Multidimensional Multifractal Random Measures

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Abstract

We construct and study space homogeneous and isotropic random measures (MMRM) which generalize the so-called MRM measures constructed in [1]. Our measures satisfy an exact scale invariance equation (see equation (1) below) and are therefore natural models in dimension 3 for the dissipation measure in a turbulent flow.

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1 Introduction

The purpose of this paper is to introduce a natural multidimensional generalization (MMRM) of the one dimensional multifractal random measures (MRM) introduced by Bacry and Muzy in [1]. The measures M we introduce are different from zero, homogeneous in space, isotropic and satisfy the following exact **scale invariance** relation: if T denotes some given cutoff parameter then the following equality in distribution holds for all $\lambda \in]0,1]$:

$$(M(\lambda A))_{A\subset B(0,T)} \overset{(law)}{=} \lambda^d e^{\Omega_\lambda}(M(A))_{A\subset B(0,T)}, \tag{1}$$

where $B(0,T)$ is the euclidian ball of radius T in $\mathbb{R}^d$ and Ω_λ is an infinitely divisible random variable independent of $(M(A))_{A\subset B(0,T)}$. Let us note that a multi-dimensional generalization of MRM has already been proposed in the litterature ([5]). However, this generalization is not exactly scale invariant. Let us stress that to our knowledge equation (1) has never been studied mathematically.

In dimension 1, MRM are non trivial stationary solutions to (1) for all $\lambda \in]0,1]$. In dimension $d \geq 2$, MMRM are isotropic and homogeneous solutions to (1) for all $\lambda \in]0,1]$. If we consider a non negative random variable Y independent from M then the random measure $(YM(A))_{A\subset \mathbb{R}^d}$ is also solution to (1) for all $\lambda \in]0,1]$. This leads to the following open problem (unicity):

Open problem 1. *Consider two homogeneous and isotropic random Radon measures M and M' on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. We suppose that there exists some cutoff parameter T such that M and M' satisfy (1) for all $\lambda \in]0,1]$. Can one find (in a possibly extended probability space) a non negative random variable Y (Y') independent of M (M') such that the following equality in law holds:*

$$(YM(A))_{A\subset B(0,T)} \overset{(law)}{=} (Y'M'(A))_{A\subset B(0,T)}$$

The study of equation (1) is justified on mathematical and physical grounds. Before reviewing in some detail the physical motivation for constructing the MRMM measures and studying equation (1), let us just mention that equation (1) has recently been used in the probabilistic derivation of the KPZ (Knizhnik-Polyakov-Zamolodchikov) equation introduced initially in [11] (see [6], [15]).

1.1 MMRM in dimension 3: a model for the energy dissipation in a turbulent flow

Equations similar to (1) were first proposed in [3] in the context of fully developed turbulence. In [3], the authors conjectured that the following relation should hold between the increments of the longitudinal velocity δv at two scales $l, l' < T$, where T is an integral scale characteristic of the turbulent flow: if $P_l(\delta v)$, $P_{l'}(\delta v)$ denote the probability density functions (p.d.f.) of the longitudinal velocity difference between two points separated by a distance l, l', one has:

$$P_l(\delta v) = \int_{\mathbb{R}} G_{l,l'}(x) P_{l'}(e^{-x}\delta v) e^{-x} dx, \tag{2}$$

where $G_{l,l'}$ is a p.d.f. If one makes the assumption that the $G_{l,l'}$ depend only on the factor $\lambda = l/l'$ (scale invariance), then it is easy to show that the $G_{l,l'}$ are the p.d.f.'s of infinitely divisible laws. Under the assumption of scale invariance, if we take $d = 1$, $A = [0, l']$, $\lambda = l/l'$ and $\Omega_\lambda - d\ln(\lambda)$ of p.d.f. $G_{l,l'}$ in relation (1) (valid in fact simultaneously for all A) then equation (2) is the p.d.f. equivalent to (1).

Following the standard conventions in turbulence, we note ϵ the (random) energy dissipation measure per unit mass and ϵ_l the mean energy dissipation per unit mass in a ball $B(0, l)$:

$$\epsilon_l = \frac{3}{4\pi l^3}\epsilon(B(0, l))$$

We believe the measures we consider in this paper can be used in dimension 3 to model the energy dissipation ϵ in a turbulent flow. Indeed, it is believed that the velocity field of a stationary (in time) turbulent flow at scales smaller than some integral scale T (characteristic of the turbulent flow) is homogeneous in space and isotropic (see [7]). Therefore, the measure ϵ is homogeneous and isotropic (as a function of the velocity field) and according to Kolmogorov's refined similarity hypothesis (see for instance [4], [18] for studies of this hypothesis), one has the following relation in law between the longitudinal velocity increment δv_l of two points separated by a distance l and ϵ_l:

$$\delta v_l \overset{(law)}{=} U(l\epsilon_l)^{1/3},$$

where U is a universal negatively skewed random variable independent of ϵ_l and of law independent of l. Therefore, equations similar to (2) should also hold for the p.d.f. of ϵ_l. Finally, let us note that the statistics of the velocity field and thus also those of ϵ are believed to be universal at scales smaller than T in the sense that they only depend on the average mean energy dissipation per unit mass $< \epsilon >$ defined by (note that the quantity below does not depend on l by homogeneity):

$$< \epsilon > = < \epsilon_l >$$

where $<>$ denotes an average with respect to the randomness. In particular, the law of $\frac{\epsilon}{<\epsilon>}$ and the p.d.f.'s $G_{l,l'}$ are completely universal, i.e. are the same for all flows; nevertheless, there is still a big debate in the phyics community on the exact form of the p.d.f.'s $G_{l,l'}$. In dimension 3, the measures M we consider are precisely models for $\frac{\epsilon}{<\epsilon>}$.

The rest of this paper is organized as follows: in section 2, we set the notations and give the main results. In section 3, we give a few remarks concerning the important lognormal case. In section 4, we gather the proofs of the main theorems of section 2.

2 Notations and main Results

2.1 Independently scattered infinitely divisible random measures.

The characteristic function of an infinitely divisible random variable X can be written as $\mathbb{E}[e^{iqX}] = e^{\varphi(q)}$, where φ is characterized by the Lévy-Khintchine formula

$$\varphi(q) = imq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}^*} (e^{iqx} - 1 - iq\sin(x))\,\nu(dx)$$

and $\nu(dx)$ is the so-called Lévy measure. It satisfies $\int_{\mathbb{R}^*} \min(1, x^2)\,\nu(dx) < +\infty$.

Let G be the unitary group of $\mathbb{R}^d$, that is

$$G = \{M \in M_d(\mathbb{R}); MM^t = \mathrm{I}\}.$$

Since G is a compact separable topological group, we can consider the unique right translation invariant Haar measure H with mass 1 defined on the Borel σ-algebra $\mathcal{B}(G)$. Let S be the half-space

$$S = \{(t, y); t \in \mathbb{R}, y \in \mathbb{R}_+^*\}$$

with which we associate the measure (on the Borel σ-algebra $\mathcal{B}(S)$)

$$\theta(dt, dy) = y^{-2} dt\, dy.$$

We consider an independently scattered infinitely divisible random measure μ associated to $(\varphi, H \otimes \theta)$ and distributed on $G \times S$ (see [13]). More precisely, μ satisfies:

1) For every sequence of disjoint sets $(A_n)_n$ in $\mathcal{B}(G^+ \times S)$, the random variables $(\mu(A_n))_n$ are independent and

$$\mu\Big(\bigcup_n A_n\Big) = \sum_n \mu(A_n) \text{ a.s.},$$

2) for any measurable set A in $\mathcal{B}(G \times S)$, $\mu(A)$ is an infinitely divisible random variable whose characteristic function is

$$\mathbb{E}(e^{iq\mu(A)}) = e^{\varphi(q) H\otimes\theta(A)}.$$

We stress the fact that μ is not necessarily a random signed measure. Let us additionnally mention that there exists a convex function ψ defined on $\mathbb{R}$ such that for all non empty subsets A of $G \times S$:

1. $\psi(q) = +\infty$, if $\mathbb{E}(e^{q\mu(A)}) = +\infty$,
2. $\mathbb{E}(e^{q\mu(A)}) = e^{\psi(q) H\otimes\theta(A)}$ otherwise.

Let q_c be defined as $q_c = \sup\{q \geq 0; \psi(q) < +\infty\}$. For any $q \in [0, q_c[$, $\psi(q) < +\infty$ and $\psi(q) = \varphi(-iq)$.

2.2 Multidimensional Multifractal Random Measures (MMRM).

We further assume that the independently scattered infinitely divisible random measure μ associated to $(\varphi, H \otimes \theta)$ satisfies:

$$\psi(2) < +\infty,$$

and $\psi(1) = 0$. The condition $\psi(1)$ is just a normalization condition. The condtion $\psi(2) < +\infty$ is technical and can probably be relaxed to the condition used in [1]: there exists $\epsilon > 0$ such that $\psi(1 + \epsilon) < +\infty$. However, in the multidimensional setting, the situation is more complicated because there is no strict decorrelation property similar to the one dimensional setting: in dimension $d \geq 2$, there does not exist some distance R such that $M(A)$ and $M(B)$ are independent for two Lebesgue measurable sets $A, B \subset \mathbb{R}^d$ separated by a distance of at least R. Nevertheless, the condition $\psi(2) < +\infty$ is general enough to cover the cases considered in turbulence: see [3], [16], [17].

Definition 2.3. Filtration $\mathcal{F}_{\mathbf{l}}$. *Let Ω be the probability space on which μ is defined. $\mathcal{F}_l$ is defined as the σ-algebra generated by $\{\mu(A \times B); A \subset G, B \subset S, \mathrm{dist}(B, \mathbb{R}^2 \setminus S) \geq l\}$.*

Let us now define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by:

$$f(l) = \begin{cases} l, & \text{if } l \leq T \\ T & \text{if } l \geq T \end{cases}$$

The cone-like subset $A_l(t)$ of S is defined by:

$$A_l(t) = \{(s, y) \in S; y \geq l, -f(y)/2 \leq s - t \leq f(y)/2\}.$$

For forthcoming computations, we stress that for s, t real we have:

$$\theta(A_l(s) \cap A_l(t)) = g_l(|t - s|)$$

where $g_l : \mathbb{R}_+ \to \mathbb{R}$ is given by (with the notation $x^+ = \max(x, 0)$):

$$g_l(x) = \begin{cases} \ln(T/l) + 1 - \frac{x}{l}, & \text{if } x \leq l \\ \ln^+(T/x) & \text{if } x \geq l \end{cases}$$

For any $x \in \mathbb{R}^d$ and $m \in G$, we denote by x_1^m the first coordinate of the vector mx. The cone product $C_l(x)$ is then defined as:

$$C_l(x) = \{(m, t, y) \in G \times S; (t, y) \in A_l(x_1^m)\}.$$

Definition 2.4. $\omega_{\mathbf{l}}(\mathbf{x})$ process. *The process $\omega_l(x)$ is defined as $\omega_l(x) = \mu(C_l(x))$.*

Remark 2.5. *For computational purposes, it is important that $x \mapsto \omega_l(x)$ possesses a version with a minimum of regularity. In the 1d-case, ω_l is cadlag as a sum of Lévy processes (see [1]). In higher dimensions, we will prove in section 4.4 that ω_l admits a version with at most a countable sets of discontinuities.*

Definition 2.6. $\mathbf{M_l(dx)}$ **measure.** *For any $l > 0$, we define the measure $M_l(dx) = e^{\omega_l(x)}\,dx$, that is:*

$$M_l(A) = \int_A e^{\omega_l(x)}\,dx$$

for any Borel measurable subset $A \subset \mathbb{R}^d$.

Theorem 2.7. Multidimensional Multifractal Random Measure (MMRM).

With probability one, there exists a limit measure (in the sense of weak convergence of measures):

$$M(dx) = \lim_{l \to 0^+} M_l(dx).$$

This limit is called the Multidimensional Multifractal Random Measure. The scaling exponent of M is defined by

$$\forall q \geq 0, \quad \zeta(q) = dq - \psi(q).$$

Moreover:

i) a.s., $\forall x \in \mathbb{R}^d$, $M(\{x\}) = 0$, i.e. M has no atoms,

ii) for any bounded subset K of $\mathbb{R}^d$, $M(K) < +\infty$ a.s. and $\mathbb{E}[M(K)] \leq |K|$.

Proposition 2.8. Homogeneity and isotropy

1. *The measure M is homogeneous in space, i.e. the law of $(M(A))_{A \subset \mathbb{R}^d}$ coincides with the law of $(M(x + A))_{A \subset \mathbb{R}^d}$ for each $x \in \mathbb{R}^d$.*

2. *The measure M is isotropic, i.e. the law of $(M(A))_{A \subset \mathbb{R}^d}$ coincides with the law of $(M(mA))_{A \subset \mathbb{R}^d}$ for each $m \in G$.*

Proposition 2.9. Stochastic scale invariance.

1. *For any fixed $\lambda \in]0, 1]$ and $l \leq T$, the two processes $(\omega_{\lambda l}(\lambda x))_{x \subset B(0,T/2)}$ and $(\Omega_\lambda + \omega_l(x))_{x \subset B(0,T/2)}$ have the same law, where Ω_λ is an infinitely divisible random variable independent from the process $(\omega_l(x))_{x \subset B(0,T/2)}$ and its law is characterized by $\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}$.*

2. *For any $\lambda \in]0,1]$, the law of $(M(\lambda A))_{A\subset B(0,T/2)}$ is equal to the law of $(W_\lambda M(A))_{A\subset B(0,T/2)}$, where $W_\lambda = \lambda^d e^{\Omega_\lambda}$ and Ω_λ is an infinitely divisible random variable (independent of $(M(A))_{A\subset B(0,T/2)}$) and its characteristic function is:*

$$\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}.$$

3. *If $\zeta(q) \neq -\infty$ and $0 < t < T$ then:*

$$\mathbb{E}\big[M(B(0,t))^q\big] = (2t/T)^{\zeta(q)}\mathbb{E}\big[M(B(0,T/2))^q\big].$$

Proposition 2.10. Non-triviality of the MMRM.

1. *The measure M is different from 0 if and only if there exists $\epsilon > 0$ such that $\zeta(1+\epsilon) > d$; in that case, $\mathbb{E}(M(A)) = |A|$.*

2. *Let $q > 1$ and consider the unique $n \in \mathbb{N}$ such that $n < q \leq n+1$. If $\zeta(q) > d$ and $\psi(n+1) < \infty$, then $\mathbb{E}[M(A)^q] < +\infty$.*

3 The limit lognormal case

In the gaussian case, we have $\psi(q) = \gamma^2 q^2/2$ and the condition $\zeta(1+\epsilon) > d$ corresponds to $\gamma^2 < 2d$. The approximating measures M_l are thus defined as:

$$M_l(A) = \int_A e^{\gamma X_l(x) - \frac{\gamma^2 \mathbb{E}[X_l(x)^2]}{2}}\, dx$$

where X_l is a centered gaussian field (equal to $(\omega_l - \mathbb{E}[\omega_l])/\gamma$) with correlations given by:

$$\begin{aligned}\mathbb{E}[X_l(x)X_l(y)] &= H \otimes \theta(C_l(x) \cap C_l(y)) \\ &= \int_G g_l(|x_1^m - y_1^m|)H(dm).\end{aligned}$$

The limit mesures $M = \lim_{l\to 0} M_l(dx)$ we define are in the scope of the theory of gaussian multiplicative chaos developed by Kahane in [10] (see [14] for an introduction to this theory). Formally, the measure M is defined by:

$$M(A) = \int_A e^{\gamma X(x) - \frac{\gamma^2 \mathbb{E}[X(x)^2]}{2}}\, dx$$

where X is a centered "gaussian field" (in fact a random tempered distribution) with correlations given by:

$$\mathbb{E}[X(x)X(y)] = \int_G \ln^+(T/|x_1^m - y_1^m|)H(dm).$$

Let us suppose that $T = 1$ for simplicity. Using invariance of the Haar measure H by multiplication, it is plain to see that $\mathbb{E}[X(x)X(y)]$ is of the form $F(|y-x|)$ where $F : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$. We have the scaling relation $F(ab) = \ln(1/a) + F(b)$ if $a \leq 1$ and $b \leq 1$ (see also lemma 4.7 below) which entails that for $|x| \leq T$:

$$F(|x|) = \ln(1/|x|) - \int_G \ln(|e_1^m|)H(dm).$$

where $e = (1, 0, \dots, 0)$ is the first vector of the canonical basis. As a corollary, we get the existence of some constant C (take $C = -\int_G \ln(|e_1^m|)H(dm)$) such that $\ln(1/|x|) + C$ is positive definite (as a tempered distribution) in a neighborhood of 0. This easily implies that $\ln(1/|x|)$ is positive definite in a neighborhood of 0: to our knowledge, this result is new. This contrasts with the fact that $\ln^+(1/|x|)$ is positive definite in dimension $d \leq 3$ but is not positive definite for $d \geq 4$ (see [14]).

Remark 3.1. *It is plain to see that* $1 - |x|^\alpha$ *is positive definite in a neighborhhod of* 0 *as a function defined in* $\mathbb{R}$ *if* $\alpha \leq 1$. *Therefore, one can consider the isotropic and positive definite function in a neighborhhood of* 0 *in* $\mathbb{R}^d$ *defined by:*

$$F(|x|) = \int_G (1 - |x_1^m|^\alpha)H(dm).$$

By scaling, one can see that $F(|x|) = 1 - C|x|^\alpha$ *for some* $C > 0$. *It is easy to see that this entails that* $1 - |x|^\alpha$ *is positive definite in a neighborhood* V *of* 0. *Using the main theorem in [12], one can extend* $1 - |x|^\alpha$ *(defined in* V*) in an isotropic and positive definite function defined in* $\mathbb{R}^d$. *This is in contrast with the so called Kuttner-Golubov problem which is to determine the* $\alpha, \kappa > 0$ *such that* $(1 - |x|^\alpha)_+^\kappa$ *is positive definite in* $\mathbb{R}^d$. *It is known (see [8] for instance) that for* $\alpha > 0$ *the function* $(1 - |x|^\alpha)_+$ *is not positive definite in* $\mathbb{R}^d$ *if* $d \geq 3$.

4 Proofs

4.1 Proof of Theorem 2.7

The relation $\mathbb{E}[e^{\omega_l(x)}] = e^{\psi(1)H \otimes \theta(C_l(x))} = 1$ ensures that for each Borelian subset $A \subset \mathbb{R}^d$, the process $M_l(A)$ is a martingale with respect to $\mathcal{F}_l$. Existence of the MMRM then results from [9]. Properties i) and ii) result from Fatou's lemma. $\square$

4.2 Characteristic function of $\omega_l(x)$

The crucial point is to compute the characteristic function of $\omega_l(x)$. We consider $(x^1,\dots,x^q)\in(\mathbb{R}^d)^q$ and $(\lambda_1,\dots,\lambda_q)\in\mathbb{R}^q$ and we have to compute

$$\phi(\lambda)=\mathbb{E}\big[e^{i\lambda_1\omega_l(x^1)+\cdots+i\lambda_q\omega_l(x^q)}\big].$$

Let us denote by $\mathcal{S}_q$ the permutation group of the set $\{1,\dots,q\}$. For a generic element $\sigma\in\mathcal{S}_q$, we define

$$B^\sigma=\{m\in G; x_1^{\sigma(1),m}<\cdots<x_1^{\sigma(q),m}\}.$$

Finally, given $x,z\in\mathbb{R}^d$, we define the cone like subset product:

$$C_l^\sigma(x)=C_l(x)\cap B^\sigma=\{(m,t,y)\in B^\sigma\times S;(t,y)\in A_l(x_1^m)\}$$

and

$$C_l^\sigma(x,z)=\{(m,t,y)\in B^\sigma\times S;(t,y)\in A_l(x_1^m)\cap A_l(z_1^m)\}.$$

Lemma 4.3. *The characteristic function of the vector $(\omega_l(x^i))_{1\le i\le q}$ exactly matches:*

$$\mathbb{E}\Big[\exp\Big(i\lambda_1\omega_l(x^1)+\cdots+i\lambda_q\omega_l(x^q)\Big)\Big]=\exp\Big(\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^q\sum_{k=1}^j\alpha^\sigma(j,k)\rho_l^\sigma(x^{\sigma(k)}-x^{\sigma(j)})\Big)$$

where $\rho_l^\sigma(x)=H\otimes\theta(C_l^\sigma(0,x))$, and:

$$\alpha^\sigma(j,k)=\varphi(r_{k,j}^\sigma)+\varphi(r_{k+1,j-1}^\sigma)-\varphi(r_{k,j-1}^\sigma)-\varphi(r_{k+1,j}^\sigma)$$

$$r_{k,j}^\sigma=\sum_{i=k}^j\lambda_{\sigma(i)}\quad(\textit{or }0\textit{ if }k>j).$$

Moreover,

$$\sum_{j=1}^q\sum_{k=1}^j\alpha^\sigma(j,k)=\varphi\big(\sum_{k=1}^q\lambda_k\big).$$

Proof. Without loss of generality, we assume $x^i\neq x^j$ for $i\neq j$. We point out that the family $(B^\sigma)_{\sigma\in\mathcal{S}_d}$ is a partition of G up to a set of null H-measure. The function ϕ breaks down as:

$$\begin{aligned}\phi(\lambda)=&\mathbb{E}\big[e^{i\lambda_1\mu(C_l(x^1))+\cdots+i\lambda_q\mu(C_l(x^q))}\big]\\=&\mathbb{E}\big[e^{\sum_{\sigma\in\mathcal{S}_d}i\lambda_1\mu(C_l^\sigma(x^1))+\cdots+i\lambda_q\mu(C_l^\sigma(x^q))}\big]\\=&\prod_{\sigma\in\mathcal{S}_d}\mathbb{E}\big[e^{i\lambda_1\mu(C_l^\sigma(x^1))+\cdots+i\lambda_q\mu(C_l^\sigma(x^q))}\big]\end{aligned}$$

Let us fix $\sigma \in \mathcal{S}_q$. We focus on the term:

$$\phi^\sigma(\lambda) = \mathbb{E}\big[e^{i\lambda_1 \mu(C_l^\sigma(x^1)) + \cdots + i\lambda_q \mu(C_l^\sigma(x^q)))}\big] = \mathbb{E}\big[e^{i\lambda_{\sigma(1)}\mu(C_l^\sigma(x^{\sigma(1)})) + \cdots + i\lambda_{\sigma(q)}\mu(C_l^\sigma(x^{\sigma(q)})))}\big].$$

Given $\sigma \in \mathcal{S}_d$ and $p \leq q$, we further define:

$$\phi^\sigma(\lambda, p) = \mathbb{E}\big[e^{i\lambda_{\sigma(1)}\mu(C_l^\sigma(x^{\sigma(1)})) + \cdots + i\lambda_{\sigma(p)}\mu(C_l^\sigma(x^{\sigma(p)})))}\big].$$

From now on, we adapt the proof of [1, Lemma 1] and proceed recursively. We define:

$$Y_q^\sigma = \sum_{k=1}^{q} \lambda_{\sigma(k)} \mu(C_l^\sigma(x^{\sigma(k)}) \setminus C_l(x^{\sigma(q)})),$$

which stands for the contribution of the points of the above sum that do not belong to $C_l(x^{\sigma(q)})$. Moreover, the points in the set $C_l(x^{\sigma(q)})$ can be grouped into the disjoint sets:

$$C_l(x^{\sigma(k)}, x^{\sigma(q)}) \setminus C_l(x^{\sigma(k-1)}, x^{\sigma(q)}).$$

We stress that the latter assertion is valid since, for $m \in B^\sigma$, the coordinates are suitably sorted, that is: $x_1^{\sigma(1),m} < x_2^{\sigma(2),m} < \cdots < x_1^{\sigma(q),m}$. We define:

$$X_{k,q}^\sigma = \mu\big(C_l(x^{\sigma(k)}, x^{\sigma(q)}) \setminus C_l(x^{\sigma(k-1)}, x^{\sigma(q)})\big)$$

with the convention $C_l(x^{\sigma(k)}, x^{\sigma(0)}) = C_l(x^{\sigma(0)}, x^{\sigma(k)}) = \emptyset$, in such a way that one has:

$$\lambda_{\sigma(1)}\mu(C_l^\sigma(x^{\sigma(1)})) + \cdots + \lambda_{\sigma(q)}\mu(C_l^\sigma(x^{\sigma(q)})) = Y_q^\sigma + \sum_{k=1}^{q} r_{k,q}^\sigma X_{k,q}^\sigma.$$

Furthermore, since the variable Y_q and $(X_{k,q}^\sigma)_k$ are mutually independent, we get the following decomposition:

$$\phi^\sigma(\lambda) = \mathbb{E}\big[e^{iY_q^\sigma}\big] \prod_{k=1}^{q} \mathbb{E}\big[e^{ir_{k,q}^\sigma X_{k,q}^\sigma}\big]. \tag{3}$$

Similarly, one can prove:

$$\phi^\sigma(\lambda, q-1) = \mathbb{E}\big[e^{iY_q^\sigma}\big] \prod_{k=1}^{q} \mathbb{E}\big[e^{ir_{k,q-1}^\sigma X_{k,q}^\sigma}\big]. \tag{4}$$

Gathering (3) and (4) yields:

$$\phi^\sigma(\lambda, q) = \phi^\sigma(\lambda, q-1)\prod_{k=1}^{q} \frac{\mathbb{E}\big[e^{ir^\sigma_{k,q}X^\sigma_{k,q}}\big]}{\mathbb{E}\big[e^{ir^\sigma_{k,q-1}X^\sigma_{k,q}}\big]}.$$

For any $m \in B^\sigma$, one has $x_1^{\sigma(k-1),m} < x_1^{\sigma(k),m} < x_1^{\sigma(q),m}$ and therefore;

$$\begin{aligned}\mathbb{E}\big[e^{i\alpha X^\sigma_{k,q}}\big] =& e^{\varphi(\alpha)H\otimes\theta\big(C_l(x^{\sigma(k)},x^{\sigma(q)})\setminus C_l(x^{\sigma(k-1)},x^{\sigma(q)})\big)}\\ =& e^{\varphi(\alpha)\big(H\otimes\theta(C_l(x^{\sigma(k)},x^{\sigma(q)}))-H\otimes\theta(C_l(x^{\sigma(k-1)},x^{\sigma(q)}))\big)}\end{aligned}$$

Note that:

$$\begin{aligned}H\otimes\theta(C_l(x^{\sigma(i)},x^{\sigma(j)})) =& \int_{B^\sigma}\theta\big(A_l(x_1^{\sigma(i),m})\cap A_l(x_1^{\sigma(j),m})\big)H(dm)\\ =& \int_{B^\sigma}\theta\big(A_l(0)\cap A_l(x_1^{\sigma(i),m}-x_1^{\sigma(j),m})\big)H(dm)\\ =& \rho_l^\sigma(x^{\sigma(i)}-x^{\sigma(j)})\end{aligned}$$

The proof can now be completed recursively. □

4.4 Regularity of ω_l

In the previous subsection, we have computed the finite dimensional distributions of the process ω_l. To prove that the limiting measure M inherit the properties of the finite dimensional distributions of ω_l, we need to establish some regularity properties of ω_l, namely that ω_l admits a version with at most a countable set of discontinuities.

Remind that the function φ can be written as

$$imq - \frac{1}{2}\sigma^2q^2 + \int_{\mathbb{R}^*}(e^{iqx}-1-iq\sin(x))\,\nu(dx).$$

Even if it means considering another measure μ, we may (and will) assume that μ is the sum of three independent independently scattered infinitely divisible random measures μ_1, μ_2, μ_3 respectively associated with:

$$\varphi_1(q) = imq, \quad \varphi_2(q) = -\frac{1}{2}\sigma^2q^2, \quad \varphi_3(q) = \int_{\mathbb{R}^*}(e^{iqx}-1-iq\sin(x))\,\nu(dx).$$

Then $\omega_l(x)$ is the sum of three independent random variable $\mu_1(C_l(x))+\mu_2(C_l(x))+\mu_3(C_l(x))$. It is plain to see that for $p>1$:

$$\mathbb{E}\big[|\mu_1(C_l(x))-\mu_1(C_l(y))|^p\big] \leq C(l,T)|x-y|^p, \quad \mathbb{E}\big[|\mu_2(C_l(x))-\mu_2(C_l(y))|^p\big] \leq C(l,T)|x-y|^{p/2}$$

in such a way that the Kolmogorov criterion (see [19]) ensures that $\mu_1(C_l(x))$ and $\mu_2(C_l(x))$ admit a continuous version. Furthermore, the restriction of μ_3 to $\mathcal{F}_l$ is a Poisson random measure. Hence, for each realization, μ_3 is of the form $\sum_{k=0}^{Z(\omega)} \delta_{X_k}$, where Z is a random variable taking values in $\mathbb{N}$ and $(X_k)_k$ is a countable collection of iid random variables taking values in $\{(m,t,y) \in G \times S; y \geq l\}$. It is therefore plain to see that $\mu_3(C_l(x))$ admits at most a countable set of discontinuities.

4.5 Homogeneity and isotropy

Lemma 4.3 is useful to prove the main properties of the MMRM. For instance, to prove the invariance of the law of the MMRM under translations, it suffices to prove that the law of ω_l is itself invariant. This results from Lemma 4.3 since each term $\rho_l^\sigma(x^{\sigma(k)} - x^{\sigma(i)})$ is invariant under translations, that is ρ_l^σ remains unchanged when you replace $x^1, \dots, x^q$ by $x^1+z, \dots, x^q+z$ for a given $z \in \mathbb{R}^d$.

Lemma 4.3 is nevertheless not very convenient to prove the isotropy of the MMRM so that we give a proof by a direct approach. Once again, it is sufficient to prove that the characteristic function of ω_l is invariant under G. This time, we compute that characteristic function in a more direct way. We consider $x^1, \dots, x^q \in \mathbb{R}^d$, $\lambda_1, \dots, \lambda_q \in \mathbb{R}$ and $m_0 \in G$, and define the function

$$f(m,t,y) = \sum_{k=1}^q \lambda_k \mathbb{1}_{C_l(x^k)}(m,t,y).$$

We have, using the right translation invariance of the Haar measure:

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(i\lambda_1\omega_l(m_0x^1)+\dots+i\lambda_q\omega_l(m_0x^q)\Big)\Big] &= \mathbb{E}\Big[\exp\Big(i\int f(mm_0,t,y)\,\mu(dm,dt,dy)\Big)\Big] \\
&= \exp\Big(\int \varphi\circ f(mm_0,t,y)H(dm)\theta(dt,dy)\Big) \\
&= \exp\Big(\int \varphi\circ f(m,t,y)H(dm)\theta(dt,dy)\Big) \\
&= \mathbb{E}\Big[\exp\Big(i\lambda_1\omega_l(x^1)+\dots+i\lambda_q\omega_l(x^q)\Big)\Big].
\end{aligned}$$

The isotropy follows. □

4.6 Exact scaling and stochastic scale invariance

Lemma 4.7. Exact scaling of $M_l(dx)$. *For all $\lambda \in (0,1]$ and $x^1,\dots,x^q \in B(0,T/2)$, the functions ρ_l^σ satisfy the exact scaling relation*

(5)

$$\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho_{\lambda l}^\sigma(\lambda x^{\sigma(k)}-\lambda x^{\sigma(j)}) = -\ln(\lambda)+\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho_l^\sigma(x^{\sigma(k)}-x^{\sigma(j)}).$$

Proof. We remind that for x real we have $\int_{A_l(0)\cap A_l(x)}\theta(dt,dy) = g_l(|x|)$. Given $B\subset G$ and $x\in\mathbb{R}^d$, we define:

$$\rho_l^B(x) = \int \mathbb{1}_B(m)\mathbb{1}_{\{(t,y)\in A_l(0)\cap A_l(x_1^m)\}}\theta(dt,dy)H(dm).$$

Then we can compute the function ρ_l^B:

$$\begin{aligned}
\rho_l^B(x) &= \int \mathbb{1}_B(m)\mathbb{1}_{\{(t,y)\in A_l(0)\cap A_l(x_1^m)\}}H(dm)\theta(dt,dy) = \int_B\Big(\int_{A_l(0)\cap A_l(x_1^m)}\theta(dt,dy)\Big)H(dm)\\
&= \int_B\Big(\big(\ln(T/l)+1-|x_1^m|/l\big)\mathbb{1}_{|x_1^m|\le l}+\ln(T/|x_1^m|)\mathbb{1}_{l\le|x_1^m|\le T}\Big)H(dm)
\end{aligned}$$

Given $\lambda\in]0,1]$ and $x\in B(0,T)$:

$$\begin{aligned}
\rho_{\lambda l}^B(\lambda x) =& \int_B\Big(\big(\ln(\frac{T}{\lambda l})+1-\frac{|\lambda x_1^m|}{\lambda l}\big)\mathbb{1}_{|\lambda x_1^m|\le\lambda l}+\ln(\frac{T}{\lambda|x_1^m|})\mathbb{1}_{\lambda l\le|\lambda x_1^m|\le T}\Big)H(dm)\\
=& \int_B\Big(\big(\ln(\frac{T}{l})+1-\frac{|x_1^m|}{l}\big)\mathbb{1}_{|x_1^m|\le l}+\ln(\frac{T}{|x_1^m|})\mathbb{1}_{\lambda l\le|\lambda x_1^m|\le T}\Big)H(dm)\\
&-\ln(\lambda)\int_B \mathbb{1}_{|x_1^m|\le l}+\mathbb{1}_{l\le|x_1^m|\le T}H(dm)\\
=&\rho_l^B(x)-\ln(\lambda)H(B)
\end{aligned}$$

We therefore obtain:

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho^\sigma_{\lambda l}(\lambda x^{\sigma(k)}-\lambda x^{\sigma(j)})\\
&\quad=\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho^\sigma_{l}(x^{\sigma(k)}-x^{\sigma(j)})-\ln(\lambda)\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)H(B^\sigma)\\
&\quad=\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho^\sigma_{l}(x^{\sigma(k)}-x^{\sigma(j)})-\ln(\lambda)\sum_{\sigma\in\mathcal{S}_q}\varphi(\sum_{k=1}^{q}\lambda_k)H(B^\sigma)\\
&\quad=\sum_{\sigma\in\mathcal{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho^\sigma_{l}(x^{\sigma(k)}-x^{\sigma(j)})-\ln(\lambda)\varphi(\sum_{k=1}^{q}\lambda_k)\quad\square
\end{aligned}$$

From Lemma 4.3, we deduce that, for any $\lambda\in(0,1]$, there exists a random variable C_λ such that $(\omega_{\lambda l}(\lambda x))_{x\in B(0,T/2)}\overset{law}{=}(C_\lambda+\omega_{\lambda l}(\lambda x))_{x\in B(0,T/2)}$ and such that C_λ is independent of $(\omega_{\lambda l}(\lambda x))_{x\in B(0,T/2)}$ and its characteristic function is given by $\mathbb{E}[e^{iqC_\lambda}]=\lambda^{-\varphi(q)}$. By integrating the previous relation, we obtain the relation:

$$(M_{\lambda l}(\lambda A))_{A\subset B(0,T/2)}\overset{law}{=}W_\lambda(M_l(A))_{A\subset B(0,T/2)}$$

where $W_\lambda=\lambda^d e^{C_\lambda}$ is a random variable independent of $(M_l(A))_{A\subset B(0,T/2)}$.

4.8 Non-triviality of the MMRM

Suppose we can find a "cube" $C_R=[0,R]^d$ and $q>1$ such that:

$$\mathbb{E}\big[M(C_R)^q\big]<+\infty.$$

Then we can find $n\in\mathbb{N}$ such that $[0,2^{-n}R]^d\subset B(0,T/2)$. We split the cube C_R into 2^{nd} smaller cubes

$$C^{k,n}=\prod_{i=1}^{d}[k_i2^{-n}R,(k_i+1)2^{-n}R),$$

where $k=(k_1,\dots,k_d)\in N^n_d\overset{def}{=}\mathbb{N}^d\cap[0,2^n-1]^d$. For each fixed value of n, the cubes $(C^{k,n})_k$, where the index k varies in N^n_d form a partition of C_T. Thus, by using the superadditivity of the function $x \mapsto x^q$, we have:

$$\mathbb{E}\big[M(C_T)^q\big] = \mathbb{E}\Big[\Big(\sum_{k\in N_d^n} M(C^{k,n})\Big)^q\Big]$$
$$\geq \sum_{k\in N_d^n} \mathbb{E}\big[\big(M(C^{k,n})\big)^q\big]$$

By using the translation invariance and the scale invariance property of the MMRM, we deduce:

$$\mathbb{E}\big[\big(M(C^{k,n})\big)^q\big] = \mathbb{E}\big[\big(M(C^{0,n})\big)^q\big] = \mathbb{E}\big[\big(M(2^{-n}C_R)\big)^q\big] = 2^{-n\zeta(q)}\mathbb{E}\big[\big(C_R)\big)^q\big].$$

Finally, gathering the previous inequalities yields:

$$\mathbb{E}\big[\big(C_R)\big)^q\big] \geq 2^{nd-n\zeta(q)}\mathbb{E}\big[\big(C_R)\big)^q\big]$$

in such a way that, necessarily, $\zeta(q) \geq d$. □

The proof of 1. and 2. is then a consequence of the following lemma:

Lemma 4.9. *Let $q > 1$ and consider the unique $n \in \mathbb{N}$ such that $n < q \leq n+1$. If $\zeta(q) > d$ and $\psi(n+1) < \infty$, then we can find a constant C such that:*

$$\sup_l \mathbb{E}\big[M_l([0,T)^d)^q\big] \leq C.$$

Proof. The proof is an adaptation of the one in [1] (which is itself an adaptation of the corresponding result in [2]). Unfortunately, the multi-dimensional setting is a bit more complicated because there is no strict decorrelation property similar to the one dimensional setting. With no restriction, we can suppose that $T = 1$ and $d = 2$. We consider the following dyadic partition of the cube $[0,1)^2$:

$$[0,1)^2 = \underset{0\leq i,j\leq 2^m-1}{\cup} I_{i,j}^{(m)},$$

where $I_{i,j}^{(m)} = [\frac{i}{2^m}, \frac{i+1}{2^m}) \times [\frac{j}{2^m}, \frac{j+1}{2^m})$. Let us write the above decompostion in the following form:

$$[0,1)^2 = C_1 \cup C_2 \cup C_3 \cup C_4.$$

where

$$C_1 = \underset{i \text{ and } j \text{ even}}{\cup} I_{i,j}^{(m)}, \quad C_2 = \underset{i \text{ and } j \text{ odd}}{\cup} I_{i,j}^{(m)}$$

and

$$C_3 = \underset{i \text{ odd and } j \text{ even}}{\cup} I_{i,j}^{(m)}, \quad C_2 = \underset{i \text{ even and } j \text{ odd}}{\cup} I_{i,j}^{(m)}.$$

Since the measure M_l is homogeneous, we get:

$$\begin{aligned}\mathbb{E}\big[M_l([0,1)^2)^q\big] &\leq 4^{q-1}\sum_{i=1}^{4}\mathbb{E}\big[M_l(C_i)^q\big] \\ &\leq 4^q\mathbb{E}\big[M_l(C_1)^q\big].\end{aligned}$$

Now we get the following by subadditivity of $x \to x^{q/(n+1)}$:

$$\begin{aligned}\mathbb{E}\big[M_l(C_1)^q\big] &= \mathbb{E}\big[(\sum_{0\leq i,j\leq 2^{m-1}-1} M_l(I^{(m)}_{2i,2j}))^q\big] \\ &= \mathbb{E}\big[((\sum_{i,j} M_l(I^{(m)}_{2i,2j}))^{n+1})^{q/(n+1)}\big] \\ &= \mathbb{E}\big[(\sum_{i_1,j_1,\dots,i_{n+1},j_{n+1}}\prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k}))^{q/(n+1)}\big] \\ &\leq \mathbb{E}\big[\sum_{i_1,j_1,\dots,i_{n+1},j_{n+1}}(\prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k}))^{q/(n+1)}\big] \\ &= 2^{2(m-1)}\mathbb{E}\big[M_l(I^{(m)}_{0,0})^q\big] + \sum^{*}_{i_1,j_1,\dots,i_{n+1},j_{n+1}}\mathbb{E}\big[\prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k})^{q/(n+1)}\big] \\ &\leq 2^{2(m-1)}\mathbb{E}\big[M_l(I^{(m)}_{0,0})^q\big] + \sum^{*}_{i_1,j_1,\dots,i_{n+1},j_{n+1}}\mathbb{E}\big[\prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k})\big]^{q/(n+1)}\end{aligned}$$

where $\sum^{*}_{i_1,j_1,\dots i_{n+1},j_{n+1}}$ is a sum over indices $i_1, j_1, \dots i_{n+1}, j_{n+1}$ which are not all equal and the last inequality is a consequence of Jensen's inequality. Therefore each term in the above sum is of the form:

$$\mathbb{E}\big[\prod_{r=1}^{k} M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big]^{q/(n+1)} \tag{6}$$

where the sequence of positive integers $(n_r)_{1\leq r\leq k}$ satisfies $\sum_{r=1}^{k} n_r + 1$ and the $I_{2i_r,2j_r}$ are disjoint intervals wich lie at a distance of at least $\frac{1}{2^m}$. We want to show that each term of the form (6) is bounded by some quantity C_m independent of l. We get the following computation using Fubini:

$$\mathbb{E}\big[\prod_{r=1}^{k} M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big] = \int_{I^{n_1}_{2i_1,2j_1}\times\cdots\times I^{n_k}_{2i_k,2j_k}} \mathbb{E}\big[e^{\omega_l(x^1)+\cdots+\omega_l(x^{n+1})}\big]dx^1\dots dx^{n+1}.$$

We define $N_r = n_1 + \cdots + n_r$ for r in $[1,k]$ and we introduce the following set $\mathcal{A}_l = \mathcal{A}_l(x^1,\dots,x^{n+1})$:

$$\mathcal{A}_l = \cup_{r<r'} (\cup_{N_r \leq i \leq N_{r+1}-1} C_l(x^i)) \cap (\cup_{N_{r'} \leq j \leq N_{r'+1}-1} C_l(x^j)).$$

By construction of $\mathcal{A}_l$, if x^i and x^j are in two different $I_{2i_r,2j_r}$ then $\mu(C_l(x^i) \setminus \mathcal{A}_l)$ and $\mu(C_l(x^j) \setminus \mathcal{A}_l)$ are independent. Therefore we get the following factorization:

$$\begin{aligned}
\mathbb{E}\big[e^{\omega_l(x^1)+\cdots+\omega_l(x^{n+1})}\big] &= \mathbb{E}\big[e^{\psi(n+1)H\otimes\theta(\mathcal{A}_l)}\big]\mathbb{E}\big[e^{\mu(C_l(x^1)\setminus\mathcal{A}_l)+\cdots+\mu(C_l(x^{n+1})\setminus\mathcal{A}_l)}\big] \\
&= \mathbb{E}\big[e^{\psi(n+1)H\otimes\theta(\mathcal{A}_l)}\big] \prod_{r=1}^{k} \mathbb{E}\big[e^{\sum_{N_r \leq i \leq N_{r+1}-1} \mu(C_l(x^i)\setminus\mathcal{A}_l)}\big] \\
&= \frac{\mathbb{E}\big[e^{\psi(n+1)H\otimes\theta(\mathcal{A}_l)}\big]}{\prod_{r=1}^{k} \mathbb{E}\big[e^{\psi(n_r)H\otimes\theta(\mathcal{A}_l)}\big]} \prod_{r=1}^{k} \mathbb{E}\big[e^{\sum_{N_r \leq i \leq N_{r+1}-1} \omega_l(x^i)}\big].
\end{aligned}$$

We have the following inequality:

$$H \otimes \theta(\mathcal{A}_l) \leq \sum_{r<r'} \sum_{\substack{N_r \leq i \leq N_{r+1}-1 \\ N_{r'} \leq j \leq N_{r'+1}-1}} H \otimes \theta(C_l(x^i) \cap C_l(x^j)).$$

Note that for each x^i, x^j in the above sum we have $|x^i - x^j| \geq \frac{1}{2^m}$ and therefore $H \otimes \theta(C_l(x^i) \cap C_l(x^j))$ is bounded by some constant depending on m but independent of l. Indeed, using the notation of section 3 for F, we get:

$$\begin{aligned}
H \otimes \theta(C_l(x^i) \cap C_l(x^j)) &= \int_G g_l(|(x^i)_1^m - (x^j)_1^m|) H(dm) \\
&\leq F(|x^i - x^j|) \\
&\leq F(\frac{1}{2^m}).
\end{aligned}$$

In conclusion, we get the existence of some constant C_m such that:

$$\mathbb{E}\big[e^{\omega_l(x^1)+\cdots+\omega_l(x^{n+1})}\big] \leq C_m \prod_{r=1}^{k} \mathbb{E}\big[e^{\sum_{N_r \leq i \leq N_{r+1}-1} \omega_l(x^i)}\big].$$

Thus, we get by integrating the above relation:

$$\mathbb{E}\big[\prod_{r=1}^{k} M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big] \leq C_m \prod_{r=1}^{k} \mathbb{E}\big[M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big].$$

Since each n_r is less or equal to n, we get by induction that $\mathbb{E}\big[M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big]$ is bounded independently of l and so is the above product. In conclusion, we get the existence of C_m such that we have:

$$\mathbb{E}\big[M_l([0,1]^2)^q\big] \leq 4^{q-1} 2^{2m} \mathbb{E}\big[M_l(I^{(m)}_{0,0})^q\big] + C_m.$$

Using stochastic scale invariance, we get that:

$$\begin{aligned}\mathbb{E}\big[M_l([0,1]^2)^q\big] &\leq 4^{q-1} \frac{2^{2m}}{2^{m\zeta(q)}} \mathbb{E}\big[M_{l2^m}([0,1]^2)^q\big] + C_m \\ &\leq 4^{q-1} \frac{2^{2m}}{2^{m\zeta(q)}} \mathbb{E}\big[M_l([0,1]^2)^q\big] + C_m.\end{aligned}$$

Since $\zeta(q) > 2$, we can choose m such that $4^{q-1}\frac{2^{2m}}{2^{m\zeta(q)}} < 1$ and therefore we get:

$$\mathbb{E}\big[M_l([0,1]^2)^q\big] \leq \frac{C_m}{1 - 4^{q-1}\frac{2^{2m}}{2^{m\zeta(q)}}},$$

which entails the result. $\square$

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