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Eric Sopena, Jiaojiao Wu

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Éric Sopena* and Jiaojiao Wu†

Université de Bordeaux, LaBRI UMR 5800,

351, cours de la Libération,

F-33405 Talence Cedex, France

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Abstract

The square G^2 of a graph G is defined on the vertex set of G in such a way that distinct vertices with distance at most two in G are joined by an edge. We study the chromatic number of the square of the Cartesian product $C_m \square C_n$ of two cycles and show that the value of this parameter is at most 7 except when $m = n = 3$, in which case the value is 9, and when $m = n = 4$ or $m = 3$ and $n = 5$, in which case the value is 8.

Moreover, we conjecture that for every $G = C_m \square C_n$, the chromatic number of G^2 equals $\lceil mn/\alpha(G^2) \rceil$, where $\alpha(G^2)$ denotes the size of a maximal independent set in G^2.

Key words: Chromatic number, square, distance two coloring, Cartesian product of cycles.

*E-mail: sopena@labri.fr.

†E-mail: wujj0007@yahoo.com.tw. This work has been done while the author was visiting the LaBRI thanks to a postdoctoral fellowship from Bordeaux 1 University.

1 Introduction

A *k-coloring* of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a mapping c from $V(G)$ to the set $\{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever uv is an edge in $E(G)$. The *chromatic number* $\chi(G)$ of G is the smallest k for which G admits a k-coloring.

Let G and H be graphs. The *Cartesian product* $G \square H$ of G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. Let P_n and C_n denote respectively the path and the cycle on n vertices. We will denote by $G_{m,n} = P_m \square P_n$ the *grid* with m rows and n columns and by $T_{m,n} = C_m \square C_n$ the *toroidal grid* with m rows and n columns.

The *square* G^2 of a graph G is given by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $uv \in E(G)$ or there exists $w \in V(G)$ such that uw, $vw \in E(G)$. In other words, any two vertices within distance at most two in G are linked by an edge in G^2. The problem of determining the chromatic number of the square of particular graphs has attracted very much attention, with a particular focus on the square of planar graphs (see e.g. [2, 5, 6, 11, 12]), following Wegner [15] who conjectured that every planar graph with maximum degree $\Delta \geq 8$ satisfies $\chi(G^2) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$. Havet *et al.* proved in [5] that the square of any such planar graph admits a coloring using $(\frac{3}{2} + o(1))\Delta$ colors.

In [1], Chiang and Yan studied the chromatic number of the square of Cartesian products of paths and cycles and proved the following:

Theorem 1 (Chiang and Yan [1]) *Let $G = C_m \square P_n$, $m \geq 3$, $n \geq 2$. Then*

$$\chi(G^2) = \begin{cases} 4 & \text{if } n = 2 \text{ and } m \equiv 0 \pmod 4, \\ 6 & \text{if } n = 2 \text{ and } m = 3, 6, \\ 6 & \text{if } n \geq 3 \text{ and } m \not\equiv 0 \pmod 5, \\ 5 & \text{otherwise.} \end{cases}$$

Since $C_m \square P_n$ is a subgraph of $C_m \square C_n$, the previous theorem provides lower bounds for the chromatic number of the square of toroidal grids.

In [13], Pór and Wood studied the notion of *$\mathcal{F}$-free coloring* which generalizes several types of colorings and, in particular, square coloring. They obtained an upper bound on the $\mathcal{F}$-free chromatic number of cartesian products of general graphs. Moreover, in case of square coloring, they proved that the chromatic number of any graph given as the Cartesian product of d cycles is at most $6d + O(\log d)$.

An $L(p, q)$*-labeling* of a graph G is an assignment ϕ of nonnegative integers to the vertices of G in such a way that $|\phi(u) - \phi(v)| \geq p$ whenever u and v are adjacent and $|\phi(u) - \phi(v)| \geq q$ whenever u and v are at distance two in G. The λ_q^p*-number* of G is defined as the smallest k such that G admits an $L(p, q)$-labeling on the set $\{0, 1, \ldots, k\}$ (note that such a labeling uses $k + 1$ labels). It follows from the definition that any $L(1, 0)$-labeling of G is a usual coloring of G and that any $L(1, 1)$-labeling of G is a coloring of the square of G. Therefore, $\chi(G) = \lambda_0^1(G) + 1$ and $\chi(G^2) = \lambda_1^1(G) + 1$ for every graph G.

This notion was introduced by Griggs and Yeh [4] to model the *Channel assignment problem*. In the same paper, they conjectured that for every graph G with maximum degree Δ, $\lambda_1^2(G) \leq \Delta^2$. This motivated many authors to study $L(2, 1)$-labeling of some particular classes of graphs and the case of Cartesian products of graphs was investigated in [1, 3, 7, 8, 9, 10, 14, 16].

In particular, Schwartz and Traxell [14] considered $L(2, 1)$-labelings of products of cycles and proved the following:

Theorem 2 (Schwartz and Traxell [14]) *Let* $T_{m,n} = C_m \Box C_n$. *Then*

$$\lambda_1^2(T_{m,n}) = \begin{cases} 6 & \text{if } m, n \equiv 0 \pmod 7, \\ 8 & \text{if } (m, n) \in A, \\ 7 & \text{otherwise.} \end{cases}$$

where $A = \{\{3, i\} : i \geq 3,\ i \text{ odd or } i = 4, 10\} \cup \{\{5, i\} : i = 5, 6, 9, 10, 13, 17\} \cup \{\{6, 7\}, \{6, 11\}, \{7, 9\}, \{9, 10\}\}$.

Since every $L(2, 1)$-labeling is a $L(1, 1)$-labeling, we get $\lambda_1^2(G) + 1 \geq \lambda_1^1(G) + 1 = \chi(G^2)$ for every graph G. Therefore, Theorems 2 provides upper bounds on the chromatic number of the square of toroidal grids (note that the upper bounds corresponding to the three cases of Theorem 2 are 7, 9 and 8, respectively).

Our main result will improve the bounds provided by Theorems 1 and 2 and by the general result of Pór and Wood [13]:

Theorem 3 *Let* $T_{m,n} = C_m \Box C_n$. *Then* $\chi(T_{m,n}^2) \leq 7$ *except* $\chi(T_{3,3}^2) = 9$ *and* $\chi(T_{3,5}^2) = \chi(T_{4,4}^2) = 8$.

2 Coloring the square of toroidal grids

In this section, we shall prove Theorem 3 and give more precise bounds for Cartesian products of some particular cycles.

We shall construct explicit colorings by means of combinations of *patterns* given in matrix form. Each pattern can be thought of as a coloring of the square of the toroidal grid of the same size. For instance, the pattern E depicted in Figure 1 provides in an obvious way a 7-coloring of the square of $T_{3,7}$. Moreover, by repeating this pattern, we can easily obtain a 7-coloring of the square of toroidal grids of the form $T_{3m,7q}$.

Let G be a graph and c be any coloring of G. Since every color class in G is an independent set, we have the following standard observation:

Observation 4 *$\chi(G) \geq \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil$ where $\alpha(G)$ denotes the maximum size of an independent set in G.*

We shall extensively use the following result. Given two integers r and s, let $S(r, s)$ denote the set of all nonnegative integer combinations of r and s:

$$S(r, s) = \{\alpha r + \beta s : \alpha, \beta \text{ nonnegative integers}\}.$$

A standard property of the set $S(r, s)$ is the following:

Lemma 5 *If r and s are relatively prime integers greater than one, then $t \in S(r, s)$ for all $t \geq (r-1)(s-1)$, and $(r-1)(s-1) - 1 \notin S(r, s)$.*

We then have:

Theorem 6 *Let $T_{m,n} = C_m \square C_n$, $m \in S(4, 7)$ and $n \in S(3, 7)$. Then $\chi(T^2_{m,n}) \leq 7$.*

Proof. Let $m \in S(4, 7)$ and $n \in S(3, 7)$. We use the following 7×7 pattern A to prove the lemma.

$$A = \begin{array}{|ccccccc|} \hline 1 & 6 & 4 & 2 & 7 & 5 & 3 \\ 2 & 7 & 5 & 3 & 1 & 6 & 4 \\ 3 & 1 & 6 & 4 & 2 & 7 & 5 \\ 4 & 2 & 7 & 5 & 3 & 1 & 6 \\ 5 & 3 & 1 & 6 & 4 & 2 & 7 \\ 6 & 4 & 2 & 7 & 5 & 3 & 1 \\ 7 & 5 & 3 & 1 & 6 & 4 & 2 \\ \hline \end{array}$$

It is easy to check that this pattern provides a coloring of $T^2_{7,7}$. For any pattern X, let X_i, X'_j be the subpatterns of X such that X_i is obtained by taking the i first rows of X and X'_j is obtained by taking the j first columns of X. It is again easy to check that the patterns A_4, A'_3 and $(A_4)'_3$ provide colorings of $T^2_{4,7}$, $T^2_{7,3}$ and $T^2_{4,3}$, respectively. Therefore, using combinations of A and A_4, we can get a $m \times 7$ pattern Y and, using combinations of Y and Y'_3, we can get a $m \times n$ pattern which provides a 7-coloring of $T^2_{m,n}$. ∎

For example, the following pattern B provides a 7-coloring of $T^2_{11,13}$, obtained from A by using the combinations $11 = 7 + 4$ and $13 = 7 + 2 \times 3$.

$$B = \begin{array}{|ccccccc|ccc|ccc|}
\hline
1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 1 & 6 & 4 \\
2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 2 & 7 & 5 \\
3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 3 & 1 & 6 \\
4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 4 & 2 & 7 \\
5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 5 & 3 & 1 \\
6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 6 & 4 & 2 \\
7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 7 & 5 & 3 \\
\hline
1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 1 & 6 & 4 \\
2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 2 & 7 & 5 \\
3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 & 1 & 6 & 3 & 1 & 6 \\
4 & 2 & 7 & 5 & 3 & 1 & 6 & 4 & 2 & 7 & 4 & 2 & 7 \\
\hline
\end{array}$$

By Lemma 5 we then get:

Corollary 7 *Let $T_{m,n} = C_m \square C_n$, $m \geq 12$ and $n \geq 18$. Then $\chi(T^2_{m,n}) \leq 7$.*

We now consider toroidal grids with one component being a C_3. Then we have:

Theorem 8 *Let $T_{3,n} = C_3 \square C_n$. Then*

$$\chi(T^2_{3,n}) = \begin{cases} 6 & \text{if } n \text{ is even,} \\ 7 & \text{if } n \text{ is odd and } n \neq 3, 5, \\ 8 & \text{if } n = 5, \\ 9 & \text{if } n = 3. \end{cases}$$

Proof. Let C, D and E be the patterns given in Figure 1. These patterns clearly provide colorings of $T^2_{3,4}$, $T^2_{3,6}$ and $T^2_{3,7}$, respectively. For the upper

$$C = \begin{array}{|cccc|}\hline 1&4&2&5\\ 2&5&3&6\\ 3&6&1&4\\ \hline\end{array} \quad D = \begin{array}{|cccccc|}\hline 1&4&2&5&3&6\\ 2&5&3&6&1&4\\ 3&6&1&4&2&5\\ \hline\end{array} \quad E = \begin{array}{|ccccccc|}\hline 1&4&2&3&1&2&5\\ 2&5&1&4&7&3&6\\ 3&6&7&5&6&4&7\\ \hline\end{array}$$

Figure 1: Patterns for Theorem 8

bounds, we use the combinations of patterns C and D to obtain the even cases and use the combinations of patterns C, D and E to obtain the odd cases. The remainder cases are $n = 3, 5, 9$, and the following patterns provide the required colorings of $T_{3,3}$, $T_{3,5}$ and $T_{3,9}$, respectively.

$$\begin{array}{|ccc|}\hline 1&4&7\\ 2&5&8\\ 3&6&9\\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 1&4&2&3&6\\ 2&5&1&4&7\\ 3&6&7&5&8\\ \hline\end{array} \quad \begin{array}{|ccccccccc|}\hline 1&4&2&3&1&4&5&3&6\\ 2&5&1&4&2&3&6&4&7\\ 3&6&7&5&6&7&1&2&5\\ \hline\end{array}$$

For the lower bounds, notice that the intersection of any independent set I in $T_{3,n}$ with any two consecutive columns contains at most one vertex. Therefore, $\alpha(T_{3,n}^2) \leq \lfloor n/2 \rfloor$. By Observation 4, we get $\chi(T_{3,n}^2) > 6$ when n is odd; in particular, $\chi(T_{3,n}^2) > 7$ when $n = 5$ and $\chi(T_{3,n}^2) \geq 9$ when $n = 3$. ∎

As in the proof of Theorem 6, we can get colorings of $T_{3k,n}^2$, $k \geq 1$, by using combinations of the patterns given in Theorem 8. We thus get the following:

Corollary 9 *Let $T_{3k,n} = C_{3k} \square C_n$. Then*

$$\chi(T_{3k,n}^2) \leq \begin{cases} 6 & \text{if } n \text{ is even,} \\ 7 & \text{if } n \text{ is odd and } n \neq 3, 5, \\ 8 & \text{if } n = 5, \\ 9 & \text{if } n = 3. \end{cases}$$

We now consider toroidal grids with one component being a C_4. Then we have:

Theorem 10 *Let $T_{4,n} = C_4 \square C_n$. Then*

$$\chi(T_{4,n}^2) = \begin{cases} 6 & \text{if } n \equiv 0 \pmod 3, \\ 8 & \text{if } n = 4, \\ 7 & \text{otherwise.} \end{cases}$$

$$F = \begin{array}{|ccc|}\hline 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 1 \\ 4 & 6 & 2 \\ \hline\end{array} \quad G = \begin{array}{|ccccc|}\hline 1 & 3 & 2 & 4 & 6 \\ 2 & 4 & 6 & 3 & 5 \\ 3 & 5 & 7 & 2 & 1 \\ 4 & 6 & 1 & 5 & 7 \\ \hline\end{array} \quad H = \begin{array}{|ccccccc|}\hline 1 & 3 & 2 & 6 & 4 & 7 & 5 \\ 2 & 4 & 5 & 7 & 1 & 3 & 6 \\ 3 & 1 & 6 & 2 & 5 & 4 & 7 \\ 4 & 5 & 7 & 1 & 3 & 6 & 2 \\ \hline\end{array}$$

Figure 2: Patterns for Theorem 10

Proof. For $m = 3k$, this follows from Corollary 9. Let F, G and H be the patterns given in Figure 2. These patterns clearly provide colorings of $T^2_{4,3}$, $T^2_{4,5}$ and $T^2_{4,7}$, respectively. Thanks to Lemma 5, by using combinations of F, G and H, we can get a 7-coloring of $T^2_{4,n}$ except when $n = 4$. In this latter case, we can use the following pattern:

1	2	3	4
3	4	5	6
5	6	7	8
7	8	1	2

Observe now that the intersection of any independent set I in $T_{4,n}$ with any three consecutive columns contains at most two vertices. Thus, $\alpha(T^2_{4,n}) \leq \lfloor \frac{2n}{3} \rfloor$. By Observation 4, $\chi(T^2_{4,n}) > 6$ when n is not a multiple of 3 and $\chi(T^2_{4,n}) \geq 8$ when $n = 4$. ∎

Using combinations of the patterns from Theorem 10, we get the following:

Corollary 11 *Let $T_{4k,n} = C_{4k} \square C_n$, $k \geq 1$. Then*

$$\chi(T^2_{4k,n}) \leq \begin{cases} 6 & \textit{if } n \equiv 0 \pmod 3, \\ 8 & \textit{if } n = 4, \\ 7 & \textit{otherwise.} \end{cases}$$

We now consider toroidal grids with one component being a C_5. Then we have:

Theorem 12 *Let $T_{5,n} = C_5 \square C_n$, $n \geq 5$. Then*

$$\chi(T^2_{5,n}) = \begin{cases} 5 & \textit{if } n \equiv 0 \pmod 5, \\ 7 & \textit{if } n = 7, \\ 6 & \textit{otherwise.} \end{cases}$$

$I =$

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

$J =$

6	1	2	3	4	5
3	4	5	6	1	2
5	6	1	2	3	4
2	3	4	5	6	1
4	5	6	1	2	3

Figure 3: Patterns for Theorem 12

Proof. Let I and J be the patterns given in Figure 3 which provide colorings of $T^2_{5,5}$ and $T^2_{5,6}$, respectively. We use combinations of I and J to get a 5-coloring (resp. a 6-coloring) of $T^2_{5,n}$ when $n \equiv 0 \pmod 5$ (resp. when $n \in S(5,6)$ and $n \not\equiv 0 \pmod 5$).

The remainder cases are $n = 7, 8, 9, 13, 14, 16, 19$. The corresponding patterns are given below, except for $n = 16$, in which case the corresponding pattern is obtained by combining two 5×8 patterns.

1	3	2	1	7	5	4
2	4	5	3	6	1	7
3	1	6	7	5	4	6
4	2	3	4	2	7	1
5	6	7	5	3	6	2

$n = 7$

1	3	2	6	3	1	5	4
2	4	5	1	4	2	3	6
3	1	6	2	5	6	4	5
4	2	3	4	1	3	2	1
5	6	1	5	2	4	6	3

$n = 8$

1	3	5	1	4	6	5	2	4
2	4	6	3	2	1	4	3	5
3	1	2	4	6	5	2	1	6
4	6	3	5	1	4	3	5	2
5	2	4	6	3	2	1	6	3

$n = 9$

1	3	2	6	3	1	5	2	4	3	6	5	4
2	4	5	1	4	2	6	3	1	5	2	1	6
3	1	6	2	5	3	1	4	2	6	3	4	5
4	2	3	4	1	6	2	5	3	1	5	6	1
5	6	1	5	2	4	3	1	6	2	4	3	2

$n = 13$

1	3	2	6	3	1	5	2	4	3	2	5	6	4
2	4	5	1	4	2	6	3	5	6	4	1	3	5
3	1	6	2	5	3	1	4	2	1	5	2	4	6
4	2	3	4	1	6	2	5	3	4	6	3	5	1
5	6	1	5	2	4	3	1	6	5	1	4	2	3

$n = 14$

$$K = \begin{array}{|ccccccccccccc|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ \hline\end{array}$$

Figure 4: Pattern for Theorem 3

1	3	2	6	3	1	5	2	4	3	1	4	2	1	6	2	5	3	4
2	4	5	1	4	2	6	3	1	5	2	6	3	5	4	3	6	1	5
3	1	6	2	5	3	1	4	2	6	3	1	4	6	1	5	2	4	6
4	2	3	4	1	6	2	5	3	1	4	2	5	3	2	6	3	5	1
5	6	1	5	2	4	3	1	6	2	5	3	6	4	5	1	4	6	2

$n = 19$

Observe now that the intersection of any independent set I in $T_{5,n}$ with any column contains at most one vertex. That means $\alpha(T^2_{5,n}) \leq n$. Therefore, $\chi(T^2_{5,n}) \geq 5$ by Observation 4. It is finally straightforward to verify that $\alpha(T^2_{5,n}) < n$ when n is not a multiple of 5 (and thus $\chi(T^2_{5,n}) > 5$) and that $\alpha(T^2_{5,7}) = 5$ (and thus $\chi(T^2_{5,7}) \geq \frac{35}{5} = 7$). ∎

Using combinations of the patterns from Theorem 12, we get the following:

Corollary 13 *Let $T_{5k,n} = C_{5k} \square C_n$, $n \geq 5$. Then*

$$\chi(T^2_{5k,n}) \leq \begin{cases} 5 & \text{if } n \equiv 0 \pmod 5, \\ 7 & \text{if } n = 7, \\ 6 & \text{otherwise.} \end{cases}$$

At this point, we are able to prove our main result.

Proof of Theorem 3. By Corollaries 9, 11 and 13, we already proved that if one of m, n is a multiple of 3, 4, or 5, then Theorem 3 holds. By Lemma 5 and Corollary 7, the remainder cases are 11×11, 13×13, 13×17 and 17×17. Let K be the 7×13 pattern given in Figure 4. As in the proof of Theorem 6, we use combinations of K and K_3 to obtain an $m \times 13$ pattern

X for $m \in S(7,3)$. Then we use combinations of X and X'_4 to obtain an $m \times n$ pattern for $n \in S(13,4)$. In this way, we can obtain a 7-coloring of $T^2_{13,13}$, $T^2_{17,13}$ and $T^2_{17,17}$. We simply transpose the 17×13 pattern to get a 13×17 pattern. Finally, the 11×11 pattern which provides a 6-coloring of $T_{11,11}$ is as follows:

1	2	3	1	2	3	1	2	3	4	5
3	4	5	6	4	5	6	4	1	6	2
5	1	2	3	1	2	3	5	2	3	4
2	3	4	5	6	4	1	6	4	5	1
4	5	6	1	3	5	2	3	1	2	3
1	2	3	4	2	6	4	5	6	4	5
6	4	1	6	5	3	1	2	3	1	2
3	5	2	3	1	2	6	4	5	6	4
1	6	4	5	6	4	5	1	2	3	5
2	3	1	2	3	1	2	3	4	1	6
4	5	6	4	5	6	4	5	6	2	3

∎

As we have seen before, the general upper bound of 7 for $\chi(T^2_{m,n})$ given in Theorem 3 can be decreased for particular values of m and n. We now provide other cases for which this bound can be decreased to 6.

Using combinations of the 11×11 pattern above, we get:

Corollary 14 *Let* $T_{m,n} = C_m \Box C_n$, $m, n \equiv 0 \pmod{11}$. *Then* $\chi(T^2_{m,n}) \leq 6$.

The same bound can be obtained for toroidal grids with one component being a C_6:

Theorem 15 *Let* $T_{6,n} = C_6 \Box C_n$, $n \geq 6$. *Then* $\chi(T^2_{6,n}) = 6$.

Proof. Let L and M be the patterns given in Figure 5 which provide 6-colorings of $T^2_{6,4}$ and $T^2_{6,2}$, respectively. By Lemma 5, we can get a 6-coloring of $T_{6,n}$ by using combinations of patterns L and M. ∎

Using combinations of the patterns from Theorem 15, we get the following:

Corollary 16 *Let* $T_{6k,n} = C_{6k} \Box C_n$, $n \geq 6$. *Then* $\chi(T^2_{6k,n}) \leq 6$.

1	3	6	4
2	4	1	5
3	5	2	6
4	6	3	1
5	1	4	2
6	2	5	3

Pattern L

1	3	5
2	4	6
3	5	1
4	6	2
5	1	3
6	2	4

Pattern M

Figure 5: Patterns for Theorem 15

Finally, using Corollary 13 and the lower bound given by Theorem 1, we get the following

Corollary 17 *Let $T_{m,n} = C_m \Box C_n$ for $m, n \geq 3$. Then $\chi(T_{m,n}^2) \geq 5$. Moreover, $\chi(G^2) = 5$ if and only if $m, n \equiv 0 \pmod 5$.*

3 Discussion

In this paper, we have investigated the chromatic number of the square of toroidal grids, that is Cartesian products of two cycles. We obtained general upper bounds for this parameter by providing explicit colorings based on the use of specific patterns. This leads in an obvious way to a linear time algorithm for constructing such colorings.

Table 1 summarizes those of our results which give tight bounds. We also included two cases, marked by $(*)$, for which the tight bound has been obtained by a computer program. It can be observed that in all the cases for which a tight bound has been obtained, this bound matches the lower bound given by Observation 4. Therefore, we propose the following:

Conjecture 18 *For every toroidal grid $T_{m,n}$, $\chi(T_{m,n}^2) = \left\lceil \frac{|V(T_{m,n}^2)|}{\alpha(T_{m,n}^2)} \right\rceil$.*

References

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values of m and n	$\chi(T^2_{m,n})$
$m, n \equiv 0 \pmod 5$	5
$m = 3$, $n \equiv 0 \pmod 2$	6
$m = 4$, $n \equiv 0 \pmod 3$	6
$m = 6$, $n \geq 6$	6
$m = 8$, $n = 11, 13$ $(*)$	6
$m \equiv 0 \pmod 3$, $m \not\equiv 0 \pmod 5$, $n \equiv 0 \pmod 2$, $n \not\equiv 0 \pmod 5$	6
$m \equiv 0 \pmod 5$, $n \not\equiv 0 \pmod 5$, $n \geq 6$, $n \neq 7$	6
$m \equiv 0 \pmod 6$, $n \geq 6$, $n \not\equiv 0 \pmod 5$	6
$m, n \equiv 0 \pmod{11}$, $m \not\equiv 0 \pmod 5$, $n \not\equiv 0 \pmod 5$	6
$m = 3$, $n \not\equiv 0 \pmod 2$, $n \neq 3, 5$	7
$m = 4$, $n \not\equiv 0 \pmod 3$, $n \neq 4$	7
$m = 5$, $n = 7$	7
$m = 7$, $n = 7, 8$ $(*)$	7
$m = 3$, $n = 5$	8
$m = 4$, $n = 4$	8
$m = 3$, $n = 3$	9

Table 1: Summary of results on $\chi(T^2_{m,n})$

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