Author manuscript, published in "Automatica 45, 7 (2009) 1737-1742"
DOI : 10.1016/j.automatica.2009.03.011

State and input observability recovering by additional sensor implementation: A graph-theoretic approach

T. Boukhobza and F. Hamelin

Centre de Recherche en Automatique de Nancy (CRAN), Nancy–University, CNRS UMR 7039; BP 70239, 54506 Vandœuvre Cedex, Nancy, France, email: taha.boukhobza@cran.uhp-nancy.fr

Abstract

This paper deals with the problem of additional sensor location in order to recover the state and input observability for structured linear systems. The proposed method is based on a graph-theoretic approach and assumes only the knowledge of the system's structure. It allows to provide the minimal number of the required sensors and either their pertinent location or a necessary and sufficient condition which allows to check if a sensor location is adequate or not. We obtain a sensor placement procedure based on classical and well-known graph theory algorithms, which have polynomial complexity orders.
Key words: State and input observability, additional sensor location, structured linear systems, graph theory.

1 Introduction

The problem of estimating the state and the unknown input is of great interest mainly in control law synthesis, fault detection and isolation, fault tolerant control, supervision and so on. In this respect, using geometric or algebraic approaches, many works, among which we can cite (Basile and Marro, 1969; Hautus, 1983), are focused on the input and state observability analysis. When the input and state observability conditions are not satisfied, one way to recover this property is to add sensors, which must measure some pertinent variables.

Many studies reviewed in (van de Wal and de Jager, 2001) deal with the selection and sensor placement. They use, almost all, optimisation criteria related to the observability Gramian, sensitivity functions .... To apply these methods, which are based on classical algebraic and geometric tools, the exact knowledge of the state space matrices characterizing the system's model is required. However, in many modeling problems, these matrices have a number of fixed zero entries determined by the physical laws while the remaining entries are not precisely known, particularly during a conception stage. This is why, to deal with these systems in spite of poor knowledge we have on them, the idea is to consider models characterized by matrices where the fixed zeros are conserved while the non-zero entries are replaced by free parameters. Many studies on this kind of systems, called structured systems, are related to the graph-theoretic approach, and aim to analyse some of their most important properties such as controllability, observability, fault diagnosticability, reconfigurability or the solvability of several classical control problems (Dion *et al.*, 2003; Staroswiecki, 2006). It results from these works that the graph-theoretic approach provides generally simple and elegant analysis tools.

In this way, in (Commault *et al.*, 2008), the authors address the problem of additional sensor placement to recover the state observability of structured linear systems. However, these studies do not allow to tackle systems with unknown inputs which can model disturbances or faults for example. This is the first motivation of the presented work in which we consider systems with exogenous unknown inputs having unknown dynamics. More precisely, using a graph-theoretic approach, we study the number and the location of the additional sensors which are necessary and useful to recover the input and state observability conditions provided in (Boukhobza *et al.*, 2007). We try to answer to the following question: when a linear system is not input and state observable, where must we place a minimal number of additional sensors to recover this observability property ? At this aim, we give the necessary and sufficient number of additional sensors we must add, and either their exact location or a condition which allows to check if a given location is efficient. The goal is to precise as finely as possible the location of the additional sensors which makes possible the state and input observability of the system. All the proposed results are based on classical combinatorial algorithms with polynomial order complexity. This may be an important criterion when we deal with large scale systems. Moreover, since we consider structured systems, our approach can be used during a conception stage.

The paper is organised as follows: after Section 2, which is devoted to the problem formulation, a digraph representation of structured systems is given in Section 3. The main results are enounced in Section 4. A conclusion ends the paper.

2 Problem statement

In this paper, we treat systems of the form

$$\Sigma_\Lambda : \begin{cases} \dot{x}(t) = A^\lambda x(t) + B^\lambda u(t) \\ y(t) = C^\lambda x(t) + D^\lambda u(t) \end{cases}, \text{ where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^q$$

and $y \in \mathbb{R}^p$ are respectively the state vector, the unknown input vector and the output vector. A^λ, B^λ, C^λ and D^λ represent matrices which elements are either fixed to zero or assumed to be nonzero free parameters noted λ_i. The latter form a vector $\Lambda = (\lambda_1, \ldots, \lambda_h)^T \in \mathbb{R}^h$. If all parameters λ_i are numerically fixed, we obtain a so-called admissible realization of structured system Σ_Λ. We say that a property is true generically if it is true for almost all the realizations of structured system Σ_Λ. Here, "for almost all the realizations" is to be understood as "for all parameter values ($\Lambda \in \mathbb{R}^h$) except for those in some proper algebraic variety in the parameter space".

Let us now recall the definition of the generic state and input observability which is related to the strong observability and the left invertibility (Trentelman *et al.*, 2001).

Definition 1 *Structured system Σ_Λ is generically state and input observable if, for almost all its realizations, $y(t) = 0$*

for $t \geq 0$ *implies* $x(t) = 0$ *for* $t \geq 0$ *and* $u(t) = 0$ *for* $t > 0$.

Starting from a system which does not satisfy Definition 1, the aim of the paper is to propose a systematic procedure for additional sensor placement in order to recover the generic state and input observability.

3 Graphical representation of structured linear systems

3.1 Digraph definition for structured linear system

Digraph $\mathcal{G}(\Sigma_\Lambda)$, associated to Σ_Λ, is constituted by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. The vertices represent the state, the unknown input and the output components of Σ_Λ whereas the edges traduce the existence of static or dynamic relations between these variables. More precisely, $\mathcal{V} = \mathbf{X} \cup \mathbf{Y} \cup \mathbf{U}$, where $\mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ is the set of state vertices, $\mathbf{Y} = \{\mathbf{y_1}, \ldots, \mathbf{y_p}\}$ is the set of output vertices and $\mathbf{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_q}\}$ is the set of unknown input vertices. The edge set is $\mathcal{E} = A\text{-edges} \cup B\text{-edges} \cup C\text{-edges} \cup D\text{-edges}$, with $A\text{-edges} = \{(\mathbf{x_j}, \mathbf{x_i}),\ A^\lambda(i,j) \neq 0\}$, $B\text{-edges} = \{(\mathbf{u_j}, \mathbf{x_i}),\ B^\lambda(i,j) \neq 0\}$, $C\text{-edges} = \{(\mathbf{x_j}, \mathbf{y_i}),\ C^\lambda(i,j) \neq 0\}$, $D\text{-edges} = \{(\mathbf{u_j}, \mathbf{y_i}),\ D^\lambda(i,j) \neq 0\}$. Here, $M^\lambda(i,j)$ is the $(i,j)th$ element of matrix M^λ and $(\mathbf{v_1}, \mathbf{v_2})$ denotes a directed edge from vertex $\mathbf{v_1} \in \mathcal{V}$ to vertex $\mathbf{v_2} \in \mathcal{V}$.
Hereafter, we illustrate the proposed digraph representation with an example.

Example 2 *To the system defined by*

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_2 + \lambda_2 x_4, & \dot{x}_2 &= \lambda_3 x_3 \\ \dot{x}_3 &= \lambda_6 u_1, & \dot{x}_4 &= \lambda_4 x_1 + \lambda_5 x_5, \\ \dot{x}_5 &= \lambda_7 u_2 & & \\ y_1 &= \lambda_8 x_1 & y_2 &= \lambda_9 x_2 + \lambda_{10} u_1 \end{aligned}$$

we associate the digraph in Figure 1.

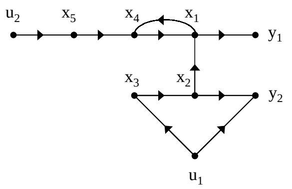

Figure 1. Digraph representing system of Example 2

3.2 Notations and definitions

• Two edges $e_1 = (\mathbf{v_1}, \mathbf{v_1'})$ and $e_2 = (\mathbf{v_2}, \mathbf{v_2'})$ are v-disjoint if $\mathbf{v_1} \neq \mathbf{v_2}$ and $\mathbf{v_1'} \neq \mathbf{v_2'}$.
In example 2, $(\mathbf{x_2}, \mathbf{y_2})$ and $(\mathbf{x_5}, \mathbf{x_4})$ as well as $(\mathbf{x_1}, \mathbf{x_4})$ and $(\mathbf{x_4}, \mathbf{x_1})$ are v-disjoint. However, neither $(\mathbf{x_2}, \mathbf{x_1})$ and $(\mathbf{x_4}, \mathbf{x_1})$ nor $(\mathbf{u_1}, \mathbf{y_2})$ and $(\mathbf{u_1}, \mathbf{x_3})$ are v-disjoint.
• Some edges are v-disjoint if they are mutually v-disjoint.
• A sequence of edges $(\mathbf{v_{r_j}}, \mathbf{v_{r_{j+1}}}) \in \mathcal{E}$ for $j = 0, \ldots, i-1$ is called a path and is denoted $P = \mathbf{v_{r_0}} \to \mathbf{v_{r_1}} \to \ldots \to \mathbf{v_{r_i}}$. We say, in this case, that P covers or contains vertices $\mathbf{v_{r_0}}, \mathbf{v_{r_1}}, \ldots, \mathbf{v_{r_i}}$.
• A path is simple when every vertex occurs only once in this path. Some paths are disjoint if they have no common vertex. Path P is an $\mathbf{Y}$-*topped* path if its end vertex belongs to $\mathbf{Y}$.
Let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote two subsets of $\mathcal{V}$. The cardinality of $\mathcal{V}_1$ is noted $\text{card}(\mathcal{V}_1)$.
• $\theta(\mathcal{V}_1, \mathcal{V}_2)$ is the maximal number of v-disjoint edges from $\mathcal{V}_1$ to $\mathcal{V}_2$.
• A simple path P is said a $\mathcal{V}_1$-$\mathcal{V}_2$ path if its begin vertex belongs to $\mathcal{V}_1$ and its end vertex belongs to $\mathcal{V}_2$. If the only vertices of P belonging to $\mathcal{V}_1 \cup \mathcal{V}_2$ are its begin and its end vertices, P is a direct $\mathcal{V}_1$-$\mathcal{V}_2$ path.
• We note by $\rho(\mathcal{V}_1, \mathcal{V}_2)$ the maximal number of disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths. A set of $\rho(\mathcal{V}_1, \mathcal{V}_2)$ disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths is called maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linking.
• The vertices which are covered by all the maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings are called the essential vertices for the $\mathcal{V}_1$-$\mathcal{V}_2$ linkings. These vertices constitute a specific subset denoted $V_{ess}(\mathcal{V}_1, \mathcal{V}_2)$ and which is defined as $V_{ess}(\mathcal{V}_1, \mathcal{V}_2) \stackrel{def}{=} \{\mathbf{v} \in \mathcal{V}\ |\mathbf{v} \text{ is covered by any maximum } \mathcal{V}_1\text{-}\mathcal{V}_2 \text{ linking}\}$.
• A subset $\mathbf{S} \subseteq \mathcal{V}$ is a separator between sets $\mathcal{V}_1$ and $\mathcal{V}_2$, if every path from $\mathcal{V}_1$ to $\mathcal{V}_2$ contains at least one vertex in $\mathbf{S}$. We call minimum separators between $\mathcal{V}_1$ and $\mathcal{V}_2$ any separators having the smallest size, which is equal to $\rho(\mathcal{V}_1, \mathcal{V}_2)$. The minimum input separator between $\mathcal{V}_1$ and $\mathcal{V}_2$ is denoted $\mathbf{S^i}(\mathcal{V}_1, \mathcal{V}_2)$ and is defined as the set of the end vertices of all direct $\mathcal{V}_1 - V_{ess}(\mathcal{V}_1, \mathcal{V}_2)$ paths.
In Example 2, $\rho(\mathbf{U}, \mathbf{Y}) = 2$ and $\theta(\mathbf{U}, \mathbf{Y}) = 1$ since there exists only one edge linking $\mathbf{U}$ to $\mathbf{Y}$ and which is $(\mathbf{u_1}, \mathbf{y_2})$. Furthermore, as the edges $(\mathbf{u_1}, \mathbf{x_3})$, $(\mathbf{x_3}, \mathbf{x_2})$, $(\mathbf{x_2}, \mathbf{y_2})$, $(\mathbf{u_2}, \mathbf{x_5})$, $(\mathbf{x_5}, \mathbf{x_4})$, $(\mathbf{x_4}, \mathbf{x_1})$ and $(\mathbf{x_1}, \mathbf{y_1})$ are v-disjoint, we have that $\theta(\mathbf{U} \cup \mathbf{X}, \mathbf{X} \cup \mathbf{Y}) = 7$. Note also that there exist two maximum linkings: $\{\mathbf{u_2} \to \mathbf{x_5} \to \mathbf{x_4} \to \mathbf{x_1} \to \mathbf{y_1}; \mathbf{u_1} \to \mathbf{y_2}\}$, $\{\mathbf{u_2} \to \mathbf{x_5} \to \mathbf{x_4} \to \mathbf{x_1} \to \mathbf{y_1}; \mathbf{u_1} \to \mathbf{x_3} \to \mathbf{x_2} \to \mathbf{y_2}\}$. The essential vertices, which are present in both the two maximum linkings, are collected in $V_{ess}(\mathbf{U}, \mathbf{Y})$ and so $V_{ess}(\mathbf{U}, \mathbf{Y}) = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{x_1}, \mathbf{x_4}, \mathbf{x_5}, \mathbf{y_1}, \mathbf{y_2}\}$. Moreover, as the input vertices are essential, we have $\mathbf{S^i}(\mathbf{U}, \mathbf{Y}) = \{\mathbf{u_1}, \mathbf{u_2}\}$.
• Finally, we define a vertex subset denoted $\mathbf{\Delta_0}$ such that

$$\mathbf{\Delta_0} \stackrel{def}{=} \Big\{\mathbf{x_i} \,|\, \rho\big(\mathbf{U} \cup \{\mathbf{x_i}\}, \mathbf{Y}\big) = \rho\big(\mathbf{U}, \mathbf{Y}\big)\Big\}$$

For a simple interpretation of $\mathbf{\Delta_0}$, note that it merges all the state vertices which are not connected to $\mathbf{Y}$, the state vertices belonging to $V_{ess}(\mathbf{U}, \mathbf{Y})$ and the state vertices from which all $\mathbf{Y}$-*topped* paths cover at least an element of $V_{ess}(\mathbf{U}, \mathbf{Y})$.
In Example 2, $\mathbf{\Delta_0} = \mathbf{X}$ because, on the one hand $\mathbf{x_1}$, $\mathbf{x_4}$, $\mathbf{x_5}$ are essential and on the other hand, all the $\mathbf{Y}$-*topped* paths from $\mathbf{x_2}$ or $\mathbf{x_3}$ arrive to $\mathbf{y_1}$ or $\mathbf{y_2}$ which are both essential. Hereafter, we illustrate the latter definitions on the following more consistent example:

Example 3 *Consider the structured linear system represented by the digraph depicted in Figure 2. There exist several max-*

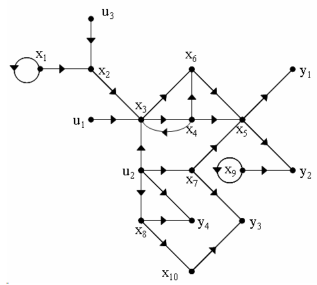

Figure 2. Digraph representing system of Example 3

imum $\mathbf{U}$-$\mathbf{Y}$ *linkings, which are of size 2 (*$\rho(\mathbf{U}, \mathbf{Y}) = 2$*):* $\{\mathbf{u_1} \to \mathbf{x_3} \to \mathbf{x_4} \to \mathbf{x_5} \to \mathbf{y_1}; \mathbf{u_2} \to \mathbf{y_4}\}$, $\{\mathbf{u_1} \to \mathbf{x_3} \to \mathbf{x_4} \to \mathbf{x_6} \to \mathbf{x_5} \to \mathbf{y_1}; \mathbf{u_2} \to \mathbf{x_8} \to \mathbf{y_4}\}$, $\{\mathbf{u_3} \to \mathbf{x_2} \to \mathbf{x_3} \to \mathbf{x_4} \to \mathbf{x_6} \to \mathbf{x_5} \to \mathbf{y_1}; \mathbf{u_2} \to \mathbf{x_7} \to \mathbf{y_3}\}, \ldots$*. Moreover,* $V_{ess}(\mathbf{U}, \mathbf{Y}) = \{\mathbf{x_3}, \mathbf{x_5}, \mathbf{u_2}\}$*. The essential vertices which are nearest the input ones constitute* $\mathbf{S^i}(\mathbf{U}, \mathbf{Y}) = \{\mathbf{x_3}, \mathbf{u_2}\}$*. Finally, all the* $\mathbf{Y}$*-rooted paths* *starting from* $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_4}$ *and* $\mathbf{x_6}$ *cover the essential vertex* $\mathbf{x_5}$. *Then,* $\mathbf{\Delta_0} = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}, \mathbf{x_5}, \mathbf{x_6}\}$.

4 Main results

4.1 Recalls on the graphical conditions for the state and input observability

Before dealing with the sensor placement problem, we recall hereafter the graphic conditions established in (Boukhobza *et al.*, 2007), which ensure the generic state and input observability of structured linear systems:

Proposition 4 *Structured linear system* Σ_Λ *is generically state and input observable iff in its associated digraph* $\mathcal{G}(\Sigma_\Lambda)$
Cond1. $\theta\big(\mathbf{X}\cup\mathbf{U},\mathbf{X}\cup\mathbf{Y}\big) = n+q$ *(maximal matching condition).*
Cond2. *every state vertex is the begin vertex of an* $\mathbf{Y}$*-topped path (output connectivity condition);*
Cond3. $\mathbf{\Delta_0} \subseteq V_{ess}(\mathbf{U},\mathbf{Y})$ *(length condition).*

These state and input observability conditions are greatly based on the more general study on the generic number of invariant zeros initiated in (van der Woude, 2000).
The conditions **Cond1.** and **Cond2.** of Proposition 4 are the two well-known state observability conditions for linear systems without unknown input recalled in (Reinschke, 1988; Dion *et al.*, 2003). The third condition of Proposition 4 is due to the fact that we consider systems with unknown input.

4.2 Global strategy for the additional sensor placement

The aim of this paper is to study additional sensor placement when the conditions of Proposition 4 are not satisfied. To do so, our proposed procedure consists of three steps which correspond one by one to the conditions of Proposition 4. In the sequel, we define a new output vector z representing the additional sensors collecting the new measurements $z(t) = H_x^\lambda x(t) + H_u^\lambda u(t)$. Hence, we denote the completed system by Σ_Λ^c:

$$\Sigma_\Lambda^c : \begin{cases} \dot{x}(t) = A^\lambda x(t) + B^\lambda u(t) \\ y(t) = C^\lambda x(t) + D^\lambda u(t) \\ z(t) = H_x^\lambda x(t) + H_u^\lambda u(t) \end{cases}$$

The additional sensor components can be represented by vertex set $\mathbf{Z}$ and edge subsets H_x-edges and H_u-edges from respectively $\mathbf{X}$ to $\mathbf{Z}$ and $\mathbf{U}$ to $\mathbf{Z}$. Since we propose three stages in our sensor implementation procedure, we subdivide vertex subset $\mathbf{Z}$ into three subsets denoted $\mathbf{Z_1}$, $\mathbf{Z_2}$ and $\mathbf{Z_3}$. Each subset $\mathbf{Z_i}$, $i = 1, 2, 3$ corresponds to the sensors possibly added at the i^{th} stage of our procedure.

4.3 Additional sensors for the maximal matching condition

The maximal matching condition is also equivalent to

$$\forall \mathbf{V_1} \subseteq \mathbf{X}\cup\mathbf{U},\ \theta\big(\mathbf{V_1},\mathbf{X}\cup\mathbf{Y}\big) = \text{card}\big(\mathbf{V_1}\big) \qquad (1)$$

This condition means that there are enough independent observation equations to determine any subset of state and input components. We say that there is a dilation in the digraph of the system when the latter condition is not satisfied for some vertex subset $\mathbf{V_1} \subseteq \mathbf{X}\cup\mathbf{U}$. The aim of additional sensors at this first stage is to eliminate all the dilations.

As in (Commault *et al.*, 2008; Staroswiecki, 2006), we also use the Dulmage-Mendelsohn decomposition (Dulmage and Mendelsohn, 1958; Murota, 1987), which is a performant tool to deal with general matching conditions. Hence, we define a bipartite graph in order to localize the dilations occurring in the digraph of the system. This bipartite graph is noted $B(\Sigma_\Lambda) = (\mathbf{V}^+,\mathbf{V}^-,W)$, where $\mathbf{V}^+$ and $\mathbf{V}^-$ are two disjoint vertex subsets and W is the edge set. More precisely, $\mathbf{V}^+ = \mathbf{X}^+\cup\mathbf{U}^+$ and $\mathbf{V}^- = \mathbf{Y}^-\cup\mathbf{X}^-$, with $\mathbf{X}^+ = \{\mathbf{x_1^+}, \mathbf{x_2^+}, \ldots, \mathbf{x_n^+}\}$, $\mathbf{U_1^+} = \{\mathbf{u_1^+}, \mathbf{u_2^+}, \ldots, \mathbf{u_q^+}\}$, $\mathbf{X}^- = \{\mathbf{x_1^-}, \mathbf{x_2^-}, \ldots, \mathbf{x_n^-}\}$, $\mathbf{Y_1^-} = \{\mathbf{y_1^-}, \mathbf{y_2^-}, \ldots, \mathbf{y_p^-}\}$. Edge set W is defined such that $(\mathbf{v_i^+},\mathbf{v_j^-}) \in W$ iff there exists an edge $(\mathbf{v_i},\mathbf{v_j})$ in the associated digraph $\mathcal{G}(\Sigma_\Lambda)$.
For the system of Example 2, the associated bipartite graph is given in Figure 3. In this bipartite graph, the edge $(\mathbf{u_1^+},\mathbf{x_3^-})$,

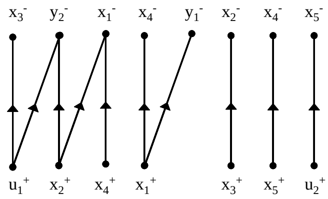

Figure 3. Bipartite graph representing the system of Example 2

for example, corresponds to the edge $(\mathbf{u_1},\mathbf{x_3})$ in the digraph associated to the system depicted in Figure 1.
A matching in a bipartite graph $B(\Sigma_\Lambda) = (\mathbf{V}^+,\mathbf{V}^-,W)$ is an edge set $M \subseteq W$ such that all the edges of M are disjoint. A matching is maximal if it has a maximal cardinality which is equal to $\theta\big(\mathbf{V}^+,\mathbf{V}^-\big)$. Yet, by construction of the digraph, we have $\theta\big(\mathbf{V}^+,\mathbf{V}^-\big) = \theta\big(\mathbf{X}\cup\mathbf{U},\mathbf{X}\cup\mathbf{Y}\big)$. Then, the fact that condition **Cond1.** of Theorem 4 is not satisfied *i.e.* $\theta\big(\mathbf{X}\cup\mathbf{U},\mathbf{X}\cup\mathbf{Y}\big) < \text{card}(\mathbf{X})+\text{card}(\mathbf{U}) = n+q = \text{card}(\mathbf{V}^+)$ or in other words $\theta\big(\mathbf{V}^+,\mathbf{V}^-\big) < \text{card}(\mathbf{V}^+)$, implies that some additional sensor vertices are needed to complete $\mathbf{V}^-$. This number is at least equal to $\text{card}(\mathbf{V}^+) - \theta\big(\mathbf{V}^+,\mathbf{V}^-\big)$. Indeed, since one sensor can augment the maximal matching at most with one unity, with less additional vertices in $\mathbf{V}^-$, it is impossible to complete the maximal matching in order to satisfy $\theta\big(\mathbf{V}^+,\mathbf{V}^-\big) = n+q = \text{card}(\mathbf{V}^+)$. In addition to the number of required sensors, another problem is to precise as finely as possible their location because obviously adding anywhere $\text{card}(\mathbf{V}^+) - \theta\big(\mathbf{V}^+,\mathbf{V}^-\big)$ sensors does not allow to recover the maximal matching condition. We use a part of the Dulmage-Mendelsohn decomposition precisely to solve this problem of sensor location. First, let us associate to each maximal matching M, a non bipartite digraph noted $B_M(\Sigma_\Lambda) = (\mathbf{V}^+,\mathbf{V}^-,\bar{W})$ where $(\mathbf{v_1},\mathbf{v_2}) \in \bar{W} \Leftrightarrow (\mathbf{v_1},\mathbf{v_2}) \in W$ or $(\mathbf{v_2},\mathbf{v_1}) \in M$.
We denote by $\partial^+\mathbf{M}$ (resp. $\partial^-\mathbf{M}$) the set of vertices in $\mathbf{V}^+$ (resp. in $\mathbf{V}^-$) covered by the edges of M. we note $\mathbf{S_0^+} = \mathbf{V}^+ \setminus \partial^+\mathbf{M}$. Then, we use the following algorithm:
↬ Find a maximal matching M in $B(\Sigma_\Lambda)$,
↬ $\mathbf{V_0^+} = \mathbf{S_0^+} \cup \{\mathbf{v} \in \mathbf{V}^+,\ \exists$ a path in $B_M(\Sigma_\Lambda)$ from $\mathbf{S_0^+}$ to $\mathbf{v}\}$
↬ $\mathbf{V_0^-} = \{\mathbf{v} \in \mathbf{V}^-,\ \exists$ a path in $B_M(\Sigma_\Lambda)$ from $\mathbf{S_0^+}$ to $\mathbf{v}\}$.
It is important to note that the obtained subsets $\mathbf{V_0^+}$ and $\mathbf{V_0^-}$ are the same whatever the choice of the maximal matching M (Dulmage and Mendelsohn, 1958; Murota, 1987). Using the previous algorithm, we have:

Proposition 5 *Consider structured linear system* Σ_Λ *represented by digraph* $\mathcal{G}(\Sigma_\Lambda)$ *and by the bipartite graph* $B(\Sigma_\Lambda)$. *To recover the maximal matching condition, the minimal num-*

ber of additional sensors, noted $\mathbf{Z_1}$ *in the digraph* $\mathcal{G}(\Sigma^c_\Lambda)$ *or* $\mathbf{Z_1^-}$ *in* $B(\Sigma^c_\Lambda)$*, is equal to* $\gamma = n + q - \theta\big(\mathbf{V}^+, \mathbf{V}^-\big) = card\big(\mathbf{V_0^+}\big) - card\big(\mathbf{V_0^-}\big)$*. These additional sensors must measure* γ *states and unknown inputs in* $\mathbf{V_0^+}$ *such that we obtain a maximal matching of size* $n+q$ *in* $B(\Sigma^c_\Lambda)$ i.e. $\theta\big(\mathbf{X}^+ \cup \mathbf{U}^+, \mathbf{X}^- \cup \mathbf{Y}^- \cup \mathbf{Z_1^-}\big) = n + q$.

Proof: the proof is similar to the one given in (Commault *et al.*, 2008).
Sufficiency: First note that the Dulmage-Mendelsohn decomposition characterizes all the maximal matchings (Murota, 1987). It follows that, if card$\big(\mathbf{V_0^+}\big)$ − card$\big(\mathbf{V_0^-}\big)$ edges are added between $\mathbf{S_0^+}$ and the additional sensors, in order to form a matching of size card$\big(\mathbf{V_0^+}\big)$ − card$\big(\mathbf{V_0^-}\big)$, then we obtain a maximal matching of size $n+q$ between $\mathbf{V_0^+}$ and $\mathbf{V_0^-} \cup \mathbf{Z_1^-}$ in $B(\Sigma^c_\Lambda)$. Thus adding γ sensors is sufficient.
Necessity: Let us consider a solution which provides a maximal matching M^c of size $n+q$ in $B(\Sigma^c_\Lambda)$. M^c covers all the state and input vertices of $\mathbf{V}^+$ and in particular the elements of $\mathbf{V_0^+}$. Thus, there are at least card$(\mathbf{V_0^+})$ − card$(\mathbf{V_0^-})$ edges from $\mathbf{V_0^+}$ to the additional sensors. △
To illustrate the previous settings, let us consider the system of Example 3. The bipartite graph used to the sensor placement is given in Figure 4.

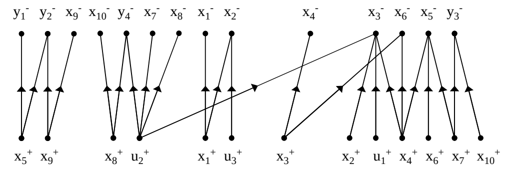

Figure 4. Bipartite graph representing the system of Example 3

We can choose as maximal matching
$M = \Big\{(\mathbf{x_5^+}, \mathbf{y_1^-}), (\mathbf{x_9^+}, \mathbf{y_2^-}), (\mathbf{x_8^+}, \mathbf{y_4^-}), (\mathbf{u_2^+}, \mathbf{x_7^-}), (\mathbf{x_1^+}, \mathbf{x_1^-}), (\mathbf{u_3^+}, \mathbf{x_2^-}), (\mathbf{x_3^+}, \mathbf{x_4^-}), (\mathbf{x_2^+}, \mathbf{x_3^-}), (\mathbf{x_4^+}, \mathbf{x_6^-}), (\mathbf{x_6^+}, \mathbf{x_5^-}), (\mathbf{x_7^+}, \mathbf{y_3^-})\Big\}$. Thus, $\mathbf{S_0^+}$, which consists of the subset of $\mathbf{V}^+$ which are not covered by M, is $\{\mathbf{u_1}^+, \mathbf{x_{10}^+}\}$. According to the definition of $B_M(\Sigma_\Lambda)$, we construct it by adding to the bipartite graph of Figure 4 the reversed edges of M *i.e.* $\Big\{(\mathbf{y_1^-}, \mathbf{x_5^+}), (\mathbf{y_2^-}, \mathbf{x_9^+}), (\mathbf{y_4^-}, \mathbf{x_8^+}), (\mathbf{x_7^-}, \mathbf{u_2^+}), (\mathbf{x_1^-}, \mathbf{x_1^+}), (\mathbf{x_2^-}, \mathbf{u_3^+}), (\mathbf{x_4^-}, \mathbf{x_3^+}), (\mathbf{x_3^-}, \mathbf{x_2^+}), (\mathbf{x_6^-}, \mathbf{x_4^+}), (\mathbf{x_5^-}, \mathbf{x_6^+}), (\mathbf{y_3^-}, \mathbf{x_7^+})\Big\}$. In the resulted graph, there exist paths from the elements of $\mathbf{S_0^+}$ to $\mathbf{x_2}^+$, $\mathbf{x_6}^+$ and $\mathbf{x_7}^+$ ($\mathbf{u_1^+} \rightarrow \mathbf{x_3^-} \rightarrow \mathbf{x_2^+}$, $\mathbf{x_{10}^+} \rightarrow \mathbf{y_3^-} \rightarrow \mathbf{x_7^+} \rightarrow \mathbf{x_5^-} \rightarrow \mathbf{x_6^+}$, and $\mathbf{x_{10}^+} \rightarrow \mathbf{y_3^-} \rightarrow \mathbf{x_7^+}$). Thus, we have that $\mathbf{V_0^+} = \{\mathbf{u_1}^+, \mathbf{x_2}^+, \mathbf{x_6}^+, \mathbf{x_7}^+, \mathbf{x_{10}^+}\}$ and $\mathbf{V_0^-} = \{\mathbf{x_3^-}, \mathbf{x_5^-}, \mathbf{y_3^-}\}$. Hence, to recover the maximal matching condition, it is necessary and sufficient to add two sensors which measure linear combinations of the state and input components associated with the vertices $\{\mathbf{u_1}, \mathbf{x_2}, \mathbf{x_6}, \mathbf{x_7}, \mathbf{x_{10}}\}$. However, all the possible linear combinations of these components do not necessarily allow to recover the maximal matching condition. Indeed, if we add a sensor which measures $\mathbf{u_1}$ and another which measures $\mathbf{x_2}$, we do not recover the maximal matching condition since we obtain $\mathbf{V_0^+} = \{\mathbf{x_6}^+, \mathbf{x_7}^+, \mathbf{x_{10}^+}\}$ and $\mathbf{V_0^-} = \{\mathbf{x_5^-}, \mathbf{y_3^-}\}$ and so $\theta\big(\mathbf{X} \cup \mathbf{U}, \mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z_1}\big) = 12 < n + q = 13$. Thus, it can be useful to specify more precisely the additional sensor locations useful to recover the maximal matching condition. At this aim, let us define by $\bar{B}(\Sigma_\Lambda)$ the non-directed graph corresponding to $B(\Sigma_\Lambda)$. We we define $\mathbf{V_0} \overset{def}{=} \mathbf{V_0^+} \cup \mathbf{V_0^-}$ and we call $\mathbf{V_0}$-path every simple path of $\bar{B}(\Sigma_\Lambda)$ which covers only vertices of $\mathbf{V_0}$. For each $\mathbf{V_0}$-path P, we define a vertex subset $\vartheta(P)$ such that $\vartheta(P) = \{\mathbf{v} \in V_0^+, \text{ such that } P \text{ covers } \mathbf{v}\}$. Finally, we say that a $\mathbf{V_0}$-path P is maximal, if there does not exist a $\mathbf{V_0}$-path P' such that $\vartheta(P) \subset \vartheta(P')$. To recover the maximal matching condition, it is necessary to have:

$$\text{for each maximal } \mathbf{V_0}\text{-path } P,\ \theta\big(\vartheta(P), \mathbf{Z_1^-}\big) \neq 0 \qquad (2)$$

In fact, after adding sensors $\mathbf{Z_1^-}$, if constraint (2) is not satisfied, then we can construct in $\bar{B}(\Sigma^c_\Lambda)$ a maximal $\mathbf{V_0}$-path which does not cover a new sensor vertex. Since P is a maximal $\mathbf{V_0}$-path then in $\bar{B}(\Sigma_\Lambda)$, $\theta\big(\vartheta(P), \mathbf{V}^-\big) = \theta\big(\vartheta(P), \mathbf{V_0^-}\big) = \text{card}\big(\vartheta(P)\big) - 1$. Yet, if $\theta\big(\vartheta(P), \mathbf{Z_1^-}\big) = 0$. Since in $B(\Sigma^c_\Lambda)$, $\mathbf{V}^-$ becomes $\mathbf{X}^- \cup \mathbf{Y}^- \cup \mathbf{Z_1^-}$, the quantity $\theta\big(\vartheta(P), \mathbf{V}^-\big)$ remains unchanged and relation (1) is no satisfied for $\mathbf{V_1} = \vartheta(P)$. It is possible to prove that, to recover the maximal matching condition, it is necessary to have, for each maximal $\mathbf{V_0}$-path P:

$$\theta\big(\mathbf{V_0^+}, \mathbf{Z_1^-} \cup \mathbf{V_0^-}\big) - \theta\big(\mathbf{V_0^+} \setminus \vartheta(P), \mathbf{Z_1^-} \cup \mathbf{V_0^-}\big) > 0 \qquad (3)$$

Conditions (2) and (3) ensure that there is at least one sensor is dedicated to take a measurement in each subset $\vartheta(P)$. Indeed, the latter subset satisfy, by construction $\theta\big(\vartheta(P), \mathbf{V_0^-}\big) = \text{card}\big(\vartheta(P)\big) - 1$ and so necessitates the addition of one sensor to recover the maximal matching condition. These conditions can be seen as complementary tools which allow to have effortlessly a better precision on the sensor location, since we obtain a kind of repartition of the required γ sensors.
For Example 3, we show previously that two sensors which measure a linear combination of the components associated with the vertices $\{\mathbf{u_1}, \mathbf{x_2}, \mathbf{x_6}, \mathbf{x_7}, \mathbf{x_{10}}\}$ are needed to recover the maximal matching condition. We can exhibit two maximal $\mathbf{V_0}$-paths: $\mathbf{u_1^+} \rightarrow \mathbf{x_3^-} \rightarrow \mathbf{x_2^+}$ and $\mathbf{x_{10}^+} \rightarrow \mathbf{y_3^-} \rightarrow \mathbf{x_7^+} \rightarrow \mathbf{x_5^-} \rightarrow \mathbf{x_6^+}$. According to relations (2) and (3), we can deduce that the additional sensors, represented by vertex subset $\mathbf{Z_1}$, must satisfy in $\mathcal{G}(\Sigma^c_\Lambda)$ the following relations:$\theta(\{\mathbf{u_1}, \mathbf{x_2}\}, \mathbf{Z_1}) \neq 0$, $\theta(\{\mathbf{x_6}, \mathbf{x_7}, \mathbf{x_{10}}\}, \mathbf{Z_1}) \neq 0$, $\theta(\{\mathbf{u_1}, \mathbf{x_2}, \mathbf{x_6}, \mathbf{x_7}, \mathbf{x_{10}}\}, \mathbf{Z_1} \cup \{\mathbf{x_3}, \mathbf{x_5}, \mathbf{y_3}\}) - \theta(\{\mathbf{x_6}, \mathbf{x_7}, \mathbf{x_{10}}\}, \mathbf{Z_1} \cup \{\mathbf{x_3}, \mathbf{x_5}, \mathbf{y_3}\}) > 0$, $\theta(\{\mathbf{u_1}, \mathbf{x_2}, \mathbf{x_6}, \mathbf{x_7}, \mathbf{x_{10}}\}, \mathbf{Z_1} \cup \{\mathbf{x_3}, \mathbf{x_5}, \mathbf{y_3}\}) - \theta(\{\mathbf{u_1}, \mathbf{x_2}\}, \mathbf{Z_1} \cup \{\mathbf{x_3}, \mathbf{x_5}, \mathbf{y_3}\}) > 0$ and $\theta(\{\mathbf{u_1}, \mathbf{x_2}, \mathbf{x_6}, \mathbf{x_7}, \mathbf{x_{10}}\}, \mathbf{Z_1}) = 2$.
Thus, we can deduce that a solution (not the unique one) could be to have two sensors $z_{1,1} = \alpha_1 x_2 + \alpha_2 u_1$ and $z_{1,2} = \beta_1 x_6 + \beta_2 x_7 + \beta_3 x_{10}$, where either α_1 or α_2 is nonzero as well as β_1, β_2 or β_3.

4.4 Additional sensors for the output connectivity condition

The problem of additional sensors for recovering the output connectivity condition has been treated and solved in (Commault *et al.*, 2008) for linear systems without unknown input. In the case where the system is submitted to unknown inputs, there are no significant differences. Hence, this subsection may be rather short. We provide hereafter some definitions and then we enounce quite immediately the conditions required on the additional sensors to recover the output connectivity condition.
Two vertices $\mathbf{v_i}$ and $\mathbf{v_j}$ are said to be strongly connected if it exists a path from $\mathbf{v_i}$ to $\mathbf{v_j}$ and a path from $\mathbf{v_j}$ to $\mathbf{v_i}$. It is assumed that a vertex is strongly connected to itself. The relation "*is strongly connected to*" is an equivalence relation and we can define its equivalence classes. We call each equivalent class a strongly connected component. These strongly components are well known in the graph theory (Murota, 1987). They can

be ordered using a partial order relation "$\preccurlyeq$" defined between two strongly connected components $\mathbf{C_i}$ and $\mathbf{C_j}$ as $\mathbf{C_i} \preccurlyeq \mathbf{C_j}$ if there exists a path from an element of $\mathbf{C_j}$ to an element of $\mathbf{C_i}$. The minimal elements with this partial order relation are the strongly connected components with no outgoing edges. We call the minimal unconnected components, the minimal strongly connected components which are not output vertices. We denote by $\bar{d}$ the number of such elements. The following Proposition is established in (Commault *et al.*, 2008) and is also true for our kind of systems:

Proposition 6 *Consider structured linear system Σ_Λ represented by digraph $\mathcal{G}(\Sigma_\Lambda)$. To recover the output connectivity condition, the additional sensors must measure at least one state in each strongly connected component constituting a minimal unconnected element.*

To illustrate the latter definitions, let us consider the system represented by the digraph depicted in Figure 5.

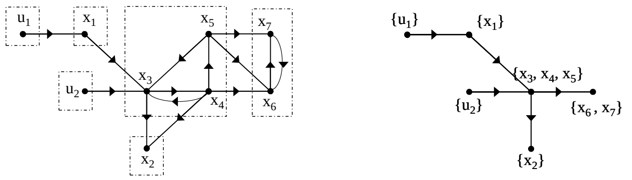

Figure 5. Example of digraph representation and its decomposition into strongly connected components

The strongly connected components are $\{\mathbf{u_1}\}$, $\{\mathbf{u_2}\}$, $\{\mathbf{x_1}\}$, $\{\mathbf{x_2}\}$, $\{\mathbf{x_3},\, \mathbf{x_4},\, \mathbf{x_5}\}$ and $\{\mathbf{x_6},\, \mathbf{x_7}\}$. We have that $\{\mathbf{x_6},\, \mathbf{x_7}\} \preccurlyeq \{\mathbf{x_3},\, \mathbf{x_4},\, \mathbf{x_5}\}$, $\{\mathbf{x_2}\} \preccurlyeq \{\mathbf{x_3},\, \mathbf{x_4},\, \mathbf{x_5}\}$, $\{\mathbf{x_3},\, \mathbf{x_4},\, \mathbf{x_5}\} \preccurlyeq \{\mathbf{x_1}\} \preccurlyeq \{\mathbf{u_1}\}$ and $\{\mathbf{x_3},\, \mathbf{x_4},\, \mathbf{x_5}\} \preccurlyeq \{\mathbf{u_2}\}$. The strong components $\{\mathbf{x_2}\}$ and $\{\mathbf{x_6},\, \mathbf{x_7}\}$ are then the two minimal elements relatively to the partial order relation "$\preccurlyeq$". Therefore, an additional sensor to recover the output connectivity condition has to measure x_2 and at least one of components x_6 or x_7 ($z_{2,1} = \alpha_1 x_2 + \alpha_2 x_6$ for example).

Proposition 6 indicates the location of the additional sensors but not their minimal number. Indeed, it is possible that only one sensor, which takes its measurement in each minimal unconnected component allows to satisfy the requirements of the output connectivity condition. In particular, we can use the sensors required to recover the maximal matching condition at this aim. Indeed, in addition to the constraints of Proposition 5, we can impose to the additional sensors represented by vertex subset $\mathbf{Z_1}$ to satisfy $\theta(\mathbf{C_i}, \mathbf{Z_1}) \neq 0$ for each minimal unconnected element $\mathbf{C_i}$. This does not increase the minimal number of required sensors which remains equal to γ and allows to satisfy condition of Proposition 6. Thus, the minimal number of additional sensors, required to recover both the maximal matching and the output connectivity conditions, is equal to $max(\gamma, 1)$, where $\gamma = n + q - \theta(\mathbf{X} \cup \mathbf{U}, \mathbf{Y} \cup \mathbf{X})$.

4.5 Additional sensors for the length condition

Assume that the two conditions **Cond1.** and **Cond2.** of Proposition 4 are verified by the system. If it is not the case, then we must add first some sensors as it is described in the two previous subsections. For the sake of clarity in the notations, since we may add some sensors to the system in order to satisfy **Cond1.** and **Cond2.**, we denote by $\tilde{\mathbf{Y}}$ the output vertex subset at the beginning of this third and last stage. To be more accurate, we redefine subset $\mathbf{\Delta_0}$ by substituting $\mathbf{Y}$ by $\tilde{\mathbf{Y}}$: $\mathbf{\Delta_0} \stackrel{def}{=} \Big\{\mathbf{x_i} \,|\, \rho\big(\mathbf{U} \cup \{\mathbf{x_i}\}, \tilde{\mathbf{Y}}\big) = \rho\big(\mathbf{U}, \tilde{\mathbf{Y}}\big)\Big\}$.

If Condition **Cond1.** is satisfied then all the state and input vertices (Dion *et al.*, 2003; Reinschke, 1988) can be covered by some disjoint simple $\tilde{\mathbf{Y}}$-topped paths and cycles. Since the input vertices have no incoming edges, they cannot be covered by cycles. Thus, if Condition **Cond1.** is satisfied, we have $\rho\big(\mathbf{U}, \tilde{\mathbf{Y}}\big) = \text{card}(\mathbf{U})$. We can deduce then, that whatever sensors z_3 represented by vertex subset $\mathbf{Z_3}$ we add to the system, we have that $V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}} \cup \mathbf{Z_3}) \subseteq V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}})$. Indeed, when we add a sensor, we do not increase the number of $\mathbf{U}$-$\tilde{\mathbf{Y}} \cup \mathbf{Z_3}$ disjoint paths, we can just add some new input-output paths and so some new maximum $\mathbf{U}$-$\tilde{\mathbf{Y}} \cup \mathbf{Z_3}$ linkings. Consequently, a state vertex which is not essential in the $\mathbf{U}$-$\tilde{\mathbf{Y}}$ linkings *i.e.* which is not covered by all the maximum $\mathbf{U}$-$\tilde{\mathbf{Y}}$ linkings, cannot become essential when we add a sensor. In other words, if an element $\mathbf{x_i}$ is in $\mathbf{\Delta_0}$ but not in $V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}})$, then adding sensors z_3 anywhere cannot make that $\mathbf{x_i} \in V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}} \cup \mathbf{Z_3})$. Consequently, the only way to ensure condition **Cond3.** is to remove from $\mathbf{\Delta_0}$ all the elements which do not belong to $V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}})$.

According to this fact, to recover condition **Cond3.**, it is necessary and sufficient to add some sensors z_3, such that for each $\mathbf{x_i} \in \mathbf{\Delta_0} \setminus V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}})$, we have $\rho\big(\mathbf{U} \cup \{\mathbf{x_i}\}, \tilde{\mathbf{Y}} \cup \mathbf{Z_3}\big) > \rho\big(\mathbf{U}, \tilde{\mathbf{Y}} \cup \mathbf{Z_3}\big)$. Moreover, as condition **Cond2.** is assumed to be satisfied at this stage, all the state components are connected to some output component. Considering virtually $\mathbf{x_i}$ as an input vertex, a necessary and sufficient condition to guarantee inequality $\rho\big(\mathbf{U} \cup \{\mathbf{x_i}\}, \tilde{\mathbf{Y}} \cup \mathbf{Z_3}\big) > \rho\big(\mathbf{U}, \tilde{\mathbf{Y}} \cup \mathbf{Z_3}\big)$ is that the added sensors z_3 must measure any vertex covered by any direct $\mathbf{U} \cup \{\mathbf{x_i}\}$-$\mathbf{S^i}(\mathbf{U} \cup \{\mathbf{x_i}\}, \tilde{\mathbf{Y}})$ path which has a nonzero length (Commault and Dion, 2007). But it is not sufficient to consider only measurement on such vertices because there can exist some edges arriving to $\mathbf{x_i}$. According to the definition of the input separator subset, all the state vertices, which are covered by any direct $\mathbf{U} \cup \{\mathbf{x_i}\}$-$\mathbf{S^i}(\mathbf{U} \cup \{\mathbf{x_i}\}, \tilde{\mathbf{Y}})$ path, are not essential and are obviously in $\mathbf{\Delta_0} \setminus V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}})$. On the other hand, if we add a new sensor to extract a component $\mathbf{x_i}$ from $\mathbf{\Delta_0}$, then all the components $\mathbf{x_k}$ which belong to a strictly inferior strongly component are also extracted from $\mathbf{\Delta_0}$. Thus, it is necessary and sufficient to consider only the state vertices of $\mathbf{\Delta_0} \setminus (V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}})$ which belong to maximal strongly connected components.

Let us denote $\mathbf{X}_\emptyset \stackrel{def}{=} \Big\{\mathbf{x_i} \in \mathbf{\Delta_0} \setminus V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}})$ and $\mathbf{x_i}$ belongs to a maximal strongly connected component$\Big\}$. To each vertex in $\mathbf{X}_\emptyset$, we define vertex subset $\delta_\mathbf{i} = \Big\{\mathbf{v_j} \in \mathbf{\Delta_0} \cup \mathbf{U},\ \ \mathbf{v_j}$ is covered by a direct $\mathbf{U} \cup \{\mathbf{x_i}\} - \Big(\mathbf{S^i}(\mathbf{U} \cup \{\mathbf{x_i}\}, \tilde{\mathbf{Y}})\Big)$ non-zero length path$\Big\} \cup \Big(\mathbf{C_i} \cap \mathbf{\Delta_0}\Big)$, where $\mathbf{C_i}$ is the strongly connected component including $\mathbf{x_i}$. For such vertex subsets, we consider as a partial order relation the subset inclusion "$\subseteq$" and we obtain the following necessary and sufficient condition for sensor placement:

Proposition 7 *Consider structured linear system Σ_Λ represented by digraph $\mathcal{G}(\Sigma_\Lambda)$. To recover the length condition, the additional sensors must measure at least one state in each subset $\delta_\mathbf{i}$, associated to each $\mathbf{x_i} \in \mathbf{X}_\emptyset$ and constituting a minimal element w.r.t relation "$\subseteq$".*

Proof: Using the discussion above, the proof of this proposition is immediate, knowing that to recover the length condition, the additional sensors must measure at least one state in each subset $\delta_\mathbf{i}$, $\mathbf{x_i} \in \mathbf{X}_\emptyset$. Using the partial order relation, it is necessary and sufficient to measure in subsets $\delta_\mathbf{i}$ which are minimal w.r.t relation "$\subseteq$". △

Consider system of Example 3 and assume that, to satisfy the maximal matching condition **Cond1.**, we add two sensors which measure respectively x_2 and x_{10}. We obtain a system which satisfies **Cond1.** and **Cond2.**. The new digraph is given on Figure 6, where the new sensors are noted $z_{1,1}$ and $z_{1,2}$.

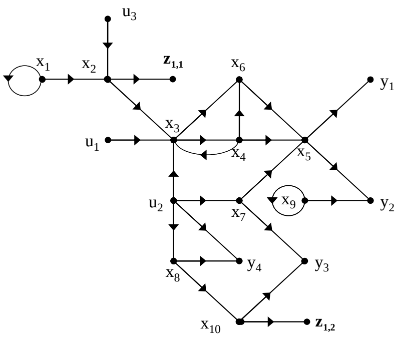

Figure 6. Digraph representing the system of Example 3 with additional sensors $\mathbf{z_{1,1}}$ and $\mathbf{z_{1,2}}$

We have that $\mathbf{\Delta_0} = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}, \mathbf{x_5}, \mathbf{x_6}\}$, $V_{ess}(\mathbf{U}, \tilde{\mathbf{Y}}) = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_5}, \mathbf{z_{1,1}}\}$. We can compute $\mathbf{X}_\emptyset = \{\mathbf{x_1}, \mathbf{x_4}, \mathbf{x_6}\}$. We compute subsets $\delta_\mathbf{1} = \{\mathbf{x_1}, \mathbf{u_3}\}$ and $\delta_\mathbf{4} = \delta_\mathbf{6} = \{\mathbf{u_1}, \mathbf{x_3}, \mathbf{x_4}, \mathbf{x_6}\}$. Therefore, an additional sensor to recover the length condition must measure (x_1 or u_3) and (u_1 or x_3 or x_4 or x_6).

Remark 8 *We establish in Proposition 5 that to satisfy the matching condition, the minimal number of required sensors represented by* $\mathbf{Z_1}$ *is equal to* $\gamma = n + q - \theta\big(\mathbf{X} \cup \mathbf{U}, \mathbf{X} \cup \mathbf{Y}\big)$. *We have also shown that, if we impose that* $\theta\big(\mathbf{C_i}, \mathbf{Z_1}\big) \neq 0$ *for each minimal unconnected element* $\mathbf{C_i}$ *and* $\theta\big(\delta_\mathbf{i}, \mathbf{Z_1}\big) \neq 0$ *for all subset* $\delta_\mathbf{i}$ *such that* $\mathbf{x_i} \in \mathbf{X}_\emptyset$*, then sensors* $\mathbf{Z_1}$ *allow to satisfy also the output connectivity and the length conditions.. Therefore, the overall minimal minimal number of additional sensors required to recover the input and state strong observability is equal to* $max(\gamma, 1)$.

5 Concluding remarks

In order to recover the generic state and input observability of structured linear systems, we provide the minimal number of the required additional sensors and either their location or necessary and sufficient conditions to be satisfied by any acceptable location. More precisely, we propose a procedure, constituted of three stages. The first stage of the proposed solution, in which we use a bipartite graph, aims to recovering the so-called maximal matching condition. The second stage concerns the output connectivity condition. Thus, our solution is similar to the one proposed in (Commault *et al.*, 2008). Finally, the last step of our procedure deals with the so-called length condition, which is specific to the systems with unknown inputs. To recover such condition, we use the notion of input separators.

From a computational point of view, our proposed approach needs few information about the system and is quite easy to check by means of well-known combinatorial techniques or simply by hand for small systems. Indeed, it uses classical programming techniques like Ford-Fulkerson algorithm to compute the input separators in a digraph and Dulmage-Mendelsohn decomposition of a bipartite graph. These algorithms are free from numerical difficulties and are classically used in structural analysis framework (Murota, 1987). More precisely, the computation of vertex subset $\mathbf{\Delta_0}$ requires $n+1$ computations of maximum linkings. Using a transformation of the problem into a Max-Flow one, the computation of these maxima linkings, which is based on the Ford-Fulkerson algorithm, requires $O(N^2\sqrt{M})$ complexity order, where M is the number of edges in the digraph and $N = n + p + q$ the number of vertices. For our digraphs, in the worst case $M = n^2 + n \cdot p + n \cdot q + q \cdot p$. The first step of our procedure requires a Dulmage-Mendelsohn decomposition which can be implemented using an algorithm with a complexity order $O(M^2) = O(n^4)$ (Lovasz and Plummer, 1986; Chen and Kanj, 2003), assuming without loss of generality that $n \geq p$ and $n \geq q$. The second step of our procedure requires the calculation of the strongly connected components which can be done using an algorithm which complexity order equals $O(Nlog(N)) = O(nlog(n))$ (Fleischer *et al.*, 2000). After finding the strongly components, we must order these components simply by comparison to find the minimal elements with a $O(n^2)$ complexity order algorithm. The third step is also based on the Ford-Fulkerson algorithm. which must be executed n times and so the complexity order for this step is $n \times O(n^3) = O(n^4)$. According to the previous settings, the proposed method can be implemented using a global algorithm with a polynomial global complexity equal to $O(n^4)$. The fact that the overall complexity order is not exponential makes the proposed method suited to deal with large scale systems.

References

Basile, G. & Marro, G. (1969). On observability of linear time-invariant systems with unknown inputs. *Journal of Optimization Theory and Applications, 3*(6), 410–415.

Boukhobza, T., Hamelin, F. & Martinez-Martinez, S. (2007). State and input observability for structured linear systems: A graph-theoretic approach. *Automatica, 43*(7), 1204–1210.

Chen, J. & Kanj, I. A. (2003). Constrained minimum vertex cover in bipartite graphs: Complexity and parameterized algorithms. *Journal of Computer and System Sciences, 67*(4), 833–847.

Commault, C. & Dion, J-M. (2007). Sensor location for diagnosis in linear Systems: a structural analysis. *IEEE Transactions on Automatic Control, 52*(2), 155–169.

Commault, C., Dion, J-M., & Trinh, D. H. (2008). Observability preservation under sensor failure. *IEEE Transactions on Automatic Control, 53*(6), 1554–1559.

Dion, J-M., Commault, C., & Van der Woude, J. (2003). Generic properties and control of linear structured systems: A survey. *Automatica, 39*(7), 1125–1144.

Dulmage, A. L. & Mendelsohn, N. S. (1958). Coverings of bipartite graphs. *Canadian Journal of Mathematics, 10* 517–534.

Fleischer, L. K., Hendrickson, B., & Pinar, A. (2000). *On Identifying Strongly Connected Components in Parallel.* Lecture Notes in Computer Science. Springer Berlin / Heidelberg.

Hautus, M.L.J. (1983). Strong detectability and observers. *Linear Algebra and its Applications, 50*, 353–360.

Lovasz, L. & Plummer, M. D. (1986). Matching Theory. *Annals of Discrete Mathematics, 29*, North-Holland, Amsterdam, Netherlands.

Murota, K. (1987). *System Analysis by Graphs and Matroids*. Springer-Verlag. New York, U.S.A.

Reinschke, K. J. (1988). *Multivariable Control. A Graph Theoretic Approach.*. Springer-Verlag. New York, U.S.A.

Staroswiecki, M. (2006). Observability and the Design of Fault Tolerant Estimation Using Structural Analysis. *Advances in Control Theory and Application*, 257–278 Springer, 2006

Trentelman, H. L., Stoorvogel, A. A. & Hautus, M. (2001). *Control Theory for Linear Systems.* Springer, London, U.K.

van de Wal, M. & de Jager,B. (2001). A review of methods for input/output selection. *Automatica, 37*(4), 487–510.

van der Woude, J. W. (2000). The generic number of invariant zeros of a structured linear system. *SIAM Journal of Control and Optimization, 38*(1), 1–21.