Logarithmic decay of the energy for an hyperbolic-parabolic coupled system

Ines Kamoun Fathallah*

Abstract

This paper is devoted to the study of a coupled system consisting in a wave and heat equations coupled through transmission condition along a steady interface. This system is a linearized model for fluid-structure interaction introduced by Rauch, Zhang and Zuazua for a simple transmission condition and by Zhang and Zuazua for a natural transmission condition.

Using an abstract Theorem of Burq and a new Carleman estimate shown near the interface, we complete the results obtained by Zhang and Zuazua and by Duyckaerts. We show, without any geometric restriction, a logarithmic decay result.

Keywords : Fluid-structure interaction; Wave-heat model; Stability; Logarithmic decay.

2000 Mathematics Subject Classification : 37L15; 35B37; 74F10; 93D20

1 Introduction and results

In this work, we are interested with a linearized model for fluid-structure interaction introduced by Zhang and Zuazua in [14] and Duyckaerts in [6]. This model consists of a wave and heat equations coupled through an interface with suitable transmission conditions. Our purpose is to analyze the stability of this system and so to determine the decay rate of energy of solution as $t \to \infty$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. Let Ω_1 and Ω_2 be two bounded open sets with smooth boundary such that $\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega \backslash \overline{\Omega}_1$. We denote by $\gamma = \partial\Omega_1 \cap \partial\Omega_2$ the interface, $\gamma \subset\subset \Omega$, $\Gamma_j = \partial\Omega_j \backslash \gamma$, $j = 1, 2$, ∂_n and $\partial_{n'}$ the unit outward normal vectors of Ω_1 and Ω_2 respectively

*Laboratoire LMV, Université de Versailles Saint-Quentin-En-Yvelines, 45 Avenue des Etats-Unis Batiment Fermat 78035 Versailles (France). E-mail address: ines.fathallah@math.uvsq.fr; Tel:+33139253629; Fax:+33139254645.

($\partial_{n'} = -\partial_n$ on γ).

$$\begin{cases} \partial_t u - \triangle u = 0 & \text{in } (0,\infty)\times\Omega_1, \\ \partial_t^2 v - \triangle v = 0 & \text{in } (0,\infty)\times\Omega_2, \\ u = 0 & \text{on } (0,\infty)\times\Gamma_1, \\ v = 0 & \text{on } (0,\infty)\times\Gamma_2, \\ u = \partial_t v, \quad \partial_n u = -\partial_{n'} v & \text{on } (0,\infty)\times\gamma, \\ u|_{t=0} = u_0 \in L^2(\Omega_1) & \text{in } \Omega_1, \\ v|_{t=0} = v_0 \in H^1(\Omega_2), \quad \partial_t v|_{t=0} = v_1 \in L^2(\Omega_2) & \text{in } \Omega_2. \end{cases} \tag{1}$$

In this system, u may be viewed as the velocity of fluid; while v and $\partial_t v$ represent respectively the displacement and velocity of the structure. That's why the transmission condition $u = \partial_t v$ is considered as the natural condition. For the modelisation subject, we refer to [11] and [14].

System (1) is introduced by Zhang and Zuazua [14]. The same system was considered by Rauch, Zhang and Zuazua in [11] but for simplified transmission condition $u = v$ on the interface instead of $u = \partial_t v$. They prove, under a suitable Geometric Control Condition (GCC) (see [1]), a polynomial decay result. Zhang and Zuazua in [14] prove, without GCC, a logarithmic decay result. Duyckaerts in [6] improves these results.

For system (1), Zhang and Zuazua in [14], show the lack of uniform decay and they prove, under GCC, a polynomial decay result. Without geometric conditions, they analyze the difficulty to prove the logarithmic decay result. This difficulty is mainly due to the lack of gain regularity of wave component v near the interface γ (see [14], Remark 19) which means that the embedding of the domain $D(\mathcal{A})$ of dissipative operator in the energy space is not compact (see [14], Theorem 1). In [6], Duyckaerts improves the polynomial decay result under GCC and confirms the same obstacle to show the logarithmic decay for solution of (1) without GCC. In this paper we are interested with this problem.

There is an extensive literature on the stabilization of PDEs and on the Logarithmic decay of the energy ([2], [3] [4], [8], [10], [12] and the references cited therein) and this paper use a part of the idea developed in [3].

Here we recall the mathematical frame work for this problem (see [14]).

Define the energy space H and the operator $\mathcal{A}$ on H, of domain $D(\mathcal{A})$ by

$$H = \left\{U_0 = (u_0, v_0, v_1) \in L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2)\right\}$$

when $H^1_{\Gamma_2}(\Omega_2)$ is defined as the space

$$H^1_{\Gamma_2}(\Omega_2) = \left\{v_0 \in H^1(\Omega_2), v_0|_{\Gamma_2} = 0\right\},$$

$$\mathcal{A}U_0 = (\triangle u_0, v_1, \triangle v_0)$$

$$D(\mathcal{A}) = \{U_0 \in H,\, u_0 \in H^1(\Omega_1),\, \triangle u_0 \in L^2(\Omega_1),$$

$$v_1 \in H^1_{\Gamma_2}(\Omega_2),\, \triangle v_0 \in L^2(\Omega_2),\, u_0|_\gamma = v_1|_\gamma,\, \partial_n u_0|_\gamma = -\partial_n v_0|_\gamma\}.$$

System (1) may thus be rewritten in the abstract form

$$\partial_t U = \mathcal{A}U, \qquad U(t) = (u(t), v(t), \partial_t v(t)).$$

For any solution $(u, v, \partial_t v)$ of system (1), we have a natural energy

$$E(t) = E(u, v, \partial_t v)(t) = \frac{1}{2}\left(\int_{\Omega_1} |u(t)|^2 \, dx + \int_{\Omega_2} |\partial_t v(t)|^2 \, dx + \int_{\Omega_2} |\nabla v(t)|^2 \, dx\right).$$

By means of the classical energy method, we have

$$\frac{d}{dt}E(t) = -\int_{\Omega_1} |\nabla u|^2 \, dx.$$

Therefore the energy of (1) is decreasing with respect to t, the dissipation coming from the heat component u. Our main goal is to prove a logarithmic decay without any geometric restrictions.

As Duyckaerts [6] did for the simplified model, the idea is, first, to use a known result of Burq (see [5]) which links, for dissipative operators, logarithmic decay to resolvent estimates with exponential loss; secondly to prove, following the work of Bellassoued in [3], a new Carleman inequality near the interface γ.

The main results are given by Theorem 1.1 for resolvent and Theorem 1.2 for decay.

Theorem 1.1 *There exists $C > 0$, such that for every $\mu \in \mathbb{R}$ with $|\mu|$ large, we have*

$$\left\|(\mathcal{A} - i\mu)^{-1}\right\|_{\mathcal{L}(H)} \leq Ce^{C|\mu|}. \tag{2}$$

Theorem 1.2 *There exists $C > 0$, such that the energy of a smooth solution of (1) decays at logarithmic speed*

$$\sqrt{E(t)} \leq \frac{C}{\log(t+2)} \|U\|_{D(\mathcal{A})}. \tag{3}$$

Burq in ([5], Theorem 3) and Duyckaerts in ([6], Section 7) show that to prove Theorem 1.2 it suffices to show Theorem 1.1.

The strategy of the proof of Theorem 1.1 is the following. A new Carleman estimate shown near the interface γ implies an interpolation inequality given by Theorem 2.2. Theorem 2.2 implies Theorem 2.1 which gives an estimate of the wave component by the heat one and which is the key point of the proof of Theorem 1.1.

The rest of this paper is organized as follows. In section 2, we show, from Theorem 2.1, Theorem 1.1 and we explain how Theorem 2.2 implies Theorem 2.1. In section 3, we begin by stating the new Carleman estimate and explain how this estimate implies Theorem 2.2. We give then the proof of this Carleman estimate. Section 4 is devoted to the proof of important estimates stated in Theorem 3.2 in the proof of Carleman estimate. Appendices A and B are devoted to prove some technical results that will be used along the paper.

Acknowledgment

Sincere thanks to professor Luc Robbiano for inspiring question, his greatly contribution to this work and for his careful reading of the manuscript. I want to thank also professor Mourad Bellassoued for his proposition to work in this domain.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We start by stating Theorem 2.1. Then we will explain how this Theorem implies Theorem 1.1. Finally, we give the proof of Theorem 2.1.

Let μ be a real number such that $|\mu|$ is large, and assume

$$F = (\mathcal{A} - i\mu)U, \quad U = (u_0, v_0, v_1) \in D(\mathcal{A}), \qquad F = (f_0, g_0, g_1) \in H \tag{4}$$

The equation (4) yields

$$\begin{cases} (\triangle - i\mu)u_0 &= f_0 & \text{in } \Omega_1, \\ (\triangle + \mu^2)v_0 &= g_1 + i\mu g_0 & \text{in } \Omega_2, \\ v_1 &= g_0 + i\mu v_0 & \text{in } \Omega_2, \end{cases} \tag{5}$$

with the following boundary conditions

$$\begin{cases} u_0|_{\Gamma_1} = 0, \ v_0|_{\Gamma_2} = 0 \\ op(b_1)u = u_0 - i\mu v_0 &= g_0|_\gamma, \\ op(b_2)u = \partial_n u_0 - \partial_n v_0 &= 0|_\gamma. \end{cases} \tag{6}$$

To proof Theorem 1.1, we begin by stating this result

Theorem 2.1 *Let $U = (u_0, v_0, v_1) \in D(\mathcal{A})$ satisfying equation (5) and (6). Then there exists constants $C > 0$, $c_1 > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$ we have the following estimate*

$$\|v_0\|^2_{H^1(\Omega_2)} \leq Ce^{c_1\mu} \left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)} \right). \tag{7}$$

Moreover, from the first equation of system (5), we have

$$\int_{\Omega_1} (-\triangle + i\mu)u_0\overline{u_0}dx = \|\nabla u_0\|^2_{L^2(\Omega_1)} + i\mu \|u_0\|^2_{L^2(\Omega_1)} - \int_\gamma \partial_n u_0 \overline{u_0} d\sigma.$$

Since $u_0|_\gamma = g_0 + i\mu v_0$ and $\partial_n u_0 = -\partial_{n'} v_0$, then

$$\int_{\Omega_1} (-\triangle + i\mu)u_0\overline{u}_0dx = \|\nabla u_0\|^2_{L^2(\Omega_1)} + i\mu \|u_0\|^2_{L^2(\Omega_1)} - i\mu \int_\gamma \partial_{n'} v_0 \overline{v}_0 d\sigma + \int_\gamma \partial_{n'} v_0 \overline{g}_0 d\sigma. \tag{8}$$

From the second equation of system (5) and multiplying by $(-i\mu)$, we obtain

$$i\mu \int_{\Omega_2} (\triangle + \mu^2)v_0\overline{v}_0dx = -i\mu \|\nabla v_0\|^2_{L^2(\Omega_2)} + i\mu^3 \|v_0\|^2_{L^2(\Omega_2)} + i\mu \int_\gamma \partial_{n'} v_0 \overline{v}_0 d\sigma. \tag{9}$$

Adding (8) and (9), we obtain

$$\int_{\Omega_1} (-\triangle + i\mu) u_0 \overline{u}_0 dx + i\mu \int_{\Omega_2} (\triangle + \mu^2) v_0 \overline{v}_0 dx =$$

$$i\mu \|u_0\|^2_{L^2(\Omega_1)} + \|\nabla u_0\|^2_{L^2(\Omega_1)} - i\mu \|\nabla v_0\|^2_{L^2(\Omega_2)} + i\mu^3 \|v_0\|^2_{L^2(\Omega_2)} + \int_\gamma \partial_{n'} v_0 \overline{g}_0 d\sigma.$$

Taking the real part of this expression, we get

$$\|\nabla u_0\|^2_{L^2(\Omega_1)} \leq \|(\triangle - i\mu)u_0\|_{L^2(\Omega_1)} \|u_0\|_{L^2(\Omega_1)} + \left\|(\triangle + \mu^2)v_0\right\|_{L^2(\Omega_2)} \|v_0\|_{L^2(\Omega_2)} + \left| \int_\gamma \partial_{n'} v_0 \overline{g}_0 d\sigma \right|. \tag{10}$$

Recalling that $\triangle v_0 = g_0 + i\mu g_0 - \mu^2 v_0$ and using the trace lemma (Lemma 3.4 in [6]), we obtain

$$\|\partial_n v_0\|_{H^{-\frac{1}{2}}(\gamma)} \leq C \left(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \right).$$

Combining with (10), we obtain

$$\begin{aligned} \|\nabla u_0\|^2_{L^2(\Omega_1)} \leq \|f_0\|_{L^2(\Omega_1)} \|u_0\|_{L^2(\Omega_1)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \|v_0\|_{L^2(\Omega_2)} \\ + \left(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \right) \|g_0\|_{H^{\frac{1}{2}}(\gamma)}. \end{aligned}$$

Then

$$\begin{aligned} \|\nabla u_0\|^2_{L^2(\Omega_1)} \leq \frac{C}{\epsilon} \|f_0\|^2_{L^2(\Omega_1)} + \epsilon \|u_0\|^2_{L^2(\Omega_1)} + \frac{C}{\epsilon} \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \epsilon \|v_0\|^2_{L^2(\Omega_2)} \\ + \left(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \right) \|g_0\|_{H^{\frac{1}{2}}(\gamma)}. \end{aligned}$$

Now we need to use this result shown in Appendix A.

Lemma 2.1 *Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^n$. Then there exists $C > 0$ such that for u and f satisfying $(\triangle - i\mu)u = f$ in $\mathcal{O}$, $\mu \geq 1$, we have the following estimate*

$$\|u\|_{H^1(\mathcal{O})} \leq C \left(\|\nabla u\|_{L^2(\mathcal{O})} + \|f\|_{L^2(\mathcal{O})} \right). \tag{11}$$

Using this Lemma, we obtain, for ϵ small enough

$$\begin{aligned} \|u_0\|^2_{H^1(\Omega_1)} \leq C \|f_0\|^2_{L^2(\Omega_1)} + C_\epsilon \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \epsilon \|v_0\|^2_{L^2(\Omega_2)} \\ + \left(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \right) \|g_0\|_{H^{\frac{1}{2}}(\gamma)}. \end{aligned}$$

Then there exists $c_3 >> c_1$ such that

$$\|u_0\|^2_{H^1(\Omega_1)} \leq C \left(\|f_0\|^2_{L^2(\Omega_1)} + \epsilon e^{-c_3\mu} \|v_0\|^2_{H^1(\Omega_2)} + C_\epsilon e^{-c_3\mu} \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + e^{c_3\mu} \|g_0\|^2_{H^1(\Omega_2)} \right). \tag{12}$$

Inserting in (7), we obtain, for ϵ small enough

$$\|v_0\|^2_{H^1(\Omega_2)} \leq C e^{c\mu} \left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} \right). \tag{13}$$

Combining (12) and (13), we obtain

$$\|u_0\|^2_{H^1(\Omega_1)} \leq Ce^{c\mu}\left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)}\right). \quad (14)$$

Recalling that $v_1 = g_0 + i\mu v_0$ and using (13), we obtain

$$\|v_1\|^2_{H^1(\Omega_1)} \leq Ce^{c\mu}\left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)}\right). \quad (15)$$

Combining (13), (14) and (15), we obtain Theorem 1.1.

□

Proof of Theorem 2.1

Estimate (7) is consequence of two important results. The first is a known result shown by Lebeau and Robbiano in [9] and the second one is given by Theorem 2.2 and proved in section 3.
Let $0 < \epsilon_1 < \epsilon_2$ and V_{ϵ_j}, $j = 1, 2$, such that $V_{\epsilon_j} = \{x \in \Omega_2,\ d(x, \gamma) < \epsilon_j\}$.
Recalling that $(\triangle + \mu^2)v_0 = g_1 + i\mu g_0$, then for all $D > 0$, there exists $C > 0$ and $\nu \in]0, 1[$ such that we have the following estimate (see [9])

$$\|v_0\|_{H^1(\Omega_2\setminus V_{\epsilon_1})} \leq Ce^{D\mu}\|v_0\|^{1-\nu}_{H^1(\Omega_2)}\left(\|g_1 + i\mu g_0\|_{L^2(\Omega_2)} + \|v_0\|_{H^1(V_{\epsilon_2})}\right)^{\nu} \quad (16)$$

Moreover we have the following result shown in section 3.

Theorem 2.2 *There exists $C > 0$, $c_1 > 0$, $c_2 > 0$, $\epsilon_2 > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$, we have the following estimate*

$$\begin{aligned}\|v_0\|^2_{H^1(V_{\epsilon_2})} &\leq Ce^{c_1\mu}\left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right] \\ &+ Ce^{-c_2\mu}\|v_0\|^2_{H^1(\Omega_2)}. \quad (17)\end{aligned}$$

Combining (16) and (17) we obtain

$$\begin{aligned}\|v_0\|^2_{H^1(\Omega_2\setminus V_{\epsilon_2})} \leq C\epsilon e^{D\mu}\|v_0\|^2_{H^1(\Omega_2)} + \frac{C}{\epsilon}\|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \frac{C}{\epsilon}e^{-c_2\mu}\|v_0\|^2_{H^1(\Omega_2)} \\ + \frac{C}{\epsilon}e^{c_1\mu}\left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right]. (18)\end{aligned}$$

Adding (17) and (18), we obtain

$$\begin{aligned}\|v_0\|^2_{H^1(\Omega_2)} \leq C\epsilon e^{D\mu}\|v_0\|^2_{H^1(\Omega_2)} + C_\epsilon\|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + C_\epsilon e^{-c_2\mu}\|v_0\|^2_{H^1(\Omega_2)} \\ + C_\epsilon e^{c_1\mu}\left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right].\end{aligned}$$

We fixe ϵ small enough and $D < c_2$, then there exists $\mu_0 > 0$ such that for any $\mu \geq \mu_0$, we obtain (7).

□

3 Carleman estimate and Consequence

In this part, we show the new Carleman estimate and we prove Theorem 2.2 which is consequence of this estimate.

3.1 State of Carleman estimate

In this subsection we state the Carleman estimate which is the starting point of the proof of the main result. Let $u = (u_0, v_0)$ satisfies the equation

$$\begin{cases} -(\triangle + \mu)u_0 = f_1 & \text{in } \Omega_1, \\ -(\triangle + \mu^2)v_0 = f_2 & \text{in } \Omega_2, \\ op(B_1)u = u_0 - i\mu v_0 = e_1 & \text{on } \gamma, \\ op(B_2)u = \partial_n u_0 - \partial_n v_0 = e_2 & \text{on} \gamma, \end{cases} \tag{19}$$

We will proceed like Bellassoued in [3], we will reduce the problem of transmission as a particular case of a diagonal system define only on one side of the interface with boundary conditions.

We define the Sobolev spaces with a parameter μ, H^s_μ by

$$u(x,\mu) \in H^s_\mu \quad \Longleftrightarrow \quad \langle \xi, \mu\rangle^s \widehat{u}(\xi,\mu) \in L^2, \quad \langle \xi, \mu\rangle^2 = |\xi|^2 + \mu^2,$$

$\widehat{u}$ denoted the partial Fourier transform with respect to x.
For a differential operator

$$P(x, D, \mu) = \sum_{|\alpha|+k\le m} a_{\alpha,k}(x)\mu^k D^\alpha,$$

we note the associated symbol by

$$p(x, \xi, \mu) = \sum_{|\alpha|+k\le m} a_{\alpha,k}(x)\mu^k \xi^\alpha.$$

The class of symbols of order m is defined by

$$S^m_\mu = \left\{ p(x,\xi,\mu) \in C^\infty, \left| D^\alpha_x D^\beta_\xi p(x,\xi,\mu)\right| \le C_{\alpha,\beta}\langle \xi,\mu\rangle^{m-|\beta|} \right\}$$

and the class of tangential symbols of order m by

$$\mathcal{T}S^m_\mu = \left\{ p(x,\xi',\mu) \in C^\infty, \left| D^\alpha_x D^\beta_{\xi'} p(x,\xi',\mu)\right| \le C_{\alpha,\beta}\langle \xi',\mu\rangle^{m-|\beta|} \right\}.$$

We denote by $\mathcal{O}^m$ (resp. $\mathcal{T}\mathcal{O}^m$) the set of differentials operators $P = op(p)$, $p \in S^m_\mu$ (resp. $\mathcal{T}S^m_\mu$).
We shall frequently use the symbol $\Lambda = \langle \xi', \mu\rangle = (|\xi'|^2 + \mu^2)^{\frac{1}{2}}$.
We shall need to use the following Gårding estimate: if $p \in \mathcal{T}S^2_\mu$ satisfies for $C_0 > 0$, $p(x,\xi',\mu) + \overline{p}(x,\xi',\mu) \ge C_0\Lambda^2$, then

$$\exists\, C_1 > 0,\ \exists\, \mu_0 > 0,\ \forall \mu > \mu_0, \forall u \in C^\infty_0(K),\ Re(P(x,D',\mu)u,u) \ge C_1 \|op(\Lambda)u\|^2_{L^2}. \tag{20}$$

Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. In the normal geodesic system given locally by

$$\Omega_2 = \{x \in \mathbb{R}^n, x_n > 0\}, \qquad x_n = dist(x, \partial\Omega_1) = dist(x, x'),$$

the Laplacian is written in the form

$$\triangle = -A_2(x, D) = -\left(D_{x_n}^2 + R(+x_n, x', D_{x'})\right).$$

The Laplacian on Ω_1 can be identified locally to an operator in Ω_2 gives by

$$\triangle = -A_1(x, D) = -\left(D_{x_n}^2 + R(-x_n, x', D_{x'})\right).$$

We denote the operator, with C^∞ coefficients defined in $\Omega_2 = \{x_n > 0\}$, by

$$A(x, D) = \text{diag}\Big(A_1(x, D_x), A_2(x, D_x)\Big)$$

and the tangential operator by

$$R(x, D_{x'}) = \text{diag}\Big(R(-x_n, x', D_{x'}), R(+x_n, x', D_{x'})\Big) = \text{diag}\Big(R_1(x, D_{x'}), R_2(x, D_{x'})\Big).$$

The principal symbol of the differential operator $A(x, D)$ satisfies $a(x, \xi) = \xi_n^2 + r(x, \xi')$, where $r(x, \xi') = \text{diag}\Big(r_1(x, \xi'), r_2(x, \xi')\Big)$ is the principal symbol of $R(x, D_{x'})$ and the quadratic form $r_j(x, \xi')$, $j = 1, 2$, satisfies

$$\exists\, C > 0, \quad \forall (x, \xi'), \quad r_j(x, \xi') \geq C\,|\xi'|^2, \quad j = 1, 2.$$

We denote $P(x, D)$ the matrix operator with C^∞ coefficients defined in $\Omega_2 = \{x_n > 0\}$, by

$$P(x, D) = \text{diag}(P_1(x, D), P_2(x, D)) = \begin{pmatrix} A_1(x, D) - \mu & 0 \\ 0 & A_2(x, D) - \mu^2 \end{pmatrix}.$$

Let $\varphi(x) = \text{diag}(\varphi_1(x), \varphi_2(x))$, with φ_j, $j = 1, 2$, are C^∞ functions in Ω_j. For μ large enough, we define the operator

$$A(x, D, \mu) = e^{\mu\varphi} A(x, D) e^{-\mu\varphi} := op(a)$$

where $a \in S_\mu^2$ is the principal symbol given by

$$a(x, \xi, \mu) = \left(\xi_n + i\mu\frac{\partial\varphi}{\partial x_n}\right)^2 + r\left(x, \xi' + i\mu\frac{\partial\varphi}{\partial x'}\right).$$

Let

$$op(\tilde{q}_{2,j}) = \frac{1}{2}(A_j + A_j^*), \qquad op(\tilde{q}_{1,j}) = \frac{1}{2i}(A_j - A_j^*), \qquad j = 1, 2$$

its real and imaginary part. Then we have

$$\begin{cases} A_j = op(\tilde{q}_{2,j}) + i op(\tilde{q}_{1,j}), \\ \tilde{q}_{2,j} = \xi_n^2 + q_{2,j}(x, \xi', \mu), \quad \tilde{q}_{1,j} = 2\mu\frac{\partial\varphi_j}{\partial x_n}\xi_n + 2\mu q_{1,j}(x, \xi', \mu), \quad j = 1, 2, \end{cases} \tag{21}$$

where $q_{1,j} \in \mathcal{T}S^1_\mu$ and $q_{2,j} \in \mathcal{T}S^2_\mu$ are two tangential symbols given by

$$\left\{ \begin{array}{l} q_{2,j}(x,\xi',\mu) = r_j(x,\xi') - (\mu\frac{\partial\varphi_j}{\partial x_n})^2 - \mu^2 r_j(x,\frac{\partial\varphi_j}{\partial x'}), \\ \\ q_{1,j}(x,\xi',\mu) = \tilde{r}_j(x,\xi',\frac{\partial\varphi_j}{\partial x'}), \quad j=1,2, \end{array} \right. \tag{22}$$

with $\tilde{r}(x,\xi',\eta')$ is the bilinear form associated to the quadratic form $r(x,\xi')$.

In the next, $P(x,D,\mu)$ is the matrix operator with C^∞ coefficients defined in $\Omega_2 = \{x_n > 0\}$ by

$$P(x,D,\mu) = \mathrm{diag}(P_1(x,D,\mu), P_2(x,D,\mu)) = \begin{pmatrix} A_1(x,D,\mu) - \mu & 0 \\ 0 & A_2(x,D,\mu) - \mu^2 \end{pmatrix} \tag{23}$$

and $u = (u_0, v_0)$ satisfies the equation

$$\left\{ \begin{array}{ll} Pu = f & \text{in } \{x_n > 0\}, \\ op(b_1)u = u_0|_{x_n=0} - i\mu v_0|_{x_n=0} = e_1 & \text{on } \{x_n = 0\}, \\ op(b_2)u = \left(D_{x_n} + i\mu\frac{\partial\varphi_1}{\partial x_n}\right)u_0|_{x_n=0} + \left(D_{x_n} + i\mu\frac{\partial\varphi_2}{\partial x_n}\right)v_0|_{x_n=0} = e_2 & \text{on } \{x_n = 0\}, \end{array} \right. \tag{24}$$

where $f = (f_1, f_2)$, $e = (e_1, e_2)$ and $B = (op(b_1), op(b_2))$. We note $p_j(x,\xi,\mu)$, $j = 1,2$, the associated symbol of $P_j(x,D,\mu)$.

We suppose that φ satisfies

$$\left\{ \begin{array}{ll} \varphi_1(x) = \varphi_2(x) & \text{on } \{x_n = 0\} \\ \\ \dfrac{\partial\varphi_1}{\partial x_n} > 0 & \text{on } \{x_n = 0\} \\ \\ \left(\dfrac{\partial\varphi_1}{\partial x_n}\right)^2 - \left(\dfrac{\partial\varphi_2}{\partial x_n}\right)^2 > 1 & \text{on } \{x_n = 0\} \end{array} \right. \tag{25}$$

and the following condition of hypoellipticity of Hörmander: $\exists\, C > 0,\ \forall x \in K$ $\forall \xi \in \mathbb{R}^n \backslash \{0\}$,

$$\left(\mathrm{Re}p_j = 0 \quad \text{et} \quad \frac{1}{2\mu}\mathrm{Im}p_j = 0\right) \Rightarrow \left\{\mathrm{Re}p_j, \frac{1}{2\mu}\mathrm{Im}p_j\right\} \geq C\langle\xi,\mu\rangle^2, \tag{26}$$

where $\{f,g\}(x,\xi) = \sum\left(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial\xi_j}\right)$ is the Poisson bracket of two functions $f(x,\xi)$ and $g(x,\xi)$ and K is a compact in Ω_2.

We denote by

$$\|u\|_{L^2(\Omega_2)} = \|u\|, \quad \|u\|^2_{k,\mu} = \sum_{j=0}^{k} \mu^{2(k-j)} \|u\|^2_{H^j(\Omega_2)}, \quad \|u\|^2_k = \left\|op(\Lambda^k)u\right\|^2,$$

$$|u|^2_{k,\mu} = \|u|_{x_n=0}\|^2_{k,\mu}, \quad |u|^2_k = |u|_{x_n=0}|^2_k,\ k \in \mathbb{R} \quad \text{and} \quad |u|^2_{1,0,\mu} = |u|^2_1 + |D_{x_n}u|^2.$$

We are now ready to state our result.

Theorem 3.1 *Let φ satisfies (25) and (26). Let $w \in C_0^\infty(\overline{\Omega}_2)$ and $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\chi = 1$ in the support of w. Then there exists constants $C > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$ we have the following estimate*

$$\mu \left\| w \right\|_{1,\mu}^2 + \mu^2 \left| w \right|_{\frac{1}{2}}^2 + \mu^2 \left| D_{x_n} w \right|_{-\frac{1}{2}}^2$$
$$\leq C \left(\left\| P(x, D, \mu) w \right\|^2 + \left| op(b_1) w \right|_{\frac{1}{2}}^2 + \mu \left| op(b_2) w \right|^2 \right). \quad (27)$$

Corollary 3.1 *Let φ satisfies (25) and (26). Then there exists constants $C > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$ we have the following estimate*

$$\mu \left\| e^{\mu\varphi} h \right\|_{H^1}^2 \leq C \left(\left\| e^{\mu\varphi} P(x, D) h \right\|^2 + \left| e^{\mu\varphi} op(B_1) h \right|_{H^{\frac{1}{2}}}^2 + \mu \left| e^{\mu\varphi} op(B_2) h \right|^2 \right), \quad (28)$$

for any $h \in C_0^\infty(\overline{\Omega}_2)$.

Proof.
Let $w = e^{\mu\varphi} h$. Recalling that $P(x, D, \mu) w = e^{\mu\varphi} P(x, D) e^{-\mu\varphi} w$ and using (27), we obtain (28).

3.2 Proof of Theorem 2.2

We denote $x = (x', x_n)$ a point in Ω. Let $x_0 = (0, -\delta)$, $\delta > 0$. We set

$$\psi(x) = \left| x - x_0 \right|^2 - \delta^2 \qquad \text{and}$$

$$\varphi_1(x) = e^{-\beta\psi(x', -x_n)}, \quad \varphi_2(x) = e^{-\beta(\psi(x) - \alpha x_n)}, \quad \beta > 0, \quad \text{and} \quad \frac{\delta}{2} < \alpha < 2\delta.$$

The weight function $\varphi = diag(\varphi_1, \varphi_2)$ has to satisfy (25) and (26). With these choices, we have $\varphi_1|_{x_n=0} = \varphi_2|_{x_n=0}$ and $\frac{\partial\varphi_1}{\partial x_n}|_{x_n=0} > 0$. It remains to verify

$$\left(\frac{\partial\varphi_1}{\partial x_n} \right)^2 - \left(\frac{\partial\varphi_2}{\partial x_n} \right)^2 > 1 \quad \text{on } \{x_n = 0\} \quad (29)$$

and the condition (26). We begin by condition (26) and we compute for φ_1 and p_1 (the computation for φ_2 and p_2 is made in the same way). Recalling that

$$\left\{ \text{Re} p_1, \frac{1}{2\mu} \text{Im} p_1 \right\}(x, \xi) = \frac{\text{Im}}{2\mu} \left[\partial_\xi p_1(x, \xi - i\mu\varphi_1'(x)) \, \partial_x p_1(x, \xi + i\mu\varphi_1'(x)) \right]$$
$$+ {}^t \left[\partial_\xi p_1(x, \xi - i\mu\varphi_1'(x)) \right] \varphi_1''(x) \left[\partial_\xi p_1(x, \xi - i\mu\varphi_1'(x)) \right].$$

We replace $\varphi_1(x)$ by $\varphi_1(x) = e^{-\beta\psi(x', -x_n)}$, $\beta > 0$, we obtain, by noting $\xi = -\beta\varphi_1(x)\eta$

$$\left\{ \text{Re} p_1, \frac{1}{2\mu} \text{Im} p_1 \right\}(x, \xi)$$
$$= (-\beta\varphi_1)^3 \left[\left\{ \text{Re} p_1(x, \eta - i\mu\psi'), \frac{1}{2\mu} \text{Im} p_1(x, \eta + i\mu\psi') \right\}(x, \eta) - \beta \left| \psi'(x) \partial_\eta p_1(x, \eta + i\mu\psi') \right|^2 \right]$$

and

$$|\psi'(x)\partial_\eta p_1(x,\eta+i\mu\psi')|^2 = 4\left[\mu^2 |p_1(x,\psi')|^2 + |\tilde{p}_1(x,\eta,\psi')|^2\right]$$

where $\tilde{p}_1(x,\eta,\psi')$ is the bilinear form associated to the quadratic form $p_1(x,\eta)$. We have

$$\left(\mathrm{Re}p_1 = 0 \quad \text{et} \quad \frac{1}{2\mu}\mathrm{Im}p_1 = 0\right) \Longleftrightarrow p_1(x,\eta+i\mu\psi') = 0,$$

- If $\mu = 0$, we have $p_1(x,\xi) = 0$ which is impossible. Indeed, we have $p_1(x,\xi) \geq C|\xi|^2$, $\forall(x,\xi) \in K \times \mathbb{R}^n$, K compact in Ω_2.
- If $\mu \neq 0$, we have $\tilde{p}_1(x,\eta,\psi') = 0$.
  Then $|\psi'(x)\partial_\eta p_1(x,\eta+i\mu\psi')|^2 = 4\mu^2 |p_1(x,\psi')|^2 > 0$. On the other hand, we have

  $$\left\{\mathrm{Re}p_1(x,\eta-i\mu\psi'), \frac{1}{2\mu}\mathrm{Im}p_1(x,\eta+i\mu\psi')\right\}(x,\eta) \leq C_1(|\eta|^2 + \mu^2|\psi'|^2)$$

  where C_1 is a positive constant independent of ψ'. Then for $\beta \geq C_1$, we satisfy the condition (26).

Now let us verify (29). We have, on $\{x_n = 0\}$,

$$\left(\frac{\partial\varphi_1}{\partial x_n}\right)^2 - \left(\frac{\partial\varphi_2}{\partial x_n}\right)^2 = \beta^2\alpha(4\delta-\alpha)e^{-2\beta\psi}.$$

Then to satisfy (29), it suffices to choose $\beta = \frac{M}{\delta}$ where $M > 0$ such that $\frac{M}{\delta} \geq C_1$.

We now choose $r_1 < r_1' < r_2 < 0 = \psi(0) < r_2' < r_3 < r_3'$. We denote

$$w_j = \{x \in \Omega,\ r_j < \psi(x) < r_j'\} \quad \text{and} \quad T_{x_0} = w_2 \cap \Omega_2.$$

We set $R_j = e^{-\beta r_j}$, $R_j' = e^{-\beta r_j'}$, $j = 1,2,3$.
Then $R_3' < R_3 < R_2' < R_2 < R_1' < R_1$. We need also to introduce a cut-off function $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{n+1})$ such that

$$\tilde{\chi}(\rho) = \begin{cases} 0 & \text{if} \quad \rho \leq r_1, \quad \rho \geq r_3' \\ 1 & \text{if} \quad \rho \in [r_1', r_3]. \end{cases}$$

Let $\tilde{u} = (\tilde{u}_0, \tilde{v}_0) = \tilde{\chi}u = (\tilde{\chi}u_0, \tilde{\chi}v_0)$. Then we get the following system

$$\begin{cases} (\triangle - i\mu)\tilde{u}_0 &=& \tilde{\chi}f_0 + [\triangle - i\mu, \tilde{\chi}]u_0 \\ (\triangle + \mu^2)\tilde{v}_0 &=& \tilde{\chi}(g_1 + i\mu g_0) + [\triangle + \mu^2, \tilde{\chi}]v_0, \\ \tilde{v}_1 &=& g_0 + i\mu\tilde{v}_0, \end{cases}$$

with the following boundary conditions

$$\begin{cases} \tilde{u}_0|_{\Gamma_1} = \tilde{v}_0|_{\Gamma_2} &=& 0, \\ op(b_1)\tilde{u} = \tilde{u}_0 - i\mu\tilde{v}_0 &=& (\tilde{\chi}g_0)|_\gamma, \\ op(b_2)\tilde{u} &=& ([\partial_n, \tilde{\chi}]u_0 - [\partial_n, \tilde{\chi}]v_0)|_\gamma. \end{cases}$$

From the Carleman estimate of Corollary 3.1 , we have

$$\mu \left\| e^{\mu\varphi}\tilde{u} \right\|_{H^1}^2 \leq C \left(\left\| e^{\mu\varphi_1}(\triangle - i\mu)\tilde{u}_0 \right\|^2 + \left\| e^{\mu\varphi_2}(\triangle + \mu^2)\tilde{v}_0 \right\|^2 + \left| e^{\mu\varphi} op(b_1)\tilde{u} \right|_{H^{\frac{1}{2}}}^2 + \mu \left| e^{\mu\varphi} op(b_2)\tilde{u} \right|^2 \right). \tag{30}$$

Using the fact $[\triangle - i\mu, \tilde{\chi}]$ is the first order operator supported in $(w_1 \cup w_3) \cap \Omega_1$, we have

$$\left\| e^{\mu\varphi_1}(\triangle - i\mu)\tilde{u}_0 \right\|^2 \leq C \left(e^{2\mu R_1} \left\| f_0 \right\|_{L^2(\Omega_1)}^2 + e^{2\mu R_1} \left\| u_0 \right\|_{H^1(\Omega_1)}^2 \right). \tag{31}$$

Recalling that $[\triangle + \mu^2, \tilde{\chi}]$ is the first order operator supported in $(w_1 \cup w_3) \cap \Omega_2$, we show

$$\left\| e^{\mu\varphi_2}(\triangle + \mu^2)\tilde{v}_0 \right\|^2 \leq C \left(e^{2\mu} \left\| g_1 + i\mu g_0 \right\|_{L^2(\Omega_2)}^2 + e^{2\mu R_3} \left\| v_0 \right\|_{H^1(\Omega_2)}^2 \right). \tag{32}$$

From the trace formula and recalling that $op(b_2)\tilde{u}$ is an operator of order zero and supported in $\{x_n = 0\} \cap w_3$, we show

$$\mu \left| e^{\mu\varphi} op(b_2)\tilde{u} \right|^2 \leq C e^{2\mu R_3} \left\| u \right\|_{H^1(\Omega)}^2 \leq C \left(e^{2\mu R_3} \left\| u_0 \right\|_{H^1(\Omega_1)}^2 + e^{2\mu R_3} \left\| v_0 \right\|_{H^1(\Omega_2)}^2 \right). \tag{33}$$

Now we need to use this result shown in Appendix B

Lemma 3.1 *There exists $C > 0$ such that for all $s \in \mathbb{R}$ and $u \in C_0^\infty(\Omega)$, we have*

$$\left\| op(\Lambda^s) e^{\mu\varphi} u \right\| \leq C e^{\mu C} \left\| op(\Lambda^s) u \right\|. \tag{34}$$

Following this Lemma, we obtain

$$\left| e^{\mu\varphi} op(b_1)\tilde{u} \right|_{H^{\frac{1}{2}}}^2 \leq C e^{2\mu c} \left| g_0 \right|_{H^{\frac{1}{2}}}^2 \leq C e^{2\mu c} \left\| g_0 \right\|_{H^1(\Omega_2)}^2. \tag{35}$$

Combining (30), (31), (32), (33) and (35), we obtain

$$C\mu e^{2\mu R_2'} \left\| u_0 \right\|_{H^1(w_2 \cap \Omega_1)}^2 + C\mu e^{2\mu R_2'} \left\| v_0 \right\|_{H^1(T_{x_0})}^2 \leq C(e^{2\mu R_1} \left\| f_0 \right\|_{L^2(\Omega_1)}^2 + e^{2\mu R_1} \left\| u_0 \right\|_{H^1(\Omega_1)}^2$$

$$+ e^{2\mu} \left\| g_1 + i\mu g_0 \right\|_{L^2(\Omega_2)}^2 + e^{2\mu R_3} \left\| v_0 \right\|_{H^1(\Omega_2)}^2 + e^{2\mu R_3} \left\| u_0 \right\|_{H^1(\Omega_1)}^2 + e^{2\mu c} \left\| g_0 \right\|_{H^1(\Omega_2)}^2).$$

Since $R_2' < R_1$. Then we have

$$\begin{aligned} \left\| v_0 \right\|_{H^1(T_{x_0})}^2 &\leq C e^{c_1\mu} \left[\left\| f_0 \right\|_{L^2(\Omega_1)}^2 + \left\| g_1 + i\mu g_0 \right\|_{L^2(\Omega_2)}^2 + \left\| g_0 \right\|_{H^1(\Omega_2)}^2 + \left\| u_0 \right\|_{H^1(\Omega_1)}^2 \right] \\ &+ C e^{-c_2\mu} \left\| v_0 \right\|_{H^1(\Omega_2)}^2. \end{aligned} \tag{36}$$

Since γ is compact, then there exists a finite number of T_{x_0}. Let $V_{\epsilon_2} \subset \cup T_{x_0}$. Then we obtain (17)

3.3 Proof of Carleman estimate (Theorem 3.1)

In the first step, we state the following estimates

Theorem 3.2 *Let φ satisfies (25) and (26). Then there exists constants $C > 0$ and μ_0 such that for any $\mu \geq \mu_0$ we have the following estimates*

$$\mu \left\|u\right\|^2_{1,\mu} \leq C \left(\left\|P(x, D, \mu)u\right\|^2 + \mu \left|u\right|^2_{1,0,\mu} \right) \tag{37}$$

and

$$\mu \left\|u\right\|^2_{1,\mu} + \mu \left|u\right|^2_{1,0,\mu} \leq C \left(\left\|P(x, D, \mu)u\right\|^2 + \mu^{-1} \left|op(b_1)u\right|^2_1 + \mu \left|op(b_2)u\right|^2 \right), \tag{38}$$

for any $u \in C^\infty_0(\overline{\Omega}_2)$.

In the second step, we need to prove this Lemma

Lemma 3.2 *There exists constants $C > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$ we have the following estimate*

$$\begin{aligned} &\left\|D^2_{x_n} op(\Lambda^{-\frac{1}{2}})u\right\|^2 + \left\|D_{x_n} op(\Lambda^{\frac{1}{2}})u\right\|^2 + \left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 + \mu \left|u\right|^2_{1,0,\mu} \\ &\qquad \leq C \left(\left\|P(x, D, \mu)u\right\|^2 + \mu^{-1} \left|op(b_1)u\right|^2_1 + \mu \left|op(b_2)u\right|^2 \right), \end{aligned} \tag{39}$$

for any $u \in C^\infty_0(\overline{\Omega}_2)$.

Proof.
We have

$$P(x, D, \mu) = D^2_{x_n} + R + \mu C_1 + \mu^2 C_0,$$

where $R \in \mathcal{TO}^2$, $C_1 = c_1(x)D_{x_n} + T_1$, with $T_1 \in \mathcal{TO}^1$ and $C_0 \in \mathcal{TO}^0$. Then we have

$$\begin{aligned} &\left\|(D^2_{x_n} + R)op(\Lambda^{-\frac{1}{2}})u\right\|^2 \\ &\quad \leq C \left(\left\|Pop(\Lambda^{-\frac{1}{2}})u\right\|^2 + \mu^2 \left\|op(\Lambda^{\frac{1}{2}})u\right\|^2 + \mu^2 \left\|D_{x_n} op(\Lambda^{-\frac{1}{2}})u\right\|^2 + \mu^4 \left\|op(\Lambda^{-\frac{1}{2}})u\right\|^2 \right). \end{aligned}$$

Since

$$\begin{aligned} &\mu^4 \left\|op(\Lambda^{-\frac{1}{2}})u\right\|^2 \leq C\mu^3 \left\|u\right\|^2, \\ &\mu^2 \left\|D_{x_n} op(\Lambda^{-\frac{1}{2}})u\right\|^2 \leq C\mu \left\|D_{x_n} u\right\|^2 \quad \text{and} \\ &\mu^2 \left\|op(\Lambda^{\frac{1}{2}})u\right\|^2 = \mu^2 (\tfrac{1}{\sqrt{\mu}} op(\Lambda)u, \sqrt{\mu} u) \leq C \left(\mu \left\|op(\Lambda)u\right\|^2 + \mu^3 \left\|u\right\|^2 \right). \end{aligned}$$

Using the fact that $\left\|u\right\|^2_{1,\mu} \simeq \left\|op(\Lambda)u\right\|^2 + \left\|D_{x_n} u\right\|^2$, we obtain

$$\left\|(D^2_{x_n} + R)op(\Lambda^{-\frac{1}{2}})u\right\|^2 \leq C \left(\left\|Pop(\Lambda^{-\frac{1}{2}})u\right\|^2 + \mu \left\|u\right\|^2_{1,\mu} \right).$$

Following (37), we have

$$\left\|(D_{x_n}^2+R)op(\Lambda^{-\frac{1}{2}})u\right\|^2 \leq C\left(\left\|Pop(\Lambda^{-\frac{1}{2}})u\right\|^2+\|Pu\|^2+\mu\,|u|_{1,0,\mu}^2\right). \qquad (40)$$

We can write

$$\begin{array}{rcl} Pop(\Lambda^{-\frac{1}{2}})u &=& op(\Lambda^{-\frac{1}{2}})Pu+[P,op(\Lambda^{-\frac{1}{2}})]u \\ &=& op(\Lambda^{-\frac{1}{2}})Pu+[R,op(\Lambda^{-\frac{1}{2}})]u \\ &+& \mu[C_1,op(\Lambda^{-\frac{1}{2}})]u+\mu^2[C_0,op(\Lambda^{-\frac{1}{2}})]u \\ &=& op(\Lambda^{-\frac{1}{2}})Pu+t_1+t_2+t_3. \end{array} \qquad (41)$$

Let us estimate t_1, t_2 and t_3. We have $[R,op(\Lambda^{-\frac{1}{2}})]\in\mathcal{TO}^{\frac{1}{2}}$, then following (37), we have

$$\|t_1\|^2 \leq C\left\|op(\Lambda^{\frac{1}{2}})u\right\|^2 \leq C\left(\|op(\Lambda)u\|^2+\|u\|^2\right) \leq C\left(\|Pu\|^2+\mu\,|u|_{1,0,\mu}^2\right). \qquad (42)$$

We have $t_3=\mu[C_1,op(\Lambda^{-\frac{1}{2}})]u=\mu[c_1(x)D_{x_n},op(\Lambda^{-\frac{1}{2}})]u+\mu[T_1,op(\Lambda^{-\frac{1}{2}})]u$. Then following (37), we obtain

$$\|t_2\|^2 \leq C\left(\mu^{-1}\|D_{x_n}u\|^2+\mu\,\|u\|^2\right) \leq C\left(\|Pu\|^2+\mu\,|u|_{1,0,\mu}^2\right). \qquad (43)$$

We have $[C_0,op(\Lambda^{-\frac{1}{2}})]\in\mathcal{TO}^{-\frac{3}{2}}$, then following (37), we obtain

$$\left\|\mu^2[C_0,op(\Lambda^{-\frac{1}{2}})]u\right\|^2 \leq C\mu\,\|u\|^2 \leq C\left(\|Pu\|^2+\mu\,|u|_{1,0,\mu}^2\right) \qquad (44)$$

From (41), (42), (43) and (44), we have

$$\left\|Pop(\Lambda^{-\frac{1}{2}})u\right\|^2 \leq C\left(\|Pu\|^2+\mu\,|u|_{1,0,\mu}^2\right).$$

Inserting this inequality in (40), we obtain

$$\left\|(D_{x_n}^2+R)op(\Lambda^{-\frac{1}{2}})u\right\|^2 \leq C\left(\|Pu\|^2+\mu\,|u|_{1,0,\mu}^2\right). \qquad (45)$$

Moreover, we have

$$\left\|(D_{x_n}^2+R)op(\Lambda^{-\frac{1}{2}})u\right\|^2=\left\|D_{x_n}^2op(\Lambda^{-\frac{1}{2}})u\right\|^2+\left\|Rop(\Lambda^{-\frac{1}{2}})u\right\|^2+2\mathcal{R}e(D_{x_n}^2op(\Lambda^{-\frac{1}{2}})u,Rop(\Lambda^{-\frac{1}{2}})u),$$

where $(.,.)$ denoted the scalar product in L^2. By integration by parts, we find

$$\begin{array}{l} \left\|(D_{x_n}^2+R)op(\Lambda^{-\frac{1}{2}})u\right\|^2=\left\|D_{x_n}^2op(\Lambda^{-\frac{1}{2}})u\right\|^2+\left\|Rop(\Lambda^{-\frac{1}{2}})u\right\|^2 \\ \quad +2\mathcal{R}e\left(i(D_{x_n}u,Rop(\Lambda^{-1})u)_0+i(D_{x_n}u,[op(\Lambda^{-\frac{1}{2}}),R]op(\Lambda^{-\frac{1}{2}})u)_0\right) \\ \quad +2\mathcal{R}e\left((RD_{x_n}op(\Lambda^{-\frac{1}{2}})u,D_{x_n}op(\Lambda^{-\frac{1}{2}})u)+(D_{x_n}op(\Lambda^{-\frac{1}{2}})u,[D_{x_n},R]op(\Lambda^{-\frac{1}{2}})u)\right). \end{array} \qquad (46)$$

Since, we have

$$\left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 = (op(\Lambda^2)op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u) = \sum_{j\leq n-1}(D_j^2 op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u) + \mu^2(op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u).$$

By integration by parts, we find

$$\left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 = \sum_{j\leq n-1}(D_j op(\Lambda^{\frac{1}{2}})u, D_j op(\Lambda^{\frac{1}{2}})u) + \mu^2\left\|op(\Lambda^{\frac{1}{2}})u\right\|^2 = k + \mu^2\left\|op(\Lambda^{\frac{1}{2}})u\right\|^2. \tag{47}$$

Let $\chi_0 \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\chi_0 = 1$ in the support of u. We have

$$k = \sum_{j\leq n-1}(\chi_0 D_j op(\Lambda^{\frac{1}{2}})u, D_j op(\Lambda^{\frac{1}{2}})u) + \sum_{j\leq n-1}((1-\chi_0)D_j op(\Lambda^{\frac{1}{2}})u, D_j op(\Lambda^{\frac{1}{2}})u).$$

Recalling that $\chi_0 u = u$, we obtain

$$k = \sum_{j\leq n-1}(\chi_0 D_j op(\Lambda^{\frac{1}{2}})u, D_j op(\Lambda^{\frac{1}{2}})u) + \sum_{j\leq n-1}([(1-\chi_0), D_j op(\Lambda^{\frac{1}{2}})]u, D_j op(\Lambda^{\frac{1}{2}})u) = k' + k". \tag{48}$$

Using the fact that $[(1-\chi_0), D_j op(\Lambda^{\frac{1}{2}})] \in \mathcal{TO}^{\frac{1}{2}}$ and $D_j op(\Lambda^{\frac{1}{2}}) \in \mathcal{TO}^{\frac{3}{2}}$, we show

$$k" \leq C\left\|op(\Lambda)u\right\|^2. \tag{49}$$

Using the fact that $\sum_{j,k\leq n-1}\chi_0 a_{j,k} D_j v\overline{D_k v} \geq \delta\chi_0\sum_{j\leq n-1}|D_j v|^2$, $\delta > 0$, we obtain

$$\begin{aligned} k' &\leq C\sum_{j,k\leq n-1}(\chi_0 a_{jk} D_j op(\Lambda^{\frac{1}{2}})u, D_k op(\Lambda^{\frac{1}{2}})u) \\ &\leq C\sum_{j,k\leq n-1}([\chi_0, a_{jk}D_j op(\Lambda^{\frac{1}{2}})]u, D_k op(\Lambda^{\frac{1}{2}})u) + \sum_{j,k\leq n-1}(a_{jk}D_j op(\Lambda^{\frac{1}{2}})u, D_k op(\Lambda^{\frac{1}{2}})u). \end{aligned}$$

Using the fact that $[\chi_0, a_{jk}D_j op(\Lambda^{\frac{1}{2}})] \in \mathcal{TO}^{\frac{1}{2}}$ and $D_k op(\Lambda^{\frac{1}{2}})u \in \mathcal{TO}^{\frac{3}{2}}$, we obtain

$$k' \leq C\left(\sum_{j,k\leq n-1}(a_{jk}D_j op(\Lambda^{\frac{1}{2}})u, D_k op(\Lambda^{\frac{1}{2}})u) + \left\|op(\Lambda)u\right\|^2\right). \tag{50}$$

By integratin by parts and recalling that $R = \sum_{j,k\leq n-1} a_{j,k}D_jD_k$, we have

$$\begin{aligned} \sum_{j,k\leq n-1}(a_{jk}D_j op(\Lambda^{\frac{1}{2}})u, D_k op(\Lambda^{\frac{1}{2}})u) &= (R op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u \\ &+ \sum_{j,k\leq n-1}([D_k, a_{jk}]D_j op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u). \end{aligned} \tag{51}$$

Since $[D_k, a_{jk}]D_j op(\Lambda^{\frac{1}{2}}) \in \mathcal{TO}^{\frac{3}{2}}$, then

$$\sum_{j,k\leq n-1}([D_k, a_{jk}]D_j op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u) \leq C\left\|op(\Lambda)u\right\|^2.$$

Following (51), we obtain

$$\sum_{j,k\leq n-1}(a_{jk}D_j op(\Lambda^{\frac{1}{2}})u, D_k op(\Lambda^{\frac{1}{2}})u) \leq C\left((R op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u) + \|op(\Lambda)u\|^2\right). \tag{52}$$

Since

$$(R op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u) = (R op(\Lambda^{-\frac{1}{2}})u, op(\Lambda^{\frac{3}{2}})u) + ([op(\Lambda^{-1}), R]op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{3}{2}})u)).$$

Using the fact that $[op(\Lambda^{-1}), R]op(\Lambda^{\frac{1}{2}}) \in \mathcal{TO}^{\frac{1}{2}}$ and the Cauchy Schwartz inequality, we obtain

$$(R op(\Lambda^{\frac{1}{2}})u, op(\Lambda^{\frac{1}{2}})u) \leq \epsilon C\left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 + \frac{C}{\epsilon}\left\|R op(\Lambda^{-\frac{1}{2}})u\right\|^2 + C\|op(\Lambda)u\|^2 \tag{53}$$

Combining (47), (48), (49), (50), (52) and (53), we obtain

$$\left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 \leq \epsilon C\left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 + \frac{C}{\epsilon}\left\|R op(\Lambda^{-\frac{1}{2}})u\right\|^2 + C\|op(\Lambda)u\|^2.$$

For ϵ small enough, we obtain

$$\left\|R op(\Lambda^{-\frac{1}{2}})u\right\|^2 \geq C\left(\left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 - \mu^2\left\|op(\Lambda^{\frac{1}{2}})u\right\|^2\right). \tag{54}$$

Using the same computations, we show

$$(R D_{x_n} op(\Lambda^{-\frac{1}{2}})u, D_{x_n} op(\Lambda^{-\frac{1}{2}})u) \geq C\left(\left\|D_{x_n} op(\Lambda^{\frac{1}{2}})u\right\|^2 - \mu\|D_{x_n}u\|^2\right). \tag{55}$$

Combining (46), (54) and (55), we obtain

$$\begin{aligned}\left\|(D_{x_n}^2 + R)op(\Lambda^{-\frac{1}{2}})u\right\|^2 &+ \left|(D_{x_n}u, R op(\Lambda^{-1})u)_0\right| + \left|(D_{x_n}u, [op(\Lambda^{-\frac{1}{2}}), R]op(\Lambda^{-\frac{1}{2}})u)_0\right| \\ &+ \left|(D_{x_n} op(\Lambda^{-\frac{1}{2}})u, [D_{x_n}, R]op(\Lambda^{-\frac{1}{2}})u)\right| + \mu\|u\|_{1,\mu}^2 \qquad (56) \\ &\geq C\left(\left\|D_{x_n}^2 op(\Lambda^{-\frac{1}{2}})u\right\|^2 + \left\|D_{x_n} op(\Lambda^{\frac{1}{2}})u\right\|^2 + \left\|op(\Lambda^{\frac{3}{2}})u\right\|^2\right).\end{aligned}$$

Since

$$\left|(D_{x_n}u, R op(\Lambda^{-1})u)_0\right| + \left|(D_{x_n}u, [op(\Lambda^{-\frac{1}{2}}), R]op(\Lambda^{-\frac{1}{2}})u)_0\right| \leq C\left(|D_{x_n}u|^2 + |u|_1^2\right) = C|u|_{1,0,\mu}^2 \tag{57}$$

and

$$\left|(D_{x_n} op(\Lambda^{-\frac{1}{2}})u, [D_{x_n}, R]op(\Lambda^{-\frac{1}{2}})u)\right| \leq C\mu\|u\|_{1,\mu}^2. \tag{58}$$

From (45), (56), (57), (58) and (37), we obtain

$$\begin{aligned}&\left\|D_{x_n}^2 op(\Lambda^{-\frac{1}{2}})u\right\|^2 + \left\|D_{x_n} op(\Lambda^{\frac{1}{2}})u\right\|^2 + \left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 \\ &\qquad\leq C\left(\|P(x,D,\mu)u\|^2 + \mu|u|_{1,0,\mu}^2\right).\end{aligned}$$

Following (38), we obtain (39).

□

We are now ready to prove Theorem 3.1.

Let $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\chi = 1$ in the support of w and $u = \chi op(\Lambda^{-\frac{1}{2}})w$. Then

$$\begin{aligned} Pu &= op(\Lambda^{-\frac{1}{2}})Pw + [P, op(\Lambda^{-\frac{1}{2}})]w + P[\chi, op(\Lambda^{-\frac{1}{2}})]w \\ &= op(\Lambda^{-\frac{1}{2}})Pw + [P, op(\Lambda^{-\frac{1}{2}})]w + D_{x_n}^2[\chi, op(\Lambda^{-\frac{1}{2}})]w \\ &+ R[\chi, op(\Lambda^{-\frac{1}{2}})]w + \mu c_1(x) D_{x_n}[\chi, op(\Lambda^{-\frac{1}{2}})]w \\ &+ \mu T_1[\chi, op(\Lambda^{-\frac{1}{2}})]w + \mu^2 C_0[\chi, op(\Lambda^{-\frac{1}{2}})]w \\ &= op(\Lambda^{-\frac{1}{2}})Pw + [P, op(\Lambda^{-\frac{1}{2}})]w + a_1 + a_2 + a_3 + a_4 + a_5. \end{aligned} \tag{59}$$

Let us estimate a_1, a_2, a_3, a_4 and a_5. Recalling that $[\chi, op(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{-\frac{3}{2}}$ and $\chi w = w$. Using the fact that $[D_{x_n}, T_k] \in \mathcal{TO}^k$ for all $T_k \in \mathcal{TO}^k$, we show

$$\|a_1\|^2 \le C\left(\left\|D_{x_n}^2 op(\Lambda^{-\frac{3}{2}})w\right\|^2 + \left\|D_{x_n} op(\Lambda^{-\frac{3}{2}})w\right\|^2 + \left\|op(\Lambda^{-\frac{3}{2}})w\right\|^2\right) \tag{60}$$

and

$$\|a_3\|^2 \le C\left(\mu^2 \left\|D_{x_n} op(\Lambda^{-\frac{3}{2}})w\right\|^2 + \mu^2 \left\|op(\Lambda^{-\frac{3}{2}})w\right\|^2\right). \tag{61}$$

We have $R[\chi, op(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{\frac{1}{2}}$, $T_1[\chi, op(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{-\frac{1}{2}}$ and $C_0[\chi, op(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{-\frac{3}{2}}$. Then we obtain

$$\|a_2\|^2 + \|a_4\|^2 + \|a_5\|^2 \le C\left\|op(\Lambda^{\frac{1}{2}})w\right\|^2. \tag{62}$$

Using the same computations made in the proof of Lemma 3.2 (cf t_1, t_2 and t_3 of (41)), we show

$$\left\|[P, op(\Lambda^{-\frac{1}{2}})]w\right\|^2 \le C\left(\left\|op(\Lambda^{\frac{1}{2}})w\right\|^2 + \mu^{-1}\|D_{x_n}w\|^2\right). \tag{63}$$

Following (59), (60), (61), (62) and (63), we obtain

$$\|Pu\|^2 \le C\left(\mu^{-1}\|Pw\|^2 + \left\|op(\Lambda^{\frac{1}{2}})w\right\|^2 + \mu^{-1}\|D_{x_n}w\|^2 + \mu^{-1}\left\|D_{x_n}^2 op(\Lambda^{-1})w\right\|^2\right). \tag{64}$$

We have

$$op(b_1)u = op(b_1)\chi op(\Lambda^{-\frac{1}{2}})w = op(\Lambda^{-\frac{1}{2}})op(b_1)w + op(b_1)[\chi, op(\Lambda^{-\frac{1}{2}})]w.$$

Recalling that $op(b_1) \in \mathcal{TO}^1$, we obtain

$$\mu^{-1}|op(b_1)u|_1^2 = \mu^{-1}|op(\Lambda)op(b_1)u|^2 \le C\left(\mu^{-1}\left|op(\Lambda^{\frac{1}{2}})op(b_1)w\right|^2 + \mu^{-1}\left|op(\Lambda^{\frac{1}{2}})w\right|^2\right). \tag{65}$$

We have

$$op(b_2)u = op(b_2)\chi op(\Lambda^{-\frac{1}{2}})w = op(\Lambda^{-\frac{1}{2}})op(b_2)w + op(b_2)[\chi, op(\Lambda^{-\frac{1}{2}})]w + [op(b_2), op(\Lambda^{-\frac{1}{2}})]w.$$

Recalling that $op(b_2) \in D_{x_n} + \mathcal{TO}^1$, we obtain

$$\mu \left|op(b_2)u\right|^2 \leq C\left(\mu \left|op(\Lambda^{-\frac{1}{2}})op(b_2)w\right|^2 + \mu\left|op(\Lambda^{-\frac{1}{2}})w\right|^2 + \mu\left|D_{x_n}op(\Lambda^{-\frac{3}{2}})w\right|^2\right). \tag{66}$$

Moreover, we have

$$\mu \left|u\right|^2_{1,0,\mu} = \mu\left|u\right|^2_1 + \mu\left|D_{x_n}u\right|^2 = \mu\left|op(\Lambda)u\right|^2 + \mu\left|D_{x_n}u\right|^2.$$

We can write

$$op(\Lambda)u = op(\Lambda)\chi op(\Lambda^{-\frac{1}{2}})w = op(\Lambda^{\frac{1}{2}})w + op(\Lambda)[\chi, op(\Lambda^{-\frac{1}{2}})]w.$$

Then

$$\mu\left|op(\Lambda)u\right|^2 \geq \mu\left|op(\Lambda^{\frac{1}{2}})w\right|^2 - C\mu\left|op(\Lambda^{-\frac{1}{2}})w\right|^2 \geq \mu\left|op(\Lambda^{\frac{1}{2}})w\right|^2 - C\mu^{-1}\left|op(\Lambda^{\frac{1}{2}})w\right|^2.$$

For μ large enough, we obtain

$$\mu\left|op(\Lambda)u\right|^2 \geq C\mu\left|op(\Lambda^{\frac{1}{2}})w\right|^2. \tag{67}$$

By the same way, we prove, for μ large enough

$$\mu\left|D_{x_n}u\right|^2 \geq C\mu\left|D_{x_n}op(\Lambda^{-\frac{1}{2}})w\right|^2. \tag{68}$$

Combining (67) and (68), we obtain

$$\mu\left|u\right|^2_{1,0,\mu} \geq C\left(\mu\left|op(\Lambda^{\frac{1}{2}})w\right|^2 + \mu\left|D_{x_n}op(\Lambda^{-\frac{1}{2}})w\right|^2\right). \tag{69}$$

By the same way, we prove

$$\left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 \geq \left\|op(\Lambda)w\right\|^2 - C\left\|w\right\|^2, \tag{70}$$

$$\left\|D_{x_n}op(\Lambda^{\frac{1}{2}})u\right\|^2 \geq \left\|D_{x_n}w\right\|^2 - C\left\|op(\Lambda^{-1})D_{x_n}w\right\|^2 - C\left\|op(\Lambda^{-1})w\right\|^2 \tag{71}$$

and

$$\left\|D^2_{x_n}op(\Lambda^{-\frac{1}{2}})u\right\|^2 \geq \left\|D^2_{x_n}op(\Lambda^{-1})w\right\|^2 - C\left\|D^2_{x_n}op(\Lambda^{-2})w\right\|^2 - C\left\|D_{x_n}op(\Lambda^{-2})w\right\|^2 - C\left\|op(\Lambda^{-2})w\right\|^2. \tag{72}$$

Combining (70), (71) and (72), we obtain for μ large enough

$$\left\|D^2_{x_n}op(\Lambda^{-\frac{1}{2}})u\right\|^2 + \left\|D_{x_n}op(\Lambda^{\frac{1}{2}})u\right\|^2 + \left\|op(\Lambda^{\frac{3}{2}})u\right\|^2 \geq C\left(\left\|D^2_{x_n}op(\Lambda^{-1})w\right\|^2 + \left\|D_{x_n}w\right\|^2 + \left\|op(\Lambda)w\right\|^2\right). \tag{73}$$

Combining (39), (64), (65), (66), (69) and (73), we obtain (27), for μ large enough.

□

4 Proof of Theorem 3.2

This section is devoted to the proof of Theorem 3.2.

4.1 Study of the eigenvalues

The proof is based on a cutting argument related to the nature of the roots of the polynomial $p_j(x,\xi',\xi_n,\mu)$, $j=1,2$, in ξ_n. On $x_n=0$, we note

$$q_1(x',\xi',\mu)=q_{1,1}(0,x',\xi',\mu)=q_{1,2}(0,x',\xi',\mu).$$

Let us introduce the following micro-local regions

$$\mathcal{E}^+_{1/_2}=\left\{(x,\xi',\mu)\in K\times\mathbb{R}^n,\quad q_{2,1/_2}+\frac{q_1^2}{(\frac{\partial\varphi_{1/_2}}{\partial x_n})^2}>0\right\},$$

$$\mathcal{Z}_{1/_2}=\left\{(x,\xi',\mu)\in K\times\mathbb{R}^n,\quad q_{2,1/_2}+\frac{q_1^2}{(\frac{\partial\varphi_{1/_2}}{\partial x_n})^2}=0\right\},$$

$$\mathcal{E}^-_{1/_2}=\left\{(x,\xi',\mu)\in K\times\mathbb{R}^n,\quad q_{2,1/_2}+\frac{q_1^2}{(\frac{\partial\varphi_{1/_2}}{\partial x_n})^2}<0\right\}.$$

(Here and in the following the index $1/_2$ using for telling 1 or 2).
We decompose $p_{1/_2}(x,\xi,\mu)$ as a polynomial in ξ_n. Then we have the following lemma describing the various types of the roots of $p_{1/_2}$.

Lemma 4.1 *We have the following*

1. *For $(x,\xi',\mu)\in\mathcal{E}^+_{1/_2}$, the roots of $p_{1/_2}$ denoted $z^\pm_{1/_2}$ satisfy $\pm\, Im\, z^\pm_{1/_2}>0$.*
2. *For $(x,\xi',\mu)\in\mathcal{Z}_{1/_2}$, one of the roots of $p_{1/_2}$ is real.*
3. *For $(x,\xi',\mu)\in\mathcal{E}^-_{1/_2}$, the roots of $p_{1/_2}$ are in the half- plane $Im\xi_n>0$ if $\frac{\partial\varphi_{1/_2}}{\partial x_n}<0$ (resp. in the half-plane $Im\xi_n<0$ if $\frac{\partial\varphi_{1/_2}}{\partial x_n}>0$).*

Proof.
Using (21) and (22), we can write

$$\begin{cases} p_1(x',\xi,\mu)=\left(\xi_n+i\mu\dfrac{\partial\varphi_1}{\partial x_n}-i\alpha_1\right)\left(\xi_n+i\mu\dfrac{\partial\varphi_1}{\partial x_n}+i\alpha_1\right),\\ \\ p_2(x',\xi,\mu)=\left(\xi_n+i\mu\dfrac{\partial\varphi_2}{\partial x_n}-i\alpha_2\right)\left(\xi_n+i\mu\dfrac{\partial\varphi_2}{\partial x_n}+i\alpha_2\right),\end{cases}\tag{74}$$

where $\alpha_j\in\mathbb{C}$, $j=1,2$, defined by

$$\begin{cases}\alpha_1^2(x',\xi',\mu)=\left(\mu\dfrac{\partial\varphi_1}{\partial x_n}\right)^2+q_{2,1}+2i\mu q_1,\\ \\ \alpha_2^2(x',\xi',\mu)=\left(\mu\dfrac{\partial\varphi_2}{\partial x_n}\right)^2-\mu^2+q_{2,1}+2i\mu q_1.\end{cases}\tag{75}$$

We set

$$z^{\pm}_{1/2} = -i\mu \frac{\partial \varphi_{1/2}}{\partial x_n} \pm i\alpha_{1/2}, \tag{76}$$

the roots of $p_{1/2}$. The imaginary parts of the roots of $p_{1/2}$ are

$$-\mu \frac{\partial \varphi_{1/2}}{\partial x_n} - \operatorname{Re} \alpha_{1/2}, \quad -\mu \frac{\partial \varphi_{1/2}}{\partial x_n} + \operatorname{Re} \alpha_{1/2}.$$

The signs of the imaginary parts are opposite if $\left|\partial \varphi_{1/2}/\partial x_n\right| < \left|\operatorname{Re} \alpha_{1/2}\right|$, equal to the sign of $-\partial \varphi_{1/2}/\partial x_n$ if $\left|\partial \varphi_{1/2}/\partial x_n\right| > \left|\operatorname{Re} \alpha_{1/2}\right|$ and one of the imaginary parts is null if $\left|\partial \varphi_{1/2}/\partial x_n\right| = \left|\operatorname{Re} \alpha_{1/2}\right|$. However the lines $\operatorname{Re} z = \pm \mu\, \partial \varphi_{1/2}/\partial x_n$ change by the application $z \mapsto z' = z^2$ into the parabolic curve $\operatorname{Re} z' = \left|\mu\, \partial \varphi_{1/2}/\partial x_n\right|^2 - \left|\operatorname{Im} z'\right|^2 / 4(\mu\, \partial \varphi_{1/2}/\partial x_n)^2$. Thus we obtain the lemma by replacing z' by $\alpha^2_{1/2}$. □

Lemma 4.2 *If we assume that the function φ satisfies the following condition*

$$\left(\frac{\partial \varphi_1}{\partial x_n}\right)^2 - \left(\frac{\partial \varphi_2}{\partial x_n}\right)^2 > 1, \tag{77}$$

then the following estimate holds

$$q_{2,2} - \mu^2 + \frac{q_1^2}{\left(\partial \varphi_2/\partial x_n\right)^2} > q_{2,1} + \frac{q_1^2}{\left(\partial \varphi_1/\partial x_n\right)^2}. \tag{78}$$

Proof.
Following (22), on $\{x_n = 0\}$, we have

$$q_{2,2}(x, \xi', \mu) - q_{2,1}(x, \xi', \mu) = \left(\mu \frac{\partial \varphi_1}{\partial x_n}\right)^2 - \left(\mu \frac{\partial \varphi_2}{\partial x_n}\right)^2. \tag{79}$$

Using (77), we have (78). □

Remark 4.1 *The result of this lemma imply that $\mathcal{E}^+_1 \subset \mathcal{E}^+_2$.*

4.2 Estimate in $\mathcal{E}^+_1$

In this part we study the problem in the elliptic region $\mathcal{E}^+$. In this region we can inverse the operator and use the Calderon projectors. Let $\chi^+(x, \xi', \mu) \in \mathcal{T}S^0_\mu$ such that in the support of χ^+ we have $q_{2,1} + \frac{q_1^2}{(\partial \varphi_1/\partial x_n)^2} \geq \delta > 0$. Then we have the following partial estimate.

Proposition 4.1 *There exists a constant $C > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$, we have*

$$\mu^2 \left\| op(\chi^+) u \right\|^2_{1,\mu} \leq C \left(\left\| P(x, D, \mu) u \right\|^2 + \left\| u \right\|^2_{1,\mu} + \mu \left| u \right|^2_{1,0,\mu} \right), \tag{80}$$

for any $u \in C^\infty_0(\overline{\Omega}_2)$.

If we suppose moreover that φ satisfies (77) then the following estimate holds

$$\mu \left|op(\chi^+)u\right|^2_{1,0,\mu} \leq C\left(\|P(x,D,\mu)u\|^2 + \mu^{-1}\left|op(b_1)u\right|_1^2 + \mu\left|op(b_2)u\right|^2 + \|u\|^2_{1,\mu} + \mu^{-2}\left|u\right|^2_{1,0,\mu}\right), \tag{81}$$

for any $u \in C_0^\infty(\overline{\Omega}_2)$ and b_j, $j=1,2$, defined in (24).

Proof
Let $\tilde{u} = op(\chi^+)u$. Then we get

$$\begin{cases} P\tilde{u} = \tilde{f} & \text{in } \{x_n > 0\}, \\ op(b_1)\tilde{u} = \tilde{u}_0|_{x_n=0} - i\mu\tilde{v}_0|_{x_n=0} = \tilde{e}_1 & \text{on } \{x_n = 0\}, \\ op(b_2)\tilde{u} = \left(D_{x_n} + i\mu\frac{\partial\varphi_1}{\partial x_n}\right)\tilde{u}_0|_{x_n=0} + \left(D_{x_n} + i\mu\frac{\partial\varphi_2}{\partial x_n}\right)\tilde{v}_0|_{x_n=0} = \tilde{e}_2 & \text{on } \{x_n = 0\}, \end{cases} \tag{82}$$

with $\tilde{f} = op(\chi^+)f + [P, op(\chi^+)]\,u$. Since $[P, op(\chi^+)] \in (\mathcal{TO}^0)D_{x_n} + \mathcal{TO}^1$, we have

$$\|\tilde{f}\|^2_{L^2} \leq C\left(\|P(x,D,\mu)u\|^2_{L^2} + \|u\|^2_{1,\mu}\right) \tag{83}$$

and $\tilde{e}_1 = op(\chi^+)e_1$ satisfying

$$\left|\tilde{e}_1\right|_1^2 \leq C\left|e_1\right|_1^2 \tag{84}$$

and

$$\tilde{e}_2 = \left[(D_{x_n} + i\mu\tfrac{\partial\varphi_1}{\partial x_n}), op(\chi^+)\right]u_0|_{x_n=0} + \left[(D_{x_n} + i\mu\tfrac{\partial\varphi_2}{\partial x_n}), op(\chi^+)\right]v_0|_{x_n=0} + op(\chi^+)e_2.$$

Since $[D_{x_n}, op(\chi^+)] \in \mathcal{TO}^0$, we have

$$\left|\tilde{e}_2\right|^2 \leq C\left(|u|^2 + |e_2|^2\right). \tag{85}$$

Let $\underline{\tilde{u}}$ the extension of $\tilde{u}$ by 0 in $x_n < 0$. According to (21), (22) and (23), we obtain, by noting $\partial\varphi/\partial x_n = \operatorname{diag}(\partial\varphi_1/\partial x_n, \partial\varphi_2/\partial x_n)$, $\gamma_j(\tilde{u}) = {}^t\left(D^j_{x_n}(\tilde{u}_0)\,|_{x_n=0^+}, D^j_{x_n}(\tilde{v}_0)\,|_{x_n=0^+}\right)$, $j=0,1$ and $\delta^{(j)} = (d/dx_n)^j\,(\delta_{x_n=0})$,

$$P\underline{\tilde{u}} = \underline{\tilde{f}} - \gamma_0(\tilde{u}) \otimes \delta' + \frac{1}{i}\left(\gamma_1(\tilde{u}) + 2i\mu\frac{\partial\varphi}{\partial x_n}\right) \otimes \delta \tag{86}$$

Let $\chi(x,\xi,\mu) \in S^0_\mu$ equal to 1 for sufficiently large $|\xi| + \mu$ and in a neighborhood of $\operatorname{supp}(\chi^+)$ and satisfies that in the support of χ we have p is elliptic. These conditions are compatible from the choice made for $\operatorname{supp}(\chi^+)$ and Remark 4.1. Let m large enough chosen later, by the ellipticity of p on $\operatorname{supp}(\chi)$ there exists $E = op(e)$ a parametrix of P. We recall that $e \in S^{-2}_\mu$, of the form $e(x,\xi,\mu) = \sum_{j=0}^m e_j(x,\xi,\mu)$, where $e_0 = \chi p^{-1}$ and $e_j = \operatorname{diag}(e_{j,1}, e_{j,2}) \in S^{-2-j}_\mu$ such that $e_{j,1}$ and $e_{j,2}$ are rational fractions in ξ_n. Then we have

$$EP = op(\chi) + R_m, \quad R_m \in \mathcal{O}^{-m-1}. \tag{87}$$

Following (86) and (87), we obtain

$$\begin{cases} \underline{\tilde{u}} = E\underline{\tilde{f}} + E\left[-h_1 \otimes \delta' + \frac{1}{i} h_0 \otimes \delta\right] + w_1, \\ h_0 = \gamma_1(\tilde{u}) + 2i\mu \frac{\partial \varphi}{\partial x_n} \gamma_0(\tilde{u}), \quad h_1 = \gamma_0(\tilde{u}), \\ w_1 = (\mathrm{Id} - op(\chi))\, \underline{\tilde{u}} - R_m \underline{\tilde{u}}. \end{cases} \tag{88}$$

Using the fact that $\mathrm{supp}(1-\chi) \cap \mathrm{supp}(\chi^+) = \varnothing$ and symbolic calculus (See Lemma 2.10 in [7]), we have $(\mathrm{Id} - op(\chi))\, op(\chi^+) \in \mathcal{O}^{-m}$, then we obtain

$$\|w_1\|_{2,\mu}^2 \leq C\mu^{-2} \|u\|_{L^2}^2 . \tag{89}$$

Now, let us look at this term $E\left[-h_1 \otimes \delta' + \frac{1}{i} h_0 \otimes \delta\right]$. For $x_n > 0$, we get

$$\begin{cases} E\left[-h_1 \otimes \delta' + \frac{1}{i} h_0 \otimes \delta\right] = \hat{T}_1 h_1 + \hat{T}_0 h_0, \\ \hat{T}_j(h) = \left(\frac{1}{2\pi}\right)^{n-1} \int e^{i(x'-y')\xi'} \hat{t}_j(x,\xi',\mu) h(y') dy' d\xi' = op(\hat{t}_j) h \\ \hat{t}_j = \frac{1}{2\pi i} \int_\gamma e^{i x_n \xi_n} e(x,\xi,\mu) \xi_n^j d\xi_n \end{cases}$$

where γ is the union of the segment $\{\xi_n \in \mathbb{R},\, |\xi_n| \leq c_0 \sqrt{|\xi'|^2 + \mu^2}\}$ and the half circle $\{\xi_n \in \mathbb{C},\, |\xi_n| = c_0 \sqrt{|\xi'|^2 + \mu^2},\, Im \xi_n > 0\}$, where the constant c_0 is chosen sufficiently large so as to have the roots z_1^+ and z_2^+ inside the domain with boundary γ (If c_0 is large enough, the change of contour $\mathbb{R} \longrightarrow \gamma$ is possible because the symbol $e(x,\xi,\mu)$ is holomorphic for large $|\xi_n|$; $\xi_n \in C$). In particular we have in $x_n \geq 0$

$$\left|\partial_{x_n}^k \partial_{x'}^\alpha \partial_{\xi'}^\beta \hat{t}_j\right| \leq C_{\alpha,\beta,k} \langle \xi', \mu\rangle^{j-1-|\beta|+k}, \quad j = 0, 1. \tag{90}$$

We now choose $\chi_1(x,\xi',\mu) \in \mathcal{T}S_\mu^0$, satisfying the same requirement as χ^+, equal to 1 in a neighborhood of $\mathrm{supp}(\chi^+)$ and such that the symbol χ be equal to 1 in a neighborhood of $\mathrm{supp}(\chi_1)$. We set $t_j = \chi_1 \hat{t}_j$, $j = 0, 1$. Then we obtain

$$\underline{\tilde{u}} = E\underline{\tilde{f}} + op(t_0) h_0 + op(t_1) h_1 + w_1 + w_2 \tag{91}$$

where $w_2 = op((1-\chi_1)\hat{t}_0) h_0 + op((1-\chi_1)\hat{t}_1) h_1$. By using the composition formula of tangential operator, estimate (90), the fact that $\mathrm{supp}(1-\chi_1) \cap \mathrm{supp}(\chi^+) = \varnothing$ and the following trace formula

$$|\gamma_0(u)|_j \leq C\mu^{-\frac{1}{2}} \|u\|_{j+1,\mu}, \quad j \in \mathbb{N}, \tag{92}$$

we obtain

$$\|w_2\|_{2,\mu}^2 \le C\mu^{-2}\left(\|u\|_{1,\mu}^2 + |u|_{1,0,\mu}^2\right). \tag{93}$$

Since $\chi = 1$ in the support of χ_1, we have $e(x,\xi,\mu)$ is meromorphic w.r.t ξ_n in the support of χ_1. $z_{1/2}^+$ are in $\mathrm{Im}\xi_n \ge c_1\sqrt{|\xi'|^2+\mu^2}$ $(c_1>0)$. If c_1 is small enough we can choose $\gamma_{1/2}$ in $\mathrm{Im}\xi_n \ge \frac{c_1}{2}\sqrt{|\xi'|^2+\mu^2}$ and we can write

$$t_j = \mathrm{diag}(t_{j,1}, t_{j,2}), \quad t_{j,1/2}(x,\xi',\mu) = \chi_1(x,\xi',\mu)\frac{1}{2\pi i}\int_{\gamma_{1/2}} e^{ix_n\xi_n} e_{1/2}(x,\xi,\mu)\xi_n^j d\xi_n, \quad j=0,1. \tag{94}$$

Then there exists $c_2>0$ such that in $x_n \ge 0$, we obtain

$$\left|\partial_{x_n}^k \partial_{x'}^\alpha \partial_{\xi'}^\beta t_j\right| \le C_{\alpha,\beta,k} e^{-c_2 x_n \langle \xi',\mu\rangle} \langle \xi',\mu\rangle^{j-1-|\beta|+k}. \tag{95}$$

In particular, we have $e^{c_2 x_n \mu}(\partial_{x_n}^k)t_j$ is bounded in $\mathcal{T}S_\mu^{j-1+k}$ uniformly w.r.t $x_n \ge 0$. Then

$$\|\partial_{x'} op(t_j)h_j\|_{L^2}^2 + \|op(t_j)h_j\|_{L^2}^2 \le C\int_{x_n>0} e^{-2c_2x_n\mu}\left|op(e^{c_2x_n\mu}t_j)h_j\right|_1^2(x_n)dx_n \le C\mu^{-1}|h_j|_j^2$$

and

$$\|\partial_{x_n} op(t_j)h_j\|_{L^2}^2 \le C\int_{x_n>0} e^{-2c_2x_n\mu}\left|op(e^{c_2x_n\mu}\partial_{x_n}t_j)h_j\right|_{L^2}^2(x_n)dx_n \le C\mu^{-1}|h_j|_j^2.$$

Using the fact that $h_0 = \gamma_1(\tilde{u}) + 2i\mu\frac{\partial\varphi}{\partial x_n}\gamma_0(\tilde{u})$ and $h_1 = \gamma_0(\tilde{u})$, we obtain

$$\|op(t_j)h_j\|_{1,\mu}^2 \le C\mu^{-1}|u|_{1,0,\mu}^2. \tag{96}$$

From (91) and estimates (83), (89), (93) and (96), we obtain (80).
It remains to proof (81). We recall that, in $\mathrm{supp}(\chi_1)$, we have

$$e_0 = \mathrm{diag}\left(e_{0,1}, e_{0,2}\right) = \mathrm{diag}\left(\frac{1}{p_1}, \frac{1}{p_2}\right) = \mathrm{diag}\left(\frac{1}{(\xi_n - z_1^+)(\xi_n - z_1^-)}, \frac{1}{(\xi_n - z_2^+)(\xi_n - z_2^-)}\right).$$

Using the residue formula, we obtain

$$e^{-ix_n z_{1/2}^+} t_{j,1/2} = \chi_1 \frac{(z_{1/2}^+)^j}{z_{1/2}^+ - z_{1/2}^-} + \lambda_{1/2}, \quad j=0,1, \quad \lambda_{1/2} \in \mathcal{T}S_\mu^{-2+j}. \tag{97}$$

Taking the traces of (91), we obtain

$$\gamma_0(\tilde{u}) = op(c)\gamma_0(\tilde{u}) + op(d)\gamma_1(\tilde{u}) + w_0, \tag{98}$$

where $w_0 = \gamma_0(E\underline{\tilde{f}} + w_1 + w_2)$ satisfies, according to the trace formula (92), the estimates (83), (89) and (93), the following estimate

$$\mu\left|w_0\right|_1^2 \le C\left(\|P(x,D,\mu)u\|^2 + \|u\|_{1,\mu}^2 + \mu^{-2}\left|u\right|_{1,0,\mu}^2\right) \tag{99}$$

and following (96), c and d are two tangential symbols of order respectively 0 and -1 given by

$$c_0 = \mathrm{diag}(c_{0,1}, c_{0,2}) \qquad \text{with} \quad c_{0,1/2} = -\left(\chi_1 \frac{z^-_{1/2}}{z^+_{1/2} - z^-_{1/2}}\right),$$

$$d_{-1} = \mathrm{diag}(d_{-1,1}, d_{-1,2}) \quad \text{with} \quad d_{-1,1/2} = \left(\chi_1 \frac{1}{z^+_{1/2} - z^-_{1/2}}\right).$$

Following (82), the transmission conditions give

$$\begin{cases} \gamma_0(\tilde{u}_0) - i\mu\gamma_0(\tilde{v}_0) = \tilde{e}_1 \\ \gamma_1(\tilde{u}_0) + \gamma_1(\tilde{v}_0) + i\mu\frac{\partial\varphi_1}{\partial x_n}\gamma_0(\tilde{u}_0) + i\mu\frac{\partial\varphi_2}{\partial x_n}\gamma_0(\tilde{v}_0) = \tilde{e}_2. \end{cases} \tag{100}$$

We recall that $\tilde{u} = (\tilde{u}_0, \tilde{v}_0)$, combining (98) and (100) we show that

$$op(k)\,{}^t\left(\gamma_0(\tilde{u}_0), \gamma_0(\tilde{v}_0), \Lambda^{-1}\gamma_1(\tilde{u}_0), \Lambda^{-1}\gamma_1(\tilde{v}_0)\right) = w_0 + \frac{1}{\mu} op\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\tilde{e}_1 + op\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\Lambda^{-1}\tilde{e}_2, \tag{101}$$

where k is a 4×4 matrix, with principal symbol defined by

$$k_0 + \frac{1}{\mu}r_0 = \begin{pmatrix} 1 - c_{0,1} & 0 & -\Lambda\, d_{-1,1} & 0 \\ 0 & 1 - c_{0,2} & 0 & -\Lambda\, d_{-1,2} \\ 0 & -i & 0 & 0 \\ i\mu\Lambda^{-1}\dfrac{\partial\varphi_1}{\partial x_n} & i\mu\Lambda^{-1}\dfrac{\partial\varphi_2}{\partial x_n} & 1 & 1 \end{pmatrix} + \frac{1}{\mu}r_0,$$

where r_0 is a tangential symbol of order 0.
We now choose $\chi_2(x, \xi', \mu) \in \mathcal{T}S^0_\mu$, satisfying the same requirement as χ^+, equal to 1 in a neighborhood of $\mathrm{supp}(\chi^+)$ and such that the symbol χ_1 be equal to 1 in a neighborhood of $\mathrm{supp}(\chi_2)$. In $\mathrm{supp}(\chi_2)$, we obtain

$$k_0|_{\mathrm{supp}(\chi_2)} = \begin{pmatrix} \dfrac{z_1^+}{z_1^+ - z_1^-} & 0 & -\dfrac{\Lambda}{z_1^+ - z_1^-} & 0 \\ 0 & \dfrac{z_2^+}{z_2^+ - z_2^-} & 0 & -\dfrac{\Lambda}{z_2^+ - z_2^-} \\ 0 & -i & 0 & 0 \\ i\mu\Lambda^{-1}\dfrac{\partial\varphi_1}{\partial x_n} & i\mu\Lambda^{-1}\dfrac{\partial\varphi_2}{\partial x_n} & 1 & 1 \end{pmatrix}.$$

Then, following (76),

$$\det(k_0)|_{\mathrm{supp}(\chi_2)} = -\left(z_1^+ - z_1^-\right)^{-1}\left(z_2^+ - z_2^-\right)^{-1}\Lambda\,\alpha_1.$$

To prove that there exists $c>0$ such that $\left|\det(k_0)|_{\mathrm{supp}(\chi_2)}\right| \geq c$, by homogeneity it suffices to prove that $\det(k_0)|_{\mathrm{supp}(\chi_2)} \neq 0$ if $|\xi'|^2+\mu^2=1$.
If we suppose that $\det(k_0)|_{\mathrm{supp}(\chi_2)} = 0$, we obtain $\alpha_1=0$ and then $\alpha_1^2=0$.
Following (75),we obtain

$$q_1 = 0 \quad \text{and} \quad \left(\mu\frac{\partial\varphi_1}{\partial x_n}\right)^2 + q_{2,1} = 0.$$

Combining with the fact that $q_{2,1} + \frac{q_1^2}{(\partial\varphi_1/\partial x_n)^2} > 0$, we obtain

$$-\left(\mu\frac{\partial\varphi_1}{\partial x_n}\right)^2 > 0.$$

Therefore $\det(k_0)|_{\mathrm{supp}(\chi_2)} \neq 0$. It follows that, for large μ, $k = k_0 + \frac{1}{\mu}r_0$ is elliptic in $\mathrm{supp}(\chi_2)$. Then there exists $l \in \mathcal{T}S^0_\mu$, such that

$$op(l)op(k) = op(\chi_2) + \tilde{R}_m,$$

with $\tilde{R}_m \in \mathcal{T}\mathcal{O}^{-m-1}$, for m large. This yields

$$\begin{aligned} {}^t\left(\gamma_0(\tilde{u}_0),\gamma_0(\tilde{v}_0),\Lambda^{-1}\gamma_1(\tilde{u}_0),\Lambda^{-1}\gamma_1(\tilde{v}_0)\right) &= op(l)w_0 + \tfrac{1}{\mu}op(l)op\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\tilde{e}_1 + op(l)op\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\Lambda^{-1}\tilde{e}_2 \\ &\quad + \left(op(1-\chi_2)-\tilde{R}_m\right){}^t\left(\gamma_0(\tilde{u}_0),\gamma_0(\tilde{v}_0),\Lambda^{-1}\gamma_1(\tilde{u}_0),\Lambda^{-1}\gamma_1(\tilde{v}_0)\right). \end{aligned}$$

Since $\mathrm{supp}(1-\chi_2)\cap\mathrm{supp}(\chi^+) = \varnothing$ and by using (99), we obtain

$$\mu|\tilde{u}|^2_{1,0,\mu} \leq C\left(\mu^{-1}|\tilde{e}_1|_1^2 + \mu|\tilde{e}_2| + \|P(x,D,\mu)u\|^2_{L^2} + \|u\|^2_{1,\mu} + \mu^{-2}|u|^2_{1,0,\mu}\right).$$

From estimates (84) and (85) and the trace formula (92), we obtain (81).

□

4.3 Estimate in $\mathcal{Z}_1$

The aim of this part is to prove the estimate in the region $\mathcal{Z}_1$. In this region, if φ satisfies (77), the symbol $p_1(x,\xi,\mu)$ admits a real roots and $p_2(x,\xi,\mu)$ admits two roots $z_2^\pm$ satisfy $\pm\,\mathrm{Im}(z_2^\pm) > 0$. Let $\chi^0(x,\xi',\mu) \in \mathcal{T}\mathcal{S}^0_\mu$ equal to 1 in $\mathcal{Z}_1$ and such that in the support of χ^0 we have $q_{2,2} - \mu^2 + \frac{q_1^2}{(\partial\varphi_2/\partial x_n)^2} \geq \delta > 0$. Then we have the following partial estimate.

Proposition 4.2 *There exists constants $C > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$ we have the following estimate*

$$\mu \left\|op(\chi^0)u\right\|^2_{1,\mu} \leq C \left(\|P(x, D, \mu)u\|^2 + \mu \left|u\right|^2_{1,0,\mu} + \|u\|^2_{1,\mu} \right), \tag{102}$$

for any $u \in C_0^\infty(\overline{\Omega}_2)$.

If we assume moreover that φ satisfies (77) then we have

$$\mu \left|op(\chi^0)u\right|^2_{1,0,\mu} \leq C \left(\|P(x, D, \mu)u\|^2 + \mu^{-1} \left|op(b_1)u\right|^2_1 + \mu \left|op(b_2)u\right|^2 + \|u\|^2_{1,\mu} + \mu^{-2} \left|u\right|^2_{1,0,\mu} \right), \tag{103}$$

for any $u \in C_0^\infty(\overline{\Omega}_2)$ and b_j, $j = 1, 2$, defined in (24).

4.3.1 Preliminaries

Let $u \in C_0^\infty(K)$, $\tilde{u} = op(\chi^0)u$ and P the differential operator with principal symbol given by

$$p(x, \xi, \mu) = \xi_n^2 + q_1(x, \xi', \mu)\xi_n + q_2(x, \xi', \mu)$$

where $q_j = \mathrm{diag}(q_{j,1}, q_{j,2})$, $j = 1, 2$. Then we have the following system

$$\begin{cases} P\tilde{u} = \tilde{f} & \text{in } \{x_n > 0\}, \\ B\tilde{u} = \tilde{e} = (\tilde{e}_1, \tilde{e}_2) & \text{on } \{x_n = 0\}, \end{cases} \tag{104}$$

where $\tilde{f} = op(\chi^0)f + [P, op(\chi^0)]\, u$. Since $[P, op(\chi^0)] \in (\mathcal{T}\mathcal{O}^0)D_{x_n} + \mathcal{T}\mathcal{O}^1$, we have

$$\|\tilde{f}\|^2_{L^2} \leq C \left(\|P(x, D, \mu)u\|^2_{L^2} + \|u\|^2_{1,\mu} \right), \tag{105}$$

B defined in (24) and $\tilde{e}_1 = op(\chi^0)e_1$ satisfying

$$|\tilde{e}_1|^2_1 \leq C\, |e_1|^2_1 \tag{106}$$

and

$$\tilde{e}_2 = \left[(D_{x_n} + i\mu\tfrac{\partial\varphi_1}{\partial x_n}), op(\chi^0)\right] u_0|_{x_n=0} + \left[(D_{x_n} + i\mu\tfrac{\partial\varphi_2}{\partial x_n}), op(\chi^0)\right] v_0|_{x_n=0} + op(\chi^0)e_2.$$

Since $[D_{x_n}, op(\chi^+)] \in \mathcal{T}\mathcal{O}^0$, we have

$$|\tilde{e}_2|^2 \leq C \left(|u|^2 + |e_2|^2 \right). \tag{107}$$

Let us reduce the problem (104) to a first order system. Put $v =^t (\langle D', \mu\rangle\tilde{u}, D_{x_n}\tilde{u})$. Then we obtain the following system

$$\begin{cases} D_{x_n}v - op(\mathcal{P})v = F & \text{in } \{x_n > 0\}, \\ op(\mathcal{B})v = (\frac{1}{\mu}\Lambda\tilde{e}_1, \tilde{e}_2) & \text{on } \{x_n = 0\}, \end{cases} \tag{108}$$

where $\mathcal{P}$ is a 4×4 matrix, with principal symbol defined by

$$\mathcal{P}_0 = \begin{pmatrix} 0 & \Lambda\,\mathrm{Id}_2 \\ \Lambda^{-1}q_2 & -q_1 \end{pmatrix}, \quad \left(\Lambda = \langle \xi', \mu\rangle = \left(|\xi'|^2 + \mu^2\right)^{\frac{1}{2}}\right),$$

$\mathcal{B}$ is a tangential symbol of order 0, with principal symbol given by

$$\mathcal{B}_0 + \frac{1}{\mu} r_0 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i\mu\Lambda^{-1}\frac{\partial\varphi_1}{\partial x_n} & i\mu\Lambda^{-1}\frac{\partial\varphi_2}{\partial x_n} & 1 & 1 \end{pmatrix} + \frac{1}{\mu} r_0$$

(r_0 a tangential symbol of order 0) and $F =^t (0, \tilde{f})$.

For a fixed (x_0, ξ_0', μ_0) in suppχ_0, the generalized eigenvalues of the matrix $\mathcal{P}$ are the zeroes in ξ_n of p_1 and p_2 i.e $z_1^{\pm} = -i\mu\frac{\partial\varphi_1}{\partial x_n} \pm i\alpha_1$ and $z_2^{\pm} = -i\mu\frac{\partial\varphi_2}{\partial x_n} \pm i\alpha_2$ with $\pm\mathrm{Im}(z_2^{\pm}) > 0$ and $z_1^{+} \in \mathbb{R}$.

We note $s(x, \xi', \mu) = (s_1^-, s_2^-, s_1^+, s_2^+)$ a basis of the generalized eigenspace of $\mathcal{P}(x_0, \xi_0', \mu_0)$ corresponding to eigenvalues with positive or negative imaginary parts. $s_j^{\pm}(x, \xi', \mu)$, $j = 1, 2$ is a C^∞ function on a conic neighborhood of (x_0, ξ_0', μ_0) of a degree zero in (ξ', μ). We denote $op(s)(x, D_{x'}, \mu)$ the pseudo-differential operator associated to the principal symbol $s(x, \xi', \mu) = \left(s_1^-(x, \xi', \mu), s_2^-(x, \xi', \mu), s_1^+(x, \xi', \mu), s_2^+(x, \xi', \mu)\right)$. Let $\hat{\chi}(x, \xi', \mu) \in \mathcal{T}S_\mu^0$ equal to 1 in a conic neighborhood of (x_0, ξ_0', μ_0) and in a neighborhood of supp(χ^0) and satisfies that in the support of $\hat{\chi}$, s is elliptic. Then there exists $n \in \mathcal{T}S_\mu^0$, such that

$$op(s)op(n) = op(\hat{\chi}) + \hat{R}_m,$$

with $\hat{R}_m \in \mathcal{T}\mathcal{O}^{-m-1}$, for m large.
Let $V = op(n)v$. Then we have the following system

$$\begin{cases} D_{x_n}V = GV + AV + F_1 & \text{in } \{x_n > 0\}, \\ op(\mathcal{B}_1)V = (\frac{1}{\mu}\Lambda\tilde{e}_1, \tilde{e}_2) + v_1 & \text{on } \{x_n = 0\}, \end{cases} \tag{109}$$

where $G = op(n)op(\mathcal{P})op(s)$, $A = [D_{x_n}, op(n)]\, op(s)$,
$F_1 = op(n)F + op(n)op(\mathcal{P})(op(1-\hat{\chi}) - \hat{R}_m)v + [D_{x_n}, op(n)]\,(op(1-\hat{\chi}) - \hat{R}_m)v$,
$op(\mathcal{B}_1) = op(\mathcal{B})op(s)$ and $v_1 = op(\mathcal{B})(op(\hat{\chi}-1) + \hat{R}_m)v$.
Using the fact that supp$(1-\hat{\chi}) \cap$ supp$(\chi^0) = \varnothing$, $\hat{R}_m \in \mathcal{T}\mathcal{O}^{-m-1}$, for m large and estimate (105), we show

$$\|F_1\|^2 \le C\left(\|P(x, D, \mu)u\|_{L^2}^2 + \|u\|_{1,\mu}^2\right). \tag{110}$$

Using the fact that supp$(1-\hat{\chi}) \cap$ supp$(\chi^0) = \varnothing$, $\hat{R}_m \in \mathcal{T}\mathcal{O}^{-m-1}$, for m large and the trace formula (92), we show

$$\mu\,|v_1|^2 \le C\left(\mu^{-2}\,|u|_{1,0,\mu}^2 + \|u\|_{1,\mu}^2\right). \tag{111}$$

Here we need to recall an argument shown in Taylor [13] given by this lemma

Lemma 4.3 *Let v solves the system*

$$\frac{\partial}{\partial y}v = Gv + Av$$

where $G = \begin{pmatrix} E & \\ & F \end{pmatrix}$ *and A are pseudo-differential operators of order 1 and 0, respectively. We suppose that the symbols of E and F are two square matrices and have disjoint sets of eigenvalues. Then there exists a pseudo-differential operator K of order -1 such that $w = (I + K)v$ satisfies*

$$\frac{\partial}{\partial y}w = Gw + \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} w + R_1 w + R_2 v$$

where α_j and R_j, $j = 1, 2$ are pseudo-differential operators of order 0 and $-\infty$, respectively.

By this argument, there exists a pseudo-differential operator $K(x, D_{x'}, \mu)$ of order -1 such that the boundary problem (109) is reduced to the following

$$\begin{cases} D_{x_n} w - op(\mathcal{H})w = \tilde{F} & \text{in } \{x_n > 0\}, \\ \\ op(\tilde{\mathcal{B}})w = (\frac{1}{\mu}\Lambda \tilde{e}_1, \tilde{e}_2) + v_1 + v_2 & \text{on } \{x_n = 0\}, \end{cases} \tag{112}$$

where $w = (I+K)V$, $\tilde{F} = (I+K)F_1$, $op(\mathcal{H})$ is a tangential of order 1 with principal symbol $\mathcal{H} = \mathrm{diag}(\mathcal{H}^-, \mathcal{H}^+)$ and $-\mathrm{Im}(\mathcal{H}^-) \geq C\Lambda$, $op(\tilde{\mathcal{B}}) = op(\mathcal{B}_1)(I + K')$ with K' is such that $(I + K')(I + K) = Id + R'_m$ ($R'_m \in \mathcal{O}^{-m-1}$, for m large) and $v_2 = op(\mathcal{B}_1)R'_m V$.
According to (110), we have

$$\|\tilde{F}\|^2 \leq C\left(\|P(x, D, \mu)u\|^2_{L^2} + \|u\|^2_{1,\mu}\right). \tag{113}$$

Using the fact that $R'_m \in \mathcal{O}^{-m-1}$, for m large, the trace formula (92) and estimates (106), (107) and (111), we show

$$\mu \left|op(\tilde{\mathcal{B}})w\right|^2 \leq C\left(\frac{1}{\mu}|e_1|^2_1 + \mu|e_2|^2 + \mu^{-2}|u|^2_{1,0,\mu} + \|u\|^2_{1,\mu}\right). \tag{114}$$

Lemma 4.4 *Let $\mathcal{R} = diag(-\rho Id_2, 0)$, $\rho > 0$. Then there exists $C > 0$ such that*

1. *$Im(\mathcal{RH}) = diag(e(x, \xi', \mu), 0)$, with $e(x, \xi', \mu) = -\rho Im(\mathcal{H}^-)$,*
2. *$e(x, \xi', \mu) \geq C\Lambda$ in supp (χ^0),*
3. *$-\mathcal{R} + \tilde{\mathcal{B}}^\star \tilde{\mathcal{B}} \geq C.Id$ on $\{x_n = 0\} \cap supp\,(\chi^0)$.*

Proof

Denote the principal symbol $\tilde{\mathcal{B}}$ of the boundary operator $op(\tilde{\mathcal{B}})$ by $\left(\tilde{\mathcal{B}}^-, \tilde{\mathcal{B}}^+\right)$ where $\tilde{\mathcal{B}}^+$ is the restriction of $\tilde{\mathcal{B}}$ to subspace generated by $\left(s_1^+, s_2^+\right)$. We begin by proving that $\tilde{\mathcal{B}}^+$ is an isomorphism. Denote

$$w_1 =^t (1,0) \quad \text{and} \quad w_2 =^t (0,1).$$

Then

$$\begin{cases} s_1^+ = \left(w_1, z_1^+ \Lambda^{-1} w_1\right) \\ \\ s_2^+ = \left(w_2, z_2^+ \Lambda^{-1} w_2\right) \end{cases}$$

are eigenvectors of z_1^+ and z_2^+. We have $\tilde{\mathcal{B}}^+ = (\mathcal{B}_0 + \frac{1}{\mu} r_0)(s_1^+ \, s_2^+) = \mathcal{B}_0^+ + \frac{1}{\mu} r_0^+$. To proof that $\tilde{\mathcal{B}}^+$ is an isomorphism it suffices, for large μ, to proof that $\mathcal{B}_0^+$ is an isomorphism. Following (76), we obtain

$$\mathcal{B}_0^+ = \begin{pmatrix} 0 & -i \\ \Lambda^{-1} i\alpha_1 & \Lambda^{-1} i\alpha_2 \end{pmatrix}.$$

Then

$$\det(\mathcal{B}_0^+) = -\Lambda^{-1}\alpha_1.$$

If we suppose that $\det(\mathcal{B}_0^+) = 0$, we obtain $\alpha_1 = 0$ and then $\alpha_1^2 = 0$. Following (75), we obtain

$$q_1 = 0 \quad \text{and} \quad \left(\mu \frac{\partial \varphi_1}{\partial x_n}\right)^2 + q_{2,1} = 0.$$

Combining with the fact that $q_{2,1} + \frac{q_1^2}{(\partial\varphi_1/\partial x_n)^2} = 0$, we obtain $\left(\mu \frac{\partial \varphi_1}{\partial x_n}\right)^2 = 0$, that is impossible because following (77), we have $\left(\frac{\partial \varphi_1}{\partial x_n}\right)^2 \neq 0$ and following (22), we have $\mu \neq 0$. We deduce that $\tilde{\mathcal{B}}^+$ is an isomorphism.

Let us show the Lemma 4.4. We have

$$\operatorname{Im}(\mathcal{RH}) = \operatorname{diag}\left(-\rho \operatorname{Im}(\mathcal{H}^-), 0\right) = \operatorname{diag}\left(e(x,\xi',\mu), 0\right), \tag{115}$$

where $e(x,\xi',\mu) = -\rho \operatorname{Im}(\mathcal{H}^-) \geq C\Lambda$, $C > 0$. It remains to proof 3.

Let $w = (w^-, w^+) \in \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. Then we have $\tilde{\mathcal{B}} w = \tilde{\mathcal{B}}^- w^- + \tilde{\mathcal{B}}^+ w^+$. Since $\tilde{\mathcal{B}}^+$ is an isomorphism, then there exists a constant $C > 0$ such that

$$\left|\tilde{\mathcal{B}}^+ w^+\right|^2 \geq C \left|w^+\right|^2.$$

Therefore, we have

$$\left|w^+\right|^2 \leq C\left(\left|\tilde{\mathcal{B}} w\right|^2 + \left|w^-\right|^2\right).$$

We deduce

$$-(\mathcal{R}w, w) = \rho\left|w^-\right|^2 \geq \frac{1}{C}\left|w^+\right|^2 + (\rho - 1)\left|w^-\right|^2 - \left|\tilde{\mathcal{B}} w\right|^2.$$

Then, we obtain the result, if ρ is large enough.

□

4.3.2 Proof of proposition 4.2

We start by showing (102). We have

$$\begin{aligned}\|P_1(x,D,\mu)u_0\|^2 &= \|(\mathrm{Re}P_1)u_0\|^2+\|(\mathrm{Im}P_1)u_0\|^2\\ &+ i\Big[\Big((\mathrm{Im}P_1)u_0,(\mathrm{Re}P_1)u_0\Big)-\Big((\mathrm{Re}P_1)u_0,(\mathrm{Im}P_1)u_0\Big)\Big].\end{aligned}$$

By integration by parts we find

$$\|P_1(x,D,\mu)u_0\|^2=\|(\mathrm{Re}P_1)u_0\|^2+\|(\mathrm{Im}P_1)u_0\|^2+i\Big(\left[\mathrm{Re}P_1,\mathrm{Im}P_1\right]u_0,u_0\Big)+\mu Q_0(u_0),$$

where

$$\left\{\begin{aligned} Q_0(u_0) &= (-2\tfrac{\partial\varphi_1}{\partial x_n}D_{x_n}u_0,D_{x_n}u_0)_0+(op(r_1)u_0,D_{x_n}u_0)_0\\ &+ (op(r_1')D_{x_n}u_0,u_0)_0+(op(r_2)u_0,u_0)_0+\mu(\tfrac{\partial\varphi_1}{\partial x_n}u_0,u_0)_0,\\ & r_1=r_1'=2q_{1,1},\qquad r_2=-2\tfrac{\partial\varphi_1}{\partial x_n}q_{2,1}.\end{aligned}\right.$$

Then we have

$$|Q_0(u_0)|^2\le C\,|u_0|^2_{1,0,\mu}\,.$$

We show the same thing for $P_2(x,D,\mu)v_0$. In addition we know that the principal symbol of the operator $[\mathrm{Re}P_j,\mathrm{Im}P_j]$, $j=1,2$, is given by $\{\mathrm{Re}P_j,\mathrm{Im}P_j\}$. Proceeding like Lebeau and Robbiano in paragraph 3 in [9], we obtain (102).

It remains to prove (103). Following Lemma 4.4, let $G(x_n)=d/dx_n(op(\mathcal{R})w,w)_{L^2(\mathbb{R}^{n-1})}$. Using $D_{x_n}w-op(\mathcal{H})=\tilde{F}$, we obtain

$$G(x_n)=-2\,\mathrm{Im}(op(\mathcal{R})\tilde{F},w)-2\,\mathrm{Im}(op(\mathcal{R})op(\mathcal{H})w,w).$$

The integration in the normal direction gives

$$(op(\mathcal{R})w,w)_0=\int_0^\infty \mathrm{Im}(op(\mathcal{R})op(\mathcal{H})w,w)dx_n+2\int_0^\infty \mathrm{Im}(op(\mathcal{R})\tilde{F},w)dx_n. \quad (116)$$

From Lemma 4.4 and the Gårding inequality, we obtain, for μ large,

$$\mathrm{Im}(op(\mathcal{R})op(\mathcal{H})w,w)\ge C\left|w^-\right|^2_{\frac12}, \quad (117)$$

moreover we have for all $\epsilon>0$

$$\int_0^\infty\left|(op(\mathcal{R})\tilde{F},w)\right|dx_n\le\epsilon C\mu\left\|w^-\right\|^2+\frac{C_\epsilon}{\mu}\|\tilde{F}\|^2. \quad (118)$$

Applying Lemma 4.4 and the Gårding inequality, we obtain, for μ large,

$$-(op(\mathcal{R})w,w)+|op(\tilde{\mathcal{B}})w|^2\ge C\,|w|^2\,. \quad (119)$$

Combining (119), (118), (117) and (116), we get

$$C\left|w^-\right|^2_{\frac{1}{2}} + C\left|w\right|^2 \leq \frac{C}{\mu}\|\tilde{F}\|^2 + |op(\tilde{\mathcal{B}})w|^2. \tag{120}$$

Then

$$\mu\left|w\right|^2 \leq C\|\tilde{F}\|^2 + \mu|op(\tilde{\mathcal{B}})w|^2.$$

Recalling that $w = (I+K)V$, $V = op(n)v$, $v = ^t(\langle D',\mu\rangle\tilde{u}, D_{x_n}\tilde{u})$ and $\tilde{u} = op(\chi^0)u$ and using estimates (113) and (114), we prove (103).

□

4.4 Estimate in $\mathcal{E}_1^-$

This part is devoted to estimate in region $\mathcal{E}_1^-$.
Let $\chi^-(x,\xi',\mu) \in \mathcal{T}S^0_\mu$ equal to 1 in $\mathcal{E}_1^-$ and such that in the support of χ^- we have $q_{2,1} + \frac{q_1^2}{(\partial\varphi_1/\partial x_n)^2} \leq -\delta < 0$. Then we have the following partial estimate.

Proposition 4.3 *There exists constants $C > 0$ and $\mu_0 > 0$ such that for any $\mu \geq \mu_0$ we have the following estimate*

$$\mu\left\|op(\chi^-)u\right\|^2_{1,\mu} \leq C\left(\|P(x,D,\mu)u\|^2 + \mu\left|u\right|^2_{1,0,\mu} + \left\|u\right\|^2_{1,\mu}\right), \tag{121}$$

for any $u \in C_0^\infty(\overline{\Omega}_2)$.
If we assume moreover that $\frac{\partial\varphi_1}{\partial x_n} > 0$ then we have

$$\mu\left|op(\chi^-)u_0\right|^2_{1,0,\mu} \leq C\left(\|P(x,D,\mu)u\|^2 + \mu^{-2}\left|u\right|^2_{1,0,\mu} + \left\|u\right\|^2_{1,\mu}\right) \tag{122}$$

for any $u = (u_0, v_0) \in C_0^\infty(\overline{\Omega}_2)$.

Proof.
Let $\tilde{u} = op(\chi^-)u = (op(\chi^-)u_0, op(\chi^-)v_0) = (\tilde{u}_0, \tilde{v}_0)$.
In this region we have not a priori information for the roots of $p_2(x,\xi,\mu)$. Using the same technique of the proof of (102), we obtain

$$\mu\left\|op(\chi^-)v_0\right\|^2_{1,\mu} \leq C\left(\|P(x,D,\mu)v_0\|^2 + \mu\left|v_0\right|^2_{1,0,\mu} + \left\|v_0\right\|^2_{1,\mu}\right) \tag{123}$$

In supp(χ^-) the two roots $z_1^\pm$ of $p_1(x,\xi,\mu)$ are in the half-plane $Im\xi_n < 0$. Then we can use the Calderon projectors. By the same way that the proof of (80) and using the fact that the operators $t_{0,1}$ and $t_{1,1}$ vanish in $x_n > 0$ (because the roots are in $Im\xi_n < 0$, see (94)), the counterpart of (91) is then

$$\tilde{u}_0 = E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1}, \quad \text{for } x_n > 0. \tag{124}$$

We then obtain (see proof of (80))

$$\mu^2\left\|op(\chi^-)u_0\right\|^2_{1,\mu} \leq C\left(\|P_1(x,D,\mu)u_0\|^2 + \mu\left|u_0\right|^2_{1,0,\mu} + \left\|u_0\right\|^2_{1,\mu}\right). \tag{125}$$

Combining (123) and (125), we obtain (121).
It remains to proof (122). We take the trace at $x_n = 0^+$ of (124),

$$\gamma_0(\tilde{u}_0) = w_{0,1} = \gamma_0(E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1}),$$

which, by the counterpart of (99), gives

$$\mu \left|\gamma_0(\tilde{u}_0)\right|_1^2 \leq C \left(\|P_1(x, D, \mu)u_0\|^2 + \|u_0\|_{1,\mu}^2 + \mu^{-2} \left|u_0\right|_{1,0,\mu}^2 \right). \tag{126}$$

From (124) we also have

$$D_{x_n}\tilde{u}_0 = D_{x_n}E\underline{\tilde{f}}_1 + D_{x_n}w_{1,1} + D_{x_n}w_{2,1}, \quad \text{for } x_n > 0.$$

We take the trace at $x_n = 0^+$ and obtain

$$\gamma_1(\tilde{u}_0) = \gamma_0(D_{x_n}(E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1})).$$

Using the trace formula (92), we obtain

$$\left|\gamma_1(\tilde{u}_0)\right|^2 \leq C\mu^{-1} \left\| D_{x_n}(E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1}) \right\|_{1,\mu}^2 \leq C\mu^{-1} \left\| E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1} \right\|_{2,\mu}^2$$

and, by the counterpart of (83), (89) and (93), this yields

$$\mu \left|\gamma_1(\tilde{u}_0)\right|^2 \leq C \left(\|P_1(x, D, \mu)u_0\|^2 + \|u_0\|_{1,\mu}^2 + \mu^{-2} \left|u_0\right|_{1,0,\mu}^2 \right). \tag{127}$$

Combining (126) and (127), we obtain

$$\mu \left|op(\chi^-)u_0\right|_{1,0,\mu}^2 \leq C \left(\|P_1(x, D, \mu)u_0\|^2 + \|u_0\|_{1,\mu}^2 + \mu^{-2} \left|u_0\right|_{1,0,\mu}^2 \right).$$

Then we have (122). □

4.5 End of the proof

We choose a partition of unity $\chi^+ + \chi^0 + \chi^- = 1$ such that χ^+, χ^0 and χ^- satisfy the properties listed in proposition 4.1, 4.2 and 4.3 respectively. We have

$$\|u\|_{1,\mu}^2 \leq \left\|op(\chi^+)u\right\|_{1,\mu}^2 + \left\|op(\chi^0)u\right\|_{1,\mu}^2 + \left\|op(\chi^-)u\right\|_{1,\mu}^2.$$

Combining this inequality and (80), (102) and (121), we obtain, for large μ, the first estimate (37) of Theorem 3.2. i.e.

$$\mu \|u\|_{1,\mu}^2 \leq C \left(\|P(x, D, \mu)u\|^2 + \mu \left|u\right|_{1,0,\mu}^2 \right).$$

It remains to estimate $\mu \left|u\right|_{1,0,\mu}^2$. We begin by giving an estimate of $\mu \left|u_0\right|_{1,0,\mu}^2$.
We have

$$\left|u_0\right|_{1,0,\mu}^2 \leq \left|op(\chi^+)u_0\right|_{1,0,\mu}^2 + \left|op(\chi^0)u_0\right|_{1,0,\mu}^2 + \left|op(\chi^-)u_0\right|_{1,0,\mu}^2,$$

$$\left|op(\chi^+)u_0\right|^2_{1,0,\mu} \leq \left|op(\chi^+)u\right|^2_{1,0,\mu}$$

and

$$\left|op(\chi^0)u_0\right|^2_{1,0,\mu} \leq \left|op(\chi^0)u\right|^2_{1,0,\mu}.$$

Combining these inequalities, (81), (103), (122) and the fact that $\mu^{-2}\left|u\right|^2_{1,0,\mu} = \mu^{-2}\left|u_0\right|^2_{1,0,\mu} + \mu^{-2}\left|v_0\right|^2_{1,0,\mu}$, we obtain, for large μ

$$\mu\left|u_0\right|^2_{1,0,\mu} \leq C\left(\|P(x,D,\mu)u\|^2 + \mu^{-1}\left|op(b_1)u\right|^2_1 + \mu\left|op(b_2)u\right|^2 + \mu^{-2}\left|v_0\right|^2_{1,0,\mu} + \|u\|^2_{1,\mu}\right). \tag{128}$$

For estimate $\mu\left|v_0\right|^2_{1,0,\mu}$, we need to use the transmission conditions given by (24). We have

$$op(b_1)u = u_0|_{x_n=0} - i\mu v_0|_{x_n=0} \quad \text{on } \{x_n = 0\}.$$

Then

$$\mu\left|v_0\right|^2_1 \leq C\left(\mu^{-1}\left|u_0\right|^2_1 + \mu^{-1}\left|op(b_1)u\right|^2_1\right).$$

Since we have $\mu^{-1}\left|u_0\right|^2_1 \leq \mu\left|u_0\right|^2_{1,0,\mu}$. Then using (128), we obtain

$$\mu\left|v_0\right|^2_1 \leq C\left(\|P(x,D,\mu)u\|^2 + \mu^{-1}\left|op(b_1)u\right|^2_1 + \mu\left|op(b_2)u\right|^2 + \mu^{-2}\left|v_0\right|^2_{1,0,\mu} + \|u\|^2_{1,\mu}\right). \tag{129}$$

We have also

$$op(b_2)u = \left(D_{x_n} + i\mu\frac{\partial\varphi_1}{\partial x_n}\right)u_0|_{x_n=0} + \left(D_{x_n} + i\mu\frac{\partial\varphi_2}{\partial x_n}\right)v_0|_{x_n=0} \quad \text{on } \{x_n = 0\}.$$

Then

$$\mu\left|D_{x_n}v_0\right|^2 \leq C\left(\mu\left|op(b_2)u\right|^2 + \mu\left|D_{x_n}u_0\right|^2 + \mu^3\left|u_0\right|^2 + \mu^3\left|v_0\right|^2\right).$$

Using the fact that $\left|u\right|_{k-1} \leq \mu^{-1}\left|u\right|_k$, we obtain

$$\mu\left|D_{x_n}v_0\right|^2 \leq C\left(\mu\left|op(b_2)u\right|^2 + \mu\left|D_{x_n}u_0\right|^2 + \mu\left|u_0\right|^2_1 + \mu\left|v_0\right|^2_1\right).$$

Since we have $\mu\left|u_0\right|^2_{1,0,\mu} = \mu\left|D_{x_n}u_0\right|^2 + \mu\left|u_0\right|^2_1$. Then using (128) and (129), we obtain

$$\mu\left|D_{x_n}v_0\right|^2 \leq C\left(\|P(x,D,\mu)u\|^2 + \mu^{-1}\left|op(b_1)u\right|^2_1 + \mu\left|op(b_2)u\right|^2 + \mu^{-2}\left|v_0\right|^2_{1,0,\mu} + \|u\|^2_{1,\mu}\right). \tag{130}$$

Combining (129) and (130), we have

$$\mu\left|v_0\right|^2_{1,0,\mu} \leq C\left(\|P(x,D,\mu)u\|^2 + \mu^{-1}\left|op(b_1)u\right|^2_1 + \mu\left|op(b_2)u\right|^2 + \|u\|^2_{1,\mu}\right). \tag{131}$$

Combining (128) and (131), we obtain

$$\mu\left|u\right|^2_{1,0,\mu} \leq C\left(\|P(x,D,\mu)u\|^2 + \mu^{-1}\left|op(b_1)u\right|^2_1 + \mu\left|op(b_2)u\right|^2 + \|u\|^2_{1,\mu}\right). \tag{132}$$

Inserting (132) in (37) and for large μ, we obtain (38).

□

Appendix A

This appendix is devoted to prove Lemma 2.1. For this, we need to distinguish two cases.

1. **Inside $\mathcal{O}$**
   To simplify the writing, we note $\|u\|_{L^2(\mathcal{O})} = \|u\|$.
   Let $\chi \in C_0^\infty(\mathcal{O})$. We have by integration by part
   $$((\triangle - i\mu)u, \chi^2 u) = (-\nabla u, \chi^2 \nabla u) - (\nabla u, \nabla(\chi^2)u) - i\mu \|\chi u\|^2 .$$
   Then
   $$\mu \|\chi u\|^2 \le C \left(\|f\| \left\|\chi^2 u\right\| + \|\nabla u\|^2 + \|\nabla u\| \|\chi u\| \right).$$
   Then
   $$\mu \|\chi u\|^2 \le C \left(\frac{1}{\epsilon} \|f\|^2 + \epsilon \left\|\chi^2 u\right\| + \|\nabla u\|^2 + \frac{1}{\epsilon} \|\nabla u\|^2 + \epsilon \|\chi u\|^2 \right).$$
   Recalling that $\mu \ge 1$, we have for ϵ small enough
   $$\|\chi u\|^2 \le C \left(\|\nabla u\|^2 + \|f\|^2 \right). \tag{133}$$
   Hence the result inside $\mathcal{O}$.

2. **In the neighborhood of the boundary**
   Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then
   $$\partial\mathcal{O} = \{x \in \mathbb{R}^n,\ x_n = 0\}.$$
   Let $\epsilon > 0$ such that $0 < x_n < \epsilon$. Then we have
   $$u(x', \epsilon) - u(x', x_n) = \int_{x_n}^{\epsilon} \partial_{x_n} u(x', \sigma) d\sigma.$$
   Then
   $$|u(x', x_n)|^2 \le 2 |u(x', \epsilon)|^2 + 2 \left(\int_{x_n}^{\epsilon} |\partial_{x_n} u(x', \sigma)| \, d\sigma \right)^2 .$$
   Using the Cauchy Schwartz inequality, we obtain
   $$|u(x', x_n)|^2 \le 2 |u(x', \epsilon)|^2 + 2\epsilon^2 \int_0^{\epsilon} |\partial_{x_n} u(x', x_n)|^2 \, dx_n.$$
   Integrating with regard to x', we obtain
   $$\int_{|x'|<\epsilon} |u(x', x_n)|^2 \, dx' \le 2 \int_{|x'|<\epsilon} |u(x', \epsilon)|^2 \, dx' + 2\epsilon^2 \int_{|x'|<\epsilon,\, |x_n|<\epsilon} \left(|\partial_{x_n} u(x', x_n)|^2 \, dx_n \right) dx'. \tag{134}$$
   Using the trace Theorem, we have
   $$\int_{|x'|<\epsilon} |u(x', \epsilon)|^2 \, dx' \le C \int_{|x'|<2\epsilon,\, |x_n - \epsilon| < \frac{\epsilon}{2}} (|u(x)|^2 + |\nabla u(x)|^2) dx. \tag{135}$$

Now we need to introduce the following cut-off functions

$$\chi_1(x) = \begin{cases} 1 & \text{if} \quad 0 < x_n < \frac{\epsilon}{2}, \\ \\ 0 & \text{if} \quad x_{n>}\epsilon \end{cases}$$

and

$$\chi_2(x) = \begin{cases} 1 & \text{if} \quad \frac{\epsilon}{2} < x_n < \frac{3\epsilon}{2}, \\ \\ 0 & \text{if} \quad x_n < \frac{\epsilon}{4}, \ x_{n>}2\epsilon. \end{cases}$$

Combining (134) and (135), we obtain for ϵ small enough

$$\|\chi_1 u\|^2 \leq C\left(\|\chi_2 u\|^2 + \|\nabla u\|^2\right). \tag{136}$$

Since following (133), we have

$$\|\chi_2 u\|^2 \leq C\left(\|f\|^2 + \|\nabla u\|^2\right).$$

Inserting in (136), we obtain

$$\|\chi_1 u\|^2 \leq C\left(\|f\|^2 + \|\nabla u\|^2\right). \tag{137}$$

Hence the result in the neighborhood of the boundary.

Following (133), we can write

$$\|(1-\chi_1)u\|^2 \leq C\left(\|f\|^2 + \|\nabla u\|^2\right). \tag{138}$$

Adding (137) and (138), we obtain

$$\|u\|^2 \leq C\left(\|f\|^2 + \|\nabla u\|^2\right).$$

Hence the result.

Appendix B: Proof of Lemma 3.1

This appendix is devoted to prove Lemma 3.1.
Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ in the support of u. We want to show that $op(\Lambda^s)e^{\mu\varphi}\chi op(\Lambda^{-s})$ is bounded in L^2. Recalling that for all u and $v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\mathcal{F}(uv)(\xi') = (\frac{1}{2\pi})^{n-1}\mathcal{F}(u) * \mathcal{F}(v)(\xi'), \qquad \forall \xi' \in \mathbb{R}^{n-1}.$$

Then

$$\begin{aligned}\mathcal{F}(op(\Lambda^s)e^{\mu\varphi}\chi op(\Lambda^{-s})v)(\xi',\mu) &= \langle \xi',\mu\rangle^s \mathcal{F}(e^{\mu\varphi}\chi op(\Lambda^{-s})v)(\xi',\mu)\\ &= (\frac{1}{2\pi})^{n-1}\langle \xi',\mu\rangle^s (g(\xi',\mu) * \langle \xi',\mu\rangle^{-s}\mathcal{F}(v))(\xi',\mu),\end{aligned}$$

where $g(\xi',\mu) = \mathcal{F}(e^{\mu\varphi}\chi)(\xi',\mu)$. Then we have

$$\mathcal{F}(op(\Lambda^s)e^{\mu\varphi}\chi op(\Lambda^{-s})v)(\xi',\mu) = \int g(\xi'-\eta',\mu)\langle \xi',\mu\rangle^s\langle \eta',\mu\rangle^{-s}\mathcal{F}(v)(\eta',\mu)d\eta'.$$

Let $k(\xi',\eta') = g(\xi'-\eta',\mu)\langle \xi',\mu\rangle^s\langle \eta',\mu\rangle^{-s}$. Our goal is to show that $\int K(\xi',\eta')\mathcal{F}(v)(\eta',\mu)d\eta'$ is bounded in L^2. To do it, we will use Lemma of Schur. It suffices to prove that there exists $M > 0$ and $N > 0$ such that

$$\int |K(\xi',\eta')|\,d\xi' \le M \qquad \text{and} \qquad \int |K(\xi',\eta')|\,d\eta' \le N.$$

In the sequel, we suppose $s \ge 0$ (the case where $s < 0$ is treated in the same way). For $R > 0$, we have

$$\begin{aligned}\langle \xi',\mu\rangle^{2R} g(\xi',\mu) &= \int \langle \xi',\mu\rangle^{2R} e^{-ix'\xi'}\xi(x)e^{\mu\varphi(x)}dx'\\ &= \int (1-\Delta+\mu^2)^R(e^{-ix'\xi'})\chi(x)e^{\mu\varphi(x)}dx'\\ &= \int e^{-ix'\xi'}(1-\Delta+\mu^2)^R(\chi(x)e^{\mu\varphi(x)})dx'.\end{aligned}$$

Then there exists $C > 0$, such that

$$\left|\langle \xi',\mu\rangle^{2R} g(\xi',\mu)\right| \le Ce^{C\mu}. \tag{139}$$

Moreover, we can write

$$\int |K(\xi',\eta')|\,d\xi' = \int \left| g(\xi'-\eta',\mu)\langle \xi'-\eta',\mu\rangle^{2R}\frac{\langle \xi',\mu\rangle^s\langle \eta',\mu\rangle^{-s}}{\langle \xi'-\eta',\mu\rangle^{2R}}\right| d\xi'.$$

Using (139), we obtain

$$\int |K(\xi',\eta')|\,d\xi' \le Ce^{C\mu}\int \frac{\langle \xi',\mu\rangle^s\langle \eta',\mu\rangle^{-s}}{\langle \xi'-\eta',\mu\rangle^{2R}}d\xi'.$$

Since

$$\int \frac{\langle \xi', \mu\rangle^s \langle \eta', \mu\rangle^{-s}}{\langle \xi' - \eta', \mu\rangle^{2R}} d\xi' = \int_{|\xi'| \le \frac{1}{\epsilon}|\eta'|} \frac{\langle \xi', \mu\rangle^s \langle \eta', \mu\rangle^{-s}}{\langle \xi' - \eta', \mu\rangle^{2R}} d\xi' + \int_{|\eta'| \le \epsilon |\xi'|} \frac{\langle \xi', \mu\rangle^s \langle +\eta', \mu\rangle^{-s}}{\langle \xi' - \eta', \mu\rangle^{2R}} d\xi', \quad \epsilon > 0.$$

If $|\xi'| \le \frac{1}{\epsilon} |\eta'|$, we have

$$\frac{\langle \xi', \mu\rangle^s \langle \eta', \mu\rangle^{-s}}{\langle \xi' - \eta', \mu\rangle^{2R}} \le C \frac{\langle \eta', \mu\rangle^s \langle \eta', \mu\rangle^{-s}}{\langle \xi' - \eta', \mu\rangle^{2R}} \le \frac{C}{\langle \xi' - \eta', \mu\rangle^{2R}} \quad \in L^1 \quad \text{if} \quad 2R > n - 1.$$

If $|\eta'| \le \epsilon |\xi'|$, i.e $\langle \xi' - \eta', \mu\rangle \ge \delta \langle \xi', \mu\rangle$, $\delta > 0$, we have

$$\frac{\langle \xi', \mu\rangle^s \langle \eta', \mu\rangle^{-s}}{\langle \xi' - \eta', \mu\rangle^{2R}} \le \frac{C}{\langle \xi' - \eta', \mu\rangle^{2R-s}} \quad \in L^1 \quad \text{if} \quad 2R - s > n - 1.$$

Then there exists $M > 0$, such that

$$\int |K(\xi', \eta')| \, d\xi' \le M e^{C\mu}.$$

By the same way, we show that there exists $N > 0$, such that

$$\int |K(\xi', \eta')| \, d\eta' \le N e^{C\mu}.$$

Using Lemma of Schur, we have $(op(\Lambda^s) e^{\mu\varphi} \chi op(\Lambda^{-s}))$ is bounded in L^2 and

$$\left\| op(\Lambda^s) e^{\mu\varphi} \chi op(\Lambda^{-s}) \right\|_{\mathcal{L}(L^2)} \le C e^{C\mu}.$$

Applying in $op(\Lambda^s)u$, we obtain the result.

References

[1] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[2] M. Bellassoued. Distribution of resonances and decay rate of the local energy for the elastic wave equation. *Comm. Math. Phys.*, 215(2):375–408, 2000.

[3] M. Bellassoued. Carleman estimates and distribution of resonances for the transparent obstacle and application to the stabilization. *Asymptot. Anal.*, 35(3-4):257–279, 2003.

[4] M. Bellassoued. Decay of solutions of the elastic wave equation with a localized dissipation. *Ann. Fac. Sci. Toulouse Math. (6)*, 12(3):267–301, 2003.

[5] N. Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.*, 180(1):1–29, 1998.

[6] T. Duyckaerts. Optimal decay rates of the energy of a hyperbolic-parabolic system coupled by an interface. *Asymptot. Anal.*, 51(1):17–45, 2007.

[7] J. Le Rousseau and L. Robbiano. Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations. *HAL : hal-00193885.*

[8] G. Lebeau. Équation des ondes amorties. In *Algebraic and geometric methods in mathematical physics (Kaciveli, 1993)*, volume 19 of *Math. Phys. Stud.*, pages 73–109. Kluwer Acad. Publ., Dordrecht, 1996.

[9] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[10] G. Lebeau and L. Robbiano. Stabilisation de l'équation des ondes par le bord. *Duke Math. J.*, 86(3):465–491, 1997.

[11] J. Rauch, X. Zhang, and E. Zuazua. Polynomial decay for a hyperbolic-parabolic coupled system. *J. Math. Pures Appl. (9)*, 84(4):407–470, 2005.

[12] L. Robbiano. Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptotic Anal.*, 10(2):95–115, 1995.

[13] M. E. Taylor. Reflection of singularities of solutions to systems of differential equations. *Comm. Pure Appl. Math.*, 28(4):457–478, 1975.

[14] X. Zhang and E. Zuazua. Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction. *Arch. Ration. Mech. Anal.*, 184(1):49–120, 2007.