
ON PROPER $\mathbb{R}$-ACTIONS ON HYPERBOLIC STEIN SURFACES

by

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Abstract. — In this paper we investigate proper $\mathbb{R}$–actions on hyperbolic Stein surfaces and prove in particular the following result: Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy which admits a proper $\mathbb{R}$–action by holomorphic transformations. The quotient $D/\mathbb{Z}$ with respect to the induced proper $\mathbb{Z}$–action is a Stein manifold. A normal form for the domain D is deduced.

1. Introduction

Let X be a Stein manifold endowed with a real Lie transformation group G of holomorphic automorphisms. In this situation it is natural to ask whether there exists a G–invariant holomorphic map $\pi\colon X \to X/\!\!/G$ onto a complex space $X/\!\!/G$ such that $\mathcal{O}_{X/\!\!/G} = (\pi_*\mathcal{O}_X)^G$ and, if yes, whether this quotient $X/\!\!/G$ is again Stein. If the group G is compact, both questions have a positive answer as is shown in [**Hei91**].

For non-compact G even the existence of a complex quotient in the above sense of X by G cannot be guaranteed. In this paper we concentrate on the most basic and already non-trivial case $G = \mathbb{R}$. We suppose that G acts properly on X. Let $\Gamma = \mathbb{Z}$. Then X/Γ is a complex manifold and if, moreover, it is Stein, we can define $X/\!\!/G := (X/\Gamma)/\!\!/(G/\Gamma)$. The following was conjectured by Alan Huckleberry.

Let X be a contractible bounded domain of holomorphy in $\mathbb{C}^n$ with a proper action of $G = \mathbb{R}$. Then the complex manifold X/Γ is Stein.

In [**FI01**] this conjecture is proven for the unit ball and in [**Mie08**] for arbitrary bounded homogeneous domains in $\mathbb{C}^n$. In this paper we make a first step towards a proof in the general case by showing

Theorem. — *Let D be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$. Suppose that the group $\mathbb{R}$ acts properly by holomorphic transformations on D. Then the complex manifold $D/\mathbb{Z}$ is Stein. Moreover, $D/\mathbb{Z}$ is biholomorphically equivalent to a domain of holomorphy in $\mathbb{C}^2$.*

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As an application of this theorem we deduce a normal form for domains of holomorphy whose identity component of the automorphism group is non-compact as well as for proper $\mathbb{R}$–actions on them. Notice that we make no assumption on smoothness of their boundaries.

We first discuss the following more general situation. Let X be a hyperbolic Stein manifold with a proper $\mathbb{R}$–action. Then there is an induced local holomorphic $\mathbb{C}$–action on X which can be globalized in the sense of [**HI97**]. The following result is central for the proof of the above theorem.

Theorem. — *Let X be a hyperbolic Stein surface with a proper $\mathbb{R}$–action. Suppose that either X is taut or that it admits the Bergman metric and $H^1(X,\mathbb{R}) = 0$. Then the universal globalization X^* of the induced local $\mathbb{C}$–action is Hausdorff and $\mathbb{C}$ acts properly on X^*. Furthermore, for simply-connected X one has that $X^* \to X^*/\mathbb{C}$ is a holomorphically trivial $\mathbb{C}$–principal bundle over a simply-connected Riemann surface.*

Finally, we discuss several examples of hyperbolic Stein manifolds X with proper $\mathbb{R}$–actions such that $X/\mathbb{Z}$ is not Stein. If one does not require the existence of an $\mathbb{R}$–action, there are bounded Reinhardt domains in $\mathbb{C}^2$ with proper $\mathbb{Z}$–actions for which the quotients are not Stein.

2. Hyperbolic Stein $\mathbb{R}$–manifolds

In this section we present the general set-up.

2.1. The induced local $\mathbb{C}$–action and its globalization. — Let X be a hyperbolic Stein manifold. It is known that the group $\mathrm{Aut}(X)$ of holomorphic automorphisms of X is a real Lie group with respect to the compact-open topology which acts properly on X (see [**Kob98**]). Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a closed one parameter subgroup of $\mathrm{Aut}(D)$. Consequently, the action $\mathbb{R} \times X \to X$, $t \cdot x := \varphi_t(x)$, is proper. By restriction, we obtain also a proper $\mathbb{Z}$–action on X. Since every such action must be free, the quotient $X/\mathbb{Z}$ is a complex manifold. This complex manifold $X/\mathbb{Z}$ carries an action of $S^1 \cong \mathbb{R}/\mathbb{Z}$ which is induced by the $\mathbb{R}$–action on X.

Integrating the holomorphic vector field on X which corresponds to this $\mathbb{R}$–action we obtain a local $\mathbb{C}$–action on X in the following sense. There are an open neighborhood $\Omega \subset \mathbb{C} \times X$ of $\{0\} \times X$ and a holomorphic map $\Phi\colon \Omega \to X$, $\Phi(t,x) =: t \cdot x$, such that the following holds:

(1) For every $x \in X$ the set $\Omega(x) := \big\{t \in \mathbb{C};\ (t,x) \in \Omega\big\} \subset \mathbb{C}$ is connected;
(2) for all $x \in X$ we have $0 \cdot x = x$;
(3) we have $(t + t') \cdot x = t \cdot (t' \cdot x)$ whenever both sides are defined.

Following [**Pal57**] (compare [**HI97**] for the holomorphic setting) we say that a globalization of the local $\mathbb{C}$–action on X is an open $\mathbb{R}$–equivariant holomorphic embedding $\iota\colon X \hookrightarrow X^*$ into a (not necessarily Hausdorff) complex manifold X^* endowed with a holomorphic $\mathbb{C}$–action such that $\mathbb{C} \cdot \iota(X) = X^*$. A globalization $\iota\colon X \hookrightarrow X^*$ is called *universal* if for every $\mathbb{R}$–equivariant holomorphic map $f\colon X \to Y$ into a holomorphic $\mathbb{C}$–manifold Y there exists a holomorphic $\mathbb{C}$–equivariant map $F\colon X^* \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "f"'] & & X^* \arrow[dl, "F"] \\ & Y & \end{tikzcd}$$

commutes. It follows that a universal globalization is unique up to isomorphism if it exists.

Since X is Stein, the universal globalization X^* of the induced local $\mathbb{C}$–action exists as is proven in [**HI97**]. We will always identify X with its image $\iota(X) \subset X^*$. Then the local $\mathbb{C}$–action on X coincides with the restriction of the global $\mathbb{C}$–action on X^* to X.

Recall that X is said to be orbit-connected in X^* if for every $x \in X^*$ the set $\Sigma(x) := \{t \in \mathbb{C};\ t \cdot x \in X\}$ is connected. The following criterion for a globalization to be universal is proven in [**CTIT00**].

Lemma 2.1. — *Let X^* be any globalization of the induced local $\mathbb{C}$–action on X. Then X^* is universal if and only if X is orbit-connected in X^*.*

Remark. — The results about (universal) globalizations hold for a bigger class of groups ([**CTIT00**]). However, we will need it only for the groups $\mathbb{C}$ and $\mathbb{C}^*$ and thus will not give the most general formulation.

For later use we also note the following

Lemma 2.2. — *The $\mathbb{C}$–action on X^* is free.*

Proof. — Suppose that there exists a point $x \in X^*$ such that $\mathbb{C}_x$ is non-trivial. Because of $\mathbb{C} \cdot X = X^*$ we can assume that $x \in X$ holds. Since $\mathbb{C}_x$ is a non-trivial closed subgroup of $\mathbb{C}$, it is either a lattice of rank 1 or 2, or $\mathbb{C}$. The last possibility means that x is a fixed point under $\mathbb{C}$ which is not possible since $\mathbb{R}$ acts freely on X.

We observe that the lattice $\mathbb{C}_x$ is contained in the connected $\mathbb{R}$–invariant set $\Sigma(x) = \{t \in \mathbb{C};\ t \cdot x \in X\}$. By $\mathbb{R}$–invariance $\Sigma(x)$ is a strip. Since X is hyperbolic, this strip cannot coincide with $\mathbb{C}$. The only lattice in $\mathbb{C}$ which can possibly be contained in such a strip is of the form $\mathbb{Z}r$ for some $r \in \mathbb{R}$. Since this contradicts the fact that $\mathbb{R}$ acts freely on X, the lemma is proven. □

Note that we do not know whether X^* is Hausdorff. In order to guarantee the Hausdorff property of X^*, we make further assumptions on X. The following result is proven in [**Ian03**] and [**IST04**].

Theorem 2.3. — *Let X be a hyperbolic Stein manifold with a proper $\mathbb{R}$–action. Suppose in addition that X is taut or admits the Bergman metric. Then X^* is Hausdorff. If X is simply-connected, then the same is true for X^*.*

We refer the reader to Chapter 4.10 and Chapter 5 in [**Kob98**] for the definitions and examples of tautness and the Bergman metric.

Remark. — Every bounded domain in $\mathbb{C}^n$ admits the Bergman metric.

2.2. The quotient $X/\mathbb{Z}$.

— We assume from now on that X fulfills the hypothesis of Theorem 2.3. Since X^* is covered by the translates $t \cdot X$ for $t \in \mathbb{C}$ and since the action of $\mathbb{Z}$ on each domain $t \cdot X$ is proper, we conclude that the quotient $X^*/\mathbb{Z}$ fulfills all axioms of a complex manifold except for possibly not being Hausdorff.

We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X^* \arrow[d] \\
X/\mathbb{Z} \arrow[r] & X^*/\mathbb{Z}.
\end{tikzcd}$$

Note that the group $\mathbb{C}^* = (S^1)^{\mathbb{C}} \cong \mathbb{C}/\mathbb{Z}$ acts on $X^*/\mathbb{Z}$. Concretely, if we identify $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^*$ via $\mathbb{C} \to \mathbb{C}^*$, $t \mapsto e^{2\pi i t}$, the quotient map $p\colon X^* \to X^*/\mathbb{Z}$ fulfills $p(t \cdot x) = e^{2\pi i t} \cdot p(x)$.

Lemma 2.4. — *The induced map $X/\mathbb{Z} \hookrightarrow X^*/\mathbb{Z}$ is the universal globalization of the local $\mathbb{C}^*$–action on $X/\mathbb{Z}$.*

Proof. — The open embedding $X \hookrightarrow X^*$ induces an open embedding $X/\mathbb{Z} \hookrightarrow X^*/\mathbb{Z}$. This embedding is S^1–equivariant and we have $\mathbb{C}^* \cdot X/\mathbb{Z} = X^*/\mathbb{Z}$. This implies that $X^*/\mathbb{Z}$ is a globalization of the local $\mathbb{C}^*$–action on $X/\mathbb{Z}$.

In order to prove that this globalization is universal, by the globalization theorem in [**CTIT00**] it is enough to show that $X/\mathbb{Z}$ is orbit-connected in $X^*/\mathbb{Z}$. Hence, we must show that for every $[x] \in X/\mathbb{Z}$ the set $\Sigma\bigl([x]\bigr) := \{t \in \mathbb{C}^*;\ t \cdot [x] \in X/\mathbb{Z}\}$ is connected in $\mathbb{C}^*$. For this we consider the set $\Sigma(x) = \{t \in \mathbb{C};\ t \cdot x \in X\}$. Since the map $X \to X/\mathbb{Z}$ intertwines the local $\mathbb{C}$– and $\mathbb{C}^*$–actions, we conclude that $t \in \Sigma(x)$ holds if and only if $e^{2\pi i t} \in \Sigma\bigl([x]\bigr)$ holds. Since X^* is universal, $\Sigma(x)$ is connected which implies that $\Sigma\bigl([x]\bigr)$ is likewise connected. Thus $X^*/\mathbb{Z}$ is universal. ☐

Remark. — The globalization $X^*/\mathbb{Z}$ is Hausdorff if and only if $\mathbb{Z}$ or, equivalently, $\mathbb{R}$ act properly on X^*. As we shall see in Lemma 3.3, this is the case if X is taut.

2.3. A sufficient condition for $X/\mathbb{Z}$ to be Stein.

— If $\dim X = 2$, we have the following sufficient condition for $X/\mathbb{Z}$ to be a Stein surface.

Proposition 2.5. — *If the $\mathbb{C}$–action on X^* is proper and if the Riemann surface $X^*/\mathbb{C}$ is not compact, then $X/\mathbb{Z}$ is Stein.*

Proof. — Under the above hypothesis we have the $\mathbb{C}$–principal bundle $X^* \to X^*/\mathbb{C}$. If the base $X^*/\mathbb{C}$ is not compact, then this bundle is holomorphically trivial, i. e. X^* is biholomorphic to $\mathbb{C} \times R$ where R is a non-compact Riemann surface. Since R is Stein, the same is true for X^* and for $X^*/\mathbb{Z} \cong \mathbb{C}^* \times R$. Since $X/\mathbb{Z}$ is locally Stein, see [**Mie08**], in the Stein manifold $X^*/\mathbb{Z}$, the claim follows from [**DG60**]. ☐

Therefore, the crucial step in the proof of our main result consists in showing that $\mathbb{C}$ acts properly on X^* under the assumption $\dim X = 2$.

3. Local properness

Let X be a hyperbolic Stein $\mathbb{R}$–manifold. Suppose that X is taut or that it admits the Bergman metric and $H^1(X,\mathbb{R}) = \{0\}$. We show that then $\mathbb{C}$ acts locally properly on X^*.

3.1. Locally proper actions. — Recall that the action of a Lie group G on a manifold M is called locally proper if every point in M admits a G–invariant open neighborhood on which the G acts properly.

Lemma 3.1. — *Let $G \times M \to M$ be locally proper.*

1. *For every $x \in M$ the isotropy group G_x is compact.*
2. *Every G–orbit admits a geometric slice.*
3. *The orbit space M/G is a smooth manifold which is in general not Hausdorff.*
4. *All G–orbits are closed in M.*
5. *The G–action on M is proper if and only if M/G is Hausdorff.*

Proof. — The first claim is elementary to check. The second claim is proven in [**DK00**]. The third one is a consequence of (2) since the slices yield charts on M/G which are smoothly compatible because the transitions are given by the smooth action of G on M. Assertion (4) follows from (3) because in locally Euclidian topological spaces points are closed. The last claim is proven in [**Pal61**]. □

Remark. — Since $\mathbb{R}$ acts properly on X, the $\mathbb{R}$-action on X^* is locally proper.

3.2. Local properness of the $\mathbb{C}$–action on X^*. — Recall that we assume that

$$X \text{ is taut} \tag{3.1}$$

or that

$$X \text{ admits the Bergman metric and } H^1(X,\mathbb{R}) = \{0\}. \tag{3.2}$$

We first show that assumption (3.1) implies that $\mathbb{C}$ acts locally properly on X^*.

Since X^* is the universal globalization of the induced local $\mathbb{C}$–action on X, we know that X is orbit-connected in X^*. This means that for every $x \in X^*$ the set $\Sigma(x) = \{t \in \mathbb{C};\ t\cdot x \in X\}$ is a strip in $\mathbb{C}$. In the following we will exploit the properties of the thickness of this strip.

Since $\Sigma(x)$ is $\mathbb{R}$–invariant, there are "numbers" $u(x) \in \mathbb{R}\cup\{-\infty\}$ and $o(x) \in \mathbb{R}\cup\{\infty\}$ for every $x \in X^*$ such that

$$\Sigma(x) = \big\{t \in \mathbb{C};\ u(x) < \mathrm{Im}(t) < o(x)\big\}.$$

The functions $u\colon X^* \to \mathbb{R}\cup\{-\infty\}$ and $o\colon X^* \to \mathbb{R}\cup\{\infty\}$ so obtained are upper and lower semicontinuous, respectively. Moreover, u und o are $\mathbb{R}$–invariant and $i\mathbb{R}$–equivariant:

$$u(it\cdot x) = u(x) - t \quad \text{and} \quad o(it\cdot x) = o(x) - t.$$

Proposition 3.2. — *The functions $u, -o\colon X^* \to \mathbb{R}\cup\{-\infty\}$ are plurisubharmonic. Moreover, u and o are continuous on $X^*\setminus\{u=-\infty\}$ and $X^*\setminus\{o=\infty\}$, respectively.*

Proof. — It is proven in [**For96**] that u and $-o$ are plurisubharmonic on X. By equivariance, we obtain this result for X^*.

Now we prove that the function $u\colon X\setminus\{u=-\infty\}\to\mathbb{R}$ is continuous which was remarked without complete proof in [**Ian03**]. For this let (x_n) be a sequence in X which converges to $x_0\in X\setminus\{u=-\infty\}$. Since u is upper semi-continuous, we have $\limsup_{n\to\infty}u(x_n)\leq u(x_0)$. Suppose that u is not continuous in x_0. Then, after replacing (x_n) by a subsequence, we find $\varepsilon>0$ such that $u(x_n)\leq u(x_0)-\varepsilon<u(x_0)$ holds for all $n\in\mathbb{N}$. Consequently, we have $\Sigma(x_0)=\big\{t\in\mathbb{C};\ u(x_0)<\operatorname{Im}(t)<o(x_0)\big\}\subset\Sigma:=\big\{t\in\mathbb{C};\ u(x_0)-\varepsilon<\operatorname{Im}(t)<o(x_0)\big\}\subset\Sigma(x_n)$ for all n and hence obtain the sequence of holomorphic functions $f_n\colon\Sigma\to X$, $f_n(t):=t\cdot x_n$. Since X is taut and $f_n(0)=x_n\to x_0$, the sequence (f_n) has a subsequence which compactly converges to a holomorphic function $f_0\colon\Sigma\to X$. Because of $f_0\big(iu(x_0)\big)=\lim_{n\to\infty}f_n\big(iu(x_0)\big)=\lim_{n\to\infty}iu(x_0)\cdot x_n=iu(x_0)\cdot x_0\notin X$ we arrive at a contradiction. Thus the function $u\colon X\setminus\{u=-\infty\}\to\mathbb{R}$ is continuous. By $(i\mathbb{R})$–equivariance, u is also continuous on $X^*\setminus\{u=-\infty\}$. A similar argument shows continuity of $-o\colon X^*\setminus\{o=\infty\}\to\mathbb{R}$. ☐

Let us consider the sets

$$\mathcal{N}(o):=\big\{x\in X^*;\ o(x)=0\big\}\quad\text{and}\quad\mathcal{P}(o):=\big\{x\in X^*;\ o(x)=\infty\big\}.$$

The sets $\mathcal{N}(u)$ and $\mathcal{P}(u)$ are similarly defined. Since $X=\big\{x\in X^*;\ u(x)<0<o(x)\big\}$, we can recover X from X^* with the help of u and o.

Lemma 3.3. — *The action of $\mathbb{R}$ on X^* is proper.*

Proof. — Let ∂^*X denote the boundary of X in X^*. Since the functions u and $-o$ are continuous on $X^*\setminus\mathcal{P}(u)$ and $X^*\setminus\mathcal{P}(o)$ one verifies directly that $\partial^*X=\mathcal{N}(u)\cup\mathcal{N}(o)$ holds. As a consequence, we note that if $x\in\partial^*X$, then for every $\varepsilon>0$ the element $(i\,\varepsilon)\cdot x$ is not contained in ∂^*X.

Let (t_n) and (x_n) be sequences in $\mathbb{R}$ and X^* such that $(t_n\cdot x_n,x_n)$ converges to (y_0,x_0) in $X^*\times X^*$. We may assume without loss of generality that x_0 and hence x_n are contained in X for all n. Consequently, we have $y_0\in X\cup\partial^*X$. If $y_0\in\partial^*X$ holds, we may choose an $\varepsilon>0$ such that $(i\,\varepsilon)\cdot y_0$ and $(i\,\varepsilon)\cdot x_0$ lie in X. Since the $\mathbb{R}$–action on X is proper, we find a convergent subsequence of (t_n) which was to be shown. ☐

Lemma 3.4. — *We have:*

1. *$\mathcal{N}(u)$ and $\mathcal{N}(o)$ are $\mathbb{R}$–invariant.*
2. *We have $\mathcal{N}(u)\cap\mathcal{N}(o)=\emptyset$.*
3. *The sets $\mathcal{P}(u)$ and $\mathcal{P}(o)$ are closed, $\mathbb{C}$–invariant and pluripolar in X^*.*
4. *$\mathcal{P}(u)\cap\mathcal{P}(o)=\emptyset$.*

Proof. — The first claim follows from the $\mathbb{R}$–invariance of u and o.

The second claim follows from $u(x)<o(x)$.

The third one is a consequence of the $\mathbb{R}$–invariance and $i\mathbb{R}$–equivariance of u and o.

If there was a point $x\in\mathcal{P}(u)\cap\mathcal{P}(o)$, then $\mathbb{C}\cdot x$ would be a subset of X which is impossible since X is hyperbolic. ☐

Lemma 3.5. — *If o is not identically ∞, then the map*

$$\varphi\colon i\mathbb{R}\times\mathcal{N}(o)\to X^*\setminus\mathcal{P}(o),\quad\varphi(it,z)=it\cdot z,$$

is an $i\mathbb{R}$–equivariant homeomorphism. Since $\mathbb{R}$ acts properly on $\mathcal{N}(o)$, it follows that $\mathbb{C}$ acts properly on $X^* \setminus \mathcal{P}(o)$. The same holds when o is replaced by u.

Proof. — The inverse map φ^{-1} is given by $x \mapsto \big(-io(x), io(x)\cdot x\big)$. □

Corollary 3.6. — *The $\mathbb{C}$–action on X^* is locally proper. If $\mathcal{P}(o) = \emptyset$ or $\mathcal{P}(u) = \emptyset$ hold, then $\mathbb{C}$ acts properly on X^*.*

From now on we suppose that X fulfills the assumption (3.2). Recall that the Bergman form ω is a Kähler form on X invariant under the action of $\mathrm{Aut}(X)$. Let ξ denote the complete holomorphic vector field on X which corresponds to the $\mathbb{R}$–action, i. e. we have $\xi(x) = \frac{\partial}{\partial t}\big|_0 \varphi_t(x)$. Hence, $\iota_\xi\omega = \omega(\cdot,\xi)$ is a 1–form on X and since $H^1(X,\mathbb{R}) = \{0\}$ there exists a function $\mu^\xi \in \mathcal{C}^\infty(X)$ with $d\mu^\xi = \iota_\xi\omega$.

Remark. — This means that μ^ξ is a momentum map for the $\mathbb{R}$–action on X.

Lemma 3.7. — *The map $\mu^\xi \colon X \to \mathbb{R}$ is an $\mathbb{R}$–invariant submersion.*

Proof. — The claim follows from $d\mu^\xi(x)J\xi_x = \omega_x(J\xi_x, \xi_x) > 0$. □

Proposition 3.8. — *The $\mathbb{C}$–action on X^* is locally proper.*

Proof. — Since μ^ξ is a submersion, the fibers $(\mu^\xi)^{-1}(c)$, $c \in \mathbb{R}$, are real hypersurfaces in X. Then

$$\frac{d}{dt}\bigg|_0 \mu^\xi(it\cdot x) = \omega_x(J\xi_x, \xi_x) > 0$$

implies that every $i\mathbb{R}$–orbit intersects $(\mu^\xi)^{-1}(c)$ transversally. Since X is orbit-connected in X^*, the map $i\mathbb{R} \times (\mu^\xi)^{-1}(c) \to X^*$ is injective and therefore a diffeomorphism onto its open image. Together with the fact that $(\mu^\xi)^{-1}(c)$ is $\mathbb{R}$–invariant this yields the existence of differentiable local slices for the $\mathbb{C}$–action. □

3.3. A necessary condition for $X/\mathbb{Z}$ to be Stein.

— We have the following necessary condition for $X/\mathbb{Z}$ to be a Stein manifold.

Proposition 3.9. — *If the quotient manifold $X/\mathbb{Z}$ is Stein, then X^* is Stein and the $\mathbb{C}$–action on X^* is proper.*

Proof. — Suppose that $X/\mathbb{Z}$ is a Stein manifold. By [**CTIT00**] this implies that X^* is Stein as well.

Next we will show that the $\mathbb{C}^*$–action on $X^*/\mathbb{Z}$ is proper. For this we will use as above a moment map for the S^1–action on $X^*/\mathbb{Z}$.

By compactness of S^1 we may apply the complexification theorem from [**Hei91**] which shows that $X^*/\mathbb{Z}$ is also a Stein manifold and in particular Hausdorff. Hence, there exists a smooth strictly plurisubharmonic exhaustion function $\rho\colon X^*/\mathbb{Z} \to \mathbb{R}^{>0}$ invariant under S^1. Consequently, $\omega := \frac{i}{2}\partial\overline{\partial}\rho \in \mathcal{A}^{1,1}(X^*)$ is an S^1–invariant Kähler form. Associated to ω we have the S^1–invariant moment map

$$\mu\colon X^*/\mathbb{Z} \to \mathbb{R}, \quad \mu^\xi(x) := \frac{d}{dt}\bigg|_0 \rho\big(\exp(it\xi)\cdot x\big),$$

where ξ is the complete holomorphic vector field on $X^*/\mathbb{Z}$ which corresponds to the S^1–action. Now we can apply the same argument as above in order to deduce that $\mathbb{C}^*$ acts locally properly on $X^*/\mathbb{Z}$.

We still must show that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. To see this, let $\mathbb{C}^* \cdot x_j$, $j = 0, 1$, be two different orbits in $X^*/\mathbb{Z}$. Since $\mathbb{C}^*$ acts locally properly, these are closed and therefore there exists a function $f \in \mathcal{O}(X^*/\mathbb{Z})$ with $f|_{\mathbb{C}^* \cdot x_j} = j$ for $j = 0, 1$. Again we may assume that f is S^1– and consequently $\mathbb{C}^*$–invariant. Hence, there is a continuous function on $(X^*/\mathbb{Z})/\mathbb{C}^*$ which separates the two orbits, which implies that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. This proves that $\mathbb{C}^*$ acts properly on $X^*/\mathbb{Z}$.

Since we know already that the $\mathbb{C}$–action on X^* is locally proper, it is enough to show that $X^*/\mathbb{C}$ is Hausdorff. But this follows from the properness of the $\mathbb{C}^*$–action on $X^*/\mathbb{Z}$ since $X^*/\mathbb{C} \cong (X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. □

4. Properness of the $\mathbb{C}$–action

Let X be a hyperbolic Stein $\mathbb{R}$–manifold. Suppose that X fulfills (3.1) or (3.2). We have seen that $\mathbb{C}$ acts locally properly on X^*. In this section we prove that under the additional assumption $\dim X = 2$ the orbit space $X^*/\mathbb{C}$ is Hausdorff. This implies that $\mathbb{C}$ acts properly on X^* if $\dim X = 2$.

4.1. Stein surfaces with $\mathbb{C}$–actions. — For every function $f \in \mathcal{O}(\Delta)$ which vanishes only at the origin, we define

$$X_f := \big\{(x, y, z) \in \Delta \times \mathbb{C}^2;\ f(x)y - z^2 = 1\big\}.$$

Since the differential of the defining equation of X_f is given by $\big(f'(x)y \ f(x) \ -2z\big)$, we see that 1 is a regular value of $(x, y, z) \mapsto f(x)y - z^2$. Hence, X_f is a smooth Stein surface in $\Delta \times \mathbb{C}^2$.

There is a holomorphic $\mathbb{C}$–action on X_f defined by

$$t \cdot (x, y, z) := \big(x, y + 2tz + t^2 f(x), z + tf(x)\big).$$

One can directly check that this defines an action.

Lemma 4.1. — *The $\mathbb{C}$–action on X_f is free, and all orbits are closed.*

Proof. — Let $t \in \mathbb{C}$ such that $\big(x, y + 2tz + t^2 f(x), z + tf(x)\big) = (x, y, z)$ for some $(x, y, z) \in X_f$. If $f(x) \neq 0$, then $z + tf(x) = z$ implies $t = 0$. If $f(x) = 0$, then $z \neq 0$ and $y + 2tz = y$ gives $t = 0$.

The map $\pi \colon X_f \to \Delta$, $(x, y, z) \mapsto x$, is $\mathbb{C}$–invariant. If $a \in \Delta^*$, then $f(a) \neq 0$ and we have

$$\frac{z}{f(a)} \cdot \big(a, f(a)^{-1}, 0\big) = (a, y, z) \in X_f,$$

which implies $\pi^{-1}(a) = \mathbb{C} \cdot \big(a, f(a)^{-1}, 0\big)$. A similar calculation gives $\pi^{-1}(0) = \mathbb{C} \cdot p_1 \cup \mathbb{C} \cdot p_2$ with $p_1 = (0, 0, i)$ and $p_2 = (0, 0, -i)$. Consequently, every $\mathbb{C}$–orbit is closed. □

Remark. — The orbit space $X_f/\mathbb{C}$ is the unit disc with a doubled origin and in particular not Hausdorff.

We calculate slices at the point p_j, $j = 1, 2$, as follows. Let $\varphi_j\colon \Delta\times\mathbb{C} \to X_f$ be given by $\varphi_1(z,t) := t\cdot(z,0,i)$ and $\varphi_2(w,s) = s\cdot(w,0,-i)$. Solving the equation $s\cdot(w,0,-i) = t\cdot(z,0,i)$ for (w,s) yields the transition function $\varphi_{12} = \varphi_2^{-1}\circ\varphi_1\colon \Delta^*\times\mathbb{C} \to \Delta^*\times\mathbb{C}$,

$$(z,t) \mapsto \left(z, t + \frac{2i}{f(z)}\right).$$

The function $\frac{1}{f}$ is a meromorphic function on Δ without zeros and with the unique pole 0.

Lemma 4.2. — *Let $\mathbb{R}$ act on X_f via $\mathbb{R}\hookrightarrow\mathbb{C}$, $t\mapsto ta$, for some $a\in\mathbb{C}^*$. Then there is no $\mathbb{R}$–invariant domain $D\subset X_f$ with $D\cap\mathbb{C}\cdot p_j \neq \emptyset$ for $j = 1, 2$ on which $\mathbb{R}$ acts properly.*

Proof. — Suppose that $D\subset X_f$ is an $\mathbb{R}$–invariant domain with $D\cap\mathbb{C}\cdot p_j\neq\emptyset$ for $j = 1, 2$. Without loss of generality we may assume that $p_1\in D$ and $\zeta\cdot p_2 = (0,-2\zeta i,-i)\in D$ for some $\zeta\in\mathbb{C}$. We will show that the orbits $\mathbb{R}\cdot p_1$ and $\mathbb{R}\cdot(\zeta\cdot p_2)$ cannot be separated by $\mathbb{R}$–invariant open neighborhoods.

Let $U_1\subset D$ be an $\mathbb{R}$–invariant open neighborhood of p_1. Then there are $r, r' > 0$ such that $\Delta_r^*\times\Delta_{r'}\times\{i\}\subset U_1$ holds. Here, $\Delta_r = \{z\in\mathbb{C};\ |z| < r\}$. For $(\varepsilon_1,\varepsilon_2)\in\Delta_r^*\times\Delta_{r'}$ and $t\in\mathbb{R}$ we have

$$t\cdot(\varepsilon_1,\varepsilon_2,i) = \big(\varepsilon_1, \varepsilon_2 + 2(ta)i + (ta)^2 f(\varepsilon_1), i + (ta)f(\varepsilon_1)\big)\in U_1.$$

We have to show that for all $r_2, r_3 > 0$ there exist $(\widetilde{\varepsilon_2},\widetilde{\varepsilon_3})\in\Delta_{r_2}\times\Delta_{r_3}$, $(\varepsilon_1,\varepsilon_2)\in\Delta_r^*\times\Delta_{r'}$ and $t\in\mathbb{R}$ such that

$$\big(\varepsilon_1, \varepsilon_2 + 2(ta)i + (ta)^2 f(\varepsilon_1), i + (ta)f(\varepsilon_1)\big) = (\varepsilon_1, -2\zeta i + \widetilde{\varepsilon_2}, -i + \widetilde{\varepsilon_3}) \tag{4.1}$$

holds.

Let $r_2, r_3 > 0$ be given. From (4.1) we obtain $\widetilde{\varepsilon_3} = taf(\varepsilon_1) + 2i$ or, equivalently, $ta = \frac{\widetilde{\varepsilon_3}-2i}{f(\varepsilon_1)}$. Setting $\widetilde{\varepsilon_2} = \varepsilon_2$ we obtain from $2(ta)i + (ta)^2 f(\varepsilon_1) = -2\zeta i$ the equivalent expression

$$f(\varepsilon_1) = -2i\frac{\zeta + ta}{(ta)^2}. \tag{4.2}$$

for $t\neq 0$. Choosing a real number $t\gg 1$, we find an $\varepsilon_1\in\Delta_r^*$ such that (4.2) is fulfilled. After possibly enlarging t we have $\widetilde{\varepsilon_3} := taf(\varepsilon_1) + 2i = -2i\frac{\zeta}{ta}\in\Delta_{r_3}$. Together with $\varepsilon_2 = \widetilde{\varepsilon_2}$ equation (4.1) is fulfilled and the proof is finished. □

Thus, the Stein surface X_f cannot be obtained as globalization of the local $\mathbb{C}$–action on any $\mathbb{R}$–invariant domain $D\subset X_f$ on which $\mathbb{R}$ acts properly.

4.2. The quotient $X^*/\mathbb{C}$ is Hausdorff. — Suppose that $X^*/\mathbb{C}$ is not Hausdorff and let $x_1, x_2\in X$ be such that the corresponding $\mathbb{C}$–orbits cannot be separated in $X^*/\mathbb{C}$. Since we already know that $\mathbb{C}$ acts locally proper on X^* we find local holomorphic slices $\varphi_j\colon\Delta\times\mathbb{C}\to U_j\subset X$, $\varphi_j(z,t) = t\cdot s_j(z)$ at each $\mathbb{C}\cdot x_j$ where $s_j\colon\Delta\to X$ is holomorphic with $s_j(0) = x_j$. Consequently, we obtain the transition function $\varphi_{12}\colon(\Delta\setminus A)\times\mathbb{C}\to(\Delta\setminus A)\times\mathbb{C}$ for some closed subset $A\subset\Delta$ which must be of the form $(z,t)\mapsto\big(z, t + f(z)\big)$ for some $f\in\mathcal{O}(\Delta\setminus A)$. The following lemma applies to show that A is discrete and that f is meromorphic on Δ. Hence, we are in one of the model cases discussed in the previous subsection.

Lemma 4.3. — *Let Δ_1 and Δ_2 denote two copies of the unit disk $\{z \in \mathbb{C};\ |z| < 1\}$. Let $U \subset \Delta_j$, $j = 1, 2$, be a connected open subset and $f\colon U \subset \Delta_1 \to \mathbb{C}$ a non-constant holomorphic function on U. Define the complex manifold*

$$M := (\Delta_1 \times \mathbb{C}) \cup (\Delta_2 \times \mathbb{C})/\!\sim,$$

where $\sim$ is the relation $(z_1, t_1) \sim (z_2, t_2) :\Leftrightarrow z_1 = z_2 =: z \in U$ and $t_2 = t_1 + f(z)$.

Suppose that M is Hausdorff. Then the complement A of U is discrete and f extends to a meromorphic function on Δ_1.

Proof. — We first prove that for every sequence (x_n), $x_n \in U$, with $\lim_{n\to\infty} x_n = p \in \partial U$, one has $\lim_{n\to\infty}|f(x_n)| = \infty \in \mathbb{P}_1(\mathbb{C})$. Assume the contrary, i.e. there is a sequence (x_n), $x_n \in U$, with $\lim_{n\to\infty} x_n = p \in \partial U$ such that $\lim_{n\to\infty} f(x_n) = a \in \mathbb{C}$. Choose now $t_1 \in \mathbb{C}$, consider the two points $(p, t_1) \in \Delta_1 \times \mathbb{C}$ and $(p, t_1 + a) \in \Delta_2 \times \mathbb{C}$ and note their corresponding points in M as q_1 and q_2. Then $q_1 \neq q_2$. The sequences $(x_n, t_1) \in \Delta_1 \times \mathbb{C}$ and $(x_n, t_1 + f(x_n)) \in \Delta_2 \times \mathbb{C}$ define the same sequence in M having q_1 and q_2 as accumulation points. So M is not Hausdorff, a contradiction.

In particular we have proved that the zeros of f do not accumulate to ∂U in Δ_1. So there is an open neighborhood V of ∂U in Δ_1 such that the restriction of f to $W := U \cap V$ does not vanish. Let $g := 1/f$ on W. Then g extends to a continuous function on V taking the value zero outside of U. The theorem of Rado implies that this function is holomorphic on V. It follows that the boundary ∂U is discrete in Δ_1 and that f has a pole in each of the points of this set, so f is a meromorphic function on Δ_1. ☐

Theorem 4.4. — *The orbit space $X^*/\mathbb{C}$ is Hausdorff. Consequently, $\mathbb{C}$ acts properly on X^*.*

Proof. — By virtue of the above lemma, in a neighborhood of two non-separable $\mathbb{C}$–orbits X is isomorphic to a domain in one of the model Stein surfaces discussed in the previous subsection. Since we have seen there that these surfaces are never globalizations, we arrive at a contradiction. Hence, all $\mathbb{C}$–orbits are separable. ☐

5. Examples

In this section we discuss several examples which illustrate our results.

5.1. Hyperbolic Stein surfaces with proper $\mathbb{R}$–actions. — Let R be a compact Riemann surface of genus $g \geq 2$. It follows that the universal covering of R is given by the unit disc $\Delta \subset \mathbb{C}$ and hence that R is hyperbolic. The fundamental group $\pi_1(R)$ of R contains a normal subgroup N such that $\pi_1(R)/N \cong \mathbb{Z}$. Let $\widetilde{R} \to R$ denote the corresponding normal covering. Then $\widetilde{R}$ is a hyperbolic Riemann surface with a holomorphic $\mathbb{Z}$–action such that $\widetilde{R}/\mathbb{Z} = R$. Note that $\mathbb{Z}$ is not contained in a one parameter group of automorphisms of $\widetilde{R}$.

We have two mappings

$$\begin{tikzcd} X := \mathbb{H} \times_{\mathbb{Z}} \widetilde{R} \arrow[r, "q"] \arrow[d, "p"'] & \widetilde{R}/\mathbb{Z} = R \\ \mathbb{H}/\mathbb{Z} \cong \Delta \setminus \{0\}. & \end{tikzcd}$$

The map $p\colon X \to \Delta \setminus \{0\}$ is a holomorphic fiber bundle with fiber $\widetilde{R}$. Since the Serre problem has a positive answer if the fiber is a non-compact Riemann surface ([**Mok82**]), the suspension $X = \mathbb{H} \times_{\mathbb{Z}} \widetilde{R}$ is a hyperbolic Stein surface. The group $\mathbb{R}$ acts on $\mathbb{H} \times \widetilde{R}$ by $t \cdot (z, x) = (z + t, x)$ and this action commutes with the diagonal action of $\mathbb{Z}$. Consequently, we obtain an action of $\mathbb{R}$ on X.

Lemma 5.1. — *The universal globalization of the local $\mathbb{C}$–action on X is given by $X^* = \mathbb{C} \times_{\mathbb{Z}} \widetilde{R}$. Moreover, $\mathbb{C}$ acts properly on X^*.*

Proof. — One checks directly that $t \cdot [z, x] := [z + t, x]$ defines a holomorphic $\mathbb{C}$–action on $X^* = \mathbb{C} \times_{\mathbb{Z}} \widetilde{R}$ which extends the $\mathbb{R}$–action on X. We will show that X is orbit-connected in X^*: Since $[z + t, x]$ lies in X if and only if there exist elements $(z', x') \in \mathbb{H} \times \widetilde{R}$ and $m \in \mathbb{Z}$ such that $(z + t, x) = \big(z' + m, m \cdot x'\big)$, we conclude $\mathbb{C}[z, x] = \big\{t \in \mathbb{C};\ \mathrm{Im}(t) > -\mathrm{Im}(z)\big\}$ which is connected.

In order to show that $\mathbb{C}$ acts properly on X^* it is sufficient to show that $\mathbb{C} \times \mathbb{Z}$ acts properly on $\mathbb{C} \times \widetilde{R}$. Hence, we choose sequences $\{t_n\}$ in $\mathbb{C}$, $\{m_n\}$ in $\mathbb{Z}$ and $\big\{(z_n, x_n)\big\}$ in $\mathbb{C} \times \widetilde{R}$ such that

$$\big((t_n, m_n) \cdot (z_n, x_n), (z_n, x_n)\big) = \big((z_n + t_n + m_n, m_n \cdot x_n), (z_n, x_n)\big) \to \big((z_1, x_1), (z_0, x_0)\big)$$

holds. Since $\mathbb{Z}$ acts properly on $\widetilde{R}$, it follows that $\{m_n\}$ has a convergent subsequence, which in turn implies that $\{t_n\}$ has a convergent subsequence. Hence, the lemma is proven. ☐

Proposition 5.2. — *The quotient $X/\mathbb{Z} \cong \Delta^* \times R$ is not holomorphically separable and in particular not Stein. The quotient $X^*/\mathbb{C}$ is biholomorphically equivalent to $\widetilde{R}/\mathbb{Z} = R$.*

Proof. — It is sufficient to note that the map $\Phi\colon X = \mathbb{H} \times_{\mathbb{Z}} \widetilde{R} \to \Delta^* \times R$, $Phi[z, x] := \big(e^{2\pi i z}, [x]\big)$, induces a biholomorphic map $X/\mathbb{Z} \to \Delta^* \times R$. ☐

Proposition 5.3. — *The quotient $X/\mathbb{Z} \cong \Delta^* \times R$ is not holomorphically separable and in particular not Stein.*

Thus we have found an example for a hyperbolic Stein surface X endowed with a proper $\mathbb{R}$–action such that the associated $\mathbb{Z}$–quotient is not holomorphically separable. Moreover, the $\mathbb{R}$–action on X extends to a proper $\mathbb{C}$–action on a Stein manifold X^* containing X as an orbit-connected domain such that $X^*/\mathbb{C}$ is any given compact Riemann surface of genus $g \geq 2$.

5.2. Counterexamples with domains in $\mathbb{C}^n$. — There is a bounded Reinhardt domain D in $\mathbb{C}^2$ endowed with a holomorphic action of $\mathbb{Z}$ such that $D/\mathbb{Z}$ is not Stein. However, this $\mathbb{Z}$–action does not extend to an $\mathbb{R}$–action. We give quickly the construction.

Let $\lambda := \frac{1}{2}(3+\sqrt{5})$ and

$$D := \{(x,y) \in \mathbb{C}^2 \mid |x| > |y|^\lambda, |y| > |x|^\lambda\}.$$

It is obvious that D is a bounded Reinhardt domain in $\mathbb{C}^2$ avoiding the coordinate hyperplanes. The holomorphic automorphism group of D is a semidirect product $\Gamma \ltimes (S^1)^2$, where the group $\Gamma \simeq \mathbb{Z}$ is generated by the automorphism $(x,y) \mapsto (x^3y^{-1}, x)$ and $(S^1)^2$ is the rotation group. Therefore the group Γ is not contained in a one-parameter group. Furthermore the quotient D/Γ is the (non-Stein) complement of the singular point in a 2-dimensional normal complex Stein space, a so-called "cusp singularity". These singularities are intensively studied in connection with Hilbert modular surfaces and Inoue-Hirzebruch surfaces, see e.g. [**vdG88**] and [**Zaf01**].

In the rest of this subsection we give an example of a hyperbolic domain of holomorphy in a 3–dimensional Stein solvmanifold endowed with a proper $\mathbb{R}$–action such that the $\mathbb{Z}$–quotient is not Stein. While this domain is not simply-connected, its fundamental group is much simpler than the fundamental groups of our two-dimensional examples.

Let $G := \left\{ \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) ;\ a,b,c \in \mathbb{C} \right\}$ be the complex Heisenberg group and let us consider its discrete subgroup

$$\Gamma := \left\{ \begin{pmatrix} 1 & m & \frac{m^2}{2} + 2\pi i k \\ 0 & 1 & m + 2\pi i l \\ 0 & 0 & 1 \end{pmatrix} ;\ m,k,l \in \mathbb{Z} \right\}.$$

Note that Γ is isomorphic to $\mathbb{Z}_m \ltimes \mathbb{Z}^2_{(k,l)}$. We let Γ act on $\mathbb{C}^2$ by

$$(z,w) \mapsto \left(z + mw - \frac{m^2}{2} - 2\pi i k, w - m - 2\pi i l \right).$$

Proposition 5.4. — *The group Γ acts properly and freely on $\mathbb{C}^2$, and the quotient manifold $\mathbb{C}^2/\Gamma$ is holomorphically separable but not Stein.*

Proof. — Since $\Gamma' \cong \mathbb{Z}^2$ is a normal subgroup of Γ, we obtain $\mathbb{C}^2/\Gamma \cong (\mathbb{C}^2/\Gamma')/(\Gamma/\Gamma')$. The map $\mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}^*$, $(z,w) \mapsto \big(\exp(z), \exp(w)\big)$, identifies $\mathbb{C}^2/\Gamma'$ with $\mathbb{C}^* \times \mathbb{C}^*$. The induced action of $\Gamma/\Gamma' \cong \mathbb{Z}$ on $\mathbb{C}^* \times \mathbb{C}^*$ is given by

$$(z,w) \mapsto \left(e^{-m^2/2} z w^m, e^{-m} w \right)$$

which shows that Γ acts properly and freely on $\mathbb{C}^2$. Moreover, we obtain the commutative diagram

$$\begin{tikzcd}
\mathbb{C}^* \times \mathbb{C}^* \arrow[r] \arrow[d, "(z,w)\mapsto w"'] & Y := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z} \arrow[d] \\
\mathbb{C}^* \arrow[r] & T := \mathbb{C}^*/\mathbb{Z}.
\end{tikzcd}$$

The group $\mathbb{C}^*$ acts by multiplication in the first factor on $\mathbb{C}^* \times \mathbb{C}^*$ and this action commutes with the $\mathbb{Z}$–action. One checks directly that the joint $(\mathbb{C}^* \times \mathbb{Z})$–action on $\mathbb{C}^* \times \mathbb{C}^*$ is proper which implies that the map $Y \to T$ is a $\mathbb{C}^*$–principal bundle. Conseqently, Y is not Stein.

In order to show that Y is holomorphically separable, note that by [**Oel92**] this $\mathbb{C}^*$–principal bundle $Y \to T$ extends to a line bundle $p\colon L \to T$ with first Chern class $c_1(L) = -1$. Therefore the zero section of $p\colon L \to T$ can be blown down and we obtain a singular normal Stein space $\overline{Y} = Y \cup \{y_0\}$ where $y_0 = \mathrm{Sing}(\overline{Y})$ is the blown down zero section. Thus Y is holomorphically separable. □

Let us now choose a neighborhood of the singularity $y_0 \in \overline{Y}$ biholomorphic to the unit ball and let U be its inverse image in $\mathbb{C}^2$. It follows that U is a hyperbolic domain with smooth strictly Levi-convex boundary in $\mathbb{C}^2$ and in particular Stein. In order to obtain a proper action of $\mathbb{R}$ we form the suspension $D = \mathbb{H} \times_\Gamma U$ where Γ acts on $\mathbb{H} \times U$ by $(t, z, w) \mapsto (t + m, z + mw - \frac{m^2}{2} - 2\pi ik, w - m - 2\pi il)$.

Proposition 5.5. — *The suspension $D = \mathbb{H} \times_\Gamma U$ is isomorphic to a Stein domain in the Stein manifold G/Γ.*

Proof. — We identify $\mathbb{H} \times U$ with the $\mathbb{R} \times \Gamma$–invariant domain

$$\Omega := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ;\ \mathrm{Im}(a) > 0, (c, b) \in U \right\}$$

in G.

Since $\mathbb{H} \times U$ is Stein, it follows that $\mathbb{H} \times_\Gamma U$ is locally Stein in G/Γ. Hence, by virtue of [**DG60**] we only have to show that G/Γ is Stein.

For this we note first that G is a closed subgroup of $\mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{C}^2$ which implies that G/Γ is a closed complex submanifold of $X := \big(\mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{C}^2\big)/\Gamma$. By [**Oel92**] the manifold X is holomorphically separable, hence G/Γ is holomorphically separable. Since G is solvable, a result of Huckleberry and Oeljeklaus ([**HO86**]) yields the Steinness of G/Γ.

One checks directly that the action of $\mathbb{R} \times \Gamma$ on $\mathbb{H} \times U$ is proper which implies that $\mathbb{R}$ acts properly on $\mathbb{H} \times_\Gamma U$. □

Because of $(\mathbb{H} \times_\Gamma U)/\mathbb{Z} \cong \Delta^* \times (U/\Gamma)$ this quotient manifold is not Stein but holomorphically separable.

6. Bounded domains with proper $\mathbb{R}$–actions

In this section we give the proof of our main result.

6.1. Proper $\mathbb{R}$–actions on D.

— Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\mathrm{Aut}(D)^0$ be the connected component of the identity in $\mathrm{Aut}(D)$.

Lemma 6.1. — *A proper $\mathbb{R}$–action by holomorphic transformations on D exists if and only if the group $\mathrm{Aut}(D)^0$ is non-compact.*

The proof follows from the existence of a diffeomorphism $K \times V \longrightarrow \mathrm{Aut}(D)^0$ where K is a maximal compact subgroup of $\mathrm{Aut}(D)^0$ and V is a linear subspace of the Lie algebra of $\mathrm{Aut}(D)^0$.

6.2. Steinness of $D/\mathbb{Z}$.

— Now we give the proof of our main result.

Theorem 6.2. — *Let D be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$. Suppose that the group $\mathbb{R}$ acts properly by holomorphic transformations on D. Then the complex manifold $D/\mathbb{Z}$ is biholomorphically equivalent to a domain of holomorphy in $\mathbb{C}^2$.*

Proof. — Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy. Since the Serre problem is solvable if the fiber is D, see [**Siu76**], the universal globalization D^* is a simply-connected Stein surface, [**CTIT00**]. Moreover, we have shown in Theorem 4.4, that $\mathbb{C}$ acts properly on D^*. Since the Riemann surface $D^*/\mathbb{C}$ is also simply-connected, it must be Δ, $\mathbb{C}$ or $\mathbb{P}_1(\mathbb{C})$. In all three cases the bundle $D^* \to D^*/\mathbb{C}$ is holomorphically trivial. So we can exclude the case that $D^*/\mathbb{C}$ is compact and it follows that $D/\mathbb{Z} \cong \mathbb{C}^* \times (D^*/\mathbb{C})$ is a Stein domain in $\mathbb{C}^2$. □

6.3. A normal form for domains with non-compact $\mathrm{Aut}(D)^0$.

— Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy such that the identity component of its automorphism group is non-compact. As we have seen, this yields a proper $\mathbb{R}$–action on D by holomorphic transformations and the universal globalization of the induced local $\mathbb{C}$–action on D is isomorphic to $\mathbb{C} \times S$ where S is either Δ or $\mathbb{C}$ and where $\mathbb{C}$ acts by translation in the first factor.

Moreover, there are plurisubharmonic functions $u, -o\colon \mathbb{C} \times S \to \mathbb{R} \cup \{-\infty\}$ which fulfill

$$u\big(t \cdot (z_1, z_2)\big) = u(z_1, z_2) - \mathrm{Im}(t) \quad \text{and} \quad o\big(t \cdot (z_1, z_2)\big) = o(z_1, z_2) - \mathrm{Im}(t)$$

such that $D = \big\{(z_1, z_2) \in \mathbb{C} \times S;\ u(z_1, z_2) < 0 < o(z_1, z_2)\big\}$. From this we conclude $u(z_1, z_2) = u(0, z_2) - \mathrm{Im}(z_1)$, $o(z_1, z_2) = o(0, z_2) - \mathrm{Im}(z_1)$ and define $u'(z_2) := u(0, z_2)$, $o'(z_2) := o(0, z_2)$.

We summarize our remarks in the following

Theorem 6.3. — *Let D be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$ admitting a non-compact connected identity component of its automorphism group. Then D is biholomorphic to a domain of the form*

$$\widetilde{D} = \big\{(z_1, z_2) \in \mathbb{C} \times S;\ u'(z_2) < \mathrm{Im}(z_1) < o'(z_2)\big\},$$

where the functions $u', -o'$ are subharmonic in S.

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