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NEW PERMUTATION CODING AND EQUIDISTRIBUTION OF SET-VALUED STATISTICS

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Abstract. A new coding for permutations is explicitly constructed and its association with the classical Lehmer coding provides a bijection of the symmetric group onto itself serving to show that six bivariable set-valued statistics are equidistributed on that group. This extends a recent result due to Cori valid for integer-valued statistics.

1. Introduction

In a recent paper Cori [Cor08] updates a classical algorithm constructed by Ossona de Mendez and Rosenstiehl [OR04] that provides a one-to-one correspondence between rooted hypermaps and indecomposable permutations. He further constructs a bijection of the symmetric group $\mathfrak{S}_n$ onto itself that maps each permutation having p cycles and q left-to-right maxima onto another one having q cycles and p left-to-right maxima. Moreover, by using an encoding of permutations by Dyck paths due to Roblet and Viennot [RV96] he also shows that three bivariable *integer-valued* statistics, introduced in the next paragraph, are equidistributed on $\mathfrak{S}_n$. The purpose of this paper is to show that all those results can be extended to *set-valued* statistics and that the construction of the underlying bijection is very simple; it involves two permutation codings called the A-*code* and the B-*code*.

The first one, classically referred to as the *Lehmer code* [Le60] or the *inversion table*, goes back, in fact, to more ancient authors (Rothe, Rodrigues, Netto), as knowledgeably stated by Knuth ([Kn98], Ex. 5.1.1-7, p. 14). The second one is a *new* coding that takes the cycle decomposition of permutations into account. Although the motivation of the paper was to prove the equidistribution of several set-valued statistics, its novelty is to fully describe that B-code and exploit its basic properties.

The set-valued statistics in question are introduced as follows. Let $w = x_1 x_2 \cdots x_n$ be a word of length n, whose letters are positive integers. The **L**eft to right **ma**ximum **p**lace set, $\mathrm{Lmap}\, w$, of w is defined to be the set of all *places* i such that $x_j < x_i$ for all $j < i$, while the **R**ight to left **mi**nimum **l**etter set, $\mathrm{Rmil}\, w$, of w is the set of all *letters* x_i such that $x_j > x_i$ for all $j > i$.

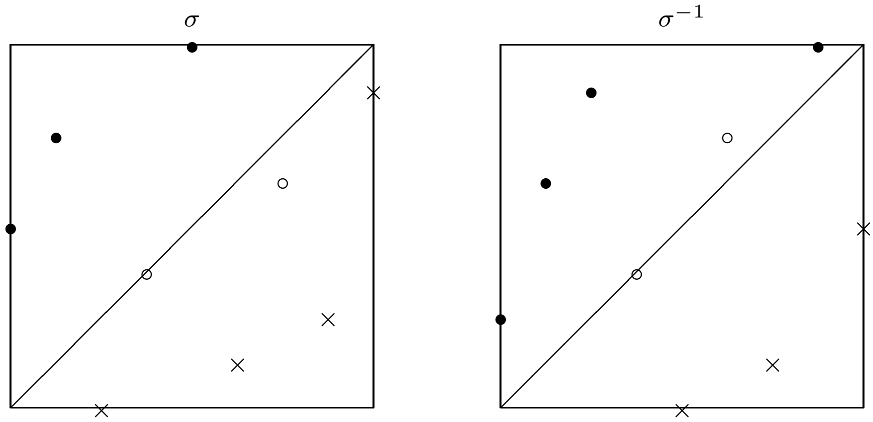
When the word w is a permutation of $12 \cdots n$ that we shall preferably denote by $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and the bijection $i \mapsto \sigma(i)$ $(1 \le i \le n)$ has

r disjoint cylces, whose *minimum* elements are $c_1, c_2, \ldots, c_r$, respectively, define $\operatorname{Cyc}\sigma$ to be the set

$$\operatorname{Cyc}\sigma := \{c_1, c_2, \ldots, c_r\}.$$

When σ is a permutation, the *cardinalities* of $\operatorname{Lmap}\sigma$, $\operatorname{Rmil}\sigma$ and $\operatorname{Cyc}\sigma$ are denoted by $\operatorname{lmap}\sigma$, $\operatorname{rmil}\sigma$ and $\operatorname{cyc}\sigma$, respectively, and classically referred to as the *number of left-to-right maxima*, *number of right-to-left minima*, *number of cycles*.

In Fig. 1 the graphs of the permutation $\sigma = 5, 7, 1, 4, 9, 2, 6, 3, 8$ and its inverse $\sigma^{-1} = 3, 6, 8, 4, 1, 7, 2, 9, 5$ have been drawn. The set $\operatorname{Lmap}\sigma$ (resp. $\operatorname{Lmap}\sigma^{-1}$) is the set of the *abscissas* of the "bullets," while $\operatorname{Rmil}\sigma$ (resp. $\operatorname{Rmil}\sigma^{-1}$) is the set of the *ordinates* of the "crosses." The set-valued statistics "Leh," "Rmil Leh" and "Max Leh" will be further introduced. Notice that $\operatorname{lmap}\sigma = \operatorname{rmil}\sigma^{-1} = 3$, $\operatorname{rmil}\sigma = \operatorname{lmap}\sigma^{-1} = 4$. As σ is the product of the disjoint cycles $(1\,5\,9\,8\,3)(4)(2\,7\,6)$, we have $\operatorname{Cyc}\sigma = \operatorname{Cyc}\sigma^{-1} = \{1, 2, 4\}$ and $\operatorname{cyc}\sigma = \operatorname{cyc}\sigma^{-1} = 3$.



$\sigma =$	5	7	1	4	9	2	6	3	8	$\sigma^{-1} =$	3	6	8	4	1	7	2	9	5
Lmap	1	2			5						1	2	3					8	
Rmil			1			2		3	8						1		2		5
Leh	1	2	1	2	5	2	5	3	8		1	2	3	2	1	5	2	8	5
Rmil Leh			1			2		3	8						1		2		5
Max Leh	1	2			5						1	2	3					8	

Fig. 1. Graphs of σ and of its inverse σ^{-1}

First, recall Cori's result [Cor08].

The three pairs of integer-valued statistics $(\operatorname{rmil}, \operatorname{cyc})$, $(\operatorname{cyc}, \operatorname{rmil})$ *and* $(\operatorname{lmap}, \operatorname{rmil})$ *are equidistributed on* $\mathfrak{S}_n$.

The equidistribution of the first two pairs (resp. of the last two ones) is proved by updating the Ossona-de-Mendez-Rosenstiehl algorithm [OR04] on hypermaps (resp. by using the Roblet-Viennot Dyck path encoding [RV96]). Second, the set-valued statistics "Cyc" and "Rmil" (or "Lmap") are known to be equidistributed on $\mathfrak{S}_n$. This is one of the properties of the first fundamental transformation [Lo83, chap. 10]. Our main result is the following theorem.

Theorem 1. *The six bivariable set-valued statistics* $(\mathrm{Cyc},\mathrm{Rmil})$, $(\mathrm{Cyc},\mathrm{Lmap})$, $(\mathrm{Rmil},\mathrm{Lmap})$, $(\mathrm{Rmil},\mathrm{Cyc})$, $(\mathrm{Lmap},\mathrm{Rmil})$, $(\mathrm{Lmap},\mathrm{Cyc})$ *are all equidistributed on* $\mathfrak{S}_n$.

Based on two permutation codings, the A-code and B-code, introduced in Sections 2 and 3, respectively, we construct a bijection ϕ of $\mathfrak{S}_n$ onto itself (see (4.1)) having the following property:

$$(\mathrm{Lmap},\mathrm{Rmil})\,\sigma = (\mathrm{Lmap},\mathrm{Cyc})\,\phi(\sigma) \qquad (\sigma\in\mathfrak{S}_n). \tag{1.1}$$

Let $\mathbf{i}:\sigma\mapsto\sigma^{-1}$. As

$$\mathrm{Cyc}\,\mathbf{i}\,\sigma = \mathrm{Cyc}\,\sigma; \tag{1.2}$$

$$\mathrm{Rmil}\,\mathbf{i}\,\sigma = \mathrm{Lmap}\,\sigma; \tag{1.3}$$

(see Fig. 1 for the second relation), it follows from (1.1) that the chain

$$\begin{matrix} \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n & \xrightarrow{\phi^{-1}} & \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n & \xrightarrow{\phi} & \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n \\ \binom{\mathrm{Cyc}}{\mathrm{Rmil}} && \binom{\mathrm{Cyc}}{\mathrm{Lmap}} && \binom{\mathrm{Rmil}}{\mathrm{Lmap}} && \binom{\mathrm{Lmap}}{\mathrm{Rmil}} && \binom{\mathrm{Lmap}}{\mathrm{Cyc}} && \binom{\mathrm{Rmil}}{\mathrm{Cyc}} \end{matrix} \tag{1.4}$$

provides all the bijections needed to prove Theorem 1. Note that (1.1), on the one hand, and (1.2)–(1.3), on the other hand, are reproduced as

$$\begin{matrix} \mathfrak{S}_n & \xrightarrow{\phi} & \mathfrak{S}_n \\ \binom{\mathrm{Lmap}}{\mathrm{Rmil}} && \binom{\mathrm{Lmap}}{\mathrm{Cyc}} \end{matrix} \quad \text{and} \quad \begin{matrix} \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n \\ \binom{\mathrm{Cyc}}{\mathrm{Rmil}} && \binom{\mathrm{Cyc}}{\mathrm{Lmap}} \end{matrix}$$

Let $A=(I_1,I_2,\ldots,I_h)$ be an ordered partition of the set $[n]:=\{1,2,\ldots,n\}$ into disjoint non-empty *intervals*, such that $\max I_j+1=\min I_{j+1}$ for $j=1,2,\ldots,h-1$. A permutation σ from $\mathfrak{S}_n$ is said to be *A-decomposable*, if each I_j is the *smallest interval* such that $\sigma(I_j)=I_j$ (see [Com74], p. 261, exercise 14). For instance, $\sigma=\binom{1\,2\,3\,4\,5}{2\,1\,5\,4\,3}$ is A-decomposable, with $A=(\{1,2\},\{3,4,5\})$. It is convenient to write $\mathrm{Decomp}\,\sigma=A$, if σ is A-decomposable. A permutation is said to be *indecomposable*, if it is A-decomposable, with $A=([n])$. The bijection ϕ defined in (4.1) has the further property

$$\mathrm{Decomp}\,\phi(\sigma)=\mathrm{Decomp}\,\sigma \quad (\sigma\in\mathfrak{S}_{\mathbf{n}}). \tag{1.5}$$

As we evidently have

$$\mathrm{Decomp}\,\mathbf{i}\,\sigma=\mathrm{Decomp}\,\sigma, \tag{1.6}$$

the following result holds.

Theorem 2. *Let A be an ordered partition of the set $[n]$ into disjoint consecutive non-empty intervals. Then,* $(\mathrm{Cyc}, \mathrm{Rmil})$, $(\mathrm{Cyc}, \mathrm{Lmap})$, $(\mathrm{Rmil}, \mathrm{Lmap})$, $(\mathrm{Rmil}, \mathrm{Cyc})$, $(\mathrm{Lmap}, \mathrm{Rmil})$, $(\mathrm{Lmap}, \mathrm{Cyc})$ *are equidistributed on the set of all A-decomposable permutations from $\mathfrak{S}_n$.*

The next corollary is relevant to the study of hypermaps, as the set of rooted hypermaps with darts $1, 2, \ldots, n$ is in one-to-one correspondence with the subset of indecomposable permutations from $\mathfrak{S}_{n+1}$ (see [Cor08, CM92]).

Corollary 3. *The statistics* $(\mathrm{Cyc}, \mathrm{Rmil})$, $(\mathrm{Cyc}, \mathrm{Lmap})$, $(\mathrm{Rmil}, \mathrm{Lmap})$, $(\mathrm{Rmil}, \mathrm{Cyc})$, $(\mathrm{Lmap}, \mathrm{Rmil})$, $(\mathrm{Lmap}, \mathrm{Cyc})$ *are equidistributed on the set of all indecomposable permutations from $\mathfrak{S}_n$.*

The construction of the bijection ϕ together with the proofs of Theorem 2, and Corollary 3 are given in Section 4. It is followed by the algorithmic definitions of both A-code and B-code in Section 5. Tables and concluding remarks are reproduced in Section 6.

2. The A-code

The *Lehmer code* [Le60] of a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ is defined to be the sequence $\mathrm{Leh}\, w = (a_1, a_2, \ldots, a_n)$, where for each $i = 1, 2, \ldots, n$

$$a_i := \#\{j : 1 \le j \le i,\, \sigma(j) \le \sigma(i)\}.$$

The sequence $\mathrm{Leh}\, w$ belongs to SE_n of all sequences $a = (a_1, a_2, \ldots, a_n)$, called *subexcedant*, such that $1 \le a_i \le i$ for each $i = 1, 2, \ldots, n$. For such a sequence it makes sense to define the set, denoted by $\mathrm{Max}\, a$, of all letters (or places!) a_i such that $a_i = i$.

Under the graphs drawn in Fig. 1 the Lehmer codes $\mathrm{Leh}\, \sigma$ and $\mathrm{Leh}\, \sigma^{-1}$ have been calculated, as well as the four sets $\mathrm{Rmil}\, \mathrm{Leh}\, \sigma$, $\mathrm{Rmil}\, \mathrm{Leh}\, \sigma^{-1}$, $\mathrm{Max}\, \mathrm{Leh}\, \sigma$ and $\mathrm{Max}\, \mathrm{Leh}\, \sigma^{-1}$. The next Proposition is geometrically evident and given without proof. It shows that the set-valued statistics "Lmap" and "Rmip" can be directly read from the Lehmer code.

Proposition 4. *For each permutation σ we have:*

(2.1) $$\mathrm{Rmil}\, \mathrm{Leh}\, \sigma = \mathrm{Rmil}\, \sigma;$$

(2.2) $$\mathrm{Max}\, \mathrm{Leh}\, \sigma = \mathrm{Lmap}\, \sigma.$$

We then define the A-*code* of a permutation σ to be

(2.3) $$\text{A-code}\, \sigma := \mathrm{Leh}\, \mathbf{i}\, \sigma.$$

Hence, $\mathrm{Max}\, \text{A-code}\, \sigma = \mathrm{Max}\, \mathrm{Leh}\, \mathbf{i}\, \sigma = \mathrm{Lmap}\, \mathbf{i}\, \sigma = \mathrm{Rmil}\, \sigma$. Furthermore, $\mathrm{Rmil}\, \text{A-code}\, \sigma = \mathrm{Rmil}\, \mathrm{Leh}\, \mathbf{i}\sigma = \mathrm{Rmil}\, \mathbf{i}\, \sigma = \mathrm{Lmap}\, \sigma$. As Leh is a bijection of the symmetric group $\mathfrak{S}_n$ onto SE_n, we obtain the following result.

Theorem 5. *The* A-code *is a bijection of* $\mathfrak{S}_n$ *onto* SE_n *having the property:*

$$(2.4)\qquad (\mathrm{Rmil}, \mathrm{Lmap})\,\sigma = (\mathrm{Max}, \mathrm{Rmil})\ \text{A-code}\,\sigma \quad (\sigma \in \mathfrak{S}_n).$$

An algorithmic definition of the A-code will be given in Section 5.

3. The B-code

The B-code is based on the decomposition of each permutation as product of disjoint cycles. For a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and each $i = 1, 2, \ldots, n$ let $k := k(i)$ be the *smallest* integer $k \geq 1$ such that $\sigma^{-k}(i) \leq i$. Then, define

$$\text{B-code}\,\sigma = (b_1, b_2, \ldots, b_n) \quad \text{with} \quad b_i := \sigma^{-k(i)}(i) \quad (1 \leq i \leq n).$$

For example, with the permutation $\sigma = \binom{1\,2\,3\,4\,5\,6}{3\,4\,5\,2\,6\,1}$ we have:
$\sigma^{-1}(1) = 6$, $\sigma^{-2}(1) = 5$, $\sigma^{-3}(1) = 3$, $\sigma^{-4}(1) = 1$, so that $b_1 = 1$;
$\sigma^{-1}(2) = 4$, $\sigma^{-2}(2) = 2$, so that $b_2 = 2$; $\sigma^{-1}(3) = 1$, so that $b_3 = 1$;
$\sigma^{-1}(4) = 2$, so that $b_4 = 2$; $\sigma^{-1}(5) = 3$, so that $b_5 = 3$;
$\sigma^{-1}(6) = 5$, so that $b_6 = 5$. Thus, $\text{B-code}\,\sigma = (1, 2, 1, 2, 3, 5)$.

An alternate definition is the following. First, the B-code of the unique permutation from $\mathfrak{S}_1$ is defined to be the sequence $(1) \in \mathrm{SE}_1$. Let $n \geq 2$. When writing each permutation $\sigma \in \mathfrak{S}_n$ of order $n \geq 2$ as a product of its disjoint cycles, the removal of n yields a permutation σ' of order $(n-1)$. Let $b' = (b'_1, b'_2, \ldots, b'_{n-1})$ be the B-code of σ'. We define the B-code of σ to be $b := (b'_1, b'_2, \ldots, b'_{n-1}, \sigma^{-1}(n))$. By induction on n, we immediately see that the B-code is a bijection of $\mathfrak{S}_n$ onto SE_n.

The following Theorem shows that the set-valued statistics "Lmap" and "Cyc" can be directly read from the B-code.

Theorem 6. *The* B-code *is a bijection of* $\mathfrak{S}_n$ *onto* SE_n *having the property:*

$$(3.1)\qquad (\mathrm{Cyc}, \mathrm{Lmap})\,\sigma = (\mathrm{Max}, \mathrm{Rmil})\ \text{B-code}\,\sigma \quad (\sigma \in \mathfrak{S}_n).$$

Proof. By induction, suppose that $\mathrm{Lmap}\,\sigma' = \mathrm{Rmil}\,b'$ and $\mathrm{Cyc}\,\sigma' = \mathrm{Max}\,b'$. If n is a fixed point of σ, so that $\sigma^{-1}(n) = n$ and $b = (b'_1, \cdots, b'_{n-1}, n)$, then $\mathrm{Lmap}\,\sigma = \mathrm{Lmap}\,\sigma' \cup \{n\} = \mathrm{Rmil}\,b' \cup \{n\} = \mathrm{Rmil}\,\sigma$. Also, $\mathrm{Cyc}\,\sigma = \mathrm{Cyc}\,\sigma' \cup \{n\} = \mathrm{Max}\,b' \cup \{n\} = \mathrm{Max}\,b$.

When n is not a fixed point of σ, then σ is a product of the form

$$\sigma = \cdots(\cdots\sigma^{-1}(n)n\sigma(n)\cdots)\cdots$$

while σ' may be expressed as

$$\sigma' = \cdots(\cdots\sigma^{-1}(n)\sigma(n)\cdots)\cdots$$

In particular, $\sigma^{-1}(n) < n$, $\sigma(n) < n$ and $\sigma'(\sigma^{-1}(n)) = \sigma(n)$. We have $\mathrm{Cyc}\,\sigma = \mathrm{Cyc}\,\sigma' = \mathrm{Max}\,b' = \mathrm{Max}\,b$ since $\sigma^{-1}(n) < n$.

To prove $\mathrm{Lmap}\,\sigma = \mathrm{Rmil}\,b$, three cases are to be considered, (*i*) $\sigma(n) = n-1$; (*ii*) $\sigma(n) \neq n-1$ and $\sigma^{-1}(n-1) < \sigma^{-1}(n)$; (*iii*) $\sigma(n) \neq n-1$ and $\sigma^{-1}(n-1) > \sigma^{-1}(n)$, each of them materialized by the following three tableaux:

$$(i)\quad \begin{array}{rcccccc} \mathrm{Id} = & 1 & \cdots & \sigma^{-1}(n) & \cdots & n-1 & n \\ \sigma = & \sigma(1) & \cdots & n & \cdots & \sigma(n-1) & \sigma(n) = n-1 \\ \sigma' = & \sigma(1) & \cdots & \sigma(n) = n-1 & \cdots & \sigma(n-1) & * \end{array}$$

$$(ii)\quad \begin{array}{rcccccccc} \mathrm{Id} = & 1 & \cdots & \sigma^{-1}(n-1) & \cdots & \sigma^{-1}(n) & \cdots & n-1 & n \\ \sigma = & \sigma(1) & \cdots & n-1 & \cdots & n & \cdots & \sigma(n-1) & \sigma(n) \\ \sigma' = & \sigma(1) & \cdots & n-1 & \cdots & \sigma(n) & \cdots & \sigma(n-1) & * \end{array}$$

$$(iii)\quad \begin{array}{rcccccccc} \mathrm{Id} = & 1 & \cdots & \sigma^{-1}(n) & \cdots & \sigma^{-1}(n-1) & \cdots & n-1 & n \\ \sigma = & \sigma(1) & \cdots & n & \cdots & n-1 & \cdots & \sigma(n-1) & \sigma(n) \\ \sigma' = & \sigma(1) & \cdots & \sigma(n) & \cdots & n-1 & \cdots & \sigma(n-1) & * \end{array}$$

In case (*i*) we get $\mathrm{Lmap}\,\sigma = \mathrm{Lmap}\,\sigma'$, $b' = (\ldots, \sigma^{-1}(n))$ and $b = (\ldots, \sigma^{-1}(n), \sigma^{-1}(n))$, then $\mathrm{Rmil}\,b = \mathrm{Rmil}\,b'$.

In case (*ii*) we clearly have: $\mathrm{Lmap}\,\sigma = \mathrm{Lmap}\,\sigma' \cup \{\sigma^{-1}(n)\}$. Also, $b' = (\ldots, \sigma^{-1}(n-1))$ and $b = (\ldots, \sigma^{-1}(n-1), \sigma^{-1}(n))$. Hence, $\mathrm{Lmap}\,\sigma = \mathrm{Lmap}\,\sigma' \cup \{\sigma^{-1}(n)\} = \mathrm{Rmil}\,b' \cup \{\sigma^{-1}(n)\} = \mathrm{Rmil}\,b$.

Finally, comes case (*iii*), which is the hardest one. We have $\mathrm{Lmap}\,\sigma = (\mathrm{Lmap}\,\sigma' \cap [1, \sigma^{-1}(n) - 1]) \cup \{\sigma^{-1}(n)\}$, also $b' = (\ldots, b'_{n-2}, \sigma^{-1}(n-1))$, $b = (\ldots, b'_{n-2}, \sigma^{-1}(n-1), \sigma^{-1}(n))$. But as $\sigma^{-1}(n) < \sigma^{-1}(n-1)$, we have $\mathrm{Rmil}\,b = (\mathrm{Rmil}\,b' \cap [1, \sigma^{-1}(n) - 1]) \cup \{\sigma^{-1}(n)\} = (\mathrm{Lmap}\,\sigma' \cap [1, \sigma^{-1}(n) - 1]) \cup \{\sigma^{-1}(n)\} = \mathrm{Lmap}\,\sigma$. $\square$

4. The bijection ϕ

The bijection ϕ, which is the main ingredient in the chain displayed in (1.4), is simply defined as

$$\phi := (\text{B-code})^{-1} \circ \text{A-code}\,. \tag{4.1}$$

It follows from Theorems 6 and 5 that

$$\begin{aligned} (\mathrm{Cyc}, \mathrm{Lmap})\,\phi(\sigma) &= (\mathrm{Max}, \mathrm{Rmil})\ \text{B-code}\,\phi(\sigma) \\ &= (\mathrm{Max}, \mathrm{Rmil})\ \text{A-code}\,\sigma = (\mathrm{Rmil}, \mathrm{Lmap})\,\sigma. \end{aligned}$$

This proves relation (1.1) and consequently Theorem 1. It also follows from Theorem 5 and/or 6 that the distribution of each pair of statistics stated in Theorem 1 is also equal to the distribution of $(\mathrm{Max}, \mathrm{Rmil})$ on SE_n.

It remains to prove identity (1.5) to achieve the proofs of Theorem 2 and its Corollary. Let $A = ([p_1, q_1], [p_2, q_2], \ldots, [p_h, q_h])$ be an ordered partition of $[n]$ into disjoint non-empty intervals, such that $p_j + 1 = q_{j+1}$ for $j = 1, 2, \ldots, h-1$ and $p_1 = 1, q_h = n$. Let $G(\sigma) = \{(i, \sigma(i)) : 1 \le i \le n\}$ be the graph of a permutation σ from $\mathfrak{S}_n$. Referring to Fig. 2, where the square $[p_j, q_j] \times [p_j, q_j]$ has been materialized by the four points B, B'', D'', D, we see that σ is A-indecomposable, if for every $j = 1, 2, \ldots, h$

(i) the square $[BB''D''D]$ contains the subgraph $\{(i, \sigma(i)) : p_j \le i \le q_j\}$;

(ii) for every l such that $p_j + 1 \le l \le q_j$ the rectangle $[B'B''C''C']$ contains at least one element from $G(\sigma)$.

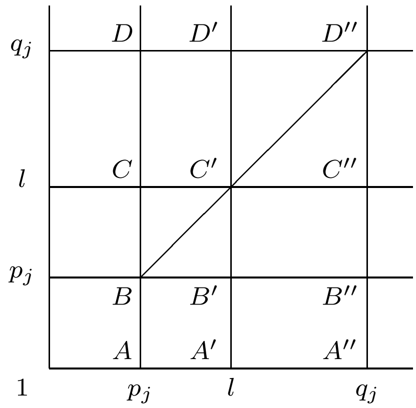


Fig. 2. Graphs of σ and c

We are then led to the following definition.

Definition. Each subexcedant sequence $c = (c_1, c_2, \ldots, c_n)$ from SE_n is said to be A-*decomposable*, if for every $j = 1, 2, \ldots, h$

(i) the *triangle* $[BB''D'']$ contains the subgraph $\{(i, c_i) : p_j \le i \le q_j\}$;

(ii) for every l such that $p_j + 1 \le l \le q_j$ the rectangle $[B'B''C''C']$ contains at least one element (i, c_i) $(l \le i \le q_j)$.

Proposition 6. *A permutation σ from $\mathfrak{S}_n$ is A-decomposable, if and only if its A-code (resp. B-code) is A-decomposable.*

Proof. Let $a = (a_1, a_2, \ldots, a_n)$ be the A-code of a permutation σ. If σ is A-decomposable, then for every $j = 1, 2, \ldots, h$ and $l = p_j, p_j + 1, \ldots, q_j$ the point $(\sigma^{-1}(l), l)$ belongs to the square $[BB''D''D]$. As a_l is equal to 1 plus the number of points $(i, \sigma(i))$ such that $1 \le i < \sigma^{-1}(l)$ and $\sigma(i) < l$, we have $a_l \ge p_j$, so that the point (l, a_l) belongs to the triangle $[BB''D'']$. Conversely, if $(l, a_l) \in [BB''D'']$, then $(\sigma^{-1}(l), l) \in [BB''D''D]$.

Now, the rectangle $[B'B''C''C']$ contains no element from $G(\sigma)$ if and only if all the points $(\sigma^{-1}(l), l)$, $\ldots$, $(\sigma^{-1}(q_j), q_j)$ are in the square $[C'C''D''D']$. This is equivalent to saying that all the quantities $\sigma^{-1}(l)$, l, $\ldots$, $\sigma^{-1}(q_j)$, q_j lie between l and q_j, which is also equivalent to the fact that $a_l, \ldots, a_{q_j}$ lie between l and q_j, that is, the rectangle $[B'B''C''C']$ has no element (i, a_i) $(l \le i \le q_j)$.

Next, let $b = (b_1, b_2, \ldots, b_n)$ be the B-code of σ. If σ is A-decomposable, the restriction of σ to the interval $[p_j, q_j]$ is a product of cycles all elements of which lie between p_j and q_j. By definition of the B-code all the terms $b_{p_j}, \ldots, b_{q_j}$ also lie between p_j and q_j and conversely, if it is the case, all the points $(p_j, \sigma(p_j)), \ldots, (q_j, \sigma(q_j))$ belong to the square $[BB''D''D]$. The same argument can be applied when all the points $(l, \sigma(l)), \ldots, (q_j, \sigma(q_j))$ belong to the square $[C'C''D''D']$. All terms $b_l, \ldots, b_{q_j}$ are greater than or equal to l and the rectangle $[B'B''C''C']$ contains no element of the form (i, b_i) with $l \le i \le q_j$. $\square$

Thus, if σ is A-decomposable, so are A-code σ and the composition product $(\text{B-code})^{-1}$ A-code$(\sigma) = \phi(\sigma)$. This proves identity (1.5) and then Theorem 2 and its corollary.

5. Algorithmic definitions and examples

Although the A-code has been greatly described in various forms (see, e.g., [Kn98], p. 14), we give a full algorithmic definition, which is to be compared with the analogous definition for the B-code.

Algorithmic definition of A-code. Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation of $12\cdots n$. By definition the A-code of σ is the sequence $a = (a_1, a_2, \ldots, a_n)$ where for each $i = 1, 2, \ldots, n$

$$a_i := \#\{j : 1 \le j \le i,\, \sigma^{-1}(j) \le \sigma^{-1}(i)\},$$

or still

$$a_i := \#\{\sigma(k) : 1 \le \sigma(k) \le i,\, k \le \sigma^{-1}(i)\}. \tag{5.1}$$

Thus, a_i is equal to 1 *plus the number of letters less than i, to the left of i, in the word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$.*

For instance, with $\sigma = 4, 6, 1, 2, 3, 5$ the A-code of σ is equal to $a = (1, 2, 3, 1, 5, 2)$: $a_1 = 1$, $a_2 = 2$ as 1 is to the left of 2, $a_3 = 3$ as 1 and 2 are to the left of 3, $a_4 = 1$, as 4 is the leftmost letter of σ, etc. Thus,

$$\text{A-code}(4, 6, 1, 2, 3, 5) = (1, 2, 3, 1, 5, 2). \tag{5.2}$$

Algorithmic definition of A-code^{-1}. Given $a = (a_1, a_2, \ldots, a_n) \in \text{SE}_n$ write a word with n empty places numbered 1 to n from left to right. First, move the letter n to the a_n-th leftmost place; let σ_n be the resulting word (having one non-empty letter!). Next, move $(n-1)$ to the place having $a_{n-1} - 1$ empty letters to its left. Let σ_{n-2} be the resulting word (having two non-empty letters). Move $(n-2)$ to the place having $a_{n-2}-1$ empty letters to its left, etc. Thus, A-code$^{-1}(a)$ is the final permutation σ_1.

For instance, start with $a=(1,2,1,2,3,5)$. We successively get:

$$\begin{array}{ll} \quad\;\; * * * * * * & \\ \sigma_6 = * * * * 6 * & a_6 = 5 \\ \sigma_5 = * * 5 * 6 * & a_5 = 3 \\ \sigma_4 = * 4 5 * 6 * & a_4 = 2 \\ \sigma_3 = 3 4 5 * 6 * & a_3 = 1 \\ \sigma_2 = 3 4 5 * 6 2 & a_2 = 2 \\ \sigma_1 = 3 4 5 1 6 2 & a_1 = 1 \end{array}$$

Thus

$$\text{A-code}^{-1}(1,2,1,2,3,5) = 3,4,5,1,6,2. \tag{5.3}$$

Algorithmic definition of B-code. Let $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)\in\mathfrak{S}_n$. Its B-code $b=(b_1,b_2,\ldots,b_n)$ is calculated as follows. First, b_n is the place occupied by n in $\sigma_n:=\sigma$. Permute the two letters n and $\sigma(n)$ in the word σ. Let σ_{n-1} be the resulting word. Then, b_{n-1} is the place occupied by $(n-1)$ in σ_{n-1}. Next, permute the two letters $(n-2)$ and $\sigma(n-2)$ in σ_{n-1} and let σ_{n-2} be the resulting word. Let b_{n-2} is the place occupied by $(n-2)$ in σ_{n-2}. Permute $(n-3)$ and $\sigma(n-3)$ in σ_{n-2}, etc. The B-code of σ is $(b_1,b_2,\cdots,b_n)$.

Start with $\sigma=3,4,5,2,6,1$. We successively get:

$$\begin{array}{ll} \mathrm{Id} = 1\,2\,3\,4\,5\,6 & \\ \sigma_6 = 3\,4\,5\,2\,6\,1 & b_6 = 5 \\ \sigma_5 = 3\,4\,5\,2\,1\,\mathbf{6} & b_5 = 3 \\ \sigma_4 = 3\,4\,1\,2\,\mathbf{5}\,\mathbf{6} & b_4 = 2 \\ \sigma_3 = 3\,2\,1\,\mathbf{4}\,\mathbf{5}\,\mathbf{6} & b_3 = 1 \\ \sigma_2 = 1\,2\,\mathbf{3}\,\mathbf{4}\,\mathbf{5}\,\mathbf{6} & b_2 = 2 \\ \sigma_1 = 1\,\mathbf{2}\,\mathbf{3}\,\mathbf{4}\,\mathbf{5}\,\mathbf{6} & b_1 = 1 \end{array}$$

Thus

$$\text{B-code}(3,4,5,2,6,1) = (1,2,1,2,3,5). \tag{5.4}$$

Algorithmic definition of B-code^{-1}. Let $b=(b_1,b_2,\ldots,b_n)\in\mathrm{SE}_n$. Start with the identity permutation $\sigma_1=1,2,\ldots,n$. In σ_1 exchange 2 and the letter at the b_2-th place. Let σ_2 be the resulting word. In σ_2 permute 3 and the letter at the b_3-th place. Let σ_3 be the resulting word. In σ_3 permute 4 and the letter at the b_4-th place, etc. The permutation $\sigma=$ B-code$^{-1}\,b$ is the permutation σ_n.

For example, starting with $b = (1, 2, 3, 1, 5, 2)$. We successively form:

$$\begin{array}{ll} \sigma_1 = 1\ \mathbf{2}\ \mathbf{3}\ \mathbf{4}\ \mathbf{5}\ \mathbf{6} & b_1 = 1 \\ \sigma_2 = 1\ 2\ \mathbf{3}\ \mathbf{4}\ \mathbf{5}\ \mathbf{6} & b_2 = 2 \\ \sigma_3 = 1\ 2\ 3\ \mathbf{4}\ \mathbf{5}\ \mathbf{6} & b_3 = 3 \\ \sigma_4 = 4\ 2\ 3\ 1\ \mathbf{5}\ \mathbf{6} & b_4 = 1 \\ \sigma_5 = 4\ 2\ 3\ 1\ 5\ \mathbf{6} & b_5 = 5 \\ \sigma_6 = 4\ 6\ 3\ 1\ 5\ 2 & b_6 = 2 \end{array}$$

Thus,

$$\text{B-code}^{-1}(1, 2, 3, 1, 5, 2) = 4, 6, 3, 1, 5, 2. \tag{5.5}$$

Let $\Phi := \mathbf{i}\,\phi\,\mathbf{i}\,\phi^{-1}\,\mathbf{i}$ be the product of the bijections occurring in (1.4). With $\sigma = 6, 4, 1, 2, 3, 5$ the computation of $\Phi(\sigma)$ can be made as follows.

$$\begin{array}{rll} \mathrm{Id} = & 1\ 2\ 3\ 4\ 5\ 6 & \\ \sigma = & 6\ 4\ 1\ 2\ 3\ 5 & \\ \mathbf{i}\,\sigma = & 3\ 4\ 5\ 2\ 6\ 1 & \\ \text{B-code}\,\mathbf{i}\,\sigma = & 1\ 2\ 1\ 2\ 3\ 5 & \text{(by (5.4))} \\ \text{A-code}^{-1}\,\text{B-code}\,\mathbf{i}\,\sigma = \phi^{-1}\,\mathbf{i}\,\sigma = & 3\ 4\ 5\ 1\ 6\ 2 & \text{(by (5.3))} \\ \mathbf{i}\,\phi^{-1}\,\mathbf{i}\,\sigma = & 4\ 6\ 1\ 2\ 3\ 5 & \\ \text{A-code}\,\mathbf{i}\,\phi^{-1}\,\mathbf{i}\,\sigma = & 1\ 2\ 3\ 1\ 5\ 2 & \text{(by (5.2))} \\ \text{B-code}^{-1}\,\text{A-code}\,\mathbf{i}\,\phi^{-1}\,\mathbf{i}\,\sigma = \phi\,\mathbf{i}\,\phi^{-1}\,\mathbf{i}\,\sigma = & 4\ 6\ 3\ 1\ 5\ 2 & \text{(by (5.5))} \\ \Phi(\sigma) = \mathbf{i}\,\phi\,\mathbf{i}\,\phi^{-1}\,\mathbf{i}\,\sigma = & 4\ 6\ 3\ 1\ 5\ 2. & \end{array}$$

We verify that

$$(\mathrm{Cyc}, \mathrm{Rmil})\,\sigma = (\mathrm{Rmil}, \mathrm{Cyc})\,\Phi(\sigma) = (\{1, 2\}, \{1, 2, 3, 5\}).$$

6. Concluding remarks and Tables

The bijection constructed by Cori [Cor08] only preserves the *cardinalities* "cyc" and "lmap", but not the sets "Cyc" and "Lmap." With the example used in his paper, the permutation

$$\theta = 6, 5, 7, 4, 2, 10, 3, 8, 9 = (1, 6, 10)(2, 5)(3, 7)(4)(8)(9)$$

is mapped onto

$$\theta' = 4, 6, 5, 7, 3, 8, 1, 9, 10, 2 = (1, 4, 7)(2, 6, 8, 9, 10)(3, 5),$$

so that $(\mathrm{Lmap}, \mathrm{Cyc})\,\theta' = (\{1, 2, 4, 6, 8, 9\}, \{1, 2, 3\}) \neq (\{1, 2, 3, 4, 8, 9\}, \{1, 3, 6\}) = (\mathrm{Cyc}, \mathrm{Lmap})\,\theta$. However, $(\mathrm{cyc}, \mathrm{lmap})\,\theta = (\mathrm{lmap}, \mathrm{cyc})\,\theta' = (6, 3)$.

In our case, we take the bijection $\phi\,\mathbf{i}\,\phi^{-1}$ that satisfies (see (1.4))

$$(\mathrm{Cyc},\mathrm{Lmap})\,\theta = (\mathrm{Lmap},\mathrm{Cyc})\phi\,\mathbf{i}\,\phi^{-1}(\theta).$$

The calculation of $\phi\,\mathbf{i}\,\phi^{-1}(\theta)$ is made for the same θ, together with the relevant set-valued statistics. We successively get:

$$\begin{aligned}
\theta &= 6,5,7,4,2,10,3,8,9\\
&= (1,6,10)(2,5)(3,7)(4)(8)(9)\\
\text{B-code}\,\theta &= 1,2,3,4,2,1,3,8,9,6\\
\text{A-code}^{-1}\,\text{B-code}\,\theta = \phi^{-1}(\theta) &= 6,1,7,5,2,10,3,4,8,9\\
\mathbf{i}\,\phi^{-1}(\theta) &= 2,5,7,8,4,1,3,9,10,6\\
\text{A-code}\ \mathbf{i}\,\phi^{-1}(\theta) &= 1,1,3,2,2,6,3,4,8,9\\
\phi\,\mathbf{i}\,\phi^{-1}(\theta) &= 2,5,7,8,4,6,3,9,10,1\\
&= (1,2,5,4,8,9,10)(3,7)(6)
\end{aligned}$$

Thus $(\mathrm{Cyc},\mathrm{Lmap})\theta = (\mathrm{Lmap},\mathrm{Cyc})\phi\,\mathbf{i}\,\phi^{-1}(\theta) = (\{1,2,3,4,8,9\},\{1,3,6\})$.

In Fig. 3 the common distribution over $\mathfrak{S}_n$ of each bivariable statistic $(\mathrm{Cyc},\mathrm{Rmil})$, $(\mathrm{Cyc},\mathrm{Lmap})$, $(\mathrm{Rmil},\mathrm{Lmap})$, $(\mathrm{Rmil},\mathrm{Cyc})$, $(\mathrm{Lmap},\mathrm{Rmil})$, $(\mathrm{Lmap},\mathrm{Cyc})$has been reproduced for $n = 1,2,3,4$. On each cell (A,B), where $A,B\subset[n]$, is written the number of permutations σ from $\mathfrak{S}_n$ such that $(\mathrm{Cyc},\mathrm{Rmil})\,\sigma = (A,B)$. In the table for $n=4$ the total sums occurring at the bottom and on the right are the numbers $\#\{\sigma\in\mathfrak{S}_4 : \mathrm{cyc}\,\sigma = k\}$ for $k=4,3,2,1$, which are the coefficients of the polynomial $x(x+1)(x+2)(x+3)$ ([Ri58], chap. 4, § 3). It will be noticed that all those tables are symmetric with respect to the main diagonal.

B=	1
A=1	1

$n=1$

B=	1, 2	2
A=1,2	1	
2		1

$n=2$

B=	1, 2, 3	1, 3	2, 3	3
A=1,2,3	1			
1,3		1		
2,3			1	1
3			1	1

$n=3$

B=	$1,2,3,4$	$1,2,4$	$1,3,4$	$2,3,4$	$1,4$	$2,4$	$3,4$	4	Σ
A=1,2,3,4	1								1
1,2,4		1							
1,3,4			1		1				6
2,3,4				1			1	1	
1,4			1		1				
2,4						1	1	1	11
3,4				1		1	2	2	
4				1		1	2	2	6
Σ	1		6			11		6	

$n = 4$

Fig. 3. Distribution of $(\mathrm{Cyc}, \mathrm{Rmil})$ over $\mathfrak{S}_n$.

There exist other bijections $\sigma \mapsto a$ such that the sum $\sum_i (a_i - 1)$ is equal to a statistic different from the inversion number "inv," but having interesting properties. Let us quote the *Tompkins-Paige method* ([To56, Le60, We61]) for generating permutations on a computer. That method was further used in [Ha92, Ha94] to show that the corresponding sum $\sum_i (a_i - 1)$ is equal to the *major index* "maj". Let us also mention the *Denert coding* [FZ90, Ha94], whose sum $\sum_i (a_i - 1)$ is equal to the *Denert statistic* "den". Those codings serve to prove that the statistics "inv," "maj" and "den" are equidistributed on $\mathfrak{S}_n$, their common distribution being called *Mahonian.*

Let $b = (b_1, b_2, \ldots, b_n)$ be the B-code of a permutation $\sigma \in \mathfrak{S}_n$. In its turn the sum $\mathrm{env}\,\sigma := \sum_i (b_i - 1)$ becomes a new Mahonian statistic. Moreover, it follows from the properties of the bijection ϕ defined in (4.1) that the two three-variable statistics (env, Cyc, Lmap) and (inv, Rmil, Lmap) are equidistributed on $\mathfrak{S}_n$.

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