

Predicate Transformers, (co)Monads and Resolutions

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Abstract. This short note contains random thoughts about a factorization theorem for closure/interior operators on a powerset which is reminiscent to the notion of *resolution* for a monad/comonad. The question originated from formal topology but is interesting in itself.

The result holds constructively (even if it classically has several variations); but usually not predicatively (in the sense that the interpolant will no be given by a set). For those not familiar with predicativity issues, we look at a “classical” version where we bound the size of the interpolant.

Introduction

A very general theorem states that any monotonic operator $F : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ can be factorized in the form $\mathcal{P}(X) \rightarrow \mathcal{P}(Z) \rightarrow \mathcal{P}(Y)$, where Z is an appropriate set; and the first predicate transformer commutes with arbitrary unions and the second commutes with arbitrary intersections.

We prove similar factorization for interior and closure operators on a powerset; the idea being to “resolve” the operator as is usually done for (co)monad in categories. We then look at the constructive version of those factorizations.

1 Relations and Predicate Transformers

We start by introducing the basic notions:

Definition 1. *If X and Y are sets, a (binary) relation between X and Y is a subset of the cartesian product $X \times Y$. The converse of a relation $r \subseteq X \times Y$ is the relation $r^\sim \subseteq Y \times X$ defined as $(y, x) \in r^\sim \equiv (x, y) \in r$.*

A predicate transformer from X to Y is an operator from the powerset $\mathcal{P}(X)$ to the powerset $\mathcal{P}(Y)$.

Since most of our predicate transformers will be monotonic (with respect to inclusion), we drop the adjective when no confusion is possible.

Definition 2. Suppose r is a relation between X and Y ; we define two monotonic predicate transformers from Y to X :

$$\begin{aligned} \langle r \rangle : \mathcal{P}(Y) &\rightarrow \mathcal{P}(X) \\ V &\mapsto \{x \in X \mid (\exists y \in Y) (x, y) \in r \ \& \ y \in V\} \end{aligned}$$

and

$$\begin{aligned} [r] : \mathcal{P}(Y) &\rightarrow \mathcal{P}(X) \\ V &\mapsto \{x \in X \mid (\forall y \in Y) (x, y) \in r \Rightarrow y \in V\}; \end{aligned}$$

and an antitonic predicate transformer:

$$\begin{aligned} [r^\perp] : \mathcal{P}(Y) &\rightarrow \mathcal{P}(X) \\ V &\mapsto \{x \in X \mid (\forall y \in V) (x, y) \in r\}. \end{aligned}$$

Concerning notation:

- $\langle r \rangle$ and $[r]$ are somewhat common in the refinement calculus, even though the main reference ([1]) uses $\{r\}$ instead of $\langle r \rangle$. The problem is that this clashes with set theoretic notation.
- In [2], Birkhoff uses V^\leftarrow for $[r]$, but this supposes that r is clear from the context. (The notation V^\rightarrow would then be $[r^\sim](U)$.)
- The linear logic community would use V^\perp for the same thing, but this also supposes that the relation is called “ \perp ”.

We will always be in a “typed” context; *i.e.* subsets will always be subset of some ambient set. We write \neg for complementation with respect to that ambient set.

Lemma 1. Suppose r is a relation between X and Y ; we have:

1. $\langle r \rangle \cdot \neg = \neg \cdot [r]$;
2. $\langle \neg r \rangle = \neg \cdot [r]$. (where $(x, y) \in \neg r \equiv (x, y) \notin r$)

A very interesting property is the following Galois connections:

Lemma 2. Suppose r is a relation between X and Y ; then $\langle r \rangle \vdash [r]$, and $[r]$ is Galois-connected to itself:

1. $\langle r \rangle(V) \subseteq U \Leftrightarrow V \subseteq [r^\sim](U)$;
2. $U \subseteq [r](V) \Leftrightarrow V \subseteq [r^\sim](U)$.

Those predicate transformers satisfy:

Lemma 3. If r is a relation between X and Y , then

- $\langle r \rangle$ commutes with arbitrary unions; (*i.e.* it is a sup-lattice morphism³)
- $[r]$ commutes with arbitrary intersections; (*i.e.* it is an inf-lattice morphism)
- $[r]$ transforms arbitrary unions into intersections.

³ All of our lattices are complete, so we do not bother writing “complete” all the time...

and moreover:

- any sup-lattice morphism from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ is of the form $\langle r \rangle$ for some $r \subseteq X \times Y$;
- any inf-lattice morphism from $\mathcal{P}(Y)$ to $\mathcal{P}(Y)$ is of the form $[r]$ for some $r \subseteq X \times Y$;
- any predicate transformer from Y to X taking arbitrary unions to intersections is of the form $[r]$ for some $r \subseteq X \times Y$.

Proof. The sup-lattice part is easy; and the rest is an application of Lemma 1. \square

Just like it is possible to factorize any relation as the composition of a total function and the inverse of a total function, it is possible to factorize an monotonic predicate transformer as the composition of a $\langle r \rangle$ and a $[s]$ (see [3] for a detailed categorical construction).

Proposition 1. *Suppose F is a monotonic predicate transformer from X to Y ; then there is an X' and there are relations $s \subseteq X' \times X$ and $r \subseteq Y \times X'$ such that $F = \langle r \rangle \cdot [s]$.*

Proof. Let F be a monotonic predicate transformer, and define $X' = \mathcal{P}(X)$ together with $(U, x) \in s \equiv x \in U$ and $(y, U) \in r \equiv y \in F(U)$. We have:

$$\begin{aligned}
 & y \in \langle r \rangle \cdot [s](U) \\
 & \Leftrightarrow \{ \text{definition of } \langle r \rangle \} \\
 & (\exists V \in X') (y, V) \in r \ \& \ V \in [s](U) \\
 & \Leftrightarrow \{ \text{definition of } X' \text{ and } r \} \\
 & (\exists V \subseteq X) y \in F(V) \ \& \ V \in [s](U) \\
 & \Leftrightarrow \{ \text{definition of } [s] \} \\
 & (\exists V \subseteq X) y \in F(V) \ \& \ (\forall x) (V, x) \in s \Rightarrow x \in U \\
 & \Leftrightarrow \{ \text{definition of } s \} \\
 & (\exists V \subseteq X) y \in F(V) \ \& \ (\forall x) V \subseteq U \\
 & \Leftrightarrow \{ \text{since } F \text{ is monotonic} \} \\
 & y \in F(U)
 \end{aligned}$$

The result thus holds, but the proof doesn't bring much information... \square

And as a direct application of Lemma 1:

Corollary 1. *Any monotonic predicate transformer can be factorized as a $[r] \cdot \langle s \rangle$ or as a $[r] \cdot [s]$.*

Similarly, any antitonic predicate transformer can be written has one of $[r] \cdot [s]$, $[r] \cdot \langle s \rangle$, $\langle r \rangle \cdot [s]$ or $[r] \cdot [s]$.

2 Interior and Closure Operators

Definition 3. If (X, \leq) is a partial order, we say that $F : X \rightarrow X$ is an interior operator if:

- F is monotonic;
- F is contractive: $F(x) \leq x$;
- $F(x) \leq FF(x)$.

We say that it is a closure operator if:

- F is monotonic;
- F is expansive: $x \leq F(x)$;
- $FF(x) \leq F(x)$.

It is well known that the composition of two Galois connected operators yield interior/closure operators, so that we have:

Lemma 4. If r is a relation between X and Y , then

- $\langle r \rangle \cdot [r^\sim]$ is an interior operator on $\mathcal{P}(X)$;
- $[r] \cdot \langle r^\sim \rangle$ is a closure operator on $\mathcal{P}(X)$;
- $[r] \cdot [r^\sim]$ is a closure operator on $\mathcal{P}(X)$.

Some other consequences of the Galois connection are listed below:

- $\langle r \rangle \cdot [r^\sim] \cdot \langle r \rangle = \langle r \rangle$;
- $[r] \cdot \langle r^\sim \rangle \cdot [r] = [r]$;
- $[r] \cdot [r^\sim] \cdot [r] = [r]$.

The problem is now to mimic Proposition 1.

Definition 4. If F is an interior operator on $\mathcal{P}(X)$, a resolution for F is given by a set Y (called the interpolant) together with a relation $r \subseteq Y \times X$ such that $F = \langle r \rangle \cdot [r^\sim]$.

Proposition 2. If F is an interior operator on $\mathcal{P}(X)$, then it has a resolution.

The proof relies on the following lemma:

Lemma 5. Let F be an interior operator on a complete sup-lattice (X, \leq, \bigvee) ; write $\mathbf{Fix}(F)$ for the collection of fixed-point for F . We have that $(\mathbf{Fix}(F), \leq, \bigvee)$ is a complete sup-lattice; and for any $x \in X$

$$F(x) = \bigvee \{y \in \mathbf{Fix}(F) \mid y \leq x\} .$$

Proof. That $(\mathbf{Fix}(F), \leq, \bigvee)$ is a complete sup-lattice is left as an easy exercise; for the second point, let $x \in X$;

- we know that $F(x)$ is a fixed point of F , and that $F(x) \leq x$. This implies that $F(x) \in \{y \in \mathbf{Fix}(F) \mid y \leq x\}$; and so $F(x) \leq \bigvee \{y \in \mathbf{Fix}(F) \mid y \leq x\}$;

- suppose $y \in \mathbf{Fix}(F)$ and $y \leq x$; this implies that $F(y) \leq F(x)$, i.e. that $y \leq F(x)$. We can conclude that $\bigvee\{y \in \mathbf{Fix}(F) \mid y \leq x\} \leq F(x)$.

□

Proof (of proposition 2). Suppose F is an interior operator on $\mathcal{P}(X)$; define $Y = \mathbf{Fix}(F)$ and $(U, x) \in r \equiv x \in U$. We have:

$$\begin{aligned}
& x \in \langle r \rangle \cdot [r^\sim](U) \\
& \Leftrightarrow \{ \text{definition of } r \} \\
& (\exists V \in \mathbf{Fix}(F)) \ x \in V \ \& \ (\forall y) \ y \in V \Rightarrow y \in U \\
& \Leftrightarrow \\
& (\exists V \in \mathbf{Fix}(F)) \ x \in V \ \& \ V \subseteq U \\
& \Leftrightarrow \\
& x \in \bigcup\{V \in \mathbf{Fix}(F) \mid V \subseteq U\} \\
& \Leftrightarrow \{ \text{Lemma 5} \} \\
& x \in F(U)
\end{aligned}$$

which concludes the proof. □

Just like for Proposition 1, the statement of the theorem is interesting, but the proof hardly tells us anything about the structure of F . To gain a little more information about F , we will try to “bound” the size of the interpolant set Y .

Definition 5. *If (X, \leq, \bigvee) is a complete sup-lattice, we say that a family $(x_i)_{i \in I}$ of element of X is a basis if, for any $y \in X$, we have*

$$y = \bigvee\{x_i \mid x_i \leq y\} .$$

Corollary 2. *Suppose F is an interior operator on $\mathcal{P}(X)$; if $(\mathbf{Fix}(F), \subseteq, \bigcup)$ has a basis of cardinality κ , then we can find a resolution of F with an interpolant Y of cardinality κ .*

Proof. It is easy to see that in the above proof of Proposition 2, we can replace $\mathbf{Fix}(F)$ by a basis of $(\mathbf{Fix}(F), \subseteq, \bigcup)$. □

In particular, if there is a basis of $\mathbf{Fix}(F)$ which has cardinality less than the cardinality of X ; we can use X as the interpolant and use a relation $r \subseteq X \times X$ to obtain a resolution of F .

We now show that this result is optimal:

Lemma 6. *Let F be an interior operator on $\mathcal{P}(X)$; and suppose there are no basis of $\mathbf{Fix}(F)$ of cardinality κ ; then there is no interpolant of cardinality less than κ .*

Proof. To show that, we will construct a basis of $\mathbf{Fix}(F)$ indexed by any interpolant for F . Suppose Y and r form a resolution for F . For any $y \in Y$, define $U_y \equiv \langle r \rangle\{y\}$. We will show that $(U_y)_{y \in Y}$ is a basis for $\mathbf{Fix}(F)$.

Each U_y is a fixed point for F :

$$\begin{aligned} U_y &= \langle r \rangle \{y\} && \{\text{definition}\} \\ &= \langle r \rangle \cdot [r^\sim] \cdot \langle r \rangle \{y\} && \{\text{second part of Lemma 4}\} \\ &= F \cdot \langle r \rangle \{y\} && \{r \text{ is a resolution of } F\} \\ &= F(U_y) && \{\text{definition}\} \end{aligned}$$

Let U be a fixed point of F :

$$\begin{aligned} U &= \langle r \rangle \cdot [r^\sim](U) && \{U \text{ is a fixed point of } F\} \\ &= \langle r \rangle (\bigcup \{ \{y\} \mid y \in [r^\sim](U) \}) && \{\text{basic logic}\} \\ &= \bigcup \{ \langle r \rangle \{y\} \mid \{y\} \subseteq [r^\sim](U) \} && \{\langle r \rangle \text{ commutes with unions}\} \\ &= \bigcup \{ \langle r \rangle \{y\} \mid \langle r \rangle \{y\} \subseteq U \} && \{\text{Galois connection between } \langle r \rangle \text{ and } [r^\sim]\} \\ &= \bigcup \{ U_y \mid U_y \subseteq U \} && \{\text{definition}\} \end{aligned}$$

which concludes the proof that $(U_y)_{y \in Y}$ is a basis for $\mathbf{Fix}(F)$. \square

Let's look at an example of interior operator on $\mathcal{P}(X)$ which *cannot* be resolved using X as an interpolant. Let X be a countable infinite set (natural numbers for example); and define $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as follows:

$$F(U) = \begin{cases} \emptyset & \text{if } U \text{ is finite} \\ U & \text{if } U \text{ is infinite} \end{cases}$$

The sup-lattice $\mathbf{Fix}(F)$ is given by the collection of infinite subsets of X ; and this lattice doesn't have a countable basis. To prove that, it is enough to do it for any particular countable infinite set. Take C to be the set of finite strings over $\{0, 1\}$. If α is an infinite string of 0's and 1's, define $U_\alpha \subseteq C$ to be the set of finite prefixes of α . Each U_α is an element of $\mathbf{Fix}(F)$; but no countable family of infinite subsets can "generate" all the U_α : since $\alpha \neq \beta$ implies that $U_\alpha \cap U_\beta$ is finite, if $V_i \subseteq U_\alpha$ and $V_j \subseteq U_\beta$ then $i \neq j$. In other words, a family which generates all the U_α 's needs to have the cardinality of the collection of the U_α 's, *i.e.* uncountable.

Using Lemma 1, we can now extend all what has been done for interior operators for closure operators: if F is a closure operator on $\mathcal{P}(X)$, then $\neg \cdot F \cdot \neg$ is an interior operator on $\mathcal{P}(X)$.

Corollary 3. *If F is a closure operator on $\mathcal{P}(X)$, then it has a resolution as a composition $[r] \cdot \langle r^\sim \rangle$ or as $[r] \cdot [r^\sim]$.*

As for interior operators, the possible cardinalities of the interpolant are given by the cardinalities of the bases for the inf-lattice $\mathbf{Fix}(F)$.

In particular, for linear logicians, it is not the case that any closure operator can be written as a biorthogonal...

3 Comonad and Monads

It seems that the traditional way to look at monads in a category is to see them as a kind of generalized monoid; at least in my part of the world. Another view⁴

⁴ which has given me a much better understanding of what (co)monads are

is to view them as a generalization of closure operators. This is the view taken in the introduction of [4].

Definition 6. A monad on a category \mathcal{C} is a morphism $F : \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations $\eta : \mathbf{Id}_{\mathcal{C}} \rightarrow F$ and $\mu : FF \rightarrow F$ s.t. some diagram commute.

If one takes the partial order category $\mathcal{P}(X)$, then a monad is an operator on $\mathcal{P}(X)$ s.t.:

- it acts on morphisms: if $i : U \subseteq V$ then $F_i : F(U) \subseteq F(V)$; i.e. F is monotonic;
- there is a natural transformation $\eta_U : U \subseteq F(U)$;
- there is a natural transformation $\mu_U : FF(U) \subseteq F(U)$.

All the “coherence” conditions are trivially satisfied in a partial order category, since every diagram commute! What we’ve just shown is that a monad on $\mathcal{P}(X)$ is nothing more than a closure operator on $\mathcal{P}(X)$; and vice and versa. Similarly, a comonad corresponds to an interior operator.

In categories, a *resolution* for a monad corresponds to factorizing the functor F as the composition of two adjoint functors. Adjointness $H \vdash G$ between the functors $G : \mathcal{D} \rightarrow \mathcal{C}$ and $H : \mathcal{C} \rightarrow \mathcal{D}$ in a locally small category means that $\mathcal{C}[A, G(B)] \simeq \mathcal{D}[H(A), B]$ which, in the case of a partial order category simplifies to “ $U \subseteq G(V)$ iff $H(U) \subseteq V$ ” i.e. is exactly the Galois connection $H \vdash G$.

Category theory tells us that there always is a resolution since we have two degenerate resolutions: something like $F = \mathbf{Id} \cdot F$ and $F = F \cdot \mathbf{Id}$. The first one is given by the Eleinberg-Moore category. In the context of order theory, it amounts to taking: interpolant category to partial orders: if F is a closure operator on X , take $\mathcal{E}(X, F)$ to be the the collection of fixed points for F , with the ordering inherited from X .⁵ We have $F : X \rightarrow \mathcal{E}(X, F)$ and $\mathbf{Id} : \mathcal{E}(X, F) \rightarrow X$ which are adjoint: $F \vdash \mathbf{Id}$.

$$\begin{aligned} x &\leq \mathbf{Id}(f) \\ &\Rightarrow \{ F \text{ is monotonic} \} \\ F(x) &\leq F(f) \\ &\Leftrightarrow \{ f \in \mathcal{E}(X, F), \text{ i.e. } f \text{ is a fixed point for } F \} \\ F(x) &\leq f \end{aligned}$$

and $F(x) \leq f \Rightarrow x \leq f = \mathbf{Id}(f)$ since $x \leq F(x)$.

The second resolution ($F \cdot \mathbf{Id}$) is obtained via the Kleisli category. In the context of order theory, it amounts to taking the order (X, \sqsubseteq) where $x \sqsubseteq y$ iff $F(x) \leq F(y)$.

What is important to us is that those resolutions are trivial and uninteresting. More abstract nonsense states that there is a category of resolutions for any given monad; and that those two trivial resolutions are respectively initial/terminal.

⁵ In a partial order, an algebra for F is just a post fixed point: $x \leq F(x)$. If F is a closure, then it is also a fixed point for F .

What we have done with Proposition 2 and Corollary 3 corresponds to finding a non-trivial *resolution* for the monad corresponding to the interior/closure operator. It is even more than a resolution, since the interpolant is itself a complete and cocomplete category (and one of the functors preserves limits while the other one preserves colimits; but this is a general fact about adjoints). The feeling is that this resolution lies “exactly in the middle” between the initial and terminal resolutions. I don’t know if this kind of “strong” resolution has been considered in category theory.⁶

The resolution constructed here is quite different from the Eleinberg-Moore resolution (even though it uses the fixed-points as a basis). We do not construct functors from $\mathcal{P}(X)$ to $\mathcal{E}(X, F)$ and back; but from $\mathcal{P}(X)$ to $\mathcal{P}(\mathcal{E}(X, F))$ and back. In particular, as noted above, the interpolant is complete and cocomplete;⁷ which is not the case for the Eleinberg-Moore category: fixed points for an interior are closed under unions but not under intersections; and conversely for closure operators.

4 Revisiting Section 2 in a Constructive Setting

4.1 Impredicative

In the previous section we used Lemma 1 to generalize results on interior to closures. It is thus natural to ask whether (1) we can make the original proof constructive; (2) we can avoid using this lemma and prove the result for closure constructively. I will not into the details but just provide some hints about that.

- Galois connections from Lemma 2 are constructive.
- Lemma 3: the first part is trivial. The second part is easy: if F commutes with unions, take $(x, y) \in r$ iff $y \in F\{x\}$; if F commutes with intersections, take $(x, y) \in r$ iff $(\forall U) y \in F(U) \Rightarrow x \in U$; and if F transforms unions into intersections, take $(x, y) \in r$ iff $y \in F\{x\}$.
- Proposition 1: the proof is constructive.
- Lemma 3 is constructive.
- Lemma 5 and the corresponding lemma for closure operator are constructive.
- Proposition 2 is constructive.
- we can mimic the proof of Proposition 2 to obtain a resolution of a closure operator as $F = [r] \cdot [r^\sim]$, but *not* to obtain a resolution as $F = [r] \cdot \langle r^\sim \rangle$.
- I doubt we can constructively obtain a resolution of a closure operator as $F = [r] \cdot \langle r^\sim \rangle$.
- all of the lemmas about the size of interpolant are constructive at least if we read them as “if B is a basis for $\mathbf{Fix}(F)$ then we can use B as an interpolant”.

As a proof of concept, all this (except the last point) has been proved in the proof assistant COQ.⁸

⁶ *i.e.* if \mathcal{C} is a complete and cocomplete category and F is a monad on \mathcal{C} , a “strong resolution” is a resolution with a complete and cocomplete interpolant. Any reference to something similar in the literature would be most welcome.

⁷ *i.e.* is a complete lattice since we deal with partial orders

⁸ proof scripts available from <http://iml.univ-mrs.fr/~hyvernac/academics.html>

4.2 Predicative

In a predicative setting like Martin-Löf type theory (see [5]) or CZF (constructive ZF, see [6]) set theory, many of the results are not provable. The reason being that we do not allow quantification on a power-set. The main result about resolution would become something like:

Proposition 3. *Suppose $\mathbf{Fix}(F)$ (proper type) has a set-indexed basis, then F as a resolution as $\langle r \rangle \cdot [r^\sim]$ (if F is an interior) or as $[r] \cdot [r^\sim]$ (if F is a closure). If F as a resolution, then $\mathbf{Fix}(F)$ as a set-index basis.*

Note that having a set-indexed basis is equivalent to being “set presented” in the terminology of P. Aczel ([6,7]). Note that in the case of an interior operator, this implies that the predicate transformer is “set-based” (in the sense that it has a factorization as in Proposition 1, where the interpolant X' is a set). It doesn't seem that the existence of a resolution for a closure operator implies that the original predicate transformer is itself set-presented.

Conclusion

Nothing revolutionary has really been done, but the statement of Propositions 1 and 2 is, in an abstract setting, quite neat. However, as the proofs show, this is mostly abstract nonsense. The best example is probably Proposition 1, where $y \in F(U)$ is factorized as “there is a V such that $s \in F(V)$ and $V \subseteq U$ ”. The proof of Proposition 2 is slightly subtler, but is hardly interesting. In the end, the most interesting and informative thing is probably Lemma 6, in its “positive” version: if F has Y as an interpolant, then $\mathbf{Fix}(F)$ has a basis indexed by Y , which is hardly a breakthrough in mathematics...

I do nevertheless hope that it might interest some people, since while Proposition 1 is known to many (especially the refinement calculus people), it seems that Proposition 2 isn't stated anywhere. I also hope the link between monad and closure operator (together with the Eleinberg-Moore category being the partial order of fixed points) will gain in popularity, as I see it as a much better way of seeing monads, at least as far as intuition is concerned.

A final word about the motivation for this: the starting point was the question about whether it is possible to represent any “basic topology” (see the forthcoming [8]) as the formal side of a basic pair, impredicatively speaking. A basic topology is a structure $(X, \mathcal{A}, \mathcal{J})$ where X is a set, and \mathcal{A} and \mathcal{J} are closure and interior operators on $\mathcal{P}(X)$ such that $\mathcal{A}(U) \setminus \mathcal{J}(V) \Rightarrow U \setminus \mathcal{J}(V)$.⁹ The answer is obviously *no* since the \mathcal{A} and \mathcal{J} arising from a formal pair are classically dual (*i.e.* $\mathcal{A} \cdot \neg = \neg \cdot \mathcal{J}$) and there are basic topologies which are provably not dual. The question then turned into: “can any interior operator be written as the formal interior of a basic pair?” and similar for closure operators. The answers are *yes* (Proposition 2) and *I don't know* (the constructive resolution of a closure is of the form $[r] \cdot [r^\sim]$, and not of the form $[r] \cdot \langle r^\sim \rangle$.)

⁹ $U \setminus \mathcal{J}V$ is the constructive version of $U \cap V \neq \emptyset$.

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