have omitted the kinetic integration element $dxdv$. Thus:

$$\int_{\phi^n_{0,s}(K)} |\partial_x f_n(s)| \leq \int_K |\partial_x f_0^n| + \|\partial_x E_n\|_{L^\infty([0,s]\times\mathbb{R})} \int_0^s \int_{\phi^n_{0,\tau}(K)} |\partial_v f_n(\tau)|\, d\tau,$$

$$\int_{\phi^n_{0,s}(K)} |\partial_x f_n(s)| \leq \int_K |\partial_x f_0^n| + \|\partial_x E\|_{L^\infty(U_T)} \int_0^s \int_{\phi^n_{0,\tau}(K)} |\partial_v f_n(\tau)|\, d\tau. \quad (15)$$

In the same way, we have $\partial_v f_n \in L_1(V_T)$ and $Y_E \partial_v f_n = -\partial_x f_n \in \mathcal{D}'(V_T)$, thus $Y_E \partial_v f_n$ lie in $L^1(V_T)$; and one shows that the trace of $\partial_v f_n$ on Π_0 is $\partial_v f_0^n$. Reasoning as above, we obtain:

$$\int_{\phi^n_{0,s}(K)} |\partial_v f_n(s)| \leq \int_K |\partial_v f_0^n| + \int_0^s \int_{\phi^n_{0,\tau}(K)} |\partial_x f_n(\tau)|\, d\tau. \quad (16)$$

We add (15) and (16):

$$\int_{\phi^n_{0,s}(K)} \{|\partial_x f_n(s)| + |\partial_v f_n(s)|\} \leq \int_K \{|\partial_v f_0^n| + |\partial_x f_0^n|\}$$
$$+ \max(\|\partial_x E\|_{L^\infty(U_T)}, 1) \int_0^s \int_{\phi^n_{0,\tau}(K)} \{|\partial_x f_n(\tau)| + |\partial_v f_n(\tau)|\}\, d\tau.$$

Then we utilize the Grönwall lemma, and we get:

$$\int_{\phi^n_{0,s}(K)} \{|\partial_x f_n(s)| + |\partial_v f_n(s)|\} \leq \exp(C(E)s) \int_K \{|\partial_v f_0^n| + |\partial_x f_0^n|\} \quad (17)$$

Therefore:

$$\int_0^T \int_{\phi^n_{0,s}(K)} |\nabla f_n(s)|\, ds \leq \frac{\exp(C(E)T) - 1}{C(E)} \int_K |\nabla f_0^n|. \quad (18)$$

Now we utilize the Dunford–Pettis weak compactness criterion in L^1, that can be found for example in [5, p. 76] or [3, p. 167]:

Theorem 8 (Dunford–Pettis)
Let $(f_n)_n$ be a bounded sequence of $L^1(\Omega)$. The sequence is weakly compact if and only if $\{f_n\}_{n\in\mathbb{N}}$ is equiintegrable, that is to say:

$$\forall\epsilon>0,\ \exists K_\epsilon \text{ compact } \subset\Omega \text{ s.t. } \sup_n \int_{\Omega\setminus K_\epsilon} |f_n|\,d\Omega<\epsilon, \quad \text{and:}$$

$$\forall\epsilon>0,\ \exists\eta>0,\ \forall\mathcal{A}\subset\Omega \text{ measurable, } \mathrm{meas}(\mathcal{A})<\eta \implies \sup_n\int_{\mathcal{A}}|f_n|\,d\Omega<\epsilon.$$

Let $\epsilon>0$. The sequences $(\partial_x f_0^n)_n$ and $(\partial_v f_0^n)_n$ converge in $L^1(\mathbb{R}^2)$, thus are weakly compact in $L^1(\mathbb{R}^2)$. By the Dunford–Pettis criterion, these sequences are equiintegrable. Thus, there exists a compact K_ϵ^0 of $\mathbb{R}^2$, and $\eta>0$ such that:

$$\sup_n\int_{\mathbb{R}^2\setminus K_\epsilon^0}\{|\partial_x f_0^n|+|\partial_v f_0^n|\}<\mathrm{e}^{-C(E)T}\,\epsilon, \quad \text{and:}$$

$$\forall\mathcal{A}\subset\mathbb{R}^2 \text{ measurable, } \mathrm{meas}(\mathcal{A})<\eta \implies \sup_n\int_{\mathcal{A}}\{|\partial_x f_0^n|+|\partial_v f_0^n|\}<\mathrm{e}^{-C(E)T}\,\epsilon.$$

Let $\mathcal{A}$ be a subset of $\mathbb{R}^2$ such that $\mathrm{meas}(\mathcal{A})\le\eta$. We have for all $n\in\mathbb{N}$, $\mathrm{meas}(\phi_{s,0}^n(\mathcal{A}))=\mathrm{meas}(\mathcal{A})\le\eta$, and we can apply the inequality (17) to get

$$\sup_n\int_{\mathcal{A}}|\nabla f_n(s)|\le\epsilon \tag{19}$$

Thus we see that the sequences $(\partial_x f_n(s))_n$ and $(\partial_v f_n(s))_n$ verify the second part of the Dunford–Pettis criterion. For the first part of this criterion, we construct a compact K_ϵ such that all the $\phi_{0,s}^n(K_\epsilon^0)\subset K_\epsilon$. Let $(X_L(\tau;0,x_0,v_0),V_L(\tau;0,x_0,v_0))$ and $\phi_{t,s}^L$ be the characteristic curves and flow associated to free transportation ($E=0$). Of course, we have : $V_L(t;0,x_0,v_0)=v_0$ and $X_L(t;0,x_0,v_0)=x_0+v_0\,t$. We denote $L_\epsilon=\phi_{0,s}^L(K_\epsilon^0)$; this set is a compact as the continuous image of a compact. Then, using the estimate on the divergence of characteristics from [6, Lemma 1] or [4, Lemma 4.8], we obtain:

$$\forall t\in[0,T],\quad |V^n(t;0,x_0,v_0)-V_L(t;0,x_0,v_0)|\le t\,\|E_n\|_{L^\infty(U_t)}\,; \tag{20}$$

$$|X^n(t;0,x_0,v_0)-X_L(t;0,x_0,v_0)|\le t^2\,\|E_n\|_{L^\infty(U_t)}. \tag{21}$$

Thus we can take for K_ϵ the compact:

$$K_\epsilon=\{(x,v)\in\mathbb{R}^2:\exists(x_1,v_1)\in L_\epsilon,\ |x-x_1|\le T\,\|E\|_{L^\infty(U_T)} \text{ and } |v-v_1|\le T^2\,\|E\|_{L^\infty(U_T)}\}.$$

We have: $\forall n\in\mathbb{N},\ \phi_{0,s}^n(K_\epsilon^0)\subset K_\epsilon$. Thus,

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^2\setminus K_\epsilon}|\nabla f_n(s)|\le\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^2\setminus\phi_{0,s}^n(K_\epsilon^0)}|\nabla f_n(s)| \tag{22}$$

$$\le\sup_n\int_{\mathbb{R}^2\setminus K_\epsilon^0}\{|\partial_x f_0^n|+|\partial_v f_0^n|\}\,\mathrm{e}^{C(E)T}\le\epsilon. \tag{23}$$

Therefore, $(\partial_x f_n(s))_n$ and $(\partial_v f_n(s))_n$ verify the Dunford-Pettis criterion and thus converge weakly (after extracting a subsequence) in $L^1(\mathbb{R}^2)$ toward some functions g and h of $L^1(\mathbb{R}^2)$.

On the other hand, we have $Y_E(f_n-f)=(E-E_n)\,\partial_v f_n$, thus f_n-f and $Y_E(f_n-f)$ are in $L^1(V_T)$. Applying point 2 of Theorem 5 and then the bound (18), we find:

$$\begin{aligned}\int_{\mathbb{R}^2}|f(s)-f_n(s)| &\le \int_{\mathbb{R}^2}|f_0-f_0^n|+\int_0^s\int_{\mathbb{R}^2}|E(\tau)-E_n(\tau)|\,|\partial_v f_n(\tau)|\,d\tau\\ &\le \int_{\mathbb{R}^2}|f_0-f_0^n|+\|E-E_n\|_{L^\infty(U_T)}\,\frac{\exp(C(E)T)-1}{C(E)}\int_{\mathbb{R}^2}|\nabla f_0^n|.\end{aligned}$$

Thus, $f_n(s)$ converges toward $f(s)$ in $L^1(\mathbb{R}^2)$. As a consequence, $\partial_x f_n(s)$ and $\partial_v f_n(s)$ converge toward $\partial_x f(s)$ and $\partial_v f(s)$ in $\mathcal{D}'(\mathbb{R}^2)$; therefore $g = \partial_x f(s)$ and $h = \partial_v f(s)$, i.e. $\partial_x f(s)$ and $\partial_v f(s)$ lie in $L^1(\mathbb{R}^2)$. In other words, $f(s)$ appears as the weak limit in $W^{1,1}(\mathbb{R}^2)$ of the sequence $(f_n(s))_n$. By passing to the limit in (17), we get:

$$\int_{\mathbb{R}^2} |\nabla f(s)| \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla f_n(s)| \leq \exp(C(E)s)\, \|\nabla f_0\|_{L^1(\mathbb{R}^2)},$$

which is (13), and yields (14) by integrating from 0 to T. □

3.2. *Construction of a contraction mapping*

We now study the non-linear Vlasov–Poisson problem. We choose $f_0 \in W^{1,1}(\mathbb{R}^2)$, and we construct a contraction mapping from a closed subset of a Banach space to itself. To this end, we define the following mappings:

- $\phi_1 : X(T) \to L^\infty(0,T; L^1(\mathbb{R}^2))$ maps $E \in X(T)$ to the unique mild solution f to the linear Vlasov equation associated to E and with initial condition f_0;
- $\phi_2 : L^\infty(0,T; L^1(\mathbb{R}^2)) \to L^\infty(U_T)$ maps $f \in L^\infty(0,T; L^1(\mathbb{R}^2))$ to the unique solution $\mathcal{E}$ to the Poisson equation (2) satisfying $\forall t \in [0,T]$, $\lim_{x\longrightarrow -\infty} \mathcal{E}(t,x) = 0$, namely:
$$\mathcal{E}(t,x) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f(t,y,v)\, dv\, dy.$$
We have in particular $\forall (t,x) \in [0,T]\times\mathbb{R},\ \mathcal{E}(t,x) \geq 0$.

The following lemma will be crucial in our proof.

Lemma 9
For $R \geq 0$, let B'_R be the closed ball of center 0 and radius R of the Banach space $(X(T), \|\cdot\|_{X(T)})$. Then, B'_R is a closed subset of the Banach space $(L^\infty(U_T), \|\cdot\|_\infty)$, hence it is complete for this norm.

Proof
Let $(E_n)_n$ be a sequence of elements of B'_R which converges in $L^\infty(U_T)$ toward $E \in L^\infty(U_T)$. Of course, there holds: $\|E\|_\infty \leq R$.

Then, $(\partial_x E_n)_n$ is a sequence of elements of the closed ball of center 0 and radius R of $L^\infty(U_T)$. Thus, by the Banach–Alaoglu theorem, $(\partial_x E_n)_n$ converges for the weak-$*$ topology (after extracting a subsequence) of $L^\infty(U_T)$ toward some $g \in L^\infty(U_T)$ with $\|g\|_\infty \leq R$. In particular, $(\partial_x E_n)_n$ converges to g in $\mathcal{D}'(U_T)$; but as $(E_n)_n$ converges to $E \in \mathcal{D}'(U_T)$, $(\partial_x E_n)_n$ converges to $\partial_x E$ in $\mathcal{D}'(U_T)$. Thus $\partial_x E = g$, i.e. $\|\partial_x E\|_\infty \leq R$. This proves $E \in B'_R$. □

3.2.1. *Stability and Lipschitz continuity of $\phi_2 \circ \phi_1$*

Let $E \in X(T)$. Theorem 7 gives $f = \phi_1(E) \in L(T)$; moreover we have:

$$\|\phi_1(E)\|_{L(T)} \leq \|f_0\|_1 \ + |f_0|_{W^{1,1}(\mathbb{R}^2)} \exp(C(E)T).$$

Let $f \in L(T)$. By the definition of ϕ_2, we have: $\partial_x\phi_2(f)(t,x) = \int_{-\infty}^{+\infty} f(t,x,v)\, dv$. But, as $f(t) \in W^{1,1}(\mathbb{R}^2)$ for a.e. $t \in [0,T]$, we deduce by Fubini's theorem that, for a.e. $(t,v) \in [0,T]\times\mathbb{R}$, the mapping $x \mapsto f(t,x,v)$ is in $W^{1,1}(\mathbb{R})$, hence it satisfies $\lim_{x\longrightarrow-\infty} f(t,x,v) = 0$. We have thus:

$$\partial_x\phi_2(f)(t,x) = \int_{-\infty}^{+\infty} f(t,x,v)\, dv = \int_{-\infty}^{+\infty}\int_{-\infty}^{x} \partial_x f(t,y,v)\, dy\, dv\,;$$

$$|\partial_x\phi_2(f)(t,x)| \leq \int_{-\infty}^{+\infty}\int_{-\infty}^{x} |\partial_x f(t,y,v)|\, dy\, dv \leq \|\partial_x f(t)\|_{L^1(\mathbb{R}^2)}\,;$$

$$\|\partial_x\phi_2(f)\|_{L^\infty([0,T]\times\mathbb{R})} \leq \|\partial_x f\|_{L^\infty([0,T],L^1(\mathbb{R}^2))}.$$

On the other hand: $\|\phi_2(f)(t)\|_{L^\infty(\mathbb{R})} = \|f(t)\|_{L^1(\mathbb{R}^2)}$; hence:

$$\|\phi_2(f)\|_{L^\infty([0,T]\times\mathbb{R})} = \|f\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \quad \text{and} \quad \|\phi_2(f)\|_{X(T)} \leq \|f\|_{L(T)}. \tag{24}$$

So, we finally have:

$$\forall E \in X(T), \quad \|\phi_2 \circ \phi_1(E)\|_{X(T)} \leq \max(\|f_0\|_1, |f_0|_{W^{1,1}(\mathbb{R}^2)}\, \mathrm{e}^{C(E)T}). \tag{25}$$

Now we show that $\phi_2 \circ \phi_1$ is a Lipschitz-continuous mapping in the norm of $L^\infty(U_T)$. Let $E_1, E_2 \in X(T)$; we denote $f_1 = \phi_1(E_1)$ and $f_2 = \phi_1(E_2)$. There holds: $Y_{E_1}(f_1 - f_2) = (Y_{E_2} - Y_{E_1})(f_2) = (E_2 - E_1)\,\partial_v f_2$. Thus, $(f_1 - f_2) \in L^1(V_T)$ and $Y_{E_1}(f_1 - f_2) \in L^1(V_T)$; we apply Theorem 5 and find:

$$\int_{\mathbb{R}^2} |f_1(s) - f_2(s)| \leq \int_0^s \int_{\mathbb{R}^2} |E_1(\tau) - E_2(\tau)|\, |\partial_v f_2(\tau)|\, d\tau.$$

Thus $\|f_1 - f_2\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq \|E_1 - E_2\|_{L^\infty(U_T)} \|\partial_v f_2\|_{L^1(V_T)}$; applying the bound (14), we obtain:

$$\|\phi_1(E_1) - \phi_1(E_2)\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq \\ \|E_1 - E_2\|_{L^\infty(U_T)}\, |f_0|_{W^{1,1}(\mathbb{R}^2)}\, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{26}$$

Now let $f_1,\ f_2 \in L(T)$. The linearity of the Poisson equation and the bound (24) allow one to write:

$$\|\phi_2(f_1) - \phi_2(f_2)\|_{L^\infty(U_T)} \leq \|f_1 - f_2\|_{L^\infty(0,T;L^1(\mathbb{R}^2))}.$$

Finally we arrive at:

$$\|\phi_2 \circ \phi_1(E_1) - \phi_2 \circ \phi_1(E_2)\|_{L^\infty(U_T)} \leq \\ \|E_1 - E_2\|_{L^\infty(U_T)}\, |f_0|_{W^{1,1}(\mathbb{R}^2)}\, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{27}$$

3.2.2. Local existence and uniqueness We now give conditions on the parameters R and T in order to have: (i) the closed ball B'_R stable by $\phi_2 \circ \phi_1$, and (ii) $\phi_2 \circ \phi_1$ a contraction mapping on B'_R. The stability estimate (25) implies (i) provided: $|f_0|_{W^{1,1}(\mathbb{R}^2)} \exp(\max(R,1)T) \leq R$ and $\|f_0\|_1 \leq R$. Thus we choose:

$$R \geq \max(|f_0|_{W^{1,1}(\mathbb{R}^2)}, \|f_0\|_1) \quad \text{and} \quad T \leq \frac{1}{\max(R,1)}\, \ln\Big(\frac{R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\Big).$$

As for the point (ii), the Lipschitz estimate (27) yields the sufficient condition $R \geq 1$ and $|f_0|_{W^{1,1}(\mathbb{R}^2)}\,(\exp(RT) - 1)/R < 1$. We take for example:

$$R \geq \max(1, |f_0|_{W^{1,1}(\mathbb{R}^2)}) \quad \text{and} \quad T < \frac{1}{R}\, \ln\left(1 + \frac{R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\right).$$

Considering the two conditions, we obtain that given

$$f_0 \in W^{1,1}(\mathbb{R}^2), \quad R \geq \max(1, |f_0|_{W^{1,1}(\mathbb{R}^2)}, \|f_0\|_1), \quad T \leq \frac{1}{R}\, \ln\Big(\frac{R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\Big),$$

the mapping $\phi_2 \circ \phi_1$ goes from B'_R into B'_R and is a contraction for the norm $\|\cdot\|_{L^\infty(U_T)}$. By Lemma 9, B'_R is a complete space for this norm. Utilizing the contraction mapping principle, the mapping $\phi_2 \circ \phi_1$ admits a unique fixed point $E \in B'_R$. If we denote $f = \phi_1(E)$, the pair $(E, f) \in X(T) \times L(T)$ is a mild solution to (1–3).

3.2.3. Estimation of the existence time Let $f_0 \in W^{1,1}(\mathbb{R}^2)$ be fixed; we define $R_0 = \max(1, \|f_0\|_1)$. The function $x \mapsto \ln(ax)/x$ admits a unique maximum at the point $x = \mathrm{e}/a$, and its value is a/e. Thus, the greatest value of expression $\frac{1}{R}\ln(\frac{R}{|f_0|_{W^{1,1}}})$ is attained at $R = \mathrm{e}\,|f_0|_{W^{1,1}}$ and equal to $(\mathrm{e}\,|f_0|_{W^{1,1}})^{-1}$.

There are two possibilities. If $|f_0|_{W^{1,1}} \geq R_0/\mathrm{e}$, we can take $R = R_1 := \mathrm{e}\,|f_0|_{W^{1,1}}$ and $T = T_1 := (\mathrm{e}\,|f_0|_{W^{1,1}})^{-1}$ in §3.2.2. The estimate (13) then shows $|f(T_1)|_{W^{1,1}} = \mathrm{e}\,|f_0|_{W^{1,1}}$. So, §3.2.2 proves the existence and uniqueness of the solution to the Vlasov–Poisson problem with initial data $f(T_1)$ during the time $T_2 := (\mathrm{e}\,|f(T_1)|_{W^{1,1}})^{-1} = (\mathrm{e}^2\,|f_0|_{W^{1,1}})^{-1}$. Thus, the solution generated by the initial data f_0 exists during $T_1 + T_2$. By induction, we obtain an existence time at least equal to:

$$\frac{1}{|f_0|_{W^{1,1}}}\left(\frac{1}{\mathrm{e}} + \frac{1}{\mathrm{e}^2} + \cdots + \frac{1}{\mathrm{e}^n} + \cdots\right) = \frac{1}{(\mathrm{e}-1)\,|f_0|_{W^{1,1}}}.$$

Now, if $|f_0|_{W^{1,1}} \leq R_0/\mathrm{e}$, the existence time given by §3.2.2 is maximal for $R = R_0$ and equal to is equal to $T_0 := \frac{1}{R_0}\ln(\frac{R_0}{|f_0|_{W^{1,1}}})$. Applying (13), we obtain $|f(T_0)|_{W^{1,1}} = \mathrm{e}^{R_0 T_0}\,|f_0|_{W^{1,1}} = R_0 > R_0/\mathrm{e}$. Thus we can use the previous argument to show that the solution to the Vlasov–Poisson problem with initial data $f(T_0)$ exists for a time at least equal to $((\mathrm{e}-1)\,R_0)^{-1}$. Finally, the total existence time is no less than

$$\frac{1}{R_0}\left(\ln\left(\frac{R_0}{|f_0|_{W^{1,1}}}\right) + \frac{1}{(\mathrm{e}-1)}\right).$$

4. Proof of Theorem 2

4.1. Preliminary results

Here we collect some well-known results on the functions of $W^{1,1}(\mathbb{R}^2)$ and $BV(\mathbb{R}^2)$. The following proposition can be found, for example, in [8, pp. 3–4]:

Proposition 10
$W^{1,1}(\mathbb{R}^2) \subset BV(\mathbb{R}^2)$ and $\forall f \in W^{1,1}(\mathbb{R}^2),\ |f|_{W^{1,1}} = TV[f]$.

The following two theorems are taken from [8], p. 7 and p. 14:

Theorem 11
Let $f \in L^1(\mathbb{R}^2)$ and $(f_n)_n$ be a sequence in $BV(\mathbb{R}^2)$ which converges to f in $L^1(\mathbb{R}^2)$. Then:

$$TV[f] \leq \liminf_{n \longrightarrow +\infty} TV[f_n].$$

Theorem 12
Let $f \in BV(\mathbb{R}^2)$. There exists a sequence $(f_n)_n$ in $C^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ such that:

$$\lim_{n \longrightarrow +\infty} \|f_n - f\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{n \longrightarrow +\infty} TV[f_n] = TV[f].$$

4.2. A priori *estimates*

Theorem 13
Let $f \in L^1(V_T)$ be the unique mild solution to the linear Vlasov equation associated to $E \in X(T)$ with initial condition $f_0 \in BV(\mathbb{R}^2)$. Then, $\forall s \in [0, T]$, $f(s) \in BV(\mathbb{R}^2)$ and

$$TV[f(s)] \leq\ TV[f_0]\ \exp(C(E)s).$$

Thus, integrating from 0 to T:

$$\int_0^T TV[f(\tau)]\,d\tau \leq TV[f_0]\,\frac{\exp(C(E)T) - 1}{C(E)}.$$

Remark that the estimate (12) is still valid, as it only uses the L^1 character of f_0 and f.

Proof
Let E, f_0, and f be as in the statement of the theorem. Theorem 12 yields the existence of a sequence $(f_0^n)_n$ in $C^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ such that:

$$\lim_{n \longrightarrow +\infty} \|f_0^n - f_0\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{n \longrightarrow +\infty} TV[f_0^n] = TV[f_0].$$

In particular, we have: $\forall n \in \mathbb{N}$, $f_0^n \in W^{1,1}(\mathbb{R}^2)$.

Let f_n be the unique mild solution to the linear Vlasov equation associated to E with initial condition f_0^n. Using Theorem 7, we get that $\forall s \in [0,T],\ f_n(s) \in W^{1,1}(\mathbb{R}^2)$, and

$$|f_n(s)|_{W^{1,1}(\mathbb{R}^2)} \leq |f_0^n|_{W^{1,1}(\mathbb{R}^2)} \ \exp(C(E)s),$$

thus, by Proposition 10:

$$TV[f_n(s)] \leq TV[f_0^n] \ \exp(C(E)s).$$

We have $Y_E(f_n - f) = Y_E(f_n) - Y_E(f) = 0$, so we can use Theorem 6 and obtain:

$$\int_{\mathbb{R}^2} |f_n(s) - f(s)| = \int_{\mathbb{R}^2} |f_0^n - f_0|.$$

Therefore, $\lim f_n(s) = f(s)$ in $L^1(\mathbb{R}^2)$, for almost every $s \in [0,T]$. Applying Theorem 11 then yields:

$$\begin{aligned} TV[f(s)] \leq \liminf_{n \longrightarrow +\infty} TV[f_n(s)] &\leq \liminf_{n \longrightarrow +\infty} TV[f_0^n] \ \exp(C(E)s) \\ &= TV[f_0] \ \exp(C(E)s), \end{aligned}$$

which implies $f(s) \in BV(\mathbb{R}^2)$. □

4.3. *Construction of a contraction mapping*

We now get down to the non-linear Vlasov–Poisson problem. We define the mappings ϕ_1 and ϕ_2 as in §3.2, and we find sufficient conditions for $\phi_2 \circ \phi_1$ to be a contraction mapping from B'_R to itself.

Let $f_0 \in BV(\mathbb{R}^2)$, $E \in X(T)$ and $f = \phi_1(E)$. By Theorem 13, $f \in L^{bv}(T)$ and

$$\|\phi_1(E)\|_{L^{bv}(T)} \leq \|f_0\|_1 + TV[f_0] \, \mathrm{e}^{C(E)T}.$$

Let us examine the mapping ϕ_2. As in §3.2.1, we find: $\|\phi_2(f)\|_{L^\infty([0,T]\times\mathbb{R})} = \|f\|_{L^\infty(0,T;L^1(\mathbb{R}^2))}$ and $\partial_x \phi_2(f)(t,x) = \int_{-\infty}^{+\infty} f(t,x,v)\, dv$. Then we state and prove the following lemma:

Lemma 14
Let $f \in BV(\mathbb{R}^2)$. We denote by $\rho[f]$ the function of $L^1(\mathbb{R})$ defined by $\forall x \in \mathbb{R},\ \rho[f](x) = \int_{-\infty}^{+\infty} f(x,v)\, dv$. Then, $\rho[f] \in L^\infty(\mathbb{R})$ and $\|\rho[f]\|_\infty \leq TV[f]$.

Proof
According to Theorem 12, there exists a sequence $(f_n)_n$ of functions in $C^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2) \subset W^{1,1}(\mathbb{R}^2)$ such that $\|f_n - f\|_1 \longrightarrow 0$ and $TV[f_n] \longrightarrow TV[f]$ when $n \longrightarrow +\infty$.

From §3.2.1, we know that $\rho[f_n](x) \leq \|\partial_x f_n\|_1 \leq TV[f_n]$. But $\rho[f_n]$ converges to $\rho[f]$ in $L^1(\mathbb{R})$, thus there exists a subsequence $\rho[f_{\sigma(n)}]$ which converges almost everywhere to $\rho[f]$. We have $\rho[f_{\sigma(n)}](x) \leq TV[f_{\sigma(n)}]$ and passing to the limit we get for a.e. $x \in \mathbb{R}$, $\rho[f](x) \leq TV[f]$. Therefore, $\rho[f] \in L^\infty(\mathbb{R})$ and $\|\rho[f]\|_\infty \leq TV[f]$. □

Lemma 14 gives: $\|\phi_2(f)\|_{X(T)} \leq \|f\|_{L^{bv}(T)}$. Thus we have:

$$\forall E \in X(T), \quad \|\phi_2 \circ \phi_1(E)\|_{X(T)} \leq \max(\|f_0\|_1, TV[f_0]\, \mathrm{e}^{C(E)T}).$$

Now we establish that the mapping $\phi_2 \circ \phi_1$ is Lipschitz continuous in the norm of $L^\infty([0,T] \times \mathbb{R})$. Let $E_1,\ E_2 \in X(T)$; we denote $f_1 = \phi_1(E_1)$ and $f_2 = \phi_1(E_2)$. Moreover, as we did in the proof of Theorem 13, we approximate f_0 by a sequence $(f_0^n)_n$ whose terms lie in $W^{1,1}(\mathbb{R}^2)$, and such that

$$\lim_{n \longrightarrow +\infty} \|f_0^n - f_0\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{n \longrightarrow +\infty} TV[f_0^n] = TV[f_0].$$

The solutions to the linear Vlasov equation with field E_1 (resp. E_2) and initial condition f_0^n will be denoted f_1^n (resp. f_2^n). Applying the $W^{1,1}$ estimate (26) to these functions yields:

$$\|f_1^n - f_2^n\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq \|E_1 - E_2\|_{L^\infty(U_T)}\, TV[f_0^n]\, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{28}$$

As seen in the proof of Theorem 13, we have

$$\|f_i^n(s) - f_i(s)\|_{L^1(\mathbb{R}^2)} = \|f_0^n - f_0\|_{L^1(\mathbb{R}^2)}, \quad \text{for a.e. } s \in [0,T], \text{ and } i = 1,\ 2.$$

Thus, f_i^n converges toward f_i in $L^\infty([0,T]; L^1(\mathbb{R}^2))$. Passing to the limit in (28), we obtain:

$$\|f_1 - f_2\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq \|E_1 - E_2\|_{L^\infty(U_T)}\, TV[f_0]\, \frac{\exp(C(E_2)T) - 1}{C(E_2)}.$$

Then, the linearity of ϕ_2 and the bound (24) imply:

$$\|\phi_2 \circ \phi_1(E_1) - \phi_2 \circ \phi_1(E_2)\|_{L^\infty(U_T)} \leq \\ \|E_1 - E_2\|_{L^\infty(U_T)}\, TV[f_0]\, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{29}$$

Reasoning like in §3.2.2, we infer that $\phi_2 \circ \phi_1$ admits a unique fixed point in B'_R for suitable values of R and T (using the contraction mapping principle of Banach), then we deduce the local existence and uniqueness of a mild solution to (1–3). The existence time is estimated as in §3.2.3.

5. Concluding remarks

We have established a result of local existence and uniqueness of a mild solution to the one-dimensional Vlasov–Poisson system. The hypotheses on the data of this problem were improved: the initial data is not assumed to have a compact support, as in [10], or an integrable majorizing function, as in [4, 6], but only to be of bounded variation. As appeared in the course of the proof, the hypothesis $f_0 \in BV(\mathbb{R}^2)$ is close to the minimal assumption guaranteeing that E and $\partial_x E$ are uniformly bounded, and thus the possibility of the existence of a mild solution.

The drawback is that we were not able to establish global existence. From the stability and continuity estimates of §§3 and 4, we see that the crucial point would be to establish that $\rho[f] = \partial_x E$ remains bounded on U_T for an arbitrary T. This is where the more restrictive assumptions made in the literature come in.

Finally, we notice that the arguments presented in this paper can be extended with slight modifications to many-species Vlasov–Poisson systems, or models featuring a neutralising background and/or a confining potential, and so on.

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