Generalization of a going-down theorem in the category of Chow-Grothendieck motives due to N. Karpenko

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June 11, 2009

Abstract

Let $\mathbb{M} := (M(X), p)$ be a direct summand of the motive associated with a geometrically split, geometrically variety over a field F satisfying the nilpotence principle. We show that under some conditions on an extension E/F, if $\mathbb{M}$ is a direct summand of another motive M over an extension E, then $\mathbb{M}$ is a direct summand of M over F.

I Introduction

Let Λ be a finite commutative ring. Our main reference on the category $CM(F;\Lambda)$ of Chow-Grothendieck motives with coefficients in Λ is [1].

The purpose of this note is to generalize the folowing theorem due to N. Karpenko ([2], proposition 4.5). Throughout this paper we understand a F-variety over a field F as a separated scheme of finite type over F.

Theorem I.1. *Let Λ be a finite commutative ring. Let X be a geometrically split, geometrically irreducible F-variety satisfying the nilpotence principle. Let $M \in CM(F;\Lambda)$ be another motive. Suppose that an extension E/F satisfies*

1. *the E-motive $M(X)_E \in CM(E;\Lambda)$ of the E-variety X_E is indecomposable;*
2. *the extension $E(X)/F(X)$ is purely transcendental;*
3. *the motive $M(X)_E$ is a direct summand of the motive M.*

Then the motive $M(X)$ is a direct summand of the motive M.

We generalize this theorem when the motive $M(X) \in CM(F;\Lambda)$ is replaced by a direct summand $(M(X), p)$ associated with a projector $p \in End_{CM(F;\Lambda)}(M(X))$. The proof given by N. Karpenko in [2] cannot be used in the case where $M(X)$ is replaced by a direct summand because of the use on the *multiplicity* ([1], §75) as the multiplicity of a projector in the category $CM(F;\Lambda)$ is not always equal to 1 (and it can even be 0). The proof given here for its generalization gives also another proof of theorem I.1.

II Suitable basis of the dual module of a geometrically split F-variety

Let X be a geometrically split, geometrically irreductible F-variety satisfying the nilpotence principle. We note $CH(\overline{X};\Lambda)$ as the colimit of the $CH(X_K;\Lambda)$ over all extensions K of F. By assumption there is a integer $n = rk(X)$ such that

$$CH(\overline{X};\Lambda) \simeq \bigoplus_{i=0}^{n} \Lambda.$$

Let $(x_i)_{i=0}^n$ be a base of the Λ-module $CH(\overline{X};\Lambda)$. Each element x_i of the basis is associated with a subvariety of X_E, where E is a splitting field of X. We note $\varphi(i)$ for the dimension of the E-variety associated to x_i.

Proposition II.1. *Let X be a geometrically split F-variety. Then the pairing*

$$\Psi \ : \ \begin{array}{ccc} CH(\overline{X};\Lambda)\times CH(\overline{X};\Lambda) & \longrightarrow & \Lambda \\ (\alpha,\beta) & \longmapsto & deg(\alpha\cdot\beta) \end{array}$$

is bilinear, symetric and non-degenerate.

The pairing Ψ induces an isomorphism between $CH(\overline{X};\Lambda)$ and its dual module $Hom_{\Lambda}(CH(\overline{X};\Lambda),\Lambda)$. This isomorphism is given by

$$: \ \begin{array}{ccc} CH(\overline{X};\Lambda) & \longrightarrow & Hom_{\Lambda}(CH(\overline{X};\Lambda),\Lambda \\ x & \longmapsto & \Psi(x,\cdot) \end{array}$$

Considering the inverse images of the dual basis of $Hom_{\Lambda}(CH(\overline{X},\Lambda);\Lambda)$ associated with the basis x_i, we get another basis $(x_i^*)_{i=0}^n$ of $CH(\overline{X};\Lambda)$ such that

$$\Psi(x_i,x_j^*)=\delta_{ij}$$

where δ_{ij} is the usual Kronecker symbol.

Proposition II.2. *Let M and N be two motives in $CM(F;\Lambda)$ such that M is split. Then there is an isomorphism*

$$CH^*(M;\Lambda)\otimes CH^*(N;\Lambda)\longrightarrow CH^*(M\otimes N;\Lambda)$$

Proof. c.f. [1] proposition 64.3. □

Let Y be a smooth complete irreducible F-variety. We note M for the motive $(M(Y),q)$ associated with a projector $q\in End(M(Y))$. Then we have the following computations.

Lemma II.3. *For any integers i, j, k and s less than $rk(X)=n$, and for any cycles y and y' in $CH(\overline{Y};\Lambda)$, with 1 being the identity class in either $CH(\overline{X};\Lambda)$ or $CH(\overline{Y};\Lambda)$ we have*

1. $(x_i\times x_j^*)\circ(x_k\times x_s^*)=\delta_{is}(x_k\times x_j^*)$
2. $(x_i\times y\times 1)\circ(x_k\times x_s^*)=\delta_{is}(x_k\times y\times 1)$
3. $(y'\times x_j^*)\circ(x_i\times y)=\deg(y'\cdot y)(x_i\times x_j^*)$

Proof. We only compute (2) (other cases are similar).

$$\begin{aligned}(x_i\times y\times 1)\circ(x_k\times x_s^*) &= ({}^{\overline{X}}p^{\overline{Y}\times\overline{X}}_{\overline{X}})_*(({}^{\overline{X}\times\overline{X}}p_{\overline{Y}\times\overline{X}})^*(x_k\times x_s^*)\cdot(p^{\overline{X}\times\overline{Y}\times\overline{X}}_{\overline{X}})^*(x_i\times y\times 1)) && \text{(II.1)}\\ &= ({}^{\overline{X}}p^{\overline{Y}\times\overline{X}}_{\overline{X}})_*((x_k\times x_s^*\times 1\times 1)\cdot(1\times x_i\times y\times 1)) && \text{(II.2)}\\ &= ({}^{\overline{X}}p^{\overline{Y}\times\overline{X}}_{\overline{X}})_*(x_k\times(x_s^*\cdot x_i)\times y\times 1) && \text{(II.3)}\\ &= \delta_{is}(x_k\times y\times 1) && \text{(II.4)}\end{aligned}$$

□

III Rational cycles of a geometrically split F-variety

Let X be a geometrically split F-variety. We note $(M(X),p)$ the direct summand of $M(X)$ associated with a projector $p\in CH_{\dim(X)}(X\times X;\Lambda)$. Considering the motive M defined in the previous section, if $(M(X_E),p_E)$ is a direct summand of M_E for some extension E/F, then there exists cycles $f\in CH(X_E\times Y_E;\Lambda)$ and $g\in CH(Y_E\times X_E;\Lambda)$ such that $f\circ g=p_E$. We can write these cycles in suitable basis of $CH(\overline{X}\times\overline{Y};\Lambda)$, $CH(\overline{Y}\times\overline{X};\Lambda)$ and $CH(\overline{X}\times\overline{X};\Lambda)$ by proposition II.2. Thus there are two subsets F and G of $\{0,\dots,n\}$, scalars f_i, g_j, p_{ij} and cycles y_i, y'_j in $CH(\overline{Y};\Lambda)$ such that

1. $\overline{f}=\sum_{i\in F}f_i(x_i\times y_i)$
2. $\overline{g}=\sum_{j\in G}g_j(y'_j\times x_j^*)$
3. $\overline{p}=\sum_{i\in F}\sum_{j\in G}p_{ij}(x_i\times x_j^*)$

With $p_{ij} = f_i g_j \deg(y'_j \cdot y_i)$ by lemma II.3 as $g \circ f = p_E$.

Notation III.1. *Let $p \in CH_{\dim(X)}(X \times X)$ be a non-zero projector. Considering $\overline{p}$, the image of p in a splitting field of the F-variety X, we can write $\overline{p} = \sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*)$. We define the* least codimension *of p (denoted $cdmin(p)$) by*

$$cdmin(p) := \min_{(i,j),\ p_{ij} \neq 0} (\dim(\overline{X}) - \varphi(i))$$

Proposition III.2. *Let $p \in CH_{\dim(X)}(X \times X)$ be a non-zero projector. We consider its decomposition $\overline{p} = \sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*)$ in a splitting field of X. Then for any $i \in P_1$ and $j \in P_2$ we have*

$$p_{ij} = \sum_{k \in P_1 \cap P_2} p_{kj} p_{ik}$$

Proof. We can assume that $\varphi(i)$ is constant on P_1. Then a straightforward computation gives

$$\overline{p} \circ \overline{p} = (\sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*)) \circ (\sum_{k \in P_1} \sum_{s \in P_2} p_{ij}(x_i \times x_j^*)) \tag{III.1}$$

$$= \sum_{i \in P_1} \sum_{j \in P_2} \sum_{k \in P_1} \sum_{s \in P_2} p_{ij} p_{ks} (x_i \times x_j^*) \circ (x_k \times x_s^*) \tag{III.2}$$

$$= \sum_{i \in P_1} \sum_{j \in P_2} \sum_{k \in P_1} \sum_{s \in P_2} p_{ij} p_{ks} \delta_{is} (x_k \times x_j^*) \tag{III.3}$$

$$= \sum_{k \in P_1} \sum_{s \in P_2} \left(\sum_{i \in P_1 \cap P_2} p_{ij} p_{ki} (x_k \times x_s^*) \right) \tag{III.4}$$

Moreover $p \circ p = p$, thus if $(k, s) \in P_1 \times P_2$ we have $p_{ks} = \sum_{i \in P_1 \cap P_2} p_{is} p_{ki}$.

□

IV General properties of Chow groups

Embedding the Chow group of the F-variety X is quite usefull for computations, but the generalization of the theorem I.1 needs a direct construction of some F-rational cycles f and g. We study in this section some properties of rationnal elements in Chow groups and how they behave when the extension $E(X)/F(X)$ is purely transcendental.

Proposition IV.1. *Let Y be an F-varieties. Let E/F be a purely transcendental extension. Then the morphism*

$$res_{E/F} : CH(Y; \Lambda) \longrightarrow CH(Y_E; \Lambda)$$

is an epimorphism.

Proof. Indeed the morphism $res_{E/F}$ coincides with the composition

$$CH(Y; \Lambda) \longrightarrow CH(Y \times \mathbb{A}^n_F; \Lambda) \longrightarrow CH(Y_E; \Lambda).$$

As the extension E/F is purely transcendental, there is an isomorphism between E and the function field of an affine space $\mathbb{A}^n_F$ for some integer n. The first map is an epimorphism by the homotopy invariance of Chow groups ([1], theorem 57.13) and the second map is an epimorphism as well ([1], corollary 57.11). □

V Generalization of the going-down theorem in the category of Chow-Grothendieck motives

We now have all the material needed to prove the generalization of theorem I.1.

Theorem V.1. *Let Λ be a finite commutative ring. Let X be a geometrically split, geometically irreducible F-variety satisfying the nilpotence principle. Let also $M \in CM(F; \Lambda)$ be a motive. Suppose that an extension E/F satisfies*

1. *the E-motive $(M(X)_E, p_E)$ associated with the E-variety X_E is indecomposable;*
2. *the extension $E(X)/F(X)$ is purely transcendental;*
3. *the motive $(M(X_E), p_E)$ is a direct summand of the E-motive M_E.*

Then the motive $(M(X), p)$ is a direct summand of the motive M.

Proof. We can consider that $M = (Y, q)$ for some smooth complete F-variety Y and a projector $q \in CH_{\dim(Y)}(Y \times Y; \Lambda)$. If p is equal to 0 then the motive $(M(X), p)$ is the 0 motive and $(M(X), p)$ is a direct summand of M. Now suppose that p is not equal to 0.

As $(M(X)_E, p_E)$ is a direct summand of M_E, there are E-rationnal cycles $f \in CH_{\dim(X_E)}(X_E \times Y_E; \Lambda)$ and $g \in CH_{\dim(Y_E)}(Y_E \times X_E; \Lambda)$ such that $g \circ f = p_E$. We can decompose the images of these cycles in a splitting field of X in suitable basis for computations :

1. $\overline{f} = \sum_{i \in F} f_i(x_i \times y_i)$
2. $\overline{g} = \sum_{j \in G} g_j(y'_j \times x^*_j)$
3. $\overline{p} = \sum_{i \in F} \sum_{j \in G} p_{ij}(x_i \times x^*_j)$

with $p_{ij} = f_i g_j \deg(y'_j \cdot y_i)$.

Splitting terms whose first codimension is minimal in $\overline{f}$ and $\overline{p}$ by introducing

$$F_1 := \{i \in F,\ \varphi(i) = cdmin(p)\}$$

we get

1. $\overline{f} = \sum_{i \in F_1} f_i(x_i \times y_i) + \sum_{i \in F \setminus F_1} f_i(x_i \times y_i)$
2. $\overline{p} = \sum_{i \in F_1} \sum_{j \in G} p_{ij}(x_i \times x^*_j) + \sum_{i \in F \setminus F_1} \sum_{j \in G} p_{ij}(x_i \times x^*_j)$

As $E(X)$ is an extension of E, the cycle $\overline{f}$ is $E(X)$-rational. Proposition IV.1 implies that the change of field $res_{E(X)/F(X)}$ is an epimorphism, hence $\overline{f}$ is an $F(X)$-rational cycle.

Considering the morphism $Spec(F(X)) \longrightarrow X$ associated with the generic point of the geometrically irreducible variety X, we get a morphism

$$\epsilon : (X \times Y)_{F(X)} \longrightarrow X \times Y \times X$$

This morphism induces a pull-back $\epsilon^* : CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda) \longrightarrow CH_{\dim(X)}(\overline{X} \times \overline{Y}; \Lambda)$ mapping any cycle of the form $\alpha \times \beta \times 1$ on $\alpha \times \beta$ and vanishing on elements $\alpha \times \beta \times \gamma$ if codim(γ)> 0. Moreover ϵ^* induces an epimorphism of F-rational cycles onto $F(X)$-rational cycles ([1], corollary 57.11). We can thus choose a F-rational cycle $f_1 \in CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda)$ such that $\epsilon^*(f_1) = \overline{f}$.

By the expression of the pull-back ϵ^* we can assume

$$\overline{f_1} = \sum_{i \in F_1} f_i(x_i \times y_i \times 1) + \sum_{i \in F \setminus F_1} f_i(x_i \times y_i \times 1) + \sum (\alpha \times \beta \times \gamma)$$

where the codimension of the cycles γ is non-zero.

Considering f_1 as a correspondance from $\overline{X}$ to $\overline{X} \times \overline{Y}$, we consider $f_2 := f_1 \circ p$ which is also a F-rational cycle. We have

$$\overline{f_2} = (\sum_{i \in F_1} f_i(x_i \times y_i \times 1)) \circ (\sum_{i \in F_1} \sum_{j \in G} p_{ij}(x_i \times x^*_j)) + \sum_{i \in F \setminus F_1} \sum_{j \in G} \lambda_{ij}(x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma} \tag{V.1}$$

$$= \sum_{i \in F_1} \sum_{j \in F_1 \cap G} f_j p_{ij}(x_i \times y_j \times 1) + \sum_{i \in F \setminus F_1} \sum_{j \in G} \lambda_{ij}(x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma} \tag{V.2}$$

where the cycles $\tilde{\gamma}$ are of non-zero codimension, the cycles $\tilde{\alpha}$ are such that $codim(\tilde{\alpha}) \geq\ cdmin(p)$ and where elements λ_{ij} are scalars.

We now consider the diagonal embedding

$$\Delta\ :\ \begin{array}{ccc} \overline{X} \times \overline{Y} & \longrightarrow & \overline{X} \times \overline{Y} \times \overline{X} \\ (x, y) & \longmapsto & (x, y, x) \end{array}$$

The morphism Δ induces a pull-back $\Delta^* : CH_{\dim(X)}(\overline{X}\times\overline{Y}\times\overline{X};\Lambda) \longrightarrow CH_{\dim(X)}(\overline{X}\times\overline{Y};\Lambda)$

We note $f_3 := \Delta^*(f_2)$, which is also a F-rational cycle and whose expression in a splitting field of X is

$$f_3 = \sum_{i\in F_1}\sum_{j\in F_1\cap G} f_j p_{ij}(x_i\times y_j) + \sum_{i\in F\setminus F_1}\sum_{j\in G}\lambda_{ij}(x_i\times y_j) + \sum(\tilde{\alpha}\cdot\tilde{\gamma})\times\tilde{\beta}$$

where $codim(\tilde{\alpha}\cdot\tilde{\gamma}) > cdmin(p)$ as $codim(\tilde{\alpha}) \geq cdmin(p)$ and $codim(\tilde{\gamma}) > 0$.

We can compute the $g\circ f_3$:

$$\overline{g}\circ\overline{f_3} = \overline{g}\circ(\sum_{i\in F_1}\sum_{j\in G} f_j p_{ij}(x_i\times y_j)) + \overline{g}\circ(\sum_{i\in F\setminus F_1}\sum_{j\in G}\lambda_{ij}(x_i\times y_j)) + \overline{g}\circ(\sum(\tilde{\alpha}\cdot\tilde{\gamma})\times\tilde{\beta})) \quad (V.3)$$

$$= \sum_{i\in F_1}\sum_{s\in G}\sum_{j\in F_1\cap G} g_s f_j p_{ij}(y'_s\times x^*_s)\circ(x_i\times y) + (\sum\overline{\alpha}\times\overline{\beta}) \quad (V.4)$$

With cycles $\overline{\alpha}$ such that $codim(\overline{\alpha}) > cdmin(p)$. Computing the component of $g\circ f_3$ for elements of the form $x_k\times x^*_s$ with $\varphi(k) = cdmin(p)$ we get

$$\overline{g}\circ\overline{f_3} = \sum_{i\in F_1}\sum_{s\in G}\sum_{j\in F_1\cap G} g_s f_j p_{ij}(y'_s\times x^*_s)\circ(x_i\times y_j) + (\sum\overline{\alpha}\times\overline{\beta}) \quad (V.5)$$

$$= \sum_{i\in F_1}\sum_{s\in G}\sum_{j\in F_1\cap G} g_s f_j p_{ij}\deg(y'_s\cdot y_j)(x_i\times x^*_s) \quad (V.6)$$

Now we can see that if $k\in F_1$, then the coefficient of $g\circ f_3$ relatively to an element $x_k\times x^*_s$ is equal to $\sum_{i\in F_1\cap G} g_s f_i p_{ki}\deg(y_i\cdot y'_s)$. Moreover proposition III.2 says that

$$\sum_{i\in F_1\cap G} g_s f_i p_{ki}\deg(y_i\cdot y'_s) = \sum_{i\in F_1\cap G} p_{is}p_{ki} = p_{ks}$$

Since p is non-zero, there exists (k,s) with $k\in F_1$ and $p_{ks}\neq 0$, thus we have shown that the cycle $g\circ f_3$ as a decomposition

$$g\circ f_3 = p_{ks}(x_k\times x^*_s) + \sum_{(i,j)\neq(k,s)} p_{ij}(x_i\times x^*_j) + \sum(\overline{\alpha}\circ\overline{\beta})$$

where $codim(\overline{\alpha}) > cdmin(p)$. Since p is a projector, for any integer n the n-th power of $g\circ f_3$ as always a non-zero component relatively to $x_k\times x^*_s$ which is equal to p_{ks}, that is to say

$$\forall n\in\mathbb{N},\ (g\circ f_3)^{\circ n} = p_{ks}(x_k\times x^*_s) + \sum_{(i,j)\neq(k,s)} p_{ij}(x_i\times x^*_j) + \sum(\overline{\alpha}\circ\overline{\beta})$$

where $codim(\overline{\alpha}) > cdmin(p)$.

As the ring Λ is finite, there is a power of $g\circ(f_3)_E$ which is a non-zero idempotent (cf [2] lemma 3.2). Since the E-motive $(M(X)_E, p_E)$ is indecomposable this power of $g\circ(f_3)_E$ is equal to p_E. Thus we have shown that there exists an integer n_1 such that

$$(g\circ(f_3)_E)^{\circ n_1} = p_E$$

In particular if $g_1 := (g\circ(f_3)_E)^{\circ n_1 - 1}\circ g$ we get $g_1\circ(f_3)_E = p_E$.

Now we can transpose the last equality and get

$$^t(f_3)_E\circ{}^tg_1 = {}^tp_E.$$

Repeating the same process as before, we get a F-rational cycle $\tilde{g}$ and an integer n_2 such that

$$(^t(f_3)_E\circ(\tilde{g})_E)^{\circ n_2} = {}^tp_E$$

Now setting $\hat{g} := ({}^t\tilde{g}\circ(f_3))^{\circ n_2 - 1}\circ{}^t\tilde{g}$, we have two F-rational cycles $\hat{g}$ and f_3 such that

$$\hat{g}_E\circ(f_3)_E = p_E$$

Using the nilpotence principle again, there is an integer $\overline{n}\in\mathbb{N}$ such that

$$(\hat{g}\circ f_3)^{\overline{n}} = p$$

Hence if $\hat{f} := f_3\circ(\hat{g}\circ f_3)^{\overline{n}-1}$, $\hat{f}$ is a F-rational cycle satisfying

$$\hat{g}\circ\hat{f} = p$$

Thus we have shown that the motive $(M(X), p)$ is a direct summand of the motive M. □

References

[1] R. Elman, N. Karpenko, and A. Merkurjev. *The Algebraic and Geometric Theory of Quadratic Forms*. American Mathematical Society, 2008.

[2] N. Karpenko. *Hyperbolicity of hermitian forms over biquaternion algebras*. 2009.