

# COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS

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## Abstract

In this paper, we establish some common fixed point theorems for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a metric space. Our results unify and extend many results existing in the literature including those of Aliouche [3], Bouhadjera [4] and Popa [18]-[23]. Also we establish another common fixed point theorem for four occasionally weakly compatible single and set-valued maps of Greguš type which improves the results of Djoudi and Nisse [5], Pathak et al. [16] and others and we end our work by giving another theorem which generalizes the results given by Elamrani and Mehdaoui [6], Mbarki [13] and references therein.

**Key words and phrases:** Occasionally weakly compatible maps, weakly compatible maps, compatible and compatible maps of type (A), (B), (C) and (P), implicit relation, common fixed point theorem, Greguš type, strict contractive condition, metric space.

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## 1 Introduction and preliminaries

Throughout this paper,  $(\mathcal{X}, d)$  denotes a metric space and  $\mathcal{P}_{fb}(\mathcal{X})$  the class of all nonempty bounded closed subsets of  $\mathcal{X}$ . We recall these usual notations: for  $x \in \mathcal{X}$  and  $A \subseteq \mathcal{X}$ ,

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let  $H$  be the associated Hausdorff metric on  $\mathcal{P}_{fb}(\mathcal{X})$ : for every  $A$  and every  $B$  in  $\mathcal{P}_{fb}(\mathcal{X})$ ,

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\right\}$$

and

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

For simplicity, we write  $\delta(a, B)$  in place of  $\delta(\{a\}, B)$ ; as well as  $\delta(A, b)$  in place of  $\delta(A, \{b\})$ .

In the following, we use small letters:  $f, g, \dots$  to denote maps from  $\mathcal{X}$  to  $\mathcal{X}$  and capital letters:  $F, G, \dots$  for set-valued maps; that is, maps from  $\mathcal{X}$  to  $\mathcal{P}_{fb}(\mathcal{X})$  and we write  $fx$  for  $f(x)$  and  $Fx$  for  $F(x)$ .

The concepts of weak commutativity, compatibility, noncompatibility and weak compatibility were frequently used to prove existence theorems in fixed and common fixed points for single and set-valued maps satisfying certain conditions in different spaces. The study of common fixed points on occasionally weakly compatible maps is new and also interesting. This notion which is defined by Al-Thagafi and Shahzad [2] and which is published in 2008, has been used by Jungck and Rhoades [11] in 2006 and by Abbas and Rhoades [1] in 2007.

We begin by a short historic of these different notions. Generalizing the concept of commuting maps, Sessa [24] introduced the concept of weakly commuting maps.  $f$  and  $g$  are weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx)$$

for all  $x \in \mathcal{X}$ , where  $f$  and  $g$  are two self-maps of  $(\mathcal{X}, d)$ .

In 1986, Jungck [7] made more generalized commuting and weakly commuting maps called compatible maps.  $f$  and  $g$  are said to be compatible if

$$(1) \quad \lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$$

whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{X}$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  for some  $t \in \mathcal{X}$ . This concept has been useful as a tool for obtaining more comprehensive fixed point theorems. Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible, but neither implication is reversible (see [7]).

Further, the same author with Murthy and Cho [9] gave another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A).  $f$  and  $g$  are said to be compatible of type (A) if in place of (1) we have the two equalities

$$\lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) = 0.$$

Obviously, weakly commuting maps are compatible of type (A). From [9] it follows that the implication is not reversible.

In their paper [15], Pathak and Khan extended type (A) maps by introducing the concept of compatible maps of type (B) and compared these maps with compatible and compatible maps of type (A) in normed spaces. To be compatible of type (B),  $f$  and  $g$  above have to satisfy, in lieu of condition (1), the

inequalities

$$\begin{aligned} \lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) &\leq \frac{1}{2} \left[ \lim_{n \rightarrow \infty} d(fgx_n, ft) + \lim_{n \rightarrow \infty} d(ft, f^2x_n) \right] \\ &\text{and} \\ \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) &\leq \frac{1}{2} \left[ \lim_{n \rightarrow \infty} d(gfx_n, gt) + \lim_{n \rightarrow \infty} d(gt, g^2x_n) \right]. \end{aligned}$$

It is clear that compatible maps of type (A) are compatible of type (B). The converse is not true ([15]).

In 1998, Pathak et al. [16] introduced an extension of compatibility of type (A) by giving the notion of compatible maps of type (C).  $f$  and  $g$  are compatible of type (C) if they satisfy the two inequalities

$$\begin{aligned} \lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) &\leq \frac{1}{3} \left[ \lim_{n \rightarrow \infty} d(fgx_n, ft) + \lim_{n \rightarrow \infty} d(ft, f^2x_n) \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} d(ft, g^2x_n) \right] \\ \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) &\leq \frac{1}{3} \left[ \lim_{n \rightarrow \infty} d(gfx_n, gt) + \lim_{n \rightarrow \infty} d(gt, g^2x_n) \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} d(gt, f^2x_n) \right]. \end{aligned}$$

The same authors gave some examples to show that compatible maps of type (C) need not be neither compatible nor compatible of type (A) (resp., type (B)).

In [14] the concept of compatible maps of type (P) was introduced and compared with compatible and compatible maps of type (A).  $f$  and  $g$  are compatible of type (P) if in lieu of (1) we have

$$\lim_{n \rightarrow \infty} d(f^2x_n, g^2x_n) = 0.$$

Note that compatibility, compatibility of type (A) (resp. (B), (C) and (P)) are equivalent if  $f$  and  $g$  are continuous.

Afterwards, Jungck [8] generalized the compatibility, the compatibility of type (A), (B), (C) and (P) by introducing the concept of weak compatibility. He defines  $f$  and  $g$  to be weakly compatible if  $ft = gt, t \in \mathcal{X}$  implies  $fgt = gft$ .

It is known that all of the above compatibility notions imply weakly compatible notion, however, there exist weakly compatible maps which are neither compatible nor compatible of type (A), (B), (C) and (P) (see [3]).

Recently in a paper submitted before 2006 but published only in 2008, Al-Thagafi and Shahzad [2] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps. Two self-maps  $f$  and  $g$  of  $\mathcal{X}$  are called occasionally weakly compatible maps (shortly owc) if there is a point  $x$  in  $\mathcal{X}$  such that  $fx = gx$  at which  $f$  and  $g$  commute. This notion is used in 2006 by Jungck and Rhoades [11] to prove some common fixed point theorems in symmetric spaces.

In their paper [12], Kaneko and Sessa extended the compatibility to the setting of single and set-valued maps as follows:  $f : \mathcal{X} \rightarrow \mathcal{X}$  and  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  are said to be compatible if  $fFx \in \mathcal{P}_{fb}(\mathcal{X})$  for all  $x \in \mathcal{X}$  and

$$\lim_{n \rightarrow \infty} H(Ffx_n, fFx_n) = 0$$

whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{X}$  such that  $fx_n \rightarrow t$ ,  $Fx_n \rightarrow A \in \mathcal{P}_{fb}(\mathcal{X})$  and  $t \in A$ .

After, in [10] Jungck and Rhoades extend the concept of compatible single and set-valued maps by giving the concept of weak compatibility. Maps  $f$  and  $F$  are weakly compatible if they commute at their coincidence points; i.e., if  $fFx = Ffx$  whenever  $fx \in Fx$ .

More recently, Abbas and Rhoades [1] extended the definition of owc maps to the setting of set-valued maps and they proved some common fixed point theorems satisfying generalized contractive condition of integral type.  $f$  and  $F$  are said to be owc if and only if there exists some point  $x$  in  $\mathcal{X}$  such that  $fx \in Fx$  and  $fFx \subseteq Ffx$ . Clearly, weakly compatible maps are occasionally weakly compatible. However, the converse is not true in general. The example below illustrate this fact.

**1.1 Example** Let  $\mathcal{X} = [1, \infty[$  with the usual metric. Define  $f : \mathcal{X} \rightarrow \mathcal{X}$  and  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  by, for all  $x \in \mathcal{X}$ ,

$$fx = 2x + 1, Fx = [1, 2x + 1].$$

$$fx = 2x + 1 \in Fx \text{ and } fFx = [3, 4x + 3] \subset Ffx = [1, 4x + 3].$$

Hence,  $f$  and  $F$  are occasionally weakly compatible but non weakly compatible.

## 2 General fixed point theorems

In this section, before giving our first main result, we recall this definition.

**2.1 Definition** Let  $F : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a set-valued map on  $\mathcal{X}$ .  $x \in \mathcal{X}$  is a fixed point of  $F$  if  $x \in Fx$ .

**2.2 Theorem** Let  $f, g : \mathcal{X} \rightarrow \mathcal{X}$  be maps and  $F, G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\varphi : (\mathbb{R}^+)^5 \rightarrow \mathbb{R}$  be a real map satisfying the following conditions:

( $\varphi_1$ ) :  $\varphi$  is nonincreasing in variables  $t_4$  and  $t_5$ ,

( $\varphi_2$ ) :  $\varphi(t, 0, 0, t, t) \geq 0 \forall t > 0$ .

If, for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$ ,

$$(2.2) \quad \varphi(d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0$$

then  $f, g, F$  and  $G$  have a unique common fixed point.

**Proof**

i) We begin to show the existence of a common fixed point.

Since the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc then, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu$  and  $gGv \subseteq Ggv$ .

First, we show that  $gv = fu$ . Suppose that is not the case, then by (2.3), we have

$$\begin{aligned} & \varphi(d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu)) \\ &= \varphi(d(fu, gv), 0, 0, d(fu, Gv), d(gv, Fu)) < 0 \end{aligned}$$

and by  $(\varphi_1)$ ,

$$\varphi(d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) < 0$$

which from  $(\varphi_2)$  gives  $d(fu, gv) = 0$ . So  $fu = gv$ .

Next, we claim that  $f^2u = fu$ . If it is not, then condition (2.3) implies that

$$\begin{aligned} & \varphi(d(f^2u, gv), d(f^2u, Ffu), d(gv, Gv), d(f^2u, Gv), d(gv, Ffu)) \\ &= \varphi(d(f^2u, fu), 0, 0, d(f^2u, Gv), d(fu, Ffu)) < 0. \end{aligned}$$

By  $(\varphi_1)$  we have

$$\varphi(d(f^2u, fu), 0, 0, d(f^2u, fu), d(f^2u, fu)) < 0$$

which, from  $(\varphi_2)$ , gives  $d(f^2u, fu) = 0$ . We have  $f^2u = fu$ .

Since  $(f, F)$  and  $(g, G)$  have the same role, we have  $gv = g^2v$ . Therefore,  $ffu = fu = gv = ggv = gfu, fu = f^2u \in fFu \subset Ffu$ , so  $fu \in Ffu$  and  $fu = gfu \in Gfu$ . Then  $fu$  is a common fixed point of  $f, g, F$  and  $G$ .

ii) Now, we show uniqueness of the common fixed point.

Put  $fu = w$  and let  $w'$  be another common fixed point of the four maps such that  $w \neq w'$ , then, by (2.3), we get

$$\begin{aligned} & \varphi(d(fw, gw'), d(fw, Fw), d(gw', Gw'), d(fw, Gw'), d(gw', Fw)) \\ &= \varphi(d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw)) < 0. \end{aligned}$$

By  $(\varphi_1)$ , we get

$$\varphi(d(fw, gw'), 0, 0, d(fw, gw'), d(fw, gw')) < 0.$$

So, by  $(\varphi_2)$ ,  $d(fw, gw') = 0$  and thus  $d(fw, gw') = d(w, w') = 0$ . ■

We can give two variants of Theorem 2.2:

**2.3 Theorem** Let  $f, g : \mathcal{X} \rightarrow \mathcal{X}$  be maps and  $F, G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\varphi : (\mathbb{R}^+)^6 \rightarrow \mathbb{R}$  be a real map satisfying the following conditions:

$(\varphi_1)$  :  $\varphi$  is nonincreasing in variables  $t_5$  and  $t_6$ ,

$(\varphi_2)$  : for every  $t'$ ,  $\varphi(t', t, 0, 0, t, t) \geq 0 \forall t > 0$ .

If, for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$ ,

$$(2.3) \quad \varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0$$

then  $f, g, F$  and  $G$  have a unique common fixed point.

**Proof**

*i)* We begin to show the existence of a common fixed point in a similar proof of Theorem 2.2.

Since the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc then, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu$  and  $gGv \subseteq Ggv$ .

First, we show that  $gv = fu$ . Suppose that is not the case, then condition (2.2) implies that

$$\begin{aligned} & \varphi(H(Fu, Gv), d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu)) \\ &= \varphi(H(Fu, Gv), d(fu, gv), 0, 0, d(fu, Gv), d(gv, Fu)) < 0. \end{aligned}$$

By  $(\varphi_1)$  we have

$$\varphi(H(Fu, Gv), d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) < 0$$

which from  $(\varphi_2)$  gives  $d(fu, gv) = 0$ . So  $fu = gv$ .

Next, we claim that  $f^2u = fu$ . If it is not, then condition (2.2) implies that

$$\begin{aligned} & \varphi(H(Ffu, Gv), d(f^2u, gv), d(f^2u, Ffu), \\ & \quad d(gv, Gv), d(f^2u, Gv), d(gv, Ffu)) \\ &= \varphi(H(Ffu, Gv), d(f^2u, fu), 0, 0, d(f^2u, Gv), d(fu, Ffu)) < 0. \end{aligned}$$

By  $(\varphi_1)$  we have

$$\varphi(H(Ffu, Gv), d(f^2u, fu), 0, 0, d(f^2u, fu), d(f^2u, fu)) < 0$$

which, from  $(\varphi_2)$ , gives  $d(f^2u, fu) = 0$ . We have  $f^2u = fu$ .

Since  $(f, F)$  and  $(g, G)$  have the same role, we have  $gv = g^2v$ . Therefore,  $ffu = fu = gv = ggv = gfu, fu = f^2u \in fFu \subset Ffu$ , so  $fu \in Ffu$  and  $fu = gfu \in Gfu$ . Then  $fu$  is a common fixed point of  $f, g, F$  and  $G$ .

*ii)* Now, we show uniqueness of the common fixed point.

Put  $fu = w$  and let  $w'$  be another common fixed point of the four maps such that  $w \neq w'$ , then, by (2.2), we get

$$\begin{aligned} & \varphi(H(Fw, Gw'), d(fw, gw'), d(fw, Fw), \\ & \quad d(gw', Gw'), d(fw, Gw'), d(gw', Fw)) \\ &= \varphi(H(Fw, Gw'), d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw)) < 0. \end{aligned}$$

By  $(\varphi_1)$ , we get

$$\varphi(H(Fw, Gw'), d(fw, gw'), 0, 0, d(fw, gw'), d(fw, gw')) < 0.$$

So, by  $(\varphi_2)$ ,  $d(fw, gw') = 0$  and thus  $d(fw, gw') = d(w, w') = 0$ . ■

**2.4 Theorem** Let  $f, g : \mathcal{X} \rightarrow \mathcal{X}$  be maps and  $F, G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\varphi : (\mathbb{R}^+)^6 \rightarrow \mathbb{R}$  be a real map satisfying the following conditions:

$(\varphi_1)$  :  $\varphi$  is nondecreasing in variable  $t_1$  and nonincreasing in variables  $t_5$  and  $t_6$ ,

$(\varphi_2)$  :  $\varphi(t, t, 0, 0, t, t) \geq 0 \forall t > 0$ .

If, for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$ ,

$$(2.4) \quad \varphi(\delta(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0$$

then  $f, g, F$  and  $G$  have a unique common fixed point.

**Proof**

*i)* We begin to show existence of a common fixed point. The beginning of the proof is similar of that of previous theorems.

With the same notations, we suppose that  $gv \neq fu$ . then condition (2.4) implies that

$$\begin{aligned} & \varphi(\delta(Fu, Gv), d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu)) \\ &= \varphi(\delta(Fu, Gv), d(fu, gv), 0, 0, d(fu, Gv), d(gv, Fu)) < 0. \end{aligned}$$

By  $(\varphi_1)$  we have

$$\varphi(d(fu, gv), d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) < 0$$

which from  $(\varphi_2)$  gives  $d(fu, gv) = 0$ . So  $fu = gv$ . Next, we claim that  $f^2u = fu$ . If it is not, then condition (2.4) implies that

$$\begin{aligned} & \varphi(\delta(Ffu, Gv), d(f^2u, gv), d(f^2u, Ffu), d(gv, Gv), d(f^2u, Gv), d(gv, Ffu)) \\ &= \varphi(\delta(Ffu, Gv), d(f^2u, fu), 0, 0, d(f^2u, Gv), d(fu, Ffu)) < 0. \end{aligned}$$

By  $(\varphi_1)$  we have

$$\varphi(d(f^2u, fu), d(f^2u, fu), 0, 0, d(f^2u, fu), d(f^2u, fu)) < 0$$

which, from  $(\varphi_2)$ , gives  $d(f^2u, fu) = 0$  which implies that  $f^2u = fu$ .

Since  $(f, F)$  and  $(g, G)$  have the same role, we have:  $g^2v = gv$ . Therefore,  $ffu = fu = gv = ggv = gfu$ ,  $fu = f^2u \in fFu \subset Ffu$ , so  $fu \in Ffu$  and  $fu = gfu \in Gfu$ . Then  $fu$  is a common fixed point of  $f, g, F$  and  $G$ .

*ii)* Now, we show uniqueness of the common fixed point.

Put  $fu = w$  and let  $w'$  be another common fixed point of the four maps such that  $w \neq w'$ , by (2.4), we get

$$\begin{aligned} & \varphi(\delta(Fw, Gw'), d(fw, gw'), d(fw, Fw), d(gw', Gw'), d(fw, Gw'), d(gw', Fw)) \\ &= \varphi(\delta(Fw, Gw'), d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw)) < 0. \end{aligned}$$

By  $(\varphi_1)$ , we get

$$\begin{aligned} & \varphi(d(fw, gw'), d(fw, gw'), 0, 0, d(fw, gw'), d(fw, gw')) \\ &= \varphi(d(w, w'), d(w, w'), 0, 0, d(w, w'), d(w, w')) < 0. \end{aligned}$$

So, by  $(\varphi_2)$ ,  $d(w, w') = 0$  and thus  $w = w'$ .

**2.5 Remark** Truly Theorems 2.3-2.4 are generalizations of corresponding theorems of [3], [4], [18]-[23] and others since we extended the setting of single-valued maps to the one of single and set-valued maps, also we deleted the compactness in [3], [21], we further add that we not required the continuity, although we used the strict contractive conditions (2.3), (2.4) which are substantially more general than the inequalities in the cited papers, and we weakened the concepts of compatibility, compatibility of type (A), compatibility of type (C), compatibility of type (P) and weak compatibility to the more general one say occasional weak compatibility. Finally we deleted some assumptions of functions  $\varphi$  which are superfluous for us but are necessary in the papers [3], [4], [18]-[23].

If we let  $f = g$  and  $F = G$  in Theorems 2.2, 2.3 and 2.4, we get different corollaries. As example, we give the following corollaries of Theorem 2.4:

**2.6 Corollary** Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  and let  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  such that the pair  $\{f, F\}$  is ovc. Let  $\varphi : (\mathbb{R}^+)^6 \rightarrow \mathbb{R}$  be a real map satisfying conditions  $(\varphi_1)$  and  $(\varphi_2)$  of Theorem 2.4 and

$$\varphi(\delta(Fx, Fy), d(fx, fy), d(fx, Fx), d(fy, Fy), d(fx, Fy), d(fy, Fx)) < 0$$

for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, fy), d(fx, Fx), d(fy, Fy)\} > 0$ , then  $f$  and  $F$  have a unique common fixed point.

Now, if we let  $f = g$ , we get the next result:

**2.7 Corollary** Let  $f$  be a self-map of a metric space  $(\mathcal{X}, d)$  and let  $F, G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps. Suppose pairs  $\{f, F\}$  and  $\{f, G\}$  are ovc and  $\varphi : (\mathbb{R}^+)^6 \rightarrow \mathbb{R}$  satisfies conditions  $(\varphi_1)$  and  $(\varphi_2)$  of Theorem 2.4 and

$$\varphi(\delta(Fx, Gy), d(fx, fy), d(fx, Fx), d(fy, Gy), d(fx, Gy), d(fy, Fx)) < 0$$

for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, fy), d(fx, Fx), d(fy, Gy)\} > 0$ , then  $f, F$  and  $G$  have a unique common fixed point.

With different choices of the real map  $\varphi$ , we obtain the following corollaries:

**2.8 Corollary** If in the hypotheses of Theorem 2.4, we have instead of (2.4) one of the following inequalities, for all  $x$  and  $y \in \mathcal{X}$  whenever the right hand side of each inequality is not zero, then the four maps have a unique common fixed point.

$$(a) \quad \delta(Fx, Gy) < k \max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}$$

where  $0 < k \leq 1$ ,

$$(b) \quad \delta^2(Fx, Gy) < c_1 \max\{d^2(fx, gy), d^2(fx, Fx), d^2(gy, Gy)\} \\ + c_2 \max\{d(fx, Fx)d(fx, Gy), d(gy, Gy)d(gy, Fx)\} \\ + c_3 d(fx, Gy)d(gy, Fx)$$

where  $c_1 > 0$ ,  $c_2, c_3 \geq 0$  and  $c_1 + c_3 \leq 1$ ,

$$(c) \quad \delta(Fx, Gy) < [\alpha\delta^{p-1}(Fx, Gy)d(fx, gy) + \beta\delta^{p-2}(Fx, Gy)d(fx, Fx)d(gy, Gy) + \gamma d^{p-1}(fx, Gy)d(gy, Fx) + \delta d(fx, Gy)d^{p-1}(gy, Fx)]^{\frac{1}{p}}$$

where  $\alpha > 0$ ,  $\beta, \gamma, \delta \geq 0$ ,  $\alpha + \gamma + \delta \leq 1$  and  $p \geq 2$ ,

$$(d) \quad \delta^2(Fx, Gy) < \frac{1}{\alpha} \left[ \beta d^2(fx, gy) + \frac{\gamma d(fx, Gy)d(gy, Fx)}{1 + \delta d^2(fx, Fx) + \epsilon d^2(gy, Gy)} \right]$$

where  $\alpha > 0$ ,  $\beta, \gamma, \delta, \epsilon \geq 0$  and  $\beta + \gamma \leq \alpha$ ,

$$(e) \quad \delta(Fx, Gy) < [\alpha d^p(fx, gy) + \beta d^p(fx, Fx) + \gamma d^p(gy, Gy)]^{\frac{1}{p}} + \delta [d(fx, Gy)d(gy, Fx)]^{\frac{1}{2}}$$

where  $0 < \alpha \leq (1 - \delta)^p$ ,  $\beta, \gamma, \delta \geq 0$  and  $p \in \mathbb{N}^* = \{1, 2, \dots\}$ .

**Proof**

For proof of (a), (b), (c), (d) and (e), we use Theorem 2.4 with the following functions  $\varphi$  which satisfy, for every case, hypothesis  $(\varphi_1)$  and  $(\varphi_2)$  for (a):

$$\begin{aligned} & \varphi(\delta(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ &= \delta(Fx, Gy) - k \max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}. \end{aligned}$$

This function  $\varphi$  is used by many authors with single maps, for example: [11] in Theorem 1, Example 3.4 in [17].

For (b):

$$\begin{aligned} & \varphi(\delta(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ &= \delta^2(Fx, Gy) - c_1 \max\{d^2(fx, gy), d^2(fx, Fx), d^2(gy, Gy)\} \\ & \quad - c_2 \max\{d(fx, Fx)d(fx, Gy), d(gy, Gy)d(gy, Fx)\} \\ & \quad - c_3 d(fx, Gy)d(gy, Fx). \end{aligned}$$

This function  $\varphi$  is Example 2 of [21].

For (c):

$$\begin{aligned} & \varphi(\delta(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ &= \delta(Fx, Gy) - [\alpha\delta^{p-1}(Fx, Gy)d(fx, gy) + \beta\delta^{p-2}(Fx, Gy)d(fx, Fx)d(gy, Gy) + \gamma d^{p-1}(fx, Gy)d(gy, Fx) + \delta d(fx, Gy)d^{p-1}(gy, Fx)]^{\frac{1}{p}}. \end{aligned}$$

For  $p = 3$ , we have Example 3.4 of [4] and Example 3 of [22]. If we take  $p = 2$ ,  $\varphi$  is Example 1 of [19].

For (d):

$$\begin{aligned} & \varphi(\delta(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ = & \delta^2(Fx, Gy) - \frac{1}{\alpha} \left[ \beta d^2(fx, gy) + \frac{\gamma d(fx, Gy) d(gy, Fx)}{1 + \delta d^2(fx, Fx) + \epsilon d^2(gy, Gy)} \right]. \end{aligned}$$

This function  $\varphi$  is that one of Example 6 of [18].

And for (e):

$$\begin{aligned} & \varphi(\delta(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ = & \delta(Fx, Gy) - [\alpha d^p(fx, gy) + \beta d^p(fx, Fx) + \gamma d^p(gy, Gy)]^{\frac{1}{p}} \\ & - \delta [d(fx, Gy) d(gy, Fx)]^{\frac{1}{2}} \end{aligned}$$

■

**2.9 Corollary** *Let  $f, g$  be two self-maps of a metric space  $(\mathcal{X}, d)$  and let  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Suppose that, for all  $x, y \in \mathcal{X}$ , we have the inequality*

$$(f) \quad \delta^p(Fx, Gy) < \alpha d^p(fx, gy) + \beta d^p(fx, Fx) + \gamma d^p(gy, Gy)$$

*such that  $0 < \alpha \leq 1$ ,  $\beta$  and  $\gamma \geq 0$  and  $p \in \mathbb{N}^* = \{1, 2, \dots\}$  whenever the right hand side of the above inequality is positive. Then  $f, g, F$  and  $G$  have a unique common fixed point.*

**Proof**

We give this corollary because it is an interesting particular case of the previous corollary. We obtain the result by using (e) in Corollary 2.8 with  $\delta = 0$ . ■

### 3 Two other type common fixed point theorems

We begin by a Greguš type common fixed point theorem. As we already said, in 1998, Pathak et al. [16] gave an extension of compatibility of type (A) by introducing the concept of compatibility of type (C) and they proved a common fixed point theorem of Greguš type for four compatible maps of type (C) in a Banach space. Further, Djoudi and Nisse [5] extended the result of [16] by weakening compatibility of type (C) to the weak one without continuity.

Our objective here is to establish a common fixed point theorem for four occasionally weakly compatible single and set-valued maps of Greguš type in a metric space which improves the results of [5], [16] and others.

**3.1 Theorem** *Let  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  be maps,  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$  and satisfying the following condition:*

$$(3.1) \quad \delta^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1 - a)d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)]$$

for all  $x$  and  $y \in \mathcal{X}$ , where  $0 < a \leq 1$  and  $p \geq 1$ . Then  $f, g, F$  and  $G$  have a unique common fixed point.

**Proof**

Since  $f, F$  and  $g, G$  are owc, as in proof of Theorem 2.3, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu, gGv \subseteq Ggv$ .

i) As in proof of Theorem 2.3, we begin to show existence of a common fixed point. We have,

$$\delta^p(Fu, Gv) \leq \Psi(ad^p(fu, gv) + (1 - a)d^{\frac{p}{2}}(gv, Fu)d^{\frac{p}{2}}(fu, Gv))$$

and by the properties of  $\delta$  and  $\Psi$ , we get

$$d^p(fu, gv) \leq \delta^p(Fu, Gv) \leq \Psi(d^p(fu, gv)).$$

So, if  $d(fu, gv) > 0$ ,  $\Psi(t) < t$  for  $t > 0$ , we obtain

$$d^p(fu, gv) \leq \delta^p(Fu, Gv) \leq \Psi(d^p(fu, gv)) < d^p(fu, gv)$$

which is a contradiction, thus we have  $d(fu, gv) = 0$ , hence  $fu = gv$ .

Again, if  $d(f^2u, fu) > 0$ , then by (3.1), we have

$$\delta^p(Ffu, Gv) \leq \Psi[ad^p(f^2u, gv) + (1 - a)d^{\frac{p}{2}}(gv, Ffu)d^{\frac{p}{2}}(f^2u, Gv)]$$

and hence

$$d^p(f^2u, fu) \leq \delta^p(Ffu, Gv) \leq \Psi[d^p(f^2u, fu)]$$

Since  $d(f^2u, fu) > 0$ , we obtain

$$d^p(f^2u, fu) \leq \delta^p(Ffu, Gv) \leq \Psi[d^p(f^2u, fu)] < d^p(f^2u, fu)$$

what it is impossible. Then we have  $d(f^2u, fu) = 0$ ; i.e.,  $f^2u = fu$ . Similarly, we can prove that  $g^2v = gv$ , let  $fu = w$  then,  $fw = w = gw, w \in Fw$  and  $w \in Gw$ , this completes the proof of the existence.

ii) For the uniqueness, let  $w'$  be a second common fixed point of  $f, g, F$  and  $G$  with  $w' \neq w$ . Then,  $d(w, w') = d(fw, gw') \leq \delta(Fw, Gw')$  and, by assumption (3.1), we obtain

$$\delta^p(Fw, Gw') \leq \Psi[ad^p(fw, gw') + (1 - a)d^{\frac{p}{2}}(fw, Gw')d^{\frac{p}{2}}(gw', Fw)]$$

and thus

$$d^p(w, w') = d^p(fw, gw') \leq \delta^p(Fw, Gw') \leq \Psi[d^p(w, w')] < d^p(w, w')$$

Since  $d(w, w') > 0$ , we have a contradiction. So,  $w = w'$ . ■

**3.2 Theorem** Let  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  be maps,  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$  and satisfying the following condition:

$$\begin{aligned} \delta^p(Fx, Gy) \leq & \Psi[ad^p(fx, gy) + (1-a)\max\{\alpha d^p(fx, Fx), \beta d^p(gy, Gy), \\ & d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(gy, Fx), d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy), \\ & \frac{1}{2}(d^p(fx, Fx) + d^p(gy, Gy))\}] \end{aligned}$$

for all  $x$  and  $y \in \mathcal{X}$ , where  $0 < a \leq 1$ ,  $0 < \alpha, \beta \leq 1$  and  $p \geq 1$ . Then  $f, g, F$  and  $G$  have a unique common fixed point.

**Proof**

Since  $f, F$  and  $g, G$  are owc, as in proof of Theorem 2.2, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu, gGv \subseteq Ggv$ . Since  $\Psi$  is a nondecreasing function and since for any real numbers  $c$  and  $d$ ,  $\frac{c+d}{2} \leq \max\{c, d\}$  we have, for all  $x, y \in \mathcal{X}$ ,

$$\begin{aligned} \delta^p(Fx, Gy) \leq & \Psi[ad^p(fx, gy) + (1-a)\max\{d^p(fx, Fx), d^p(gy, Gy) \\ & d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(gy, Fx), d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)\}] \end{aligned}$$

and, for  $u$  and  $v$ ,

$$\delta^p(Fu, Gv) \leq \Psi[ad^p(fu, gv) + (1-a)d^{\frac{p}{2}}(gv, Fu)d^{\frac{p}{2}}(fu, Gv)].$$

The continuation of the proof is identical of that of Theorem 2.3. ■

If in (3.1), we replace  $\delta$  with  $H$  and  $\Psi(t) < t$  with  $\Psi(t) \leq t$ , we can prove the

**3.3 Theorem** Let  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  be maps,  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $u$  and  $v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu, gGv \subseteq Ggv$ . Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) \leq t$  and satisfying the following condition:

$$(3.3) \quad H^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1-a)d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)]$$

for all  $x$  and  $y \in \mathcal{X}$ , where  $0 < a \leq 1$  and  $p \geq 1$ .

If  $fu = gv$  is a common fixed point of  $f$  and  $g$ , then  $fu$  is a common fixed point of  $f, g, F$  and  $G$  and  $Fu = Gv$ .

**Proof**

Since  $gv \in Gv, fu \in Fu$  and  $f^2u \in fFu \subseteq Ffu$ , we have

$$\begin{aligned} d(gv, Fu) & \leq H(Fu, Gv), \quad d(fu, Gv) \leq H(Fu, Gv), \\ d(gv, Ffu) & \leq H(Ffu, Gv) \quad \text{and} \quad d(f^2u, Gv) \leq H(Ffu, Gv). \end{aligned}$$

From the nondecrease of  $\Psi$ , we obtain

$$\begin{aligned} H^p(Ffu, Gv) & \leq \Psi[ad^p(f^2u, gv) + (1-a)d^{\frac{p}{2}}(gv, Ffu)d^{\frac{p}{2}}(f^2u, Gv)] \\ & \leq \Psi[ad^p(f^2u, gv) + (1-a)H^p(Ffu, Gv)] \end{aligned}$$

$$H^p(Fu, Gv) \leq \Psi[ad^p(fu, gv) + (1 - a)H^p(Fu, Gv)]$$

and

$$H^p(Fu, Ggv) \leq \Psi[ad^p(fu, g^2v) + (1 - a)H^p(Fu, Ggv)].$$

Now, if  $Fu \neq Gv$ , since, for every  $t > 0$ ,  $\Psi(t) \leq t$ ,

$$H^p(Fu, Gv) \leq ad^p(fu, gv) + (1 - a)H^p(Fu, Gv).$$

Consequently,  $H(Fu, Gv) \leq d(fu, gv)$  and  $fu \neq gv$ . We have shown that if  $fu = gv$ , then  $Fu = Gv$ . By similar proofs, if  $f^2u = gv$ , then  $Gv = Ffu$  and if  $fu = g^2v$ , then  $Fu = Ggv$ . The proof is finished. ■

**3.4 Remark** Obviously, Theorems 3.1 and 3.2 extend the results of [5], [16] and others to the class of four single and set-valued maps. In particular, Theorem 3.2 improves the cited results since we not required the closeness of the sets  $F(\mathcal{X})$  and  $G(\mathcal{X})$ , also we deleted the inclusions  $F(\mathcal{X}) \subset f(\mathcal{X})$  and  $G(\mathcal{X}) \subset g(\mathcal{X})$  in [5], we weakened the weak compatibility in [5] and the compatibility of type (C) in [16] to the wider one cited occasional weak compatibility and we deleted the continuity which is indispensable in [16] and the upper semicontinuity imposed on  $\Psi$  in [5].

If we put  $f = g$  in Theorem 3.1, then we get the corollary:

**3.5 Corollary** Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  be a map and let  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps. Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$ . Suppose pairs  $\{f, F\}$  and  $\{f, G\}$  are owc and satisfy the inequality

$$\delta^p(Fx, Gy) \leq \Psi[ad^p(fx, fy) + (1 - a)d^{\frac{p}{2}}(fy, Fx)d^{\frac{p}{2}}(fx, Gy)]$$

for all  $x, y \in \mathcal{X}$ , where  $0 < a \leq 1$  and  $p \geq 1$ , then  $f, F$  and  $G$  have a unique common fixed point.

If we put  $f = g$  and  $F = G$  in Theorem 3.1, then we obtain the following result:

**3.6 Corollary** Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  be a map and let  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued mapping such that  $f$  and  $F$  are owc. Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$ . If

$$\begin{aligned} \delta^p(Fx, Fy) \leq & \Psi[ad^p(fx, fy) + (1 - a)\max\{\alpha d^p(fx, Fx), \beta d^p(fy, Fy), \\ & d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(fy, Fx), d^{\frac{p}{2}}(fy, Fx)d^{\frac{p}{2}}(fx, Fy), \\ & \frac{1}{2}(d^p(fx, Fx) + d^p(fy, Fy))\}] \end{aligned}$$

for all  $x, y \in \mathcal{X}$ , where  $0 < a \leq 1$ ,  $\{\alpha, \beta\} \subset ]0, 1]$  and  $p \geq 1$ , then  $f$  and  $F$  have a unique common fixed point.

Now, we end our work by establishing a near-contractive common fixed point theorem which improves those given by Elamrani and Mehdaoui [6], Mbarki [13] and others since our version does not impose continuity and we use occasional weak compatibility which is more general than compatibility and weak compatibility; also we delete, on  $\Phi$ , some strong conditions which are necessary in papers [6] and [13] on a metric space instead of a complete metric space.

**3.7 Theorem** *Let  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  be maps,  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps and  $\Phi$  be a nondecreasing function of  $[0, \infty[$  into itself such that  $\Phi(t) = 0$  iff  $t = 0$  and satisfying inequality*

$$(3.7) \quad \begin{aligned} \Phi(\delta(Fx, Gy)) &\leq \alpha(d(fx, gy))\Phi(d(fx, gy)) \\ &\quad + \beta(d(fx, gy))[\Phi(d(fx, Gy)) + \Phi(d(gy, Gy))] \\ &\quad + \gamma(d(fx, gy))[\Phi(d(fx, Fx)) + \Phi(d(gy, Fx))] \end{aligned}$$

for all  $x, y \in \mathcal{X}$  and  $\alpha, \beta, \gamma : [0, \infty[ \rightarrow [0, 1[$  satisfying condition

$$(4) \quad \alpha(t) + \beta(t) + \gamma(t) < 1 \quad \forall t > 0.$$

If the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc, then  $f, g, F$  and  $G$  have a unique common fixed point in  $\mathcal{X}$ .

**Proof**

Since  $f, F$  and  $g, G$  are owc, as in proof of Theorem 2.2, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu, gGv \subseteq Ggv$ .

i) First we prove that  $fu = gv$ . By (3.7), we have

$$\begin{aligned} \Phi(\delta(Fu, Gv)) &\leq \alpha(d(fu, gv))\Phi(d(fu, gv)) \\ &\quad + \beta(d(fu, gv))[\Phi(d(fu, Fu)) + \Phi(d(gv, Gv))] \\ &\quad + \gamma(d(fu, gv))[\Phi(d(fu, Fu)) + \Phi(d(gv, Fu))] \\ &= \alpha(d(fu, gv))\Phi(d(fu, gv)) + \beta(d(fu, gv))\Phi(d(fu, Fu)) \\ &\quad + \gamma(d(fu, gv))\Phi(d(gv, Fu)). \end{aligned}$$

If  $d(fu, gv) > 0$ , since  $\Phi$  is nondecreasing and  $\Phi(t) = 0$  iff  $t = 0$ , from inequalities (3.7) and (4) we get

$$\begin{aligned} \Phi(d(fu, gv)) &\leq \Phi(\delta(Fu, Gv)) \\ &\leq \alpha(d(fu, gv))\Phi(d(fu, gv)) + \beta(d(fu, gv))\Phi(d(fu, Fu)) \\ &\quad + \gamma(d(fu, gv))\Phi(d(gv, Fu)) \\ &\leq [\alpha(d(fu, gv)) + \beta(d(fu, gv)) + \gamma(d(fu, gv))]\Phi(d(fu, gv)) \\ &< \Phi(d(fu, gv)) \end{aligned}$$

which is a contradiction. Hence  $d(fu, gv) = 0$  and thus  $fu = gv$ .

Now we claim that  $f^2u = fu$ . Suppose not, since  $\Phi$  is nondecreasing and  $\Phi(t) = 0$  iff  $t = 0$ , the use of (3.7) and (4) gives

$$\begin{aligned}
 \Phi(d(f^2u, fu)) &\leq \Phi(\delta(Ffu, Gv)) \\
 &\leq \alpha(d(f^2u, gv))\Phi(d(f^2u, gv)) \\
 &\quad + \beta(d(f^2u, gv))[\Phi(d(f^2u, Gv)) + \Phi(d(gv, Gv))] \\
 &\quad + \gamma(d(f^2u, gv))[\Phi(d(f^2u, Ffu)) + \Phi(d(gv, Ffu))] \\
 &= \alpha(d(f^2u, fu))\Phi(d(f^2u, fu)) + \beta(d(f^2u, fu))\Phi(d(f^2u, Gv)) \\
 &\quad + \gamma(d(f^2u, fu))\Phi(d(fu, Ffu)) \\
 &\leq [\alpha(d(f^2u, fu)) + \beta(d(f^2u, fu)) \\
 &\quad + \gamma(d(f^2u, fu))]\Phi(d(f^2u, fu)) \\
 &< \Phi(d(f^2u, fu))
 \end{aligned}$$

this contradiction implies that  $\Phi(d(f^2u, fu)) = 0$  and hence  $f^2u = fu$ . Similarly, we can prove that  $g^2v = gv$ . So, if  $w = fu = gv$  therefore  $fw = w = gw$ ,  $w \in Fw$  and  $w \in Gw$ . Existence of a common fixed point is proved.

ii) Assume that there exists a second common fixed point  $w'$  of  $f, g, F$  and  $G$  such that  $w' \neq w$ . We have  $d(w, w') = d(fw, gw') \leq \delta(Fw, Gw')$ . Since  $d(w, w') > 0$ , by inequality (3.7) and properties of functions  $\Phi, \alpha$  and  $\beta$ , we obtain

$$\begin{aligned}
 \Phi(d(w, w')) &\leq \Phi(\delta(Fw, Gw')) \\
 &\leq \alpha(d(fw, gw'))\Phi(d(fw, gw')) \\
 &\quad + \beta(d(fw, gw'))[\Phi(d(fw, Gw')) + \Phi(d(gw', Gw'))] \\
 &\quad + \gamma(d(fw, gw'))[\Phi(d(fw, Fw)) + \Phi(d(gw', Fw))] \\
 &= \alpha(d(w, w'))\Phi(d(w, w')) + \beta(d(w, w'))\Phi(d(w, Gw')) \\
 &\quad + \gamma(d(w, w'))\Phi(d(w', Fw)) \\
 &\leq [\alpha(d(w, w')) + \beta(d(w, w')) + \gamma(d(w, w'))]\Phi(d(w, w')) \\
 &< \Phi(d(w, w'))
 \end{aligned}$$

this contradiction implies that  $\Phi(d(w, w')) = 0$ , hence  $w' = w$ . ■

**3.8 Remark** The above theorem remains valid if we replace inequality (3.7) by the following one:

$$\begin{aligned}
 \Phi(\delta(Fx, Gy)) &\leq \alpha(d(fx, gy))\Phi(d(fx, gy)) \\
 &\quad + \beta(d(fx, gy)) \max\{\Phi(d(fx, Gy)), \Phi(d(gy, Gy))\} \\
 &\quad + \gamma(d(fx, gy))[\Phi(d(fx, Fx)) + \Phi(d(gy, Fx))].
 \end{aligned}$$

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