Riesz exponential families on homogeneous cones

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Abstract. In this paper, we introduce, for a multiplier χ, a notion of generalized power function $x \mapsto \Delta_\chi(x)$, defined on the homogeneous cone $\mathcal{P}$ of a Vinberg algebra $\mathcal{A}$. We then extend to $\mathcal{A}$ the famous Gindikin result, that is we determine the set of multipliers χ such that the map $\theta \mapsto \Delta_\chi(\theta^{-1})$, defined on $\mathcal{P}^*$, is the Laplace transform of a positive measure R_χ. We also determine the set of χ such that R_χ generates an exponential family, and we calculate the variance function of this family.

Key words: Homogeneous cone; multiplier; generalized power; Riesz probability distribution; exponential family; variance function.

1 Introduction

It is well known (see Casalis and Letac (1996)) that the Wishart distributions on the cone of (r, r) positive symmetric matrices or on the symmetric cone Ω of any Euclidean Jordan algebra E of rank r are the elements of the natural exponential families generated by the measures μ_p such that the Laplace transform is defined on Ω by

$$L_{\mu_p}(\theta) = (\det(\theta^{-1}))^p,$$

for p in $\{\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{r-1}{2}\} \cup]\frac{r-1}{2}, +\infty[$. The measure μ_p is absolutely continuous when $p \in]\frac{r-1}{2}, +\infty[$ and is singular concentrated on the boundary of the cone, when $p \in \{\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{r-1}{2}\}$. In 2001, Hassairi and Lajmi have introduced the Riesz distribution on Ω as an extension of the Wishart distribution. These authors have started from the fact that in a Jordan algebra, besides the real power of the determinant, there is the so called generalized power $\Delta_s(x)$ of an element x of Ω, defined for a fixed ordered Jordan frame of E and for $s = (s_1, \cdots, s_r)$ in $\mathbb{R}^r$, and they have used a remarkable result, due to Gindikin (1964), which determines the set Ξ

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of s in $\mathbb{R}^r$ such that $\Delta_s(\theta^{-1})$ is the Laplace transform of some positive measure R_s on E. The generalized power $\Delta_s(x)$ is a power function of the principal minors of x which reduces to $(\det(x))^p$ in the particular case where $s_1 = s_2 = \cdots = s_r = p$, and in this case, the measure R_s in nothing but μ_p. We mention here that Ishi (2000) has given a more detailed description of the Gindikin set Ξ based on the orbit structure of $\overline{\Omega}$ under the action of some Lie group. He has also given explicitly the measure R_s for each s in Ξ. In all these works, the definition of the Riesz measure R_s and in particular of the Riesz probability distribution is based on the choice of a totally ordered Jordan frame which allows the definition of the principal minors and of the generalized power of an element of the algebra. The fact that the order is total is a fundamental condition not only for the definition of the distribution but also in the proof of many results. To define models in which some specified conditional independencies, usually given by a graph, are taken into account, there has been an interest in probability distributions on the homogeneous cone of a Vinberg algebra. For instance, Andersson and Wojnar (2004) have defined a class of absolutely continuous "Wishart" distributions on an homogeneous cone. These distributions have been characterized by Boutouria (2005, 2007) in the Bobecka and Wesołowski (2002) way. They have also been charcterized by Boutouria and Hassairi (2008) in the way given in Olkin and Rubin (1962) for the ordinary Wishart. The aim of the present work is to use an approach similar to the one used in the definition of a Riesz exponential family on a symmetric cone to introduce a Riesz exponential family on an homogeneous cone. The distributions in these families are defined for any graph, that is for any order relation not necessary total. Some of these distributions are absolutely continuous with respect to the Lebesgue measure and some are singular concentrated on the boundary of the cone. In this connection, the Riesz distribution on the symmetric cone of a Jordan algebra may be seen as the particular one corresponding to the particular directed graph with vertex set $\{1, \cdots, r\}$ and edges defined by the usual order on integers. We first define for an element of an homogeneous cone two kinds of principal minors, minors which are said strict and minors which are said large. We then define for a multiplier χ, a notion of generalized power function $x \mapsto \Delta_\chi(x)$. One of our main results is the determination of the set of multipliers χ such that the map $\theta \mapsto \Delta_\chi(\theta^{-1})$ is the Laplace transform of a positive measure R_χ. It is a generalization of Gindikin result with a more elaborate proof adapted to the properties of the Vinberg algebra and the graph. Concerning the generated exponential families, we give a necessary and sufficient condition on χ in order that R_χ generates an exponential family and, under this condition, we determine the variance function of the family.

2 Vinberg algebras and homogeneous cones

In this section, we introduce some notations and review some basis concepts concerning Vinberg algebras and their homogeneous cones. We also introduce a useful decomposition of an element of the cone.

Throughout the paper, I denote a partially ordered finite set equipped with a relation denoted $\preceq$. We will write $i \prec j$ if $i \preceq j$ and $i \neq j$. For all pairs $(i,j) \in I \times I$ with $j \prec i$, let E_{ij} be a finite-dimensional vector space over $\mathbb{R}$ with $n_{ij} = \dim(E_{ij}) > 0$. Set

$$\mathcal{A}_{ij} = \begin{cases} \mathbb{R} & \text{for} \quad i = j \\ E_{ij} & \text{for} \quad j \succ i \quad \text{or} \quad j \prec i \\ \{0\} & \text{otherwise.} \end{cases}$$

and $\mathcal{A} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}$. An element $A \equiv (a_{ij},\ i,j \in I)$ of $\mathcal{A}$ may be seen as a matrix and so we define the trace $\mathrm{tr}A = \sum_{i \in I} a_{ii}$. We also define

$$n_{i.} = \sum_{\mu < i} n_{i\mu}, n_{.i} = \sum_{i<\mu} n_{\mu i},\ n_i = 1 + \frac{1}{2}(n_{i.} + n_{.i}), i \in I \text{ and } n_{.} = \sum_{i \in I} n_{i}. \tag{2.1}$$

Let $f_{ij} : E_{ij} \to E_{ij}$, $i \succ j$, be involutional linear mappings, i.e., $f_{ij}^{-1} = f_{ij}$. They induce an involutional mapping ($A \mapsto A^*$) of $\mathcal{A}$ given as follows: $A^* = (a^*_{ij} | (i,j) \in I \times I)$, where

$$a^*_{ij} = \begin{cases} a_{ii} & \text{for} \quad i = j \\ f_{ij}(a_{ij}) = a^*_{ij} & \text{for} \quad j \prec i \quad \text{or} \quad i \prec j \\ \{0\} & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}_u = \{A \equiv (a_{ij}) \in \mathcal{A},\ \forall i,j \in I : i \npreceq j \Rightarrow a_{ij} = 0\}$, $\mathcal{T}_l = \{A \equiv (a_{ij}) \in \mathcal{A},\ \forall i,j \in I : j \npreceq i \Rightarrow a_{ij} = 0\}$ and $\mathcal{H} = \{\mathcal{A} \in \mathcal{A},\ \mathcal{A}^* = \mathcal{A}\}$ denote respectively the set of upper triangular matrices, the lower triangular matrices and the Hermitian matrices. The sets of upper and lower triangular matrices in $\mathcal{P}$ with positive diagonal elements are respectively denoted by $\mathcal{T}_u^+$ and $\mathcal{T}_l^+$. The sets of diagonal matrices and of diagonal matrices with positive entries are denoted by $\mathcal{D}$ and $\mathcal{D}^+$, respectively.

The space $\mathcal{A}$ is equipped with a bilinear map called multiplication and denoted by $(A,B) \mapsto AB$, using bilinear mappings $\mathcal{A}_{ij} \times \mathcal{A}_{jk} \to \mathcal{A}_{ik}$, denoted by $(a_{ij}, b_{jk}) \mapsto a_{ij}b_{jk}$, such that $AB = C \equiv (c_{ij} | (i,j) \in I \times I)$ with $c_{ij} = \sum_{\mu \in I} a_{i\mu} b_{\mu j}$.
The multiplication is required to satisfy the following properties:

$$\begin{aligned}
&i)\ \ \forall A \in \mathcal{A};\ A \neq 0 \Rightarrow \mathrm{tr}(AA^*) > 0 \\
&ii)\ \ \forall A, B \in \mathcal{A};\ (AB)^* = B^*A^* \\
&iii)\ \ \forall A, B \in \mathcal{A};\ \mathrm{tr}(AB) = \mathrm{tr}(BA) \\
&iv)\ \ \forall A, B, C \in \mathcal{A};\ \mathrm{tr}(A(BC)) = \mathrm{tr}((AB)C) \\
&v)\ \ \forall U, S, T \in \mathcal{T}_l;\ (ST)U = S(TU) \\
&vi)\ \ \forall U, S, T \in \mathcal{T}_l;\ T(UU^*) = (TU)U^*.
\end{aligned}$$

An algebra $\mathcal{A}$ with the above structure and properties is called a Vinberg algebra (For more details, we can refer to Andersson and Wojnar (2004). Define the inner products $(.,.)_{ij}$ on E_{ij}, $i \succ j$ by $\| a_{ij} \|_{ij}^2 = a_{ij} f_{ij}(a_{ij})$, $a_{ij} \in E_{ij}$. Thus instead of specifying the bilinear form $(a_{ij}, b_{ji}) \mapsto a_{ij} b_{ji}$ on E_{ij} one can specify an inner product $(.,.)_{ij}$ on E_{ij}, $i \succ j$. It can be established that the following two conditions also must hold:

1. $\forall\ a_{ij} \in E_{ij},\ b_{jk} \in E_{jk} : \| a_{ij} b_{jk} \|_{ik}^2 = \| a_{ij} \|_{ij}^2 \| b_{jk} \|_{jk}^2,\ k \prec j \prec i,$

   and

2. If $a_{ik} \in E_{ik},\ b_{jk} \in E_{jk}$, with $k \prec j \prec i$ and $(a_{ik}, c_{ij} b_{jk})_{ik} = 0$ for all $c_{ij} \in E_{ij}$, then $(d_{li} a_{ik}, c_{lj} b_{jk})_{lk} = 0$ for all $l \in I$ with $i \prec l$, and all $c_{lj} \in E_{lj}$, and $d_{li} \in E_{li}$.

We consider the element $(a_{ij} | (i,j) \in I \times I)$ of $\mathcal{D}$ such that $a_{ii} = 1,\ \forall\ i \in I$ as the unit element of $\mathcal{A}$ and we denote it by e. We also define $E_k = (d_{ij})_{i,j \in I} \in \mathcal{D}$ with $d_{kk} = 1$ and $d_{jj} = 0\ \forall j \neq k$. It is clear that $\sum_{k \in I} E_k = e$.

Vinberg (1965) proved that the subset $\mathcal{P} = \{TT^* \in \mathcal{A},\ T \in \mathcal{T}_l^+\} \subset \mathcal{H} \subset \mathcal{A}$ forms a homogeneous cone, that is the action of its automorphism group is transitive. Let G be the connected component of the identity in $\mathrm{Aut}(\mathcal{P})$; the group of linear transformations leaving $\mathcal{P}$ invariant. We recall that $\chi : G \mapsto \mathbb{R}_+$ is said to be a multiplier on the group G if it is continuous, $\chi(e) = 1$ and $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$. Consider the map $\pi\ :\ T \in \mathcal{T}_l^+ \mapsto \pi(T) \in \pi(\mathcal{T}_l^+) \subset G$ such that for $X = VV^* \in \mathcal{P}$, $V \in \mathcal{T}_l^+$

$$\pi(T)(X) = (TV)(V^* T^*). \tag{2.2}$$

Andersson and Wojnar (2004) have shown that the restriction of a multiplier χ to the (lower) triangular group $\mathcal{T}_l^+$, i.e., $\chi \circ \pi : T_l^+ \to \mathbb{R}_+$ is in one to one correspondence with the set of $(\lambda_i, i \in I) \in \mathbb{R}^I$. We will then describe a multiplier χ by its corresponding point in $\mathbb{R}^I$ and we denote $\mathcal{X} = \{\chi : T_l^+ \to \mathbb{R}_+\}$.

If $\preceq^{\text{opp}}$ is the opposite ordering on the index set I, i.e., $i \preceq^{\text{opp}} j \Leftrightarrow j \preceq i$. The Vinberg algebra $\mathcal{A}^{\text{opp}} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}^{\text{opp}}$, where

$$\mathcal{A}_{ij}^{\text{opp}} = \begin{cases} \mathbb{R} & \text{for} \quad i = j \\ E_{ij} & \text{for} \quad j \succ^{\text{opp}} i \quad \text{or} \quad j \prec^{\text{opp}} i \\ \{0\} & \quad \text{otherwise,} \end{cases}$$

differs from the Vinberg algebra $\mathcal{A}$ only in the ordering of the index set I. Vinberg (1965) proved that $\mathcal{P}_{\preceq^{\text{opp}}} = \{T^* T \in \mathcal{A},\ T \in \mathcal{T}_l^+\}$ is the dual cone of $\mathcal{P}$. The inner product $(A, B) \to \mathrm{tr}(AB)$ on H identifies H with its dual H^*, i.e.,

$$\begin{array}{rcl} H & \leftrightarrow & H^* \\ A & \mapsto & (B \mapsto \mathrm{tr}(AB)), \end{array}$$

and this isomorphism identifies $\mathcal{P}_{\preceq^{\text{opp}}}$ with the dual cone $\mathcal{P}^*$ of $\mathcal{P}$.

Now, for $i \in I$, we denote $I_{\preceq i} = \{j \in I;\ j \preceq i\}$ and $I_{i\preceq} = \{j \in I;\ i \preceq j\}$ and we say that j separates i_1 and i_2 if $j \in I_{i_1\preceq} \cap I_{i_2\preceq}$ and $j \notin \{i_1,\ i_2\}$. In this case, j is called a separator. We denote $S_i = \{j \in I_{i\preceq};\ j \text{ is a separator}\}$ and $\mathcal{S} = \bigcup_{i\in I} S_i$.

If $T = (t_{ij})_{i,j\in I}$ is in $\mathcal{T}_l$, we define the element $T_{i\preceq}$ of $\mathcal{T}_l$ by

$$T_{i\preceq} = (t'_{ij})_{i,j\in I},\ \text{with } t'_{jk} = t_{jk} \text{ if } i \preceq j,\ k \text{ and } t'_{jk} = 0 \text{ otherwise,} \tag{2.3}$$

and the element $T_{i\prec}$ of $\mathcal{T}_l$ by

$$T_{i\prec} = (t'_{ij})_{i,j\in I},\ \text{with } t'_{jk} = t_{jk} \text{ if } i \prec j,\ k \text{ and } t'_{jk} = 0 \text{ otherwise.} \tag{2.4}$$

If $X = TT^*$, we denote

$$X_{i\preceq} = T_{i\preceq}T^*_{i\preceq}, \qquad\qquad X_{i\prec} = T_{i\prec}T^*_{i\prec}. \tag{2.5}$$

We also denote by $\mathcal{P}_{i\preceq}$ (resp $\mathcal{P}_{i\prec}$) the set of $X_{i\preceq} = T_{i\preceq}T^*_{i\preceq}$ (resp $X_{i\prec} = T_{i\prec}T^*_{i\prec}$) corresponding to $T \in \mathcal{T}_l^+$. It is easy to see that $\mathcal{P}_{i\preceq}$ and $\mathcal{P}_{i\prec}$ are respectively the homogeneous cones of the Vinberg subalgebras of $\mathcal{A}$ defined by $\mathcal{A}_{i\preceq} = \prod_{k,l\in I_{i\preceq}} \mathcal{A}_{kl}$ and $\mathcal{A}_{i\prec} = \prod_{k,l\in I_{i\prec}} \mathcal{A}_{kl}$. We denote by e_i and $\check{e}_{\check{i}}$ respectively, the unit element of $\mathcal{A}_{i\preceq}$ and $\mathcal{A}_{i\prec}$. We also define the rank of $\mathcal{P}_{i\preceq}$ (resp the rank of $\mathcal{P}_{i\prec}$) the cardinal of the set $\{j \in I, i \preceq j\}$ (resp the cardinal of the set $\{j \in I, i \prec j\}$). Finally, if we denote $\wp = \{i \in I;\ I_{i\prec} \neq \emptyset \text{ and } I_{\prec i} = \emptyset\}$ and if we set, for $X \in \mathcal{P}$,

$$X_i = \begin{cases} X_{i\preceq} - \sum_{s\in S_i} X_{s\preceq} & \text{if} \quad i \in \wp \\ X_{i\preceq} & \text{if} \quad i \in S \\ 0 & \qquad\text{otherwise,} \end{cases} \tag{2.6}$$

then, we have the following decomposition of X

$$X = \sum_{i\in I} X_i. \tag{2.7}$$

3 Riesz measures on an homogeneous cone

The definition of a Riesz measure on the symmetric cone of a Jordan algebra relies on the notion of generalized power of an element of the cone which is a power function of the so-called principal minors. In order to define a Riesz distribution on an homogeneous cone, we need to extend all these things to a Vinberg algebra.

3.1 Generalized power

We first introduce a notion of determinant. For $X = TT^*$, with $T = (t_{ij}) \in \mathcal{T}_l$, we define the determinants

$$\det X = \prod_{i\in I} t_{ii}^2, \qquad \det{}^{\preceq} X_{\preceq i} = \prod_{j\in I_{\preceq i}} t_{jj}^2 \quad \text{and} \quad \det{}^{\prec} X_{\prec i} = \prod_{j\in I_{\prec i}} t_{jj}^2.$$

For $X = TT^*$, with $T \in \mathcal{T}_l$, we define the strict principal minor of order k of X as

$$\Delta_{\prec k}(X) = \begin{cases} \det^{\prec}(X_{\prec k}) & \text{if} \quad I_{\prec k} \neq \emptyset \\ 1 & \text{if} \quad I_{\prec k} = \emptyset \end{cases} \tag{3.8}$$

and the large principal minor of order k of X as

$$\Delta_{\preceq k}(X) = \det^{\preceq}(X_{\preceq k}). \tag{3.9}$$

Definition 3.1 *Let $\chi = \{\lambda_i, i \in I\}$ be a multiplier and $X \in \mathcal{P}$, then the map defined by*

$$X \mapsto \Delta_{\chi}(X) = \prod_{k \in I} (\frac{\Delta_{\preceq k}(X)}{\Delta_{\prec k}(X)})^{\lambda_k}. \tag{3.10}$$

is called the generalized power function corresponding of χ.

We also denote

$$\Delta_{\chi}^{(i)}(X) = \prod_{k \in I_{i\preceq}} (\frac{\Delta_{\preceq k}(X)}{\Delta_{\prec k}(X)})^{\lambda_k}. \tag{3.11}$$

Note that, if $\lambda_i = \lambda, \ i \in I$, then $\Delta_{\chi}(X) = (\det X)^{\lambda}$. It is easy to verify that $\Delta_{\chi+\chi'}(X) = \Delta_{\chi}(X)\Delta_{\chi'}(X)$, where $\chi + \chi' = \{\lambda_i + \lambda'_i, \ i \in I\}$.

Example 3.1 *Let us consider $I = \{1, 2, 3, 4\}$ and the poset defined by*

$$1 \prec 3, \ 1 \prec 4, \ 2 \prec 3.$$

For $X = TT^* \in \mathcal{P}$, with $T = (t_{ij}) \in \mathcal{T}_l$, we have $\Delta_{\preceq 1}(X) = det^{\preceq}(X_{\preceq 1}) = t_{11}^2$, $\Delta_{\preceq 2}(X) = det^{\preceq}(X_{\preceq 2}) = t_{22}^2$, $\Delta_{\preceq 4}(X) = det^{\preceq}(X_{\preceq 4}) = t_{11}^2 t_{44}^2$ and $\Delta_{\preceq 3}(X) = det^{\preceq}(X_{\preceq 3}) = t_{11}^2 t_{22}^2 t_{33}^2$. Hence, for $\chi = \{\lambda_i = 1, \ i \in I\}$,

$$\Delta_{\chi}(X) = \frac{t_{11}^2}{1} \frac{t_{22}^2}{1} \frac{t_{11}^2 t_{44}^2}{t_{11}^2} \frac{t_{11}^2 t_{22}^2 t_{33}^2}{t_{11}^2 t_{22}^2} = t_{11}^2 t_{22}^2 t_{33}^2 t_{44}^2.$$

3.2 Orbit decomposition of the closure $\overline{\mathcal{P}}$ of $\mathcal{P}$

For $i \in \wp \cup \mathcal{S}$, we denote by ε^i the set of maps ψ defined from I into $\{0, 1\}$ as follows: If $i \in \wp$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\preceq}$ or $j \in \mathcal{S}$, and if $i \in \mathcal{S}$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\preceq}$. Similarly, we denote by $\varepsilon^{\check{i}}$ the set of maps ψ defined from I into $\{0, 1\}$. If $i \in \wp$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\prec}$ or $j \in \mathcal{S}$, and if $i \in \mathcal{S}$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\prec}$. With these notations, we define for $i \in \wp \cup \mathcal{S}$ and $\psi \in \varepsilon^i$, $e_\psi = \mathrm{diag}(\psi) = \mathrm{diag}(\psi(j), j \in I) \in \mathcal{D}$ and we denote by $\mathcal{T}_l^+.e_\psi = \{Te_\psi T^*, \ T \in \mathcal{T}_l^+\}$. We also consider the two elements of $\mathcal{A}$

$$E^i = \begin{cases} e_i & \text{in} \quad \mathcal{A}_{i\preceq} \\ 0 & \text{elsewhere} \end{cases}, \qquad \text{and} \qquad \check{E}^{\check{i}} = \begin{cases} \check{e}_{\check{i}} & \text{in} \quad \mathcal{A}_{i\prec} \\ 0 & \text{elsewhere} \end{cases}.$$

Next, we state and prove a fundamental result. It is a decomposition of $\overline{\mathcal{P}}$ in orbits.

Theorem 3.1 *i)*

$$\overline{\mathcal{P}} = \sum_{i\in\wp\cup\mathcal{S}} \overline{\mathcal{P}}_{i\preceq}\,. \tag{3.12}$$

ii) Let $i \in \wp\cup\mathcal{S}$*, then*

$$\overline{\mathcal{P}}_{i\preceq} = \bigcup_{\psi\in\varepsilon^i} \mathcal{T}_l^+.e_\psi. \tag{3.13}$$

Proof *i)* Let $Z \in \overline{\mathcal{P}}$, then there exist a sequence $\{Z^{(n)}\}_{n\in N}$ in $\mathcal{P}$ such that $Z^{(n)} \to Z$ as $n \to \infty$. Since $\{Z^{(n)}\}_{n\in N}$ is in $\mathcal{P}$, using the decomposition (2.7) we write $Z^{(n)} = \sum_{i\in I} Z_i^{(n)}$, where $Z_i^{(n)} \in \overline{\mathcal{P}}_{i\preceq}$. Hence $Z = \sum_{i\in I} Z_i$, where $Z_i \in \overline{\mathcal{P}}_{i\preceq}$ and (3.12) is proved.

ii) We will prove (3.13) by induction on the rank of the cone $\mathcal{P}_{i\preceq}$. It is obvious that (3.13) holds for $i \in \wp\cup\mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} = 1$. Suppose that (3.13) holds for any $i \in \wp\cup\mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} < l$ and let us show that it holds for i such that rank$\mathcal{P}_{i\preceq} = l$. Consider the set $M_i = \{j \in I;\ I_{\prec j} = \{i\}\}$. Then using the decomposition defined by (2.6) and (2.7) for an element of the cone $\mathcal{P}_{i\prec}$, we easily see

$$\overline{\mathcal{P}}_{i\prec} = \sum_{j\in M_i} \overline{\mathcal{P}}_{j\preceq}. \tag{3.14}$$

As rank$\mathcal{P}_{i\preceq} = l$, we have that rank$\mathcal{P}_{i\prec} = l-1$ and it follows that rank$\mathcal{P}_{j\preceq} \leq l-1$, $\forall j \in M_i$. Using the induction hypothesis, we can write

$$\overline{\mathcal{P}}_{j\preceq} = \bigcup_{\psi\in\varepsilon^j} \mathcal{T}_l^+.e_\psi, \qquad j \in M_i.$$

Now, let $\varepsilon^{i\prec} = \sum_{j\in M_i} \varepsilon^j$, then we obtain

$$\overline{\mathcal{P}}_{i\prec} = \sum_{j\in M_i} \overline{\mathcal{P}}_{j\preceq} = \sum_{j\in M_i} \Big(\bigcup_{\psi\in\varepsilon^j} \mathcal{T}_l^+.e_\psi\Big) = \bigcup_{\psi\in\varepsilon^{i\prec}} \mathcal{T}_l^+.e_\psi \subset \bigcup_{\psi\in\varepsilon^i} \mathcal{T}_l^+.e_\psi. \tag{3.15}$$

To conclude, we will verify that for $Z \in \overline{\mathcal{P}}_{i\preceq}$, there exist $\psi \in \varepsilon^i$ and $T \in \mathcal{T}_l^+$ such that $Z = T.e_\psi$. Let $Z \in \overline{\mathcal{P}}_{i\preceq}$, then there exists a sequence $\{Z^{(n)}\}_{n\in N}$ in $\mathcal{P}_{i\preceq}$ such that $Z^{(n)} \to Z$ as $n \to \infty$. As $Z^{(n)} \in \mathcal{P}_{i\preceq}$, there exists $U_{i\preceq}^{(n)} = (u_{jk}^{(n)})_{j,k\in I}$ in $\mathcal{T}_l$ such that $Z^{(n)} = U_{i\preceq}^{(n)}(U_{i\preceq}^{(n)})^*$ (see (2.3) and (2.5)). In particular, we have

$$z_{kk}^{(n)} = (u_{kk}^{(n)})^2 + \sum_{j\prec k} \parallel u_{kj}^{(n)} \parallel_{kj}^2, \tag{3.16}$$

for $k \in I_{i\preceq}$. This implies that the sequences $(u_{kk}^{(n)})_{n\in\mathbf{N}}$ and $(u_{kj}^{(n)})_{n\in\mathbf{N}}$ are bounded. Therefore there exists a subsequence of positive integers (n_m) such that $(u_{kk}^{(n_m)})_m$ and $(u_{kj}^{(n_m)})_m$ converge. Let $\widetilde{u}_{kk} = \lim_{m\to+\infty} u_{kk}^{(n_m)}$ and $\widetilde{u}_{kj} = \lim_{m\to+\infty} u_{kj}^{(n_m)}$. Then $\lim_{m\to+\infty} U_{i\preceq}^{(n_m)} = \widetilde{U}_{i\preceq}$, so that $Z = \widetilde{U}_{i\preceq}\widetilde{U}_{i\preceq}^* = (z_{kj})_{k,j\in I}$. As $z_{ii} \geq 0$, we will consider separately the case $z_{ii} = 0$ and the case $z_{ii} > 0$.

Suppose that $z_{ii} = 0$. Then $\widetilde{u}_{ii} = (z_{ii})^{1/2} = 0$, so that $z_{ki} = \widetilde{u}_{ii}\widetilde{u}_{ki} = 0,\ i \prec k$. Thus $Z = Z_{i\prec} \in \overline{\mathcal{P}}_{i\prec}$, and the result follows according to (3.15).

If $z_{ii} > 0$, we consider the elements of $\mathcal{A}_{i\preceq}$

$$\widetilde{u}^i = \begin{pmatrix} \widetilde{u}_{ii} & 0 \\ \widetilde{C}_i & e_{\check{i}} \end{pmatrix} \text{ and } \widetilde{u}_i = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{U}_{i\prec} \end{pmatrix},$$

where $\widetilde{C}_i = \sum_{i\prec j} \widetilde{u}_{ji}$ and $\widetilde{U}_{i\prec} = (\widetilde{u}_{jk})_{j,k\in I_{i\prec}}$. Then $\widetilde{u}_{i\preceq} = \widetilde{u}^i\widetilde{u}_i$ and $\widetilde{u}_{ii} = (z_{ii})^{1/2} > 0$. Let T_1 and T_2 in $\mathcal{T}_l^+$, such that $T_1 = \widetilde{u}^i$ in $\mathcal{A}_{i\preceq}$ and $T_2 = \widetilde{u}_i$ in $\mathcal{A}_{i\preceq}$, we have $T_2.\check{E}^{\check{i}} \in \overline{\mathcal{P}}_{i\prec}$. By induction hypothesis, there exists a unique $\psi_1 \in \varepsilon^i$ such that $\psi_1(i) = 0$ and there exists $\tilde{T}_2 \in \mathcal{T}_l^+$ such that $T_2.\check{E}^{\check{i}} = \tilde{T}_2.e_{\psi_1}$. Let $\psi_2 \in \varepsilon^i$, such that $\psi_2(i) = 1$ and $\psi_2(j) = 0\ \forall j \neq i$ and put $\psi = \psi_1 + \psi_2 \in \varepsilon^i$ and $T = T_1\tilde{T}_2 \in \mathcal{T}_l^+$. Then we have

$$\begin{aligned} Z &= T_1T_2.(E_i + \check{E}^{\check{i}}) \\ &= T_1.E_i + T_2.\check{E}^{\check{i}} \\ &= T_1.E_i + \tilde{T}_2.e_{\psi_1} \\ &= (T_1\tilde{T}_2).e_\psi \\ &= T.e_\psi, \end{aligned}$$

and (3.13) is proved. □

3.3 Gamma functions

We use the generalized power function to introduce a generalized gamma function on an homogeneous cone.

For $i \in \wp \cup \mathcal{S}$, $\psi \in \varepsilon^i$ and $\chi_i = \{\lambda_j,\ j \in I;\ \lambda_j = 0, \forall j \notin I_{i\preceq}\}$, we set

$$\mathcal{X}(\psi) = \{\chi_i \in \mathcal{X} \mid \lambda_j = 0, \text{ for all } j \in I_{i\preceq} \text{ such that } \psi(j) = 0\}. \tag{3.17}$$

For every $\chi_i \in \mathcal{X}(\psi)$, we define a generalized power function on $\mathcal{T}_l^+.e_\psi$ by

$$\Delta^\psi_{\chi_i}(T.e_\psi) = \Delta^{(i)}_{\chi_i}(TT^*), \qquad \forall T \in \mathcal{T}_l^+, \tag{3.18}$$

where $\Delta^{(i)}_{\chi_i}(TT^*)$ is defined by (3.11). We also define $n^\psi = (n^i_{j.}, j \in I)$ by

$$n^i_{j.} = \sum_{k\prec j} \psi(k) n_{kj} \qquad \forall j \in I_{i\preceq}. \tag{3.19}$$

When $\psi \neq 0$, we introduce the measure ν_ψ on $\mathcal{T}_l^+.e_\psi$ defined by

$$\nu_\psi(d(T.e_\psi)) = \Delta^{(i)}_{\dot{\chi}^\psi_i}(TT^*) \prod_{\substack{i \preceq j \preceq k \\ \psi(j) = 1}} dt_{kj}, \tag{3.20}$$

where $\dot{\chi}_i^{\psi} = \{\lambda_j \in \mathbb{R}, j \in I,$ such that $\lambda_j = -\psi(j)(1+n^i_{j.})/2,$ if $j \in I_{i\preceq}$ and $\lambda_j = 0$ if $j \notin I_{i\preceq}\}$, and $T = (t_{jk})_{j,k\in I} \in \mathcal{T}_l^+$. Finally, we denote by ν_0 be the Dirac measure at 0.

Theorem 3.2 *Let $i \in \wp \cup \mathcal{S}$ and $\chi_i = \{\lambda_j,\ j \in I;\ \lambda_j = 0, \forall j \notin I_{i\preceq}\} \in \mathcal{X}(\psi)$. The integral*

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = \int_{\mathcal{T}_l^+.e_\psi} \exp\{-tr(Z)\}\Delta^{\psi}_{\chi_i}(Z)\nu_\psi(dZ) \tag{3.21}$$

converges if and only if $\chi_i \in \mathcal{X}(\psi)$ satisfies the following condition:

$$\lambda_j > \frac{n^i_{j.}}{2} \qquad \forall j \in I \quad \text{such that} \quad \psi(j) = 1. \tag{3.22}$$

Moreover, under this condition, one has

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = 2^{-|\psi|}\pi^{-|n^\psi|/2} \prod_{\substack{j\in I\\ \psi(j)=1}} \Gamma(\lambda_j - \frac{n^i_{j.}}{2}), \tag{3.23}$$

where $|\psi| = \sum_{j\in I_{i\preceq}} \psi(j)$ and $|n^\psi| = \sum_{j\in I_{i\preceq}} n^i_{j.}$.

Proof If $\psi = 0$, the integral (3.21) reduces to 1. Thus (3.22) and (3.23) hold trivially. If $\psi \neq 0$, then writing $Z = U.e_\psi$, where $U = (u_{jk})_{j,k\in I} \in \mathcal{T}_l^+$, the integral (3.21) can be written

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = \int_{\mathcal{T}_l^+ e_\psi} \exp\{-(\sum_{\psi(j)=1}(u^2_{jj} + \sum_{j\prec k} \| u_{kj} \|^2_{kj}))\} \prod_{\substack{i\preceq j\prec k\\ \psi(j)=1}} u_{jj}^{2\lambda_j - n^i_{j.}-1} du_{jj}du_{kj}.$$

For $j \in I_{i\preceq}$, let

$$C_j = (s_{ln})_{l,n\in I} \in \mathcal{C}^j = \sum_{j\prec k} E_{kj},\ \text{with } s_{lj} = u_{lj} \text{ if } j \prec l \text{ and } s_{ln} = 0 \text{ otherwise.} \tag{3.24}$$

It is clear that $\| C_j \|^2 = \sum_{j\prec k} \| u_{kj} \|^2_{kj}$ and $dC_j = \prod_{j\prec k} du_{kj}$. Hence

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = \prod_{\substack{i\preceq j\\ \psi(j)=1}} \int_0^{+\infty} \exp^{-u^2_{jj}} u_{jj}^{2\lambda_j - n^i_{j.}-1} du_{jj} \prod_{\substack{i\preceq j\\ \psi(j)=1}} \int_{\mathcal{C}^j} \exp^{\{-\|C_j\|^2\}} dC_j.$$

Therefore the convergence condition is reduced to the one corresponding to the ordinary gamma functions, that is $\lambda_j > \frac{n^i_{j.}}{2}$ $\forall j \in I$ such that $\psi(j) = 1$. □

Remark 3.1 *From Theorem 3.2, we have, for $i \in I$, a relation between $\Gamma_{\mathcal{T}_l^+ . e_\psi}(\chi_i)$ and $\Gamma_{\mathcal{P}}(\chi)$. In fact, if we denote by $\mathbf{1}_i$ the element of ε^i, such that $\psi(j) = 1, \ \forall \ j \in I_{i\preceq} \setminus \mathcal{S}$, if $i \in \wp$ and such that $\psi(j) = 1, \ \forall \ j \in I_{i\preceq}$, if $i \in \mathcal{S}$, then it is clear that*

$$\mathcal{P} = \sum_{i\in\wp\cup\mathcal{S}} \mathcal{T}_l^+ . e_{\mathbf{1_i}},$$

and using (2.1), for $\chi = \sum_{i\in I} \chi_i \in \mathcal{X}$, where $\chi_i = \{\lambda_j, \ j \in I; \ \lambda_j = 0, \forall j \notin I_{i\preceq}\}$, we have

$$\prod_{i\in\wp\cup\mathcal{S}} \Gamma_{\mathcal{T}_l^+ . e_{\mathbf{1_i}}}(\chi_i) = 2^{-|I|}\pi^{\frac{n_. - |I|}{2}} \prod_{i\in I} \Gamma(\lambda_i - \frac{n_{i.}}{2}) = 2^{-|I|}\Gamma_{\mathcal{P}}(\chi)\cdot$$

3.4 Riesz measures

For the definition of the Riesz distribution, we need to introduce some other notations. Let $i \in \wp \cup \mathcal{S}$ and $\chi_i = \{\lambda_j, \ j \in I; \ \lambda_j = 0, \forall j \notin I_{i\preceq}\}$ and introduce for $\psi \in \varepsilon^i$ and $\omega \in \varepsilon^{\check{i}}$, the following sets

$$\mathcal{B}(i, \psi) = \left\{ \chi_i \in \mathcal{X}; \ \lambda_j = 0 \text{ for } j \notin I_{i\preceq}, \lambda_j = \frac{n^i_{j.}}{2} \text{ when } j \in I_{i\preceq} \text{ and } \psi(j) = 0 \right\}, \quad (3.25)$$

$$\check{\mathcal{B}}(\check{i}, \omega) = \left\{ \chi_i \in \mathcal{X}; \ \upsilon_j = 0 \text{ for } j \notin I_{i\prec}, \vartheta_j = \frac{n^i_{j.}}{2} \text{ when } j \in I_{i\prec} \text{ and } \omega(j) = 0 \right\}, \quad (3.26)$$

$$\Xi(i, \ \psi) = \left\{ \chi_i \ \in \mathcal{B}(i, \ \psi), \ \lambda_j > \frac{n^i_{j.}}{2} \text{ when } j \in I_{i\preceq} \text{ and } \psi(j) = 1 \right\}, \quad (3.27)$$

$$\check{\Xi}(\check{i}, \ \omega) = \left\{ \chi_i \ \check{\mathcal{B}}(\check{i}, \omega) \ \lambda_j > \frac{n^i_{j.}}{2} \text{ when } j \in I_{i\prec} \text{ and } \omega(j) = 1 \right\}, \quad (3.28)$$

$$\Xi(i) = \bigcup_{\psi\in\varepsilon^i} \Xi(i, \ \psi), \qquad \check{\Xi}(\check{i}) = \bigcup_{\omega\in\varepsilon^{\check{i}}} \check{\Xi}(\check{i}, \ \omega), \qquad \Xi = \sum_{i\in\wp\cup\mathcal{S}} \Xi(i)\cdot \quad (3.29)$$

For every $\chi_i = \{\lambda_j, \ j \in I\} \in \mathcal{X}$, let

$$\widetilde{\chi}_i = \{\lambda_j - (1 - \psi(j))\frac{n^i_{j.}}{2}, \text{when } j \in I_{i\preceq}, \text{ and } 0 \text{ if } j \notin I_{i\preceq}\}. \quad (3.30)$$

It is clear that if $\chi_i \in \mathcal{B}(i, \ \psi)$, then $\widetilde{\chi}_i \in \mathcal{X}(\psi)$.

In what follows, we denote the Laplace transform of a positive measure μ on the cone $\mathcal{P}$ by

$$L_\mu(\theta) = \int_{\mathcal{P}} \exp\{-\text{tr}(\theta Z)\}\mu(dZ), \qquad \theta \in \mathcal{P}^*. \quad (3.31)$$

Theorem 3.3 *There exists a positive measure R_χ such that the Laplace transform is defined on $\mathcal{P}^*$ and is equal to $\Delta_\chi(\theta^{-1})$ if and only if $\chi \in \Xi$.*

The proof of Theorem 3.3 relies on the following proposition.

Proposition 3.4 *Let $i \in \wp \cup \mathcal{S}$. Then there exists a positive measure R_{χ_i} such that the Laplace transform is defined on $\mathcal{P}^*$ and is equal to $\Delta^{(i)}_{\chi_i}(\theta^{-1})$ if and only if $\chi_i \in \Xi(i)$.*

Proof $\Leftarrow$) Let $\chi_i \in \Xi(i)$. Then there exists $\psi \in \varepsilon^i$ such that $\chi_i \in \Xi(i,\ \psi)$. It is clear that $\widetilde{\chi}_i$ defined by (3.30) satisfies (3.22). We will show that the Laplace transform of the measure

$$R_{\chi_i}(dZ) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Z)\mathbf{1}_{\mathcal{T}_l^+.e_\psi}(Z)\nu_\psi(dZ)$$

is defined on $\mathcal{P}^*$ and is given by

$$L_{R_{\chi_i}}(\theta) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\int_{\mathcal{T}_l^+.e_\psi}\exp\{-\mathrm{tr}(\theta Z)\}\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Z)\nu_\psi(dZ) = \Delta^{(i)}_{\chi_i}(\theta^{-1}).$$

In fact, as $\theta \in \mathcal{P}^*$, then $\theta_{i\preceq}$ defined by (2.5) is in the dual cone $\mathcal{P}^*_{i\preceq}$ of $\mathcal{P}_{i\preceq}$, and there exists T in $\mathcal{T}_l^+$ such that $\theta_{i\preceq} = T^*.E^i$. Let $Y = \pi(T)(Z)$, where π is defined by (2.2). As $Z \in \mathcal{T}_l^+.e_\psi$, there exists $S \in \mathcal{T}_l^+$ such that $Z = S.e_\psi$. This with (3.20) imply that

$$\nu_\psi(dY) = \nu_\psi(d(\pi(T)(Z))) = \Delta^{(i)}_{\ddot{\chi}_i^\psi}(\theta)\nu_\psi(dZ), \tag{3.32}$$

where $\ddot{\chi}_i^\psi = \{\lambda_j \in \mathbb{R}, j \in I \text{ such that } \lambda_j = (1-\psi(j))\frac{n^i_{j.}}{2}, \text{ if } j \in I_{i\preceq} \text{ and } 0 \text{ if } j \notin I_{i\preceq}\}$. Since $\widetilde{\chi}_i \in \mathcal{X}(\psi)$, then

$$\begin{aligned}\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Z) &= \Delta^{\psi}_{\underset{\sim}{\chi_i}}(\pi^{-1}(T)(Y))\\ &= \Delta^{\psi}_{\underset{\sim}{\chi_i}}(T^{-1}.e_\psi)\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Y)\\ &= \Delta^{(i)}_{\underset{\sim}{\chi_i}}(\theta^{-1})\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Y).\end{aligned} \tag{3.33}$$

Using (3.30), (3.32) and (3.33), we get

$$\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Y)\nu_\psi(dY) = \Delta^{(i)}_{\chi_i}(\theta)\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Z)\nu_\psi(dZ). \tag{3.34}$$

Then

$$\begin{aligned}L_{R_{\chi_i}}(\theta) &= \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\int_{\mathcal{T}_l^+.e_\psi}\exp\{-\mathrm{tr}(\theta\pi^{-1}(T)(Y))\}\Delta^{(i)}_{\chi_i}(\theta^{-1})\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Y)\nu_\psi(dY)\\ &= \Delta^{(i)}_{\chi_i}(\theta^{-1})\frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\int_{\mathcal{T}_l^+.e_\psi}\exp\{-\mathrm{tr}Y\}\Delta^{\psi}_{\underset{\sim}{\chi_i}}(Y)\nu_\psi(dY)\\ &= \Delta^{(i)}_{\chi_i}(\theta^{-1}).\end{aligned}$$

$\Rightarrow$) Suppose that there exists a positive measure R_{χ_i} such that the Laplace transform is defined on $\mathcal{P}^*$ and is equal to $\Delta^{(i)}_{\chi_i}(\theta^{-1})$. Our aim to show that $\chi_i \in \Xi(i)$.
For this χ_i and a ψ in ε^i, consider the generalized positive Riesz measure which we also denote R_{χ_i} defined for φ in the Schwartz space $\mathcal{S}(\mathcal{A})$ of rapidly decreasing functions on $\mathcal{A}$ by

$$R_{\chi_i}(\varphi) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)} \int_{\mathcal{T}_l^+.e_\psi} \varphi(Z)\Delta^{\psi}_{\widetilde{\chi}_i}(Z)\nu_\psi(dZ). \tag{3.35}$$

We will prove by induction on the rank of the cone $\mathcal{P}_{i\preceq}$ that $\chi_i \in \Xi(i,\ \psi)$. Suppose that rank$\mathcal{P}_{i\preceq} = 1$. Then we have either cardinality of $\wp$ equal to 1 or cardinality of $\mathcal{S}$ equal to 1. Thus R_{χ_i} coincides with the Riesz measure ρ_λ on $]0, +\infty[$ given by

$$\rho_\lambda(\varphi) = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} \varphi(u)u^{\lambda-1}du. \tag{3.36}$$

This implies that $\chi_i \equiv \lambda$ and $\lambda > 0$ which means that the result is true when rank$\mathcal{P}_{i\preceq} = 1$. Now, suppose that the claim holds for any $i \in \wp \cup \mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} \leq k-1$, and let us show that it also holds for i such that rank$\mathcal{P}_{i\preceq} = k$. Consider $i \in \wp \cup \mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} = k$. Then rank$\mathcal{P}_{i\prec} = k-1$ so that rank$\mathcal{P}_{j\preceq} \leq k-1$, $\forall j \in M_i$. Using the induction hypothesis, we have that $\forall j \in M_i$, $\xi_j = \{\beta_l \in \mathbb{R},\ l \in I,\ \beta_l = 0;\ \forall l \notin I_{j\preceq}\}$ is in $\Xi(j)$. Let R_{ξ_j} be a Riesz measure defined as in (3.35) on the cone $\mathcal{P}_{j\preceq}$ for some ψ in ε^j and let $\check{\xi}_{\check{i}} = \sum_{j\in M_i} \xi_j = \{\beta_j \in \mathbb{R}, j \in I; \beta_j = 0$ for $j \notin I_{i\prec}\}$. Then from (3.14), the measure $\check{R}_{\check{\xi}_{\check{i}}} = \prod^*_{j\in M_i} R_{\xi_j}$, where $\prod^*$ is the convolution product, is concentrated on $\mathcal{P}_{i\prec}$. Consider the sets

$$\check{\chi}_{\check{i}} = \{\lambda_j \in \mathbb{R}, j \in I; \lambda_j = 0 \text{ for } j \notin I_{i\prec}\},$$

$$\check{n}^i = \{\beta_k \in \mathbb{R},\ k \in I; \beta_k = n_{ki} \text{ for } k \in I_{i\prec} \text{ and } \beta_k = 0 \text{ for } k \notin I_{i\prec}\},$$

and

$$M(\lambda_i) = \{\alpha_k,\ k \in I\},$$

where

$$\alpha_k = \begin{cases} \lambda_i & \text{for} \quad k = i \\ \frac{n_{ki}}{2} & \text{for} \quad k \in I_{i\prec} \\ 0 & \quad\text{otherwise.} \end{cases}$$

Then it is easy to verify that $M(\lambda_i) \in \mathcal{B}(i, \psi_1)$, where $\psi_1 \in \varepsilon^i$ such that $\psi_1(i) = 1$, and $\psi_1(j) = 0\ \forall j \neq i$. Also we have $\chi_i - M(\lambda_i) = \check{\chi}_{\check{i}} - \frac{\check{n}^i}{2} \in \check{\mathcal{B}}(\check{i}, \omega)$, where $\check{\mathcal{B}}(\check{i}, \omega)$ is defined by (3.26).
Using the Laplace transforms, we obtain that

$$R_{\chi_i} = R_{M(\lambda_i)} * \check{R}_{\check{\chi}_{\check{i}} - \frac{\check{n}^i}{2}}. \tag{3.37}$$

Proceeding as in the proof of Theorem 3.2, and using (3.23), we get

$$R_{M(\lambda_i)}(\varphi) = \frac{2\pi^{-\frac{\dim \mathcal{C}^i}{2}}}{\Gamma(\lambda_i)} \int_{\mathcal{T}_l^+ e_{\psi_1}} \varphi(U.e_{\psi_1}) u_{ii}^{2\lambda_i - 1} du_{ii} dC_i,$$

where $U \in \mathcal{T}_l^+$, $U.e_{\psi_1} = (u_{jk})_{j,k\in I}$ and C_i is defined by (3.24). On the other hand, it is easy to verify that

$$U.e_{\psi_1} = u_{ii}^2 E_i + u_{ii} C_i + \sum_{i\prec j} \| u_{ji} \|_{ji}^2 \, E_j.$$

Hence, if we define

$$\begin{array}{rcl} \mathcal{Q}^i \; : \; \mathcal{C}^i \times \mathcal{C}^i & \rightarrow & \mathcal{A}_{i\prec} \\ (C_i, C_i) & \mapsto & \mathcal{Q}^i(C_i, C_i) = \sum_{i\prec j} \| u_{ji} \|_{ji}^2 \, E_j, \end{array}$$

then we have

$$U.e_{\psi_1} = u_{ii}^2 E_i + u_{ii} C_i + \mathcal{Q}^i(C_i, C_i).$$

Setting $u_{ii} = \sqrt{v}$, we get

$$R_{M(\lambda_i)}(\varphi) = \frac{2\pi^{-\frac{\dim \mathcal{C}^i}{2}}}{\Gamma(\lambda_i)} \int_0^{+\infty} \int_{\mathcal{C}^i} \varphi(vE_i + \sqrt{v} C_i + \mathcal{Q}^i(C_i, C_i)) dC_i v^{\lambda_i - 1} dv.$$

This, using (3.36), becomes

$$R_{M(\lambda_i)}(\varphi) = \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i} (\int_{\mathcal{C}^i} \varphi(vE_i + \sqrt{v} C_i + \mathcal{Q}^i(C_i, C_i)) dC_i)_v. \qquad (3.38)$$

As for $\lambda_i = 0$, ρ_0 is the Dirac measure at $v = 0$, we get

$$R_{M(0)}(\varphi) = R_{\frac{\check{n}^i}{2}}(\varphi) = \pi^{-\frac{\dim \mathcal{C}^i}{2}} \int_{\mathcal{C}^i} \varphi(\mathcal{Q}^i(C_i, C_i)) dC_i. \qquad (3.39)$$

Using (3.37) and (3.38), we obtain

$$R_{\chi_i}(\varphi) = \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i} (\int_{\mathcal{C}^i} \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}} (\varphi(vE_i + \sqrt{v} C_i + \mathcal{Q}^i(C_i, C_i) + Y))_Y dC_i))_v, \; (3.40)$$

Denote by C_c^∞ the set of C^∞ functions with compact support and consider the functions of the form

$$\varphi(Z) = \varphi_1(z_{11}) \varphi_2(Z_{i\prec})$$

where $Z = (z_{ij})_{i,j\in I} \in \mathcal{A}$, $Z_{i\prec} \in \mathcal{A}_{i\prec}$, $\varphi_1 \in C_c^\infty(\mathbb{R})$ and $\varphi_2 \in C_c^\infty(\mathcal{A}_{i\prec})$. Then by (3.39) and (3.40), we have

$$\begin{array}{rcl} R_{\chi_i}(\varphi) & = & \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i}(\varphi_1) \int_{\mathcal{C}^i} \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}} (\varphi_2(\mathcal{Q}^i(C_i, C_i) + Y))_Y dC_i \\ & = & \rho_{\lambda_i}(\varphi_1) (\check{R}_{\frac{\check{n}^i}{2}} * \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}})(\varphi_2) \\ & = & \rho_{\lambda_i}(\varphi_1) \check{R}_{\check{\chi}_i}(\varphi_2). \end{array} \qquad (3.41)$$

For a suitable choice of non-negative $\varphi_1 \in C_c^\infty(\mathbb{R})$, we have $\rho_{\lambda_i}(\varphi_1) > 0$. If $\varphi_2 \geq 0$, then using (3.40) and the positivity of R_{χ_i}, we get $\check{R}_{\check{\chi}_{\check{i}}}(\varphi_2) = (\rho_{\lambda_i}(\varphi_1))^{-1} R_{\chi_i}(\varphi) \geq 0$. Thus $\check{R}_{\check{\chi}_i}$ is positive and the induction hypothesis ensures that $\check{\chi}_{\check{i}} \in \check{\Xi}(\check{i},\ \omega)$.
Now, fix a non-negative φ_2 such that $\check{R}_{\check{\chi}_{\check{i}}}(\varphi_2)$ is strictly positive. Then using again (3.40), we get $\rho_{\lambda_i}(\varphi_1) \geq 0$ for any $\varphi_1 \geq 0$. Therefore ρ_{λ_i} is positive and we deduce that $\lambda_i \geq 0$. If $\lambda_i = 0$, then choosing a ψ in ε^i such that $\psi(i) = 0$, we get $\chi_i \in \Xi(i,\ \psi)$. To study the case $\lambda_i > 0$, we first observe that the map

$$\begin{array}{rcl}]0,+\infty[\times \mathcal{C}^i \times \mathcal{A}_{i\prec} & \to & \{X \in \mathcal{A}, x_{11} > 0\} \\ (v, C_i, Y) & \mapsto & vE_i + \sqrt{v}C_i + \mathcal{Q}^i(C_i, C_i) + Y \end{array}$$

is a diffeomorphism whose the inverse is given by

$$x \mapsto (x_{11},\ x_{11}^{-1/2} \sum_{1\prec k} X_{k1},\ x_{i\prec} - \frac{1}{x_{11}} \mathcal{Q}^i(\sum_{1\prec k} X_{k1}, \sum_{1\prec k} X_{k1}).$$

For a functions the functions $\varphi \in C_c^\infty(\mathcal{A})$ of the form

$$\varphi(Z) = \begin{cases} \varphi_1(v)\varphi_2(C_i)\varphi_3(Y) & (z_{11} > 0), \\ 0 & (z_{11} \leq 0), \end{cases}$$

with $\varphi_1 \in C_c^\infty(]0,+\infty[)$, $\varphi_2 \in C_c^\infty(\mathcal{C}^i)$ and $\varphi_3 \in C_c^\infty(\mathcal{A}_{i\prec})$, by (3.40), we have that

$$R_{\chi_i}(\varphi) = \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i}(\varphi_1) \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}}(\varphi_3) \int_{\mathcal{C}^i} \phi_2(C_i) dC_i.$$

Since $\lambda_i > 0$, the positivity assumption of R_{χ_i} yields that $\check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}}$ is positive. This by the induction hypothesis implies that $\check{\chi}_i - \frac{\check{n}^i}{2} \in \check{\Xi}(\check{i},\ \omega)$. Finally, choose a ψ in ε^i such that

$$\psi(j) = \begin{cases} \omega(j) & \text{for} \quad \forall j \neq i \\ 1 & \text{for} \quad j = i, \end{cases}$$

where $\omega \in \varepsilon^{\check{i}}$. Then, as $\check{\chi}_i - \frac{\check{n}^i}{2} \in \check{\Xi}(\check{i},\ \omega)$, for $j \in I_{i\prec}$, we have that $\lambda_j = \frac{n^i_{j.}}{2}$ if $\omega(j) = 0$ and $\lambda_j > \frac{n^i_{j.}}{2}$ if $\omega(j) = 1$. As $\lambda_i > 0$ and $n^i_{i.} = 0$, then $\lambda_i > \frac{n^i_{i.}}{2}$. This means that for such a ψ, we have that $\chi_i \in \Xi(i,\ \psi)$. Hence $\chi_i \in \Xi(i)$ and Proposition 3.4 is proved. □

Proof of Theorem 3.3 ($\Leftarrow$) Let $\chi = \sum_{i\in\wp\cup\mathcal{S}} \chi_i \in \Xi$, where $\chi_i \in \Xi(i)$ and let

$$R_\chi = \prod_{i\in\wp\cup\mathcal{S}}^{*} R_{\chi_i}, \tag{3.42}$$

R_{χ_i} is the positive measure defined from Proposition 3.4. Then, for $\theta \in \mathcal{P}^*$

$$\begin{aligned} L_{R_\chi}(\theta) &= \prod_{i\in\wp\cup\mathcal{S}} L_{R_{\chi_i}}(\theta) \\ &= \prod_{i\in\wp\cup\mathcal{S}} \Delta^{(i)}_{\chi_i}(\theta^{-1}) \\ &= \Delta^{(i)}_{\sum_{i\in\wp\cup\mathcal{S}}\chi_i}(\theta^{-1}) \\ &= \Delta_\chi(\theta^{-1}). \end{aligned}$$

($\Rightarrow$) We have $L_{R_\chi}(\theta) = \Delta_\chi(\theta^{-1})$, then using the fact $\prod_{i\in\wp\cup\mathcal{S}} \Delta^{(i)}_{\chi_i}(\theta^{-1}) = \Delta_\chi(\theta^{-1})$, Proposition 3.4, and putting $R_\chi = \prod^*_{i\in\wp\cup\mathcal{S}} R_{\chi_i}$, such that for $\theta \in \mathcal{P}^*$, $L_{R_{\chi_i}}(\theta) = \Delta^{(i)}_{\chi_i}(\theta^{-1})$, we get $\chi_i \in \Xi(i)$. Therefore $\chi = \sum_{i\in\wp\cup\mathcal{S}} \chi_i \in \Xi$. □

Following the terminology used in the paper by Hassairi and Lajmi (2001) in the case of symmetric matrices, we call the measures R_χ, defined above in terms of their Laplace transforms, Riesz measures on the homogeneous cone. These measures are divided into two classes according to the position of $\chi \in \Xi$. A class of measures which are absolutely continuous with respect to the Lebesgue measure on $\mathcal{P}$ and a class concentrated on the boundary $\partial\mathcal{P}$ of $\mathcal{P}$.

Proposition 3.5 *Let $\chi = \{\lambda_i,\ i \in I\} \in \mathcal{X}$. Then R_χ is absolutely continuous if and only if $\lambda_i > \frac{n_{i.}}{2}$, $i \in I$. In this case*

$$R_\chi(dZ) = \frac{1}{\Gamma_{\mathcal{P}}(\chi)} \Delta_{\chi+\ddot{\chi}}(Z) 1_{\mathcal{P}}(Z) dZ, \tag{3.43}$$

where $\ddot{\chi} = \{-n_i,\ i \in I\}$ and $\Gamma_{\mathcal{P}}(\chi) = \pi^{\frac{n.-|I|}{2}} \prod_{i\in I} \Gamma(\lambda_i - \frac{n_{i.}}{2})$.

Proof $\Rightarrow$) We have $\mathcal{P} = \sum_{i\in\wp\cup\mathcal{S}} \mathcal{T}_l^+.e_{\mathbf{1_i}}$, then $\widetilde{\chi}_i$ defined by (3.30) is equal to χ_i.

From (3.42), we have $R_\chi = \prod^*_{i\in\wp\cup\mathcal{S}} R_{\chi_i}$. Writing $Z = UU^* \in \mathcal{P}$, $U \in \mathcal{T}_l^+$, then $Z = \sum_{i\in I} Z_i$ where Z_i is in $\mathcal{T}_l^+.e_{\mathbf{1_i}}$, and using the proof of Proposition 3.4, we have that for $Z_i = U^1_{i\preceq} U^{1*}_{i\preceq}$, $U^1_{i\preceq} = (u_{kj})_{k,j\in I}$

$$R_{\chi_i}(dZ_i) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_{\mathbf{1_i}}}(\chi_i)} \Delta_{\chi_i}(Z_i) \nu(dZ_i),$$

where $\Delta^{\mathbf{1_i}}_{\chi_i} = \Delta_{\chi_i}$ and $\nu(dZ_i) = \nu_{\mathbf{1_i}}(dZ_i) = \Delta^{(i)}_{\dot{\chi}_i^{\mathbf{1_i}}}(Z) \prod_{\substack{i \preceq j \preceq k \\ \psi(j)=1}} du_{kj}$.

Then $\nu(dZ)=\Delta_{\dot\chi}(Z)dU$, where

$$\dot\chi=\{-(\frac{1+n_{j\cdot}}{2}),\ j\in I\} \text{ and } dU=\prod_{i\in\wp\cup\mathcal{S}}\prod_{\substack{i\preceq j\preceq k\\ \psi(j)=1}} du_{kj}.$$

Using the fact that the mapping $U\in\mathcal{T}_l^+\mapsto UU^*\in\mathcal{P}$ is a diffeomorphism, we have that $dU=2^{-|I|}\Delta_{\hat\chi}(Z)dZ$, where $\hat\chi=\{-\frac{1+n_{\cdot j}}{2},\ \forall j\in I\}$. As $n_j=1+\frac{1}{2}(n_{\cdot j}+n_{j\cdot})$, we have

$$R_\chi(dZ)=\frac{1}{\Gamma_{\mathcal{P}}(\chi)}\Delta_\chi(Z)\Delta_{\dot\chi}(Z)\Delta_{\hat\chi}(Z)dZ=\frac{1}{\Gamma_{\mathcal{P}}(\chi)}\Delta_{\chi+\ddot\chi}(Z)dZ,$$

where $\ddot\chi=\{-n_j,\ j\in I\}$ and $\Gamma_{\mathcal{P}}(\chi)=2^{|I|}\prod_{i\in\wp\cup\mathcal{S}}\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i)=\pi^{\frac{n.-|I|}{2}}\prod_{i\in I}\Gamma(\lambda_i-\frac{n_{i.}}{2})$.
Moreover, the condition $\lambda_i>\frac{n_{i.}}{2},\ i\in I$ is easily deduced from Theorem 3.2.
$\Leftarrow$) It suffices to verify that for χ such that $\lambda_i>\frac{n_{i.}}{2},\ i\in I$, the Laplace transform of the measure

$$\frac{1}{\Gamma_{\mathcal{P}}(\chi)}\Delta_{\chi+\ddot\chi}(Z)1_{\mathcal{P}}(Z)dZ$$

is equal to $\Delta_\chi(\theta^{-1})$, $\forall\theta\in\mathcal{P}^*$. In fact, let $\theta\in\mathcal{P}^*$, then there exists $T=(t_{ij})$ in $\mathcal{T}_l^+$ such that $\theta=T^*T$. Let $Y=\pi(T)(Z)$, where π is defined by (2.2). Then $dZ=\det\pi^{-1}(T)dY$ and

$$\Delta_{\chi+\ddot\chi}(Z)=\Delta_{\chi+\ddot\chi}(\pi^{-1}(T)Y)=\Delta_{\chi+\ddot\chi}(\theta^{-1}Y)=\Delta_{\chi+\ddot\chi}(\theta^{-1})\Delta_{\chi+\ddot\chi}(Y).$$

From Andersson and Wojnar (2004), we have

$$\det\pi^{-1}(T)=\prod_{i\in I}t_{ii}^{-2n_i}=\Delta_{-\ddot\chi}(\theta^{-1}),$$

where $-\ddot\chi=\{n_i,i\in I\}$. Writing $Y=\sum_{i\in I}Y_i$, where Y_i is in $\mathcal{T}_l^+.e_{\mathbf{1_i}}$, then as

$$\begin{aligned}\Gamma_{\mathcal{P}}(\chi)=2^{|I|}\prod_{i\in\wp\cup\mathcal{S}}\Gamma_{\mathcal{T}_l^+.e_{\mathbf{1_i}}}(\chi_i) &= 2^{|I|}\prod_{i\in\wp\cup\mathcal{S}}\int_{\mathcal{T}_l^+.e_{\mathbf{1_i}}}\exp\{-\mathrm{tr}Y_i\}\Delta_{\chi_i}^{\mathbf{1_i}}(Y_i)\nu_{\mathbf{1_i}}(dY_i)\\ &= \int_{\mathcal{P}}\exp-\{\mathrm{tr}Y\}\Delta_{\chi+\ddot\chi}(Y)dY,\end{aligned}$$

we obtain that

$$\begin{aligned}\frac{1}{\Gamma_{\mathcal{P}}(\chi)}\int_{\mathcal{P}}\exp\{-\mathrm{tr}(\theta Z)\}\Delta_{\chi+\ddot\chi}(Z)dZ &= \Delta_\chi(\theta^{-1})\frac{1}{\Gamma_{\mathcal{P}}(\chi)}\int_{\mathcal{P}}\exp-\{\mathrm{tr}Y\}\Delta_{\chi+\ddot\chi}(Y)dY\\ &= \Delta_\chi(\theta^{-1}).\end{aligned}$$

□

4 Riesz exponential families

In this section, we study the natural exponential family generated by a Riesz measure. We first review some basic concepts concerning exponential families and their variance functions and introduce some notations.

For a positive measure on $\mathcal{A}$, we denote

$$\Theta(\mu) = \text{interior}\{\theta \in \mathcal{A}^*;\ L_\mu(\theta) < \infty\}$$

$$k_\mu = \log L_\mu$$

where L_μ and k_μ are respectively the Laplace transform and the cumulant generating function of μ.

The set $\mathcal{M}(\mathcal{A})$ is now defined as the set of positive measures μ such that μ is not concentrated on an affine hyperplane of $\mathcal{A}$ and $\Theta(\mu)$ is not empty. For μ in $\mathcal{M}(\mathcal{A})$, the set of probability

$$F = F(\mu) = \{P(\theta,\ \mu)dX = \exp\{-\text{tr}(\theta X) - k_\mu\}\mu(dX);\ \theta \in \Theta(\mu)\}$$

To each $\mu \in \mathcal{M}(E)$ and $\theta \in \Theta(\mu)$, we associate the probability distribution on $\mathcal{A}$

$$P(\theta, \mu)(dX) = exp\left(\langle\theta, X\rangle - k_\mu(\theta)\right)\mu(dX).$$

The set

$$F = F(\mu) = \{P(\theta, \mu);\ \theta \in \Theta(\mu)\}$$

is called the natural exponential family (NEF) generated by μ. We also say that μ is a basis of F. Note that a basis of F is by no means unique. If μ and ν are in $\mathcal{M}(\mathcal{A})$, then it is easy to check that $F(\mu) = F(\nu)$ if and only if there exist $a \in \mathcal{A}$ and $b \in \mathbb{R}$ such that $d\nu(X) = \exp(\langle a, X\rangle + b)d\mu(X)$. Therefore, if μ is in $\mathcal{M}(\mathcal{A})$ and $F = F(\mu)$, then

$$\mathcal{B}_F = \{\nu \in \mathcal{M}(\mathcal{A});\ F(\nu) = F\} = \{\exp(\langle a, X\rangle + b)\mu(dX);\ (a, b) \in \mathcal{A} \times \mathbb{R}\}$$

is the set of basis of F.

The function k_μ is strictly convex and real analytic. Its first derivative k'_μ defines a diffeomorphism between $\Theta(\mu)$ and its image M_F. Since $k'_\mu(\theta) = \int XP(\theta, \mu)(dX)$, M_F is called the domain of the means of F. The inverse function of k'_μ is denoted by ψ_μ and setting $P(m, F) = P(\psi_\mu(m), \mu)$ the probability of F with mean m, we have

$$F = \{P(m, F); m \in M_F\},$$

which is the parametrization of F by the mean.

If μ and $\nu = \exp(\langle a, X\rangle + b)\mu$ are two basis of F, then for all $\theta \in D(\nu) = D(\mu) - a$,

$$k_\nu(\theta) = k_\mu(\theta + a) + b \tag{4.44}$$

and for all $m \in M_F$,

$$\psi_\nu(m) = \psi_\mu(m) - a. \tag{4.45}$$

Now the covariance operator of $P(m,F)$ is denoted by $V_F(m)$ and the map defined from M_F into $L_s(\mathcal{A})$ by $m \longmapsto V_F(m) = k''_\mu(\psi_\mu(m))$ is called the variance function of the NEF F. It is easy proved that $V_F(m) = (\psi'_\mu(m))^{-1}$ and an important feature of V_F is that it characterizes F in the following sense: If F and F' are two NEFs such that $V_F(m)$ and $V_{F'}(m)$ coincide on a nonempty open set of $M_F \cap M_{F'}$, then $F = F'$. In particular, knowledge of the variance function gives knowledge of the NEF.

Let $\varphi(X) = \delta(X) + \gamma$ be an affine transformation on $\mathcal{A}$, where $\delta \in GL(\mathcal{A})$ and $\gamma \in \mathcal{A}$ and let F be some NEF, generated by μ we denote by $\mu_1 = \varphi * \mu$ the image measure of μ by φ, then for all $\theta \in \Theta(\mu_1) = {\delta^*}^{-1}(\Theta(\mu))$,

$$k_{\mu_1}(\theta) = k_\mu(\delta^*(\theta)) + \langle \theta, \gamma \rangle \cdot \tag{4.46}$$

The following theorem gives a necessary and sufficient condition on χ so that R_χ generates a natural exponential family.

Theorem 4.1 *Let $\chi = \{\lambda_i,\ i \in I\}$ be in Ξ. Then the Riesz measure R_χ is in $\mathcal{M}(\mathcal{A})$ if and only if $\lambda_i \neq 0$, for $i \in \wp \cup \mathcal{S}$.*

Proof ($\Leftarrow$) Suppose that $\chi = \{\lambda_i,\ i \in I\}$ is in Ξ such that $\lambda_i \neq 0\ \forall i \in \wp \cup \mathcal{S}$. We have $\Theta(R_\chi)$ is not empty since it contains $\mathcal{P}^*$. We need to show that R_χ is not concentrated on an affine hyperplane of $\mathcal{A}$. Write $\chi = \sum_{i \in \wp \cup \mathcal{S}} \chi_i$ where $\chi_i \in \Xi(i)$ (see 3.29). Then $R_\chi = \prod^{*}_{i \in \wp \cup \mathcal{S}} R_{\chi_i}$, and it suffices to show that for any $i \in \wp \cup \mathcal{S}$, R_{χ_i} is not concentrated on a affine hyperplane of $\mathcal{A}_{i\preceq}$. In fact suppose that there exists $i \in \wp \cup \mathcal{S}$ such that R_{χ_i} is concentrated on a affine hyperplane H of $\mathcal{A}_{i\preceq}$. Then there exists $\psi \in \varepsilon^i$ such that $\mathcal{T}_l^+ . e_\psi \subset H$. On the other hand, there exist an element $a \in \mathcal{A}_{i\preceq}$ and an hyperplane H_0 of $\mathcal{A}_{i\preceq}$ such that $H = a + H_0$. Write

$$\mathcal{A}_{i\preceq} = H_0 + \mathbb{R}X,$$

where $X \notin H_0$. Let $(e^k_{lj})_{1 \leq k \leq n_{lj}}$ be a basis of $\mathcal{A}_{lj}$, $(l,j) \in I_{i\preceq} \times I_{i\preceq}$.
As $\mathcal{A}_{i\preceq} = \prod_{(l,j) \in I_{i\preceq} \times I_{i\preceq}} \mathcal{A}_{lj}$, we can write

$$X = \sum_{l,j \in I_{i\preceq}} \sum_{1 \leq k \leq n_{lj}} \beta^k_{lj} e^k_{lj},$$

where for $l,j \in I_{i\preceq}$ and $1 \leq k \leq n_{lj}$, β^k_{lj}, is a real number. Thus, we can write

$$\mathcal{A}_{i\preceq} = H_0 + \sum_{l,j \in I} \sum_{1 \leq k \leq n_{lj}} \mathbb{R} e^k_{lj}.$$

Since the dimension of H_0 is equal to $\dim(\mathcal{A}_{i\preceq}) - 1$, then there exist $l,j \in I_{i\preceq}$, and $1 \leq k \leq n_{lj}$ such that

$$\mathcal{A}_{i\preceq} = H_0 + \mathbb{R} e^k_{lj}.$$

Let us consider the vectors

$$A_1 = e^k_{lj} + \sum_{i \preceq j} \psi(j) e^1_{jj} \quad \text{and} \quad A_2 = 2e^k_{lj} + \sum_{i \preceq j} \psi(j) e^1_{jj} \cdot$$

It is clear that A_1 and A_2 are in $\mathcal{T}_l^+ . e_\psi \subset H$. Using the fact that for $i \in \wp \cup \mathcal{S}$, $\lambda_i \neq 0$, we have necessarily $\psi(i) \neq 0$, and we get $A_2 - A_1 = e^k_{lj}$ which is an element of H_0. This is in contradiction with the fact that $X \notin H_0$. Thus for any $i \in \wp \cup \mathcal{S}$, R_{χ_i} is not concentrated on a affine hyperplane of $\mathcal{A}_{i\preceq}$ and Thereoem 4.1 is proved.
($\Rightarrow$) Suppose that $\lambda_i = 0, \forall i \in \wp \cup \mathcal{S}$. As the support of R_χ is $\overline{\mathcal{P}}$, thus R_χ is not an element of $\mathcal{M}(\mathcal{A})$. □

Next, we give the variance function of the Riesz exponential family $F(R_\chi)$ generated by R_χ. For X and K in $\mathcal{A}$, we define the quadratic representation $P(X)$ by

$$P(X)K = X(KX) \cdot$$

It is symmetric, since we have $\langle P(X)K,\ L\rangle = \langle P(X)L,\ K\rangle$.

Theorem 4.2 *For any $m \in \mathcal{P}$,*

$$V_{F(R_\chi)}(m) = \sum_{i \in I} \frac{1}{\lambda_i}(P(m_{i\preceq}) - P(m_{i\prec})) = \sum_{i \in I} \frac{1}{\lambda_i} P_i(P(m)) \cdot$$

For the proof of this theorem we were led to establish the following intermediary result

Lemma 4.3 *The map*

$$\begin{array}{rcl} \varphi : X \to X^{-1} : \mathcal{P} & \to & I\!R \\ Y & \mapsto & tr(X^{-1}Y) \end{array}$$

is differentiable and its differential is

$$\varphi'(X)(K) = (X^{-1})'(K) = -P(X^{-1})(K),$$

that is

$$(X^{-1})'(K)(Y) = -tr((X^{-1}(KX^{-1}))Y).$$

Proof We have

$$\begin{aligned} \lim_{t \to 0} \frac{1}{t}((X + tK)^{-1} - X^{-1})(Y) &= \lim_{t \to 0} \frac{1}{t} \operatorname{tr}((X + tK)^{-1}((X - (X + tK))X^{-1})Y) \\ &= -\lim_{t \to 0} \frac{1}{t} \operatorname{tr}((X + tK)^{-1}(KX^{-1})Y) \\ &= -(X^{-1}(KX^{-1}))(Y) \end{aligned}$$

We now show that

$$(X + K)^{-1} - X^{-1} + X^{-1}(KX^{-1}) = o(K)$$

$$\begin{array}{rcl} \| (X+K)^{-1} - X^{-1} + X^{-1}(KX^{-1}) \| & = & \| ((X+K)^{-1}(X-(X+K)+(X+K)(X^{-1}K))X^{-1}) \| \\ & = & \| ((X+K)^{-1}(X-(X+K)+(X+K)(X^{-1}K))X^{-1}) \| \\ & = & \| ((X+K)^{-1}(K(X^{-1}K))X^{-1}) \| \\ & \leq & \| (X+K)^{-1} \| \| X^{-1} \| \| K^2 \| \cdot \end{array}$$

Then

$$(X^{-1}(K))' = -X^{-1}(KX^{-1}) = -P(X^{-1})(K).$$

□

Proof of Theorem 4.2 For $\theta \in \mathcal{P}^*$, we have $L_{R_\chi}(\theta) = \Delta_\chi(\theta^{-1})$. Then using (3.10), we get

$$k_{R_\chi}(\theta) = -\sum_{i \in I} \lambda_i (\log \Delta_{\preceq i}(\theta) - \log \Delta_{\prec i}(\theta))\cdot$$

Therefore

$$k'_{R_\chi}(\theta) = -\sum_{i \in I} \lambda_i ((\theta_{\preceq i})^{-1} - (\theta_{\prec i})^{-1}),$$

and from Lemma 4.3, we get

$$k''_{R_\chi}(\theta) = \sum_{i \in I} \lambda_i (P((\theta_{\preceq i})^{-1}) - P((\theta_{\prec i})^{-1})).$$

It is easy to see that $(\theta_{\preceq i})^{-1} = (\theta^{-1})_{\preceq i}$ and $(\theta_{\prec i})^{-1} = (\theta^{-1})_{\prec i}$, then

$$k'_{R_\chi}(\theta) \quad = \quad -\sum_{i \in I} \lambda_i ((\theta^{-1})_{\preceq i} - (\theta^{-1})_{\prec i})\cdot$$

For $m \in \mathcal{P}$, such that $m = k'_{R_\chi}(\theta)$, we have

$$\theta^{-1} \quad = \quad -\sum_{i \in I} \frac{1}{\lambda_i} (m_{\preceq i} - m_{\prec i}),$$

then

$$(\theta^{-1})_{i\preceq} = -\frac{1}{\lambda_i} m_{i\preceq}, \quad (\theta^{-1})_{i\prec} = -\frac{1}{\lambda_i} m_{i\prec}\cdot$$

Therefore

$$V_{F(R_\chi)}(m) = \sum_{i \in I} \frac{1}{\lambda_i} (P(m_{i\preceq}) - P(m_{i\prec}))\cdot$$

□

Note that, in the particular case of the Wishart NEF, that is when $\chi = \{\lambda, \forall i \in I\}$, we have

$$V_{F(R_\chi)}(m) = \frac{1}{\lambda} \sum_{i \in I} (P(m_{i\preceq}) - P(m_{i\prec})) = \frac{1}{\lambda} P(m)\cdot$$

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