

# LOSS OF SMOOTHNESS AND ENERGY CONSERVING ROUGH WEAK SOLUTIONS FOR THE $3d$ EULER EQUATIONS

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**ABSTRACT.** A basic example of shear flow was introduced by DiPerna and Majda to study the weak limit of oscillatory solutions of the Euler equations of incompressible ideal fluids. In particular, they proved by means of this example that weak limit of solutions of Euler equations may, in some cases, fail to be a solution of Euler equations. We use this example to provide non-generic, yet nontrivial, examples concerning the loss of smoothness of solutions of the three-dimensional Euler equations, for initial data that do not belong to  $C^{1,\alpha}$ . In addition, we use this shear flow to provide explicit examples of non-regular solutions of the three-dimensional Euler equations that conserve the energy, an issue which related to the Onsager conjecture. Moreover, we show by means of this shear flow example that, unlike to the two-dimensional case, the minimal regularity in the three-dimensional vortex sheet Kelvin-Helmholtz problem need not to be the class of real analytic solutions.

*This paper is dedicated to Professor V. Solonnikov, on the occasion of his 75th birthday, as token of friendship and admiration for his contributions to research in partial differential equations and fluid mechanics.*

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## 1. INTRODUCTION

More than 250 years after the Euler equations have been written our knowledge of their mathematical structure and their relevance to describe the complicated phenomenon of turbulence is still very incomplete, to say the least. Both in two and three dimensions certain challenging problems concerning the Euler equations remain open. In particular, we still have no idea of whether three-dimensional solutions of the Euler equations, which start with smooth initial data, remain smooth all the time or whether they may become singular in finite time. In the case of finite time singularity it would be tempting to rely on weak solution formulation. However, there is almost no construction, so far, of weak solutions for the three-dimensional Euler equations. Moreover, defining an optimal functional space in which the three-dimensional problem is *well posed in the sense of Hadamard* is also an important issue.

Let us observe that the conservation of energy in the  $3d$  Euler equations is always formally true. However, physical intuition and scaling argument, i.e. the *Kolmogorov Obukhov law*, lead to the idea that non conservation of energy in the three-dimensional Euler equations would be intimately related to the loss of regularity. Therefore, Onsager [19] conjectured the existence of a threshold in the regularity of the  $3d$  Euler equations that would distinguish between solutions which conserve energy and solutions which might dissipate energy.

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Configuration where the vorticity is concentrated, as a measure, on a curve (in  $2d$ ) or on a surface (in  $3d$ ) are called Kelvin-Helmholtz flows. They seem to play a rôle in numerical simulations and in the description of turbulence. However, mathematical analysis and experiments show that these configurations are extremely unstable.

For the above reasons we believe that the detailed study of explicit examples remains extremely insightful and useful. Therefore, this contribution is devoted to new information that can be obtained from the study of the example of shear flow that was introduced by DiPerna and Majda [9].

For simplicity we will consider solutions of Euler equations defined in the whole space  $\mathbb{R}^3$ , or subject to periodic boundary conditions of period 1 (i.e. defined in the periodic box  $(\mathbb{R}/\mathbb{Z})^3$ ).

Observe that when the functions  $u_1$  and  $u_2$  are smooth the vector field

$$u(x, t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))) \quad (1)$$

is an obvious solution of the  $3d$  incompressible Euler equations of inviscid (ideal) fluids:

$$\partial_t u + \nabla \cdot (u \otimes u) = -\nabla p \quad \text{and} \quad \nabla \cdot u = 0, \quad (2)$$

with  $p = 0$ , i.e. this is a pressureless flow.

DiPerna and Majda introduced this shear flow in their seminal paper [9] to construct a family of oscillatory solutions of the  $3d$  Euler equations whose weak limit does not satisfy the Euler equations. In this paper we will investigate properties of this shear flow to address issues related to the questions of well-posedness, stability of solutions of the Kelvin-Helmholtz problem, and conservation of energy (Onsager conjecture [19]). It is worth mentioning that this shear flow was also investigated by Yudovich [25] to show that the vorticity grows to infinity, as  $t \rightarrow \infty$ , which he calls gradual loss of smoothness. This is a completely different notion of loss of smoothness than the one presented in Theorem 2 below, where we show the instantaneous loss of smoothness of the solutions for certain class of initial data.

## 2. INSTABILITY OF CAUCHY PROBLEM AND LOSS OF SMOOTHNESS

Most of the basic existing results for the initial value problem concerning the Euler equations rely on the expression of this solution in term of the vorticity:

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \quad (3)$$

$$\nabla \cdot u = 0, \nabla \wedge u = \omega. \quad (4)$$

Equation (4) defines  $u$  in term of  $\omega$ ; that is  $u = K(\omega)$  is given by the Biot-Savart law, where  $K$  is a pseudo-differential operator of order  $-1$ , and the map  $\omega \mapsto \nabla u$  is an operator of order 0. Therefore, and as it is well known, equation (3) seems to share some similarity with the Riccati equation

$$y' = Cy^2 \quad \text{whose solution is} \quad y(t) = \frac{y(0)}{1 - Cty(0)}, \quad (5)$$

which blows up in finite time for every  $y(0) > 0$ . On one hand, there is not enough justification for this similarity to deduce from (5) some blow up property for the Euler equations. However, one can deduce some *local in time* existence and stability result in any appropriate norm  $\|\cdot\|$  which satisfies the relation:

$$\|\omega \cdot \nabla u\| = \|\omega \cdot \nabla(K(\omega))\| \leq C\|\omega\|^2. \quad (6)$$

The operator  $K$  is not continuous from  $C^0$  to  $C^1$ , therefore the  $L^\infty$  norm is not appropriate for this scenario. On the other hand, the Hölder norm, i.e.  $\omega \in C^{0,\alpha}$  or  $u \in C^{1,\alpha}$ , is convenient. With the standard Sobolev estimates the norm  $H^s$ , for  $s > \frac{5}{2}$ , i.e.  $\omega \in H^{s-1}$  or  $u \in H^s$ , would also be convenient (and leads, by virtue of common functional analysis tools, to slightly simpler proofs, see, e.g. [17]). This is fully consistent with the fact that  $H^s$ , for  $s > \frac{5}{2}$ , is continuously imbedded in  $C^{1,s-\frac{3}{2}}$ .

With this classical observations in mind we recall the following facts (see also the recent surveys for more details [1] and [6]):

- (i) For initial data  $u(x, 0) = u_0(x)$  in  $C^{1,\alpha}$  the Euler equations (2) has a unique local in time solution  $u(x, t)$  in  $C^{1,\alpha}$  (cf. [14]). The same result is valid for initial data in  $H^s$ , for  $s > \frac{5}{2}$  (cf. [2],[17]). Moreover, this unique solution conserves the energy. In spite of the fact that the above results imply the short time control of the  $L^\infty$  norm of the vorticity (which seems to be the relevant quantity) one has the following complementary statement established in [2] (see also [17]). For every initial data  $u(x, 0)$  in  $C^{1,\alpha}$  or in  $H^s$ , for  $s > \frac{5}{2}$ , the solution of the three-dimensional Euler equations exists and depends continuously on the initial data, for as long as the time integral of the  $L^\infty$  norm of the vorticity remains bounded.
- (ii) Following [8] one can construct (in any space dimension) a residual set (in  $L^2(\Omega)$ ) of initial data  $u_0$  for which the Cauchy problem has an infinite family of weak solutions of the Euler equations, in the space  $C_w(\mathbb{R}_t; L^2(\Omega))$ , with the same initial data  $u_0$ .
- (iii) Eventually one does not know the existence of a 3d regular (say in  $C^{1,\alpha}$ ) solution of the Euler equations that becomes singular in a finite time (blow up problem).

The shear flow (1) was already used by DiPerna and Lions (cf. [15] page 124, see also [25]) to add information to these issues as stated in the following theorem.

**Theorem 1** (DiPerna-Lions). *For every  $p \geq 1$ ,  $T > 0$  and  $M > 0$  given there exists a smooth shear flow solution of the form (1) for which  $\|u(x, 0)\|_{W^{1,p}} = 1$  and  $\|u(x, T)\|_{W^{1,p}} > M$ .*

In a refined version of the above theorem we show the instantaneous loss of smoothness of weak solutions for the 3d Euler equations with initial data in the Hölder space  $C^{0,\alpha}$ . In some sense this shows that  $C^1$  is the *critical case*. That is, for initial data *more regular than  $C^1$* , say in  $C^{1,\alpha}$ , one has well-posedness and for *less regular* initial data one has ill-posedness.

**Theorem 2.** (i) *For  $u_1(x), u_3(x) \in C^{1,\alpha}$  the shear flow solution (1) is in  $C^{1,\alpha}$ , for all  $t \in \mathbb{R}$ .*

(ii) *For  $u_1(x), u_3(x) \in C^{0,\alpha}$  the shear flow solution (1) is always in  $C^{0,\alpha^2}$ .*

(iii) *There exists shear flow solutions, of the form (1), which for  $t = 0$  belong to  $C^{0,\alpha}$  and which for any  $t \neq 0$  are not in  $C^{0,\beta}$ , for any  $\beta > \alpha^2$ .*

**Proof.** Observe first that regularity results of (i) are concerning only the component  $u_3$  ( $u_1$  remains  $t$  independent). The statement (i) is trivial, but it is worth noticing as it shows that our analysis is in line with the classical results of [14]. To prove (ii) we write

$$\begin{aligned} & \frac{|u_3(x_1 - tu_1(x_2 + h)) - u_3(x_1 - tu_1(x_2))|}{h^{\alpha^2}} \\ &= \frac{|u_3(x_1 - tu_1(x_2 + h)) - u_3(x_1 - tu_1(x_2))|}{|tu_1(x_2 + h) - tu_1(x_2)|^\alpha} \left( \frac{|tu_1(x_2 + h) - tu_1(x_2)|}{h^\alpha} \right)^\alpha \\ &\leq |t|^\alpha \|u_3\|_{0,\alpha} \|u_1\|_{0,\alpha}^\alpha. \end{aligned} \tag{7}$$

For the point (iii) of the statement one introduces two periodic functions  $u_1(\xi)$  and  $u_3(\xi)$  which near the point  $\xi = 0$  coincide with the function  $|\xi|^\alpha$ . Consequently, for every  $t$  given one has, for  $x_1$  and  $x_2$  small enough,  $u_3(x_1 - tu_1(x_2))$  coincides with the function

$$|x_1 - t|x_2|^\alpha|^\alpha.$$

In particular, for  $t$  given, and for  $(x_1, x_2, x_3) = (0, x_2, x_3)$ , with  $x_2$  small enough, one has

$$u_3(x_1 - tu_1(x_2)) = |t|^\alpha |x_2|^{\alpha^2},$$

and the conclusion follows.

## 3. THE KELVIN-HELMHOLTZ PROBLEM AND THE SHEAR FLOW

As we have mentioned in the introduction the Kelvin-Helmholtz problem corresponds to the situation where the vorticity is concentrated on a moving orientable curve in  $r(t, \lambda)$ , in  $2d$ , parameterized by a parameter  $\lambda \in \mathbb{R}$ , or on a moving orientable surface  $r(t, \lambda, \mu)$ , in  $3d$ , parameterized by the parameters  $(\lambda, \mu) \in \mathbb{R}^2$ .

We assume that the curves or the surfaces are  $C^1$  orientable manifolds, denoted by  $\Gamma(t)$ , with unit normal  $\vec{n}$ . For  $x \notin \Gamma(t)$ , the velocity  $u$  can be expressed explicitly in term of the vorticity by the following Biot-Savart formulas:

$$u(x, t) = \begin{cases} \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x-r(t, \lambda')}{|x-r(t, \lambda')|^2} \tilde{\omega}(t, r(t, \lambda')) |\partial_\lambda r(t, \lambda')| d\lambda' & \text{in } 2d, \\ -\frac{1}{4\pi} \int \frac{x-r(t, \lambda', \mu')}{|x-r(t, \lambda', \mu')|^3} \tilde{\omega}(t, r(t, \lambda', \mu')) |\partial_\lambda r(t, \lambda', \mu') \wedge \partial_\mu r(t, \lambda', \mu')| d\lambda' d\mu' & \text{in } 3d, \end{cases} \quad (8)$$

where  $R_{\frac{\pi}{2}}$  is the  $\frac{\pi}{2}$  rotation matrix, and  $\tilde{\omega}$  is the *vorticity density* on these manifolds.

When  $x$  converges to a point  $r \in \Gamma(t)$  the velocity  $u(x, t)$  converges to two different values, on either side of the manifold,  $u_\pm(r, t)$ . In particular, and in agreement with the divergence free condition, one has

$$u_+(r, t) \cdot \vec{n} = u_-(r, t) \cdot \vec{n}, \quad \omega(x, t) = (u_+(r, t) - u_-(r, t)) \wedge \vec{n} \otimes \delta_{\Gamma(t)}(x), \quad (9)$$

for  $r \in \Gamma(t)$  and  $x \in \mathbb{R}^d$ ,  $d = 2, 3$ .

The vorticity density  $\tilde{\omega}$  is a vector valued density. In the  $2d$  case this vector is orthogonal to the plane of the flow and therefore is identified with a scalar. Hence the vorticity density is related to the vorticity by the expressions:

$$\begin{aligned} \omega(x, t) &= (u_+(r, t) - u_-(r, t)) \wedge \vec{n} \otimes \delta_{\Gamma(t)}(x) \\ &= \begin{cases} \tilde{\omega}(t, r(t, \lambda)) |\partial_\lambda r(t, \lambda)| d\lambda & \text{in } 2d, \\ \tilde{\omega}(t, r(t, \lambda, \mu)) |\partial_\lambda r(t, \lambda, \mu) \wedge \partial_\mu r(t, \lambda, \mu)| d\lambda d\mu & \text{in } 3d. \end{cases} \end{aligned} \quad (10)$$

$$(11)$$

Formulas (8) remain valid for  $x \in \Gamma(t)$  with the integral taken in the sense of Cauchy principal value and with the left-hand side of (8) replaced by the averaged velocity

$$v = \frac{u_+ + u_-}{2} \quad (12)$$

Therefore with some hypothesis on the regularity of the solution (cf. [16] for details) the problem can be reduced to equation (8) for  $v$  with:

$$(\partial_t r - v) \cdot \vec{n} = 0, \quad (13)$$

and in  $2d$

$$\partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda} \left( \frac{\tilde{\omega}}{|r_\lambda|^2} (v - r_\lambda) \cdot r_\lambda \right) = 0 \quad (14)$$

or in  $3d$  with  $N = \partial_\lambda r(t, \lambda, \mu) \wedge \partial_\mu r(t, \lambda, \mu)$

$$\begin{aligned} \partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda} \left( \frac{\tilde{\omega}}{|N|^2} (((v - \partial_t r) \wedge \partial_\mu r) \cdot N) \right) - \frac{\partial}{\partial \mu} \left( \frac{\tilde{\omega}}{|N|^2} (((v - \partial_t r) \wedge \partial_\lambda r) \cdot N) \right) \\ = \frac{1}{|N|^2} ((\partial_\mu r \wedge N) \cdot \tilde{\omega}) \partial_\lambda v - \frac{1}{|N|^2} ((\partial_\lambda r \wedge N) \cdot \tilde{\omega}) \partial_\mu v. \end{aligned} \quad (15)$$

We recall below some classical results which contribute to the understanding of the basic properties of this problem (see also, e.g., [1]).

- (i) The initial value problem is locally, in time, well-posed in  $2d$  and  $3d$  in the class of analytic data. More precisely, for any initial curve (respectively surface)  $\Gamma(0, \lambda)$ , (respectively  $\Gamma(0, \lambda, \mu)$ ) and any initial density of vorticity  $\tilde{\omega}(0, r(0, \lambda))$  (respectively  $\tilde{\omega}(0, r(0, \lambda, \mu))$ ) which can be extended as analytic functions uniformly bounded in the strip  $|\Im \lambda| \leq c$ , in the complex plane  $\lambda \in \mathbb{C}$ , for some  $c > 0$ , (respectively  $|\Im \lambda| + |\Im \mu| \leq c$ , for  $(\lambda, \mu) \in \mathbb{C}^2$ , and for some  $c > 0$ ) there exists a finite time  $T$  and a constant  $C$  such that the initial value problem has, for  $0 \leq t < T$ , a unique solution which is analytic in the strip  $|\Im \lambda| \leq C(T-t)$  (respectively  $|\Im \lambda| + |\Im \mu| \leq C(T-t)$ ) (cf. [23]).
- (ii) There exist in  $2d$  (to the best of our knowledge this issue has not been addressed in  $3d$ ) analytic solutions that become singular in finite time. This has been first observed by numerical simulations of Baker, Meiron and Orszag [18], then Duchon and Robert [10] have shown the existence of a very large class of singularities which can be reached in a finite time by analytic solutions. Eventually, Caffisch and Orellanna [3] have constructed analytic solutions, for  $0 \leq t < T$ , which exhibit a cusp as  $t$  approaches  $T$ . Specifically, with  $0 < \nu < 1$  they have shown that their solutions satisfy:

$$\lim_{t \rightarrow T} (\Gamma(t, \cdot), \tilde{\omega}(t, r(t, \cdot))) = (\Gamma(T, \cdot), \tilde{\omega}(T, r(T, \cdot))) \begin{cases} \notin C^{1,\nu} \times C^\nu, \\ \in C^{1,\nu'} \times C^{\nu'} \text{ for every } \nu' \in (0, \nu). \end{cases}$$

- (iii) In  $2d$  : If in a  $(t, \lambda)$  neighborhood of a point  $(t_0, \lambda_0)$  the vorticity density,  $\tilde{\omega}(t, r(t, \lambda))$ , does not vanish and if the functions  $r(t, \lambda), \tilde{\omega}(t, r(t, \lambda))$  have some *limited regularity* then in fact they are analytic in this neighbourhood. By a limited regularity we mean, for instance, that in this neighborhood

$$(r(t, \cdot), \tilde{\omega}(t, r(t, \cdot))) \in C^{1,\alpha} \times C^\alpha \quad (16)$$

$$|\lambda - \lambda'| \leq C|r(t, \lambda) - r(t, \lambda')|, \text{ with some constant } C < \infty. \quad (17)$$

The hypothesis (17) is called the *chord-arc property*, and the hypothesis (16) matches perfectly the example studied in [3]. In fact under the chord-arc hypothesis a refined version of this statement has been obtained by Wu [24] which matches some numerical observations made by Krasny [12]. The consequence of this observation is that solutions *with limited regularity* do not exist in  $2d$ . That is, if at some time  $t_0$  and at some point  $\lambda_0$  the solution,  $(r(t, \lambda), \tilde{\omega}(t, r(t, \lambda)))$ , ceases to be analytic then it cannot be of limited regularity at a later time. For instance the solution of [3] is no longer in  $C^{1,\nu'} \times C^{\nu'}$ , for any  $\nu' > 0$ , for  $t > T$ .

*Remark 1.* The hypothesis that  $\tilde{\omega}(t, r(t, \lambda))$  does not vanish is natural. This is because if  $\tilde{\omega}$  vanishes near  $(t_0, \lambda_0)$  then there is no more interface, and the ellipticity as described below is lost. This will appear explicitly in formulas (25) and (26).

The clue in the above  $2d$  results lies in the fact that under the above hypothesis the problem is locally a *small* perturbation of an elliptic system: since this analysis is local one can assume, without loss of generality, that  $\Gamma(t) = (x, \epsilon y(x, t))$  is a graph. As a result, equations (8), (13) and (14) are equivalent to the system:

$$\partial_t y - v_2 = (v_1 \partial_x y), \quad (18)$$

$$\partial_t \tilde{\omega} + \partial_x (v_1 \Omega_0) = -\epsilon \partial_x (v_1 \tilde{\omega}), \quad (19)$$

$$v_1(x, t) = -\frac{1}{2\pi} P.V. \int \frac{y(x, t) - y(x', t)}{(x - x')^2 + \epsilon^2 (y(x, t) - y(x', t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx', \quad (20)$$

$$v_2(x, t) = \frac{1}{2\pi} P.V. \int \frac{x - x'}{(x - x')^2 + \epsilon^2 (y(x, t) - y(x', t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx'. \quad (21)$$

For small values of  $\epsilon$ , this system describes a small perturbations in  $\mathbb{R}^2$  about the stationary solution

$$y(x, 0) = 0, u_- = \frac{\Omega_0}{2}, u_+ = -\frac{\Omega_0}{2}.$$

Indeed, for functions  $f$  and  $y$  in  $C^1$ , with  $\frac{\partial y}{\partial x}$  bounded, the expansion

$$\begin{aligned} \frac{1}{\pi} P.V. \int \frac{f(x) - f(x')}{(x - x')^2 + \epsilon^2(y(x, t) - y(x', t))^2} dx' &= \\ \frac{1}{\pi} P.V. \int \frac{f(x) - f(x')}{(x - x')^2 + \epsilon^2(y(x, t) - y(x', t))^2} dx' &= \\ \frac{1}{\pi} P.V. \int \frac{f(x) - f(x')}{(x - x')^2} \left( 1 + \sum_{n \geq 1} (-1)^n \epsilon^{2n} \left( \frac{y(x) - y(x')}{x - x'} \right)^2 \right) dx' & \end{aligned} \quad (22)$$

leads to the introduction of the operators (Hilbert transform):

$$Hf(x) = \frac{1}{\pi} P.V. \int \frac{1}{x - x'} f(x') dx' = \mathcal{F}^{-1}(-i \operatorname{sgn}(\xi) \hat{f}(\xi)) \quad (23)$$

$$|D|f(x) = \frac{1}{\pi} P.V. \int \frac{f(x) - f(x')}{(x - x')^2} = \partial_x(Hf(x)) = \mathcal{F}^{-1}(|\xi| \hat{f}(\xi)). \quad (24)$$

This in turn gives, together with formulas (18)-(22), for the perturbation about the stationary solution the system:

$$\begin{aligned} \partial_t y_x - \Omega_0 |D| \tilde{\omega} &= \epsilon F(y_x, \tilde{\omega})_x \\ \partial_t \tilde{\omega} - |D| y_x &= \epsilon G(y_x, \tilde{\omega})_x, \end{aligned}$$

where in the right-hand side  $F$  and  $G$  are first order operators. Eventually with the introduction of the ‘‘Laplacian’’ one has:

$$\partial_{tt}(y_x) + \Omega_0^2 \partial_{xx}(y_x) = \epsilon(\partial_t(F(y_x, \tilde{\omega})_x) + |D|(\epsilon G(y_x, \tilde{\omega})_x)), \quad (25)$$

$$\partial_{tt}(\tilde{\omega}) + \Omega_0^2 \partial_{xx}(\tilde{\omega}) = \epsilon(|D|(F(y_x, \tilde{\omega})_x) + \partial_t(\epsilon G(y_x, \tilde{\omega})_x)). \quad (26)$$

Keeping in mind the above considerations we observe the following evident, but complementary, statement:

**Proposition 3.** *In 3d with the following configuration*

$$u_3(s) = \begin{cases} 1 & \text{for } s < 0 \\ 0 & \text{for } s > 0 \end{cases},$$

and  $u_1 \in C^1$ , the shear flow:

$$u(x) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$$

is a weak solution of the 3d Euler equations with a vorticity which is singular on the  $C^1$  surface

$$\Gamma(t) = \{(x_1, x_2, x_3) \mid x_1 = tu_1(x_2), \}.$$

This vorticity is given by:

$$\omega(x, t) = \nabla \wedge u(x, t) = \left( -t \frac{\partial_{x_2} u_1(x_2)}{(|t \partial_{x_2} u_1|^2 + 1)^{\frac{1}{2}}} \otimes \delta_{\Gamma(t)}(x), \frac{1}{(|t \partial_{x_2} u_1|^2 + 1)^{\frac{1}{2}}} \otimes \delta_{\Gamma(t)}(x), -\partial_{x_2} u_1(x_2) \right). \quad (27)$$

We remark that the above example, presented in Proposition 3, is not a genuine solution of the Kelvin-Helmholtz problem because of the presence of the *distributed* vorticity,  $-\partial_{x_2} u_1(x_2)$ , which is not concentrated on a surface. In order for the vorticity of the shear flow solution (1) of the three-dimensional

Euler equations to be concentrated on an interface one can easily check that the solution must be of the form:

$$u_1(s) = \begin{cases} \alpha_1 & \text{for } s < \xi_2 \\ \beta_1 & \text{for } s > \xi_2 \end{cases} \quad \text{and} \quad u_3(s) = \begin{cases} \alpha_3 & \text{for } s < \xi_1 \\ \beta_3 & \text{for } s > \xi_1 \end{cases},$$

for some fixed real parameters  $\alpha_1, \alpha_3, \beta_1, \beta_3, \xi_1, \xi_2$ , satisfying  $\alpha_1 \neq \beta_1$  and  $\alpha_3 \neq \beta_3$ . Consequently, the corresponding vorticity of the above solution is concentrated on the singular surface:

$$\Sigma(t) = \{(x_1, x_2, x_3) | x_2 = \xi_2\} \cup \{(x_1, x_2, x_3) | x_1 = \xi_1 + t\alpha_1, x_2 \leq \xi_2\} \cup \{(x_1, x_2, x_3) | x_1 = \xi_1 + t\beta_1, x_2 \leq \xi_2\}.$$

Since the vortex sheet  $\Sigma(t)$  is a singular surface the above example provides a solution, which is not analytic, and hence it is of a “*limited regularity*”, of the 3d Kelvin-Helmholtz problem. As a result, there is no hope for a possible extension of the above mentioned 2d theory (as presented, e.g., in [3],[10],[13] and [24]) to the 3d case. In fact, the explanation of the main difference of the 3d Kelvin-Helmholtz problem from the 2d one lies in the loss of ellipticity of the linearized operator. As it was done in 2d case, we consider a local perturbation about the stationary solution, in this situation we assume (following the notation of [23] or [4]) that  $\Gamma(t)$  can be parameterized in the form  $x_3 = \epsilon x(x_1, x_2, t)$ , and reduce the analysis to the properties of the *small* perturbation about the stationary state  $x_3 = 0, \tilde{\omega}^0(x_1, x_2) = (\tilde{\omega}_1^0, \tilde{\omega}_2^0, 0)$ . The leading part of the perturbed equations (as was done above in the 2d case) is the linear operator (written in the 2d Fourier variables  $k = (k_1, k_2)$ , the dual of  $(x_1, x_2)$ )

$$\partial_t \begin{pmatrix} \hat{x}_3 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{pmatrix} = \mathcal{A} \begin{pmatrix} \hat{x}_3 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{pmatrix} \quad (28)$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{i}{2} \sin \theta & -\frac{i}{2} \cos \theta & 0 \\ -\frac{i}{2} |k|^2 |\tilde{\omega}^0|^2 \sin \theta & 0 & 0 & \frac{1}{2} (k \cdot \tilde{\omega}^0) \sin \theta \\ \frac{i}{2} |k|^2 |\tilde{\omega}^0|^2 \cos \theta & 0 & 0 & -\frac{1}{2} (k \cdot \tilde{\omega}^0) \cos \theta \\ 0 & -\frac{1}{2} (k \cdot \tilde{\omega}^0) \sin \theta & \frac{1}{2} (k \cdot \tilde{\omega}^0) \cos \theta & 0 \end{pmatrix}, \quad (29)$$

with  $k = (k_1, k_2) = |k|(\cos \theta, \sin \theta)$ .

The eigenvalues of the matrix  $\mathcal{A}$  are

$$\{0, 0, -\frac{1}{2}|k \wedge \tilde{\omega}^0|, \frac{1}{2}|k \wedge \tilde{\omega}^0|\}.$$

Therefore, the first order pseudo-differential operator

$$\partial_t - \mathcal{A}$$

is no longer elliptic, as is the situation in the 2d case (see (25)-(26)). Indeed, this seems to be the basic reason of why in the 3d case solutions with limited regularity may exist. That is, limited regularity in the 3d case does not imply analyticity.

#### 4. LOSS OF REGULARITY AND ENERGY CONSERVATION

It has been conjectured by Onsager [19] that for some weak solutions of the 3d Euler equations the decay in energy would be related to some loss of regularity in these solutions. Arguing by some dimensional analysis, the Hölder exponent  $1/3$  appears to be a critical value of such regularity.

On the one hand, it has been shown rigorously in [7] that the formal conservation of energy in the 3d Euler Equations is in fact true for any weak solution which is slightly more regular than the Besov space  $\mathcal{B}_{3,\infty}^{\frac{1}{3}}$  (see also [5] and [11]). On the other hand, the existence of very weak solutions *wild solutions* that become identically 0 after a finite time has been established in [20], [21] and most recently in [8]. Moreover, it is commonly believed that for solutions which are slightly weaker than  $\mathcal{B}_{3,\infty}^{\frac{1}{3}}$  there might

be no conservation of energy. In fact Eyink [11] has constructed a function  $u_0(x) \in C^{0, \frac{1}{3}}$  which cannot be the initial data of any weak solution which conserves the energy. This, however, is not a *complete* counter example because the existence of weak solutions for the 3d Euler equations with such initial data is still an open problem.

With the shear flow

$$u(x, t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))) \quad (30)$$

in a periodic box we show that there is no hope for general theorem stating that conservation of energy implies some type of regularity. The first step consists of observing that the above conservation of energy remains true under the only assumption that  $u_1, u_3 \in L^2(\Omega)$ . This is a consequence of the following

**Lemma 4.** For  $\Omega = ((\mathbb{R}/Z))^3$ ,  $u_1(x_1), u_3(x_2) \in L^2((\mathbb{R}/Z)) \times L^2((\mathbb{R}/Z))$  and any test functions  $\phi_i, i = 1, 2, 3$  the following standard formula

$$\begin{aligned} & \iiint_{\Omega} u_3(x_1 - tu_1(x_2)) \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) dx_1 dx_2 dx_3 \\ &= \iiint_{\Omega} u_3(x_1) \phi_1(x_1 + tu_1(x_2)) \phi_2(x_2) \phi_3(x_3) dx_1 dx_2 dx_3 \end{aligned} \quad (31)$$

is valid.

From (31) one deduces the

**Theorem 5.** For  $\Omega = ((\mathbb{R}/Z))^3$  and any functions  $u_1(x_1), u_3(x_3) \in L^2((\mathbb{R}/Z)) \times L^2((\mathbb{R}/Z))$  the flow

$$u(x, t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$$

is a weak solution of (2) with constant energy, i.e.,

$$\frac{d}{dt} \iiint_{\Omega} |u(x, t)|^2 dx = \frac{d}{dt} \iiint_{\Omega} (|u_1(x_2)|^2 + |u_3(x_1 - tu_1(x_2))|^2) dx_1 dx_2 dx_3 = 0. \quad (32)$$

Observe that the hypothesis on the initial data here are much weaker than those for which the Onsager conjecture is stated in [7], [11] or [22] (see also [5]).

In [22] Shvydkoy considers the energy conservation for weak solutions of the Euler equations with singularities on a curve (in 2d) and on a surface (in 3d). This class of solutions includes the Kelvin-Helmholtz problem discussed in section 3. In fact his results turns out to be more relevant for the Kelvin-Helmholtz problem in dimension 3 rather than in dimension 2. The reason being, as we have mentioned above, that in the 2d case a minimal regularity for the Kelvin-Helmholtz problem implies analyticity; and therefore the conservation of energy follows while in the 3d case the ellipticity of the linearized operator is no longer true and there is room for less regular (non-analytic), and possibly singular, surface solution of the 3d Kelvin-Helmholtz problem. In agreement with this observation we propose the following example. Consider for  $u_1(s)$  a periodic function which coincide near 0 with the function  $\sin \frac{1}{s}$  and for  $u_3(s)$  a periodic function which near 0 coincides with  $\text{sgn}(s)$  then the shear flow

$$u(x, t) = (u_1(x_1), 0, u_3(x_1 - tu_1(x_2)))$$

is a weak solution of the 3d Euler equations which conserves the energy and which does not satisfy the hypothesis given by [22].

## 5. CONCLUSION

We have used the simplest example of a genuinely 3d flow to obtain the following observations concerning the Euler equations:

- (i) In the class of Hölder spaces the space  $C^1$  is the critical space for the initial value problem of the 3d Euler equations to be locally, in time, well-posed *in the sense of Hadamard*. Old and classical results [14] (see also [2] and [17]) have shown that the 3d Euler equations are well posed in  $C^{1,\alpha}$ , for every  $\alpha > 0$ , while we have shown here that 3d Euler equations are not well-posed in  $C^\beta$  for any  $0 < \beta < 1$ .
- (ii) The Kelvin-Helmholtz problem refers to a free boundary problem were in the 2d case limited regularity implies analyticity. We show that this result is false in the 3d case by providing an explicit solution for the 3d Kelvin-Helmholtz problem with certain degree of singularity of the vortex sheet, which persists for all time. Moreover, we give an explanation for this striking difference between the 2d and 3d Kelvin-Helmholtz problems.
- (iii) The relation between dissipation of energy and loss of regularity is an essential issue in the statistical theory of turbulence, in relation with the Kolmogorov Obukhov law. It has been shown in the deterministic framework that a regularity of this type implies conservation of energy. With the shear flow example we have shown that there is no hope for a converse statement (even in the case of solutions singular on a *slit* as in [22]). This observation may not invalidate the common *physical belief* because the Kolmogorov Oboukov law belongs to the statistical theory of turbulence, where statements and results are true in some *averaged sense*. On the other hand, our family of shear flow examples are particular enough to be of measure zero with respect to any reasonable *ensemble measure* which compatible with the statistical theory of ideal (inviscid) turbulent flows (let us recall that, to the best of our knowledge, no such measure has been constructed, up to now, with full mathematical rigor).

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