APPROXIMATION OF BOUNDARY CONTROL PROBLEMS ON CURVED DOMAINS. II - THE DIRICHLET CASE[*]

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Abstract. The influence of small boundary variations of the domain on optimal controls is investigated in this paper. The domain variations are governed by a small parameter $h \to 0$. In a previous paper we have studied the Neuman control problem. In this paper, the Dirichlet control problem is considered. The optimal solutions are compared between the problems defined in the curved domain Ω and the polygonal domains Ω_h, in the norm defined on the fixed boundary $\Gamma = \partial\Omega$ of the curved domain. To this end, an appropriate parametrization of the boundaries is introduced, and a one-to-one mapping between the boundaries Γ and Γ_h is employed. Error estimates of the order h in the norm on the fixed boundary Γ are derived for the difference of optimal controls.

Key words. Dirichlet control, error estimates, semilinear elliptic equations, second order optimality conditions

AMS subject classifications. 49J20, 35J65

1. Introduction.

In this paper we study a Dirichlet control problem (P) defined on a curved domain Ω. To solve numerically this problem, usually it is necessary to approximate Ω by a new domain (typically polygonal) Ω_h. Our goal is to analyze the effect of the domain change on the optimal control. More precisely, a new optimal control problem (P$_h$) in Ω_h is defined. The convergence of global or local solutions of problems (P$_h$) to the corresponding local or global solutions of (P) is investigated for the parameter h tending to zero. We also derive some error estimates. We restrict our study to the case of a convex domain $\Omega \subset \mathbb{R}^2$ approximated by a polygonal domain Ω_h, h being the length of the biggest edge of Ω_h. A family of infinite dimensional control problems (P$_h$) defined in Ω_h is considered and the solutions of (P$_h$) are compared with the solutions of (P). In this way, the influence of small changes in the domain on the solutions of the control problem is analyzed. The case of a Neumann control problem is studied in [6].

In this paper we do not perform the numerical analysis of the optimal control problems. We refer the reader to the related papers, [5] for the numerical discretization of a Dirichlet control problem in the case of a polygonal domain and [7] for the analysis in curved domains.

Let us describe the content of the paper. In §2 control problem (P) is introduced and analyzed. In particular, the second order sufficient optimality conditions are established. In spite of the fact that the cost functional is not of class C^2 in $L^2(\Gamma)$, we prove that the standard sufficient optimality conditions imply that the control is a strict local minimum in the $L^2(\Gamma)$ norm. This is an improvement of the known results where the optimality is established in the $L^\infty(\Gamma)$ norm. Approximations Ω_h of Ω are defined in §3 along with control problem (P$_h$). In subsequent §4, the analysis is

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performed and paper is completed with the full proof of the new error estimates in §5. The second order sufficient optimality conditions are a crucial tool for the derivation of error estimates.

2. Control Problem (P). The following control problem is considered in this paper

$$\text{(P)}\left\{\begin{array}{l} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx + \frac{N}{2}\int_\Gamma u^2(x)\,d\sigma(x) \\ \text{subject to } \ (y_u, u) \in (L^\infty(\Omega) \cap H^{1/2}(\Omega)) \times L^\infty(\Gamma), \\ \alpha \le u(x) \le \beta \quad \text{for a.e.} \quad x \in \Gamma, \end{array}\right.$$

where the state y_u associated to the control u is the solution of the Dirichlet problem

$$\left\{\begin{array}{rcll} -\Delta y + a(x,y) &=& 0 & \text{in } \Omega, \\ y &=& u & \text{on } \Gamma. \end{array}\right. \tag{2.1}$$

The following hypotheses are assumed in the whole paper.
(A1) Ω is an open, convex and bounded domain in $\mathbb{R}^2$, with the boundary Γ of class C^2. Moreover we assume that $N > 0$ and $-\infty < \alpha < \beta < +\infty$.

(A2) $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions of class C^2 with respect to the second variable, $L(\cdot, 0) \in L^1(\Omega)$, $a(\cdot, 0) \in L^{\bar p}(\Omega)$, for some $2 \le \bar p < +\infty$. Furthermore, for every $M > 0$ there exist a constant $C_{L,M} > 0$ and a function $\psi_{L,M} \in L^{\bar p}(\Omega)$ such that for almost all $x \in \Omega$ and all $|y|, |y_i| \le M$, $i = 1, 2$, the following inequalities hold

$$\left\{\begin{array}{l} \left|\dfrac{\partial L}{\partial y}(x,y)\right| \le \psi_{L,M}(x), \quad \left|\dfrac{\partial^2 L}{\partial y^2}(x,y)\right| \le C_{L,M}, \\ \left|\dfrac{\partial^2 L}{\partial y^2}(x,y_2) - \dfrac{\partial^2 L}{\partial y^2}(x,y_1)\right| \le C_{L,M}|y_2 - y_1|. \end{array}\right. \tag{2.2}$$

We also assume

$$\left\{\begin{array}{l} \dfrac{\partial a}{\partial y}(x,y) \ge 0 \quad \text{for a.e.} \quad x \in \Omega \quad \text{and for all} \quad y \in \mathbb{R}, \\ \left|\dfrac{\partial a}{\partial y}(x,y)\right| + \left|\dfrac{\partial^2 a}{\partial y^2}(x,y)\right| \le C_{a,M} \quad \text{for a.e.} \quad x \in \Omega \quad \text{and for all} \quad |y| \le M. \end{array}\right. \tag{2.3}$$

We say that an element $y_u \in L^\infty(\Omega)$ is a solution of (2.1) if the following integral identity is fulfilled

$$\int_\Omega -y\Delta w\,dx + \int_\Omega a(x,y)w\,dx = \int_\Gamma u\partial_\nu w\,d\sigma \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega), \tag{2.4}$$

where ∂_ν denotes the normal derivative on the boundary Γ. This is the classical definition of a weak solution by transposition. The following result proved by Casas and Raymond [5] is valid for any convex domain Ω. If the domain is not convex, then some smoothness of Γ is required, Γ of class $C^{1,1}$ is enough.

THEOREM 2.1. *For every $u \in L^\infty(\Gamma)$ the state equation (2.1) has a unique solution $y_u \in L^\infty(\Omega) \cap H^{1/2}(\Omega)$. Moreover, the following Lipschitz properties hold*

$$\begin{array}{l} \|y_u - y_v\|_{L^\infty(\Omega)} \le \|u - v\|_{L^\infty(\Gamma)}, \\ \|y_u - y_v\|_{H^{1/2}(\Omega)} \le C\|u - v\|_{L^2(\Gamma)} \quad \forall u, v \in L^\infty(\Gamma). \end{array} \tag{2.5}$$

Finally, if $u_n \rightharpoonup u$ *weakly** *in* $L^\infty(\Gamma)$*, then* $y_{u_n} \to y_u$ *strongly in* $L^r(\Omega)$ *for all* $r < +\infty$*.*

Under the assumptions *(A1)* and *(A2)*, it can be shown by standard arguments that problem (P) has at least one solution. Since (P) is not convex we cannot expect any uniqueness of solutions. Moreover, (P) may have some local solutions. We formulate the optimality conditions satisfied by such local solutions. To this end, we analyze the differentiability of the cost functional J.

Under the assumption *(A2)*, $J : L^\infty(\Gamma) \longrightarrow \mathbb{R}$ is of class C^2 and

$$J'(u)v = \int_\Gamma (Nu - \partial_\nu \varphi_u)\, v\, dx, \tag{2.6}$$

where y_u is the state associated to u and $\varphi_u \in H^2(\Omega)$ is the unique solution of the problem

$$\left\{\begin{array}{rcll} -\Delta\varphi + \dfrac{\partial a}{\partial y}(x,y_u)\varphi &=& \dfrac{\partial L}{\partial y}(x,y_u) & \text{in } \Omega,\\ \varphi &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{2.7}$$

Furthermore, we have
(2.8)

$$J''(u)(v_1,v_2) = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,y_u) z_{v_1} z_{v_2} - \varphi_u \frac{\partial^2 a}{\partial y^2}(x,y_u) z_{v_1} z_{v_2}\right] dx + \int_\Gamma N v_1 v_2\, dx,$$

where z_{v_i}, $i = 1,2$, satisfy

$$\left\{\begin{array}{rcll} -\Delta z_{v_i} + \dfrac{\partial a}{\partial y}(x,y_u) z_{v_i} &=& 0 & \text{in } \Omega,\\ z_{v_i} &=& v_i & \text{on } \Gamma. \end{array}\right. \tag{2.9}$$

Using (2.6) we obtain the necessary optimality conditions for (P).

THEOREM 2.2. *Let* $\bar u$ *be a local minimum of* (P)*. Then* $\bar u \in W^{1-1/\bar p, \bar p}(\Gamma)$ *and there exist elements* $\bar y \in W^{1,\bar p}(\Omega)$ *and* $\bar\varphi \in W^{2,\bar p}(\Omega)$ *such that*

$$\left\{\begin{array}{rcll} -\Delta\bar y + a(x,\bar y) &=& 0 & \textit{in } \Omega,\\ \bar y &=& \bar u & \textit{on } \Gamma, \end{array}\right. \tag{2.10}$$

$$\left\{\begin{array}{rcll} -\Delta\bar\varphi + \dfrac{\partial a}{\partial y}(x,\bar y)\bar\varphi &=& \dfrac{\partial L}{\partial y}(x,\bar y) & \textit{in } \Omega,\\ \bar\varphi &=& 0 & \textit{on } \Gamma, \end{array}\right. \tag{2.11}$$

$$\int_\Gamma (N\bar u(x) - \partial_\nu\bar\varphi(x))(v(x) - \bar u(x))\, d\sigma(x) \geq 0 \quad \textit{for all} \quad \alpha \leq v \leq \beta. \tag{2.12}$$

The proof of theorem is given in [5].

In order to establish the second order optimality conditions we define the cone of critical directions

$$C_{\bar u} = \{v \in L^2(\Gamma) \text{ satisfying (2.13) and } v(x) = 0 \text{ if } |N\bar u(x) - \partial_\nu\bar\varphi(x)| > 0\},$$

$$v(x) = \left\{\begin{array}{ll} \geq 0 & \text{if } \bar u(x) = \alpha,\\ \leq 0 & \text{if } \bar u(x) = \beta, \end{array}\right. \quad \text{for a.e. } x \in \Gamma. \tag{2.13}$$

Now we formulate the second order necessary and sufficient optimality conditions.

THEOREM 2.3. *If* $\bar u$ *is a local solution of* (P), *then* $J''(\bar u)v^2 \geq 0$ *holds for all* $v \in C_{\bar u}$. *Conversely, if* $\bar u$ *is an admissible control for problem* (P) *satisfying the first order optimality conditions given in Theorem 2.2 and the coercivity condition*

$$J''(\bar u)v^2 > 0 \quad \forall v \in C_{\bar u} \setminus \{0\}, \tag{2.14}$$

then there exist $\delta > 0$ *and* $\rho > 0$ *such that*

$$J(u) \geq J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(\Gamma)} \tag{2.15}$$

for all u *such that* $\alpha \leq u \leq \beta$ *and* $\|u - \bar u\|_{L^2(\Gamma)} \leq \rho$.

Proof. The necessary condition is easy to obtain. The inequality (2.15) is strong when compared with the corresponding inequality of [5]. Indeed, here we claim that (2.14) implies that $\bar u$ is a strict local minimum of (P) in the sense of the $L^2(\Gamma)$ topology. In [5] it is shown that condition (2.14) leads to the strict local optimality of $\bar u$ in the sense of the $L^\infty(\Gamma)$ topology. A more general result is proved in [2] for a distributed control problem, but in such a case once again only the local optimality in the sense of the $L^\infty(\Omega)$ topology is shown. Here we can improve the results because the control appears in a quadratic form within the cost functional. Let us see the precise arguments.

We proceed by contradiction. Let us assume that there is no pair (δ, ρ), with $\rho, \delta > 0$, such that (2.15) holds. Then for every integer k, there exists a feasible control of (P), $u_k \in L^2(\Gamma)$, such that

$$\|u_k - \bar u\|_{L^2(\Gamma)} < \frac{1}{k} \ \text{ and } \ J(u_k) < J(\bar u) + \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Gamma)}. \tag{2.16}$$

Let us define

$$\lambda_k = \|u_k - \bar u\|_{L^2(\Gamma)} \ \text{ and } \ v_k = \frac{1}{\lambda_k}(u_k - \bar u), \ \text{ hence } \ \|v_k\|_{L^2(\Gamma)} = 1. \tag{2.17}$$

By taking a subsequence, if necessary, there exists $v \in L^2(\Gamma)$ such that $v_k \rightharpoonup v$ weakly in $L^2(\Gamma)$. The proof is divided into three steps: first, we prove that $v \in C_{\bar u}$, then we deduce that $v = 0$ and finally we get the contradiction.

Step 1. $v \in C_{\bar u}$. Since $\alpha \leq u_k \leq \beta$, it is obvious that every v_k satisfies (2.13). Also we have that the set of functions of $L^2(\Gamma)$ satisfying (2.13) is convex and closed, therefore v satisfies (2.13) as well. This implies

$$(N\bar u(x) - \partial_\nu \bar\varphi(x))v(x) = |N\bar u(x) - \partial_\nu \bar\varphi(x)||v(x)| \quad \text{a.e. on } \Gamma \tag{2.18}$$

Indeed, it is well known that (2.12) implies that $N\bar u(x) - \partial_\nu\bar\varphi(x) \geq 0$ if $\bar u(x) = \alpha$ and $N\bar u(x) - \partial_\nu\bar\varphi(x) \leq 0$ if $\bar u(x) = \beta$. This property and (2.13) lead to (2.18).

On the other hand, from (2.16) we get

$$\frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Gamma)} > J(u_k) - J(\bar u) = J(\bar u + \lambda_k v_k) - J(\bar u) = J'(\bar u + \theta_k\lambda_k v_k)v_k, \tag{2.19}$$

for some $0 < \theta_k < 1$. From (2.6) we have that

$$J'(\bar u + \theta_k\lambda_k v_k)v_k = \int_\Gamma (N[\bar u + \theta_k\lambda_k v_k] - \partial_\nu\varphi_k)v_k \, d\sigma. \tag{2.20}$$

Let us denote by y_k the state associated to $\bar u + \theta_k\lambda_k v_k = \bar u + \theta_k(u_k - \bar u)$. Since $\alpha \le \bar u + \theta_k(u_k - \bar u) \le \beta$ and $\bar u + \theta_k(u_k - \bar u) \to \bar u$ in $L^2(\Gamma)$ for $k \to +\infty$, we deduce, in view of (2.5), that $\{y_k\}_{k=1}^\infty$ is bounded in $L^\infty(\Gamma)$ and $y_k \to \bar y$ in $H^{1/2}(\Omega)$. Therefore, the sequence of adjoint states $\{\varphi_k\}_{k=1}^\infty$ converges to $\bar\varphi$ in $H^2(\Omega)$. Hence, we can pass to the limit in (2.20) and use (2.19) to deduce that

$$\int_\Gamma (N\bar u - \partial_\nu\bar\varphi)v\,d\sigma \le 0.$$

This identity and (2.18) imply that $v(x) = 0$ if $|N\bar u(x) - \partial_\nu\bar\varphi(x)| > 0$. Thus, we have that $v \in C_{\bar u}$.

Step 2. $v = 0$. Using again (2.16) we obtain

$$\frac{\lambda_k^2}{k} = \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Gamma)} > J(u_k) - J(\bar u) = J(\bar u + \lambda_k v_k) - J(\bar u)$$

$$= \lambda_k J'(\bar u)v_k + \frac{\lambda_k^2}{2}J''(\bar u + \theta_k\lambda_k v_k)v_k^2 \ge \frac{\lambda_k^2}{2}J''(\bar u + \theta_k\lambda_k v_k)v_k^2, \tag{2.21}$$

the last inequality being a consequence of (2.12)

$$\lambda_k J'(\bar u)v_k = J'(\bar u)(u_k - \bar u) = \int_\Gamma (N\bar u - \partial_\nu\bar\varphi)(u_k - \bar u)\,d\sigma \ge 0.$$

Inequality (2.21) implies that

$$\frac{2}{k} > J''(\bar u + \theta_k\lambda_k v_k)v_k^2. \tag{2.22}$$

Once again we denote by y_k and φ_k the state and adjoint state evaluated for $\bar u + \theta_k\lambda_k v_k = \bar u + \theta_k(u_k - \bar u)$. Also we define z_k and z_v as the elements of $H^{1/2}(\Omega)$ satisfying

$$\left\{\begin{array}{rcll} -\Delta z_k + \dfrac{\partial a}{\partial y}(x, y_k)z_k &=& 0 & \text{in } \Omega,\\ z_k &=& v_k & \text{on } \Gamma, \end{array}\right. \tag{2.23}$$

and

$$\left\{\begin{array}{rcll} -\Delta z_v + \dfrac{\partial a}{\partial y}(x, \bar y)z_v &=& 0 & \text{in } \Omega,\\ z_v &=& v & \text{on } \Gamma. \end{array}\right. \tag{2.24}$$

Then $z_k \rightharpoonup z_v$ weakly in $H^{1/2}(\Omega)$, hence strongly in $L^2(\Omega)$. Moreover, $y_k \to \bar y$ in $H^{1/2}(\Omega)$ and $\varphi_k \to \bar\varphi$ in $H^2(\Omega)$. Now, recalling the expression of the second derivative of J given in (2.8) we get

$$J''(\bar u + \theta_k\lambda_k v_k)v_k^2 = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_k)z_k^2 - \varphi_k\frac{\partial^2 a}{\partial y^2}(x, y_k)z_k^2\right]dx + N\int_\Gamma v_k^2\,dx. \tag{2.25}$$

Passing to the limit in this expression and using (2.22) we obtain

$$J''(\bar u)v^2 = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, \bar y)z_v^2 - \bar\varphi\frac{\partial^2 a}{\partial y^2}(x, \bar y)z_v^2\right]dx + N\int_\Gamma v^2\,dx$$

$$\leq \liminf_{k\to\infty} J''(\bar u + \theta_k\lambda_k v_k)v_k^2 \leq 0. \tag{2.26}$$

Since $v \in C_{\bar u}$, according to (2.14) this is possible only if $v = 0$.

Step 3. Final Contradiction. Using two facts, $v_k \rightharpoonup v = 0$ and $\|v_k\|_{L^2(\Gamma)} = 1$, we deduce from (2.22) and (2.25) the following contradiction

$$0 < N \leq \liminf_{k\to\infty} J''(\bar u + \theta_k\lambda_k v_k)v_k^2 \leq 0.$$

□

We conclude this section with the following result that provides an equivalent formulation of (2.14), which is more useful for our purposes.

THEOREM 2.4. *Let $\bar u$ be a feasible control of problem* (P) *satisfying the first order optimality conditions* (2.10)-(2.12). *Then the condition* (2.14) *holds if and only if*

$$\exists\mu > 0 \text{ and } \vartheta > 0 \text{ such that } J''(\bar u)v^2 \geq \mu\|v\|^2_{L^2(\Gamma)}\ \forall v \in C^\vartheta_{\bar u}, \tag{2.27}$$

where

$$C^\vartheta_{\bar u} = \{v \in L^2(\Gamma) \text{ satisfying } (2.13) \text{ and } v(x) = 0 \text{ if } |N\bar u(x) - \partial_\nu\bar\varphi(x)| > \vartheta\}.$$

Proof. Since $C_{\bar u} \subset C^\vartheta_{\bar u}$ for any $\vartheta > 0$, it is obvious that (2.27) implies (2.14). Let us prove the reciprocal implication. We proceed again by contradiction. We assume that (2.14) holds, but there is no pair of positive numbers (μ,ϑ) such that (2.27) is fulfilled. Then for every integer k there exists and element $v_k \in C^{1/k}_{\bar u}$ such that

$$J''(\bar u)v_k^2 < \frac{1}{k}\|v_k\|^2_{L^2(\Gamma)}.$$

Dividing v_k by its norm and denoting the quotient by v_k again, and taking a subsequence if necessary, we have that

$$v_k \in C^{1/k}_{\bar u},\quad \|v_k\|_{L^2(\Gamma)} = 1,\quad v_k \rightharpoonup v \text{ in } L^2(\Gamma),\quad J''(\bar u)v_k^2 < \frac{1}{k}. \tag{2.28}$$

Arguing as in the proof of Theorem 2.3, we obtain that v satisfies (2.13). On the other hand, from the fact that $v_k \in C^{1/k}_{\bar u}$ and denoting by Γ_k the subset of Γ formed by those points x such that $|N\bar u(x) - \partial_\nu\bar\varphi(x)| \leq 1/k$, we get

$$\int_\Gamma (N\bar u - \partial_\nu\bar\varphi)v\,d\sigma = \lim_{k\to\infty}\int_\Gamma (N\bar u - \partial_\nu\bar\varphi)v_k\,d\sigma$$

$$\leq \liminf_{k\to\infty}\int_\Gamma |N\bar u - \partial_\nu\bar\varphi||v_k|\,d\sigma \leq \liminf_{k\to\infty}\frac{1}{k}\int_{\Gamma_k}|v_k|\,d\sigma = 0.$$

This inequality and the fact that v satisfies (2.13) imply that v vanishes whenever $|N\bar u(x) - \partial_\nu\bar\varphi(x)| > 0$, hence $v \in C_{\bar u}$. Now (2.14) implies that $J''(\bar u) > 0$ if $v \not\equiv 0$. But from (2.28) we deduce that

$$J''(\bar u)v^2 \leq \liminf_{k\to\infty} J''(\bar u)v_k^2 \leq 0.$$

Consequently we have that $v \equiv 0$. However, if we argue as in the proof of Theorem 2.3, we have that $0 < N \leq \liminf_{k\to\infty} J''(\bar u)v_k^2 \leq 0$, which is a contradiction. □

3. Control Problem (P$_h$). Now we define Ω_h. We follow the notation introduced in [6, Section 4]. Given a set of points $\{x_j\}_{j=1}^{N(h)} \subset \Gamma$, we put

$$h_j = |x_{j+1} - x_j|, \quad h = \max_{1\le j\le N(h)} h_j, \quad \tau_j = \frac{1}{h_j}(x_{j+1} - x_j),$$

where $x_{N(h)+1} = x_1$. Γ_h is the polygonal line defined by the nodes $\{x_j\}_{j=1}^{N(h)}$ and Ω_h is the polygon delimited by Γ_h. Since Ω is convex, then $\Omega_h \subset \Omega$. Now, for every $1 \le j \le N(h)$, we denote by $\widehat{x_j x_{j+1}}$ the arc of Γ delimited by the points x_j and x_{j+1}. Let us define $\psi_j : [0, h_j] \longrightarrow \widehat{x_j x_{j+1}} \subset \Gamma$ by

$$\psi_j(t) = x_j + t\tau_j + \phi_j(t)\nu_j,$$

where ν_j represents the unit outward normal vector to Ω_h on the boundary edge (x_j, x_{j+1}) and $\phi_j : [0, h_j] \longrightarrow [0, +\infty)$ is chosen such that $\psi_j(t) \in \Gamma$. Since Ω is convex and Γ is of class C^2, the following properties hold

1. ϕ_j is of class C^2 and $\phi_j(0) = \phi_j(h_j) = 0$.
2. There exists a constant $C_\Gamma > 0$ such that $\phi_j(t) + h|\phi_j'(t)| \le C_\Gamma h_j^2 \le C_\Gamma h^2$ for all $t \in [0, h_j]$.

Now, we define we define the one-to-one mapping $g_h : \Gamma_h \longrightarrow \Gamma$ in the following way

$$g_h|_{[x_j,x_{j+1}]}(x) = g_h|_{[x_j,x_{j+1}]}(x_j + t\tau_j) = x_j + t\tau_j + \phi_j(t)\nu_j = \psi_j(t).$$

For every point $x \in \Gamma$, $\nu(x)$ denotes the unit outward normal vector to Γ at the point x. By $\tau(x)$ is denoted the unit tangent vector to Γ at the point x such that $\{\tau(x), \nu(x)\}$ is a direct reference system in $\mathbb{R}^2$. For each point $x \in \Gamma_h$ the corresponding reference system is denoted by $\{\tau_h(x), \nu_h(x)\}$. If $x \in (x_j, x_{j+1})$ then $\nu_h(x) = \nu_j$ and $\tau_h(x) = \tau_j$. The following relations are proved in [6]

$$\max\{|\tau(g_h(x)) - \tau_h(x)|, |\nu(g_h(x)) - \nu_h(x)|\} \le (C_\Gamma^2 + 1)h \quad \forall x \in \Gamma_h,\ x \ne x_j, \tag{3.1}$$

$$\int_{\Gamma_h} |v(g_h(x))|\, d\sigma_h(x) \le \int_\Gamma |v(x)|\, d\sigma(x) \quad \forall v \in L^1(\Gamma), \tag{3.2}$$

$$\left|\int_\Gamma v(x)\, d\sigma(x) - \int_{\Gamma_h} v(g_h(x))\, d\sigma_h(x)\right| \le C_\Gamma h^2 \int_\Gamma |v(x)|\, d\sigma(x) \quad \forall v \in L^1(\Gamma), \tag{3.3}$$

and

$$\int_\Gamma v(x)\, d\sigma(x) = \int_{\Gamma_h} v(g_h(x))|Dg_h(x)\cdot\tau_h(x)|\, d\sigma_h(x) \quad \forall v \in L^1(\Gamma). \tag{3.4}$$

In the domain Ω_h we define the problem (P$_h$) as follows

$$(\mathrm{P}_h)\left\{\begin{array}{l} \min J_h(u) = \displaystyle\int_{\Omega_h} L(x, y_{h,u}(x))\, dx + \frac{N}{2}\int_{\Gamma_h} u^2(x)\, d\sigma_h(x) \\ \text{subject to}\ \ (y_{h,u}, u) \in (L^\infty(\Omega_h) \cap H^{1/2}(\Omega_h)) \times L^\infty(\Gamma_h), \\ \alpha \le u(x) \le \beta \quad \text{for a.e.}\ \ x \in \Gamma_h, \end{array}\right.$$

where $y_{h,u}$ is the solution of the problem

$$\left\{\begin{array}{rcll} -\Delta y + a(x,y) & = & 0 & \text{in } \Omega_h, \\ y & = & u & \text{on } \Gamma_h. \end{array}\right. \tag{3.5}$$

Theorem 2.1 can be applied to (3.5) to get the existence and uniqueness of a solution $y_{h,u} \in H^{1/2}(\Omega_h) \cap L^\infty(\Omega_h)$. Moreover, inequalities (2.5) hold. (P$_h$) has at least one global solution and possibly there are some other local solutions of (P$_h$). For each local solution we have the first order optimality conditions analogous to the conditions in Theorem 2.2.

Theorem 3.1. *Let $\bar u_h$ be a local minimum of* (P$_h$). *Then $\bar u_h \in H^{1/2}(\Gamma_h)$ and there exist elements $\bar y_h \in H^1(\Omega_h)$ and $\bar\varphi_h \in H^2(\Omega_h)$ such that*

$$\left\{\begin{array}{rcll} -\Delta \bar y_h + a(x,\bar y_h) & = & 0 & \text{in } \Omega_h, \\ \bar y_h & = & \bar u_h & \text{on } \Gamma_h, \end{array}\right. \tag{3.6}$$

$$\left\{\begin{array}{rcll} -\Delta \bar\varphi_h + \dfrac{\partial a}{\partial y}(x,\bar y_h)\bar\varphi_h & = & \dfrac{\partial L}{\partial y}(x,\bar y_h) & \text{in } \Omega_h, \\ \bar\varphi_h & = & 0 & \text{on } \Gamma_h, \end{array}\right. \tag{3.7}$$

$$\int_{\Gamma_h} (N\bar u_h(x) - \partial_{\nu_h}\bar\varphi_h(x))(v_h(x) - \bar u_h(x))\,d\sigma_h(x) \ge 0 \quad \text{for all} \quad \alpha \le v_h \le \beta. \tag{3.8}$$

Remark 3.2. *We observe that $\bar u_h$ is less regular than $\bar u$. The same is true for $\bar y_h$ and $\bar\varphi_h$ with respect to $\bar y$ and $\bar\varphi$. The reason of the lost of regularity is the lack of regularity of Γ_h. Γ is of class C^2 and consequently we can deduce the $W^{2,\bar p}(\Omega)$ regularity of $\bar\varphi$ (see, for instance, Grisvard [8]), which leads to the $W^{1-1/\bar p}(\Gamma)$ regularity of $\bar u$ and consequently to the $W^{1,\bar p}(\Omega)$ regularity of $\bar y$. Using the results for polygonal domains of [8], we can establish $W^{2,p}(\Omega)$ regularity of $\bar\varphi_h$ for some $2 < p \le \bar p$ (assuming $\bar p > 2$), with p depending on the angles of Ω_h. The point is that $p \to 2$ if the maximal angle of Ω_h tends to π. This is exactly the case for $h \to 0$, therefore we cannot deduce the boundedness of $\{\|\bar\varphi_h\|_{W^{2,p}(\Omega_h)}\}_{h>0}$ for any $p > 2$.*

*By using the Stampacchia approach [**?**] we can derive a bound for $\|\bar y_h\|_{L^\infty(\Omega_h)}$ which is dependent on α, β and $a(\cdot,0)$, but independent of h. Then, from* (3.7) *the boundedness of $\{\|\bar\varphi_h\|_{H^2(\Omega_h)}\}_{h>0}$ can be obtained. Now, from* (3.8) *we deduce*

$$\bar u_h(x) = \mathrm{Proj}_{[\alpha,\beta]}\left(-\frac{1}{N}\partial_{\nu_h}\bar\varphi_h(x)\right) = \max\{\alpha, \min\{-\frac{1}{N}\partial_{\nu_h}\bar\varphi_h(x), \beta\}\}, \tag{3.9}$$

which implies that $\bar u_h \in H^{1/2}(\Gamma_h)$ and the family $\{\|\bar u_h\|_{H^{1/2}(\Gamma_h)}\}_{h>0}$ is bounded. Finally, (3.6) *leads to the boundedness of $\{\|\bar y_h\|_{H^1(\Omega_h)}\}_{h>0}$ as well.*

4. Convergence Analysis.

In this section we prove the convergence of the local or global solutions of (P$_h$) to the solutions of (P) with $h \to 0$. To prove the convergence, first we establish the convergence of the solutions of the state and adjoint state equations.

Theorem 4.1. *Let $u \in H^{1/2}(\Gamma)$ and $u_h \in L^2(\Gamma_h)$, with*

$$\max\{\|u\|_{L^\infty(\Gamma)}, \|u_h\|_{L^\infty(\Gamma_h)}\} \le M. \tag{4.1}$$

Let $y_u \in H^1(\Omega) \cap L^\infty(\Omega)$ and $y_{h,u_h} \in H^{1/2}(\Omega_h) \cap L^\infty(\Omega_h)$ be the corresponding solutions of (2.1) *and* (3.5)*, respectively. Then there exists a constant $C_M > 0$ independent of h such that*

$$\|y_u - y_{h,u_h}\|_{H^{1/2}(\Omega_h)} \le C_M \left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[1 + \|u\|_{H^{1/2}(\Gamma)}]\right). \tag{4.2}$$

Proof. Let us take $y_h \in H^{1/2}(\Omega_h) \cap L^\infty(\Omega_h)$ satisfying

$$\left\{\begin{array}{rcll} -\Delta y_h + a(x, y_h) & = & 0 & \text{in } \Omega_h, \\ y_h & = & u \circ g_h & \text{on } \Gamma_h. \end{array}\right. \tag{4.3}$$

From (2.5) and (3.2) we get

$$\begin{aligned} \|y_u - y_{h,u_h}\|_{H^{1/2}(\Omega_h)} &\le \|y_u - y_h\|_{H^{1/2}(\Omega_h)} + \|y_h - y_{h,u_h}\|_{H^{1/2}(\Omega_h)} \\ &\le \|y_u - y_h\|_{H^{1/2}(\Omega_h)} + C\|u \circ g_h - u_h\|_{L^2(\Gamma_h)} \\ &\le \|y_u - y_h\|_{H^{1/2}(\Omega_h)} + C\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)}. \end{aligned} \tag{4.4}$$

Let us estimate $\phi_h = y_u - y_h$. By substraction of the equations satisfied by y_u and y_h and using the mean value theorem, we get

$$\left\{\begin{array}{rcll} -\Delta \phi_h + \dfrac{\partial a}{\partial y}(x, w_h)\phi_h & = & 0 & \text{in } \Omega_h, \\ \phi_h & = & y - u \circ g_h & \text{on } \Gamma_h, \end{array}\right. \tag{4.5}$$

where $w_h = y_h + \theta_h(y_{h,u_h} - y_h)$ and $0 < \theta_h < 1$. Now we have

$$\|\phi_h\|_{H^{1/2}(\Omega_h)} \le C\|y - u \circ g_h\|_{L^2(\Gamma_h)} = C\|y - y \circ g_h\|_{L^2(\Gamma_h)}.$$

Finally, by using the inequality (see Bramble and King [1, Lemma 1])

$$\|w - w \circ g_h\|_{L^2(\Gamma_h)} \le Ch^r \|w\|_{H^r(\Omega)} \quad \text{for all } \ 1 \le r \le 2, \tag{4.6}$$

we conclude

$$\|\phi_h\|_{H^{1/2}(\Omega_h)} \le Ch\|y\|_{H^1(\Omega)} \le Ch(1 + \|u\|_{H^{1/2}(\Gamma)}).$$

This inequality along with (4.4) proves (4.2). □

Now we proceed with the analysis of the adjoint state equation. Let $\varphi_u \in H^2(\Omega)$ and $\varphi_{h,u_h} \in H^2(\Omega_h)$ be given as the solutions of the equations

$$\left\{\begin{array}{rcll} -\Delta \varphi_u + \dfrac{\partial a}{\partial y}(x, y_u)\varphi_u & = & \dfrac{\partial L}{\partial y}(x, y_u) & \text{in } \Omega, \\ \varphi_u & = & 0 & \text{on } \Gamma, \end{array}\right. \tag{4.7}$$

and

$$\left\{\begin{array}{rcll} -\Delta \varphi_{h,u_h} + \dfrac{\partial a}{\partial y}(x, y_{h,u_h})\varphi_{h,u_h} & = & \dfrac{\partial L}{\partial y}(x, y_{h,u_h}) & \text{in } \Omega_h, \\ \varphi_{h,u_h} & = & 0 & \text{on } \Gamma_h. \end{array}\right. \tag{4.8}$$

Then we have the following estimate.

Theorem 4.2. *Let (u, y_u) and (u_h, y_{h,u_h}) be as in Theorem 4.1. Let $\varphi_u \in H^2(\Omega)$ and $\varphi_{h,u_h} \in H^2(\Omega_h)$ be the corresponding solutions of (4.7) and (4.8), respectively. Then there exists a constant $C_M > 0$, independent of h, such that the following estimate holds*

$$\|\varphi_u - \varphi_{h,u_h}\|_{H^{3/2}(\Omega_h)} \le C_M \left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[1 + \|u\|_{H^{1/2}(\Gamma)}]\right). \tag{4.9}$$

Proof. Let us define $\phi_h = \varphi_u - \varphi_{h,u_h} \in H^2(\Omega_h)$. From (4.7) and (4.8) we get

$$\left\{\begin{array}{rcll} -\Delta\phi_h + \dfrac{\partial a}{\partial y}(x, y_u)\phi_h & = & \dfrac{\partial L}{\partial y}(x, y_u) - \dfrac{\partial L}{\partial y}(x, y_{h,u_h}) & \\ & & + \left[\dfrac{\partial a}{\partial y}(x, y_{h,u_h}) - \dfrac{\partial a}{\partial y}(x, y_u)\right]\varphi_{h,u_h} & \text{in } \Omega_h \\ \phi_h & = & \varphi_u & \text{on } \Gamma_h. \end{array}\right. \tag{4.10}$$

From assumption *(A2)*, taking into account that y_u and y_{h,u_h} are bounded and using (4.2), we get (see Kenig [10])

$$\|\phi_h\|_{H^{3/2}(\Omega_h)} \leq C\left(\|y_u - y_{h,u_h}\|_{L^2(\Omega_h)} + \|\varphi_u\|_{H^1(\Gamma_h)}\right)$$

$$\leq C(M)\left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[1 + \|u\|_{H^{1/2}(\Gamma)}] + \|\varphi_u\|_{H^1(\Gamma_h)}\right). \tag{4.11}$$

Let us estimate φ_u in $H^1(\Gamma_h)$. The norm in $H^1(\Gamma_h)$ is given by

$$\|\varphi_u\|_{H^1(\Gamma_h)} = \left\{\|\varphi_u\|^2_{L^2(\Gamma_h)} + \|\partial_{\tau_h}\varphi_u\|^2_{L^2(\Gamma_h)}\right\}^{1/2},$$

where $\partial_{\tau_h}\varphi_u(x) = \nabla\varphi_u(x) \cdot \tau_h(x)$, $\tau_h(x)$ being the unit tangent vector to Γ_h at the point x; see §3. The estimate of the first term of the norm follows easily from (4.6) and the fact that $\varphi_u \circ g_h = 0$ on Γ_h

$$\|\varphi_u\|_{L^2(\Gamma_h)} = \|\varphi_u - \varphi_u \circ g_h\|_{L^2(\Gamma_h)} \leq Ch^2\|\varphi_u\|_{H^2(\Omega)} \leq C(M)h^2. \tag{4.12}$$

Now the $L^2(\Gamma)$ norm of the tangential derivative is estimated. To this end we observe that $\varphi_u = 0$ on Γ, therefore $\partial_\tau\varphi_u = 0$ on Γ as well. Thus, we also have $(\nabla\varphi_u \circ g_h) \cdot (\tau \circ g_h) = 0$ on Γ_h. Hence

$$\partial_{\tau_h}\varphi_u(x) = [\nabla\varphi_u(x) - \nabla\varphi_u(g_h(x))]\tau_h(x) + \nabla\varphi_u(g_h(x))[\tau_h(x) - \tau(g_h(x))].$$

This along with (4.6) and (3.1) leads to

$$\|\partial_{\tau_h}\varphi_u\|_{L^2(\Gamma_h)} \leq \|\nabla\varphi_u - \nabla\varphi_u \circ g_h\|_{L^2(\Gamma_h)}$$

$$+\|\varphi_u\|_{H^2(\Omega)}\|\tau_h - \tau \circ g_h\|_{L^2(\Gamma_h)} \leq C(M)h. \tag{4.13}$$

Finally, (4.9) follows from (4.11), (4.12) and (4.13). ☐

COROLLARY 4.3. *Under the assumptions of Theorem 4.2, the following inequality holds*

$$\|\partial_{\nu_h}\varphi_u - \partial_{\nu_h}\varphi_{h,u_h}\|_{L^2(\Gamma_h)} \leq C_M\left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[1 + \|u\|_{H^{1/2}(\Gamma)}]\right) \tag{4.14}$$

for some $C_M > 0$ *independent of* h.

Proof. It is enough to note that $\varphi_u - \varphi_{h,u_h} \in H^{3/2}(\Omega_h)$ and $\Delta(\varphi_u - \varphi_{h,u_h}) \in L^2(\Omega_h)$, then $\partial_{\nu_h}(\varphi_u - \varphi_{h,u_h}) \in L^2(\Gamma_h)$ and we have

$$\|\partial_{\nu_h}(\varphi_u - \varphi_{h,u_h})\|_{L^2(\Gamma_h)} \leq \|\varphi_u - \varphi_{h,u_h}\|_{H^{3/2}(\Omega_h)} + \|\Delta(\varphi_u - \varphi_{h,u_h})\|_{L^2(\Omega_h)};$$

see [**?**] and [10]. From this inequality, Assumption *(A2)*, estimates (4.2), (4.9), (4.10) and (4.11) we get

$$\|\partial_{\nu_h}(\varphi_u - \varphi_{h,u_h})\|_{L^2(\Gamma_h)} \le C(M)\left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[1 + \|u\|_{H^{1/2}(\Gamma)}]\right)$$

$$+\|\frac{\partial a}{\partial y}(x, w_h)\|_{L^2(\Omega_h)}\|\varphi_u - \varphi_{h,u_h}\|_{L^2(\Omega_h)} + \|\frac{\partial L}{\partial y}(x, y_u) - \frac{\partial L}{\partial y}(x, y_{h,u_h})\|_{L^2(\Omega_h)}$$

$$+\|\frac{\partial a}{\partial y}(x, y_u) - \frac{\partial a}{\partial y}(x, y_{h,u_h})\|_{L^2(\Omega_h)}\|\varphi_{h,u_h}\|_{L^2(\Omega_h)}$$

$$\le C_M h\left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[1 + \|u\|_{H^{1/2}(\Gamma)}]\right).$$

□

We complete this section by proving that the family of problems (P$_h$) realizes a correct approximation of (P). More precisely we prove that the solutions of problems (P$_h$) converge to the solutions of (P). Reciprocally, we also prove that any strict local solution of (P) can be approximated by a sequence of local solutions of problems (P$_h$).

Theorem 4.4. *Let $\bar u_h$ be a solution of problem* (P$_h$)*. Then $\{\bar u_h \circ g_h^{-1}\}_{h>0}$ is a bounded family in $H^{1/2}(\Gamma)$. If $\bar u$ is a weak limit for a subsequence, still denoted in the same way, $\bar u_h \circ g_h^{-1} \rightharpoonup \bar u$ weakly in $H^{1/2}(\Gamma)$ with $h \to 0$, then $\bar u$ is a solution of problem* (P)*. Moreover*

$$\lim_{h\to 0}\|\bar y - \bar y_h\|_{H^{1/2}(\Omega_h)} = 0 \quad \textit{and} \quad \lim_{h\to 0} J_h(\bar u_h) \to J(\bar u),$$

where $\bar y$ and $\bar y_h$ denote the solutions of (2.1) *and* (3.5) *corresponding to $\bar u$ and $\bar u_h$, respectively.*

Proof. First of all we recall definition of norm

$$\|\bar u_h \circ g_h^{-1}\|_{H^{1/2}(\Gamma)} = \left\{\int_\Gamma |\bar u_h(g_h^{-1}(x))|^2\, d\sigma(x)\right.$$

$$\left. + \int_\Gamma\int_\Gamma \frac{|\bar u_h(g_h^{-1}(x)) - \bar u_h(g_h^{-1}(x'))|^2}{|x - x'|^2}\, d\sigma(x)d\sigma(x')\right\}^{1/2}. \tag{4.15}$$

Let us estimate each of two integrals. In Remark 3.2, we establish the boundedness of $\{\|\bar u_h\|_{H^{1/2}(\Omega_h)}\}_{h>0}$. If we prove that

$$\|\bar u_h \circ g_h^{-1}\|_{H^{1/2}(\Gamma)} \le C\|\bar u_h\|_{H^{1/2}(\Gamma_h)},$$

then we obtain that $\{\bar u_h \circ g_h^{-1}\}_{h>0}$ is bounded in $H^{1/2}(\Gamma)$. On the other hand, from (3.4) it follows that

$$\|\bar u_h \circ g_h^{-1}\|^2_{L^2(\Gamma)} = \int_\Gamma |\bar u_h(g_h^{-1}(x))|^2\, d\sigma(x)$$

$$= \int_{\Gamma_h} |\bar u_h(x)|^2 |Dg_h(x)\cdot\tau_h(x)|\, d\sigma_h(x) \le C\|\bar u_h\|^2_{L^2(\Gamma_h)}. \tag{4.16}$$

By the change of variables in the second integral of (4.15), in view of (3.4), we get

$$\int_\Gamma\int_\Gamma \frac{|\bar u_h(g_h^{-1}(x)) - \bar u_h(g_h^{-1}(x'))|^2}{|x-x'|^2}\,d\sigma(x)d\sigma(x')$$

$$\leq C^2\int_{\Gamma_h}\int_{\Gamma_h}\frac{|\bar u_h(x)-\bar u_h(x')|^2}{|g_h(x)-g_h(x')|^2}\,d\sigma_h(x)d\sigma_h(x'). \tag{4.17}$$

Let us show that $|x-x'| \leq |g_h(x)-g_h(x')|$ for every $x, x' \in \Gamma_h$. First, we assume that $x, x' \in [x_j, x_{j+1}]$ for some $1 \leq j \leq N(h)$. Then

$$x = x_j + t\tau_j,\ g_h(x) = x = x + \phi_j(t)\nu_j,\ x' = x_j + t'\tau_j, \ \text{ and }\ g_h(x') = x' + \phi_j(t')\nu_j.$$

Therefore,

$$|g_h(x)-g_h(x')|^2 = |t-t'|^2 + |\phi_j(t)-\phi_j(t')|^2 = |x-x'|^2 + |\phi_j(t)-\phi_j(t')|^2 \geq |x-x'|^2.$$

Now, we assume that $x \in [x_j, x_{j+1}]$ and $x' \in [x_i, x_{i+1}]$, with $i \neq j$. Since Ω is convex, there exist two points $\{\hat x\} = [x_j, x_{j+1}] \cap [g_h(x), g_h(x')]$ and $\{\hat x'\} = [x_i, x_{i+1}] \cap [g_h(x), g_h(x')]$. Moreover we have

$$|g_h(x)-g_h(x')| = |g_h(x)-\hat x| + |\hat x - \hat x'| + |\hat x' - g_h(x')|. \tag{4.18}$$

On the other hand,

$$g_h(x) - \hat x = (g_h(x)-x) + (x-\hat x) = \phi_j(t)\nu_j + (t-\hat t)\tau_j,$$

which implies

$$|g_h(x)-\hat x|^2 = |g_h(x)-x|^2 + |x-\hat x|^2 \Rightarrow |g_h(x)-\hat x| \geq |x-\hat x|.$$

Analogously, we can prove that $|g_h(x')-\hat x'| \geq |x'-\hat x'|$. Finally using (4.18) we obtain

$$|g_h(x)-g_h(x')| \geq |x-\hat x| + |\hat x-\hat x'| + |x'-\hat x'| \geq |x-x'|.$$

Using this inequality in (4.17) we conclude that

$$\int_\Gamma\int_\Gamma \frac{|\bar u_h(g_h^{-1}(x)) - \bar u_h(g_h^{-1}(x'))|^2}{|x-x'|^2}\,d\sigma(x)d\sigma(x')$$

$$\leq C^2\int_{\Gamma_h}\int_{\Gamma_h}\frac{|\bar u_h(x)-\bar u_h(x')|^2}{|x-x'|^2}\,d\sigma_h(x)d\sigma_h(x'). \tag{4.19}$$

From (4.15), (4.16) and (4.19) it follows

$$\|\bar u_h \circ g_h^{-1}\|_{H^{1/2}(\Gamma)} \leq C'\|\bar u_h\|_{H^{1/2}(\Gamma_h)} \leq C''.$$

Therefore, there exists a subsequence and an element $\bar u \in H^{1/2}(\Gamma)$ such that $\bar u_h \circ g_h^{-1} \rightharpoonup \bar u$ weakly in $H^{1/2}(\Gamma)$ with $h \to 0$. Since the embedding $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ is compact, we have $\bar u_h \circ g_h^{-1} \to \bar u$ strongly in $L^2(\Gamma)$. It is obvious that $\alpha \leq \bar u \leq \beta$. Now, if we denote by $\bar{y}_h$ the states associated to $\bar{u}_h$ and by $\bar{y}$ the state associated to $\bar{u}$, we deduce from (4.2) that

$$\lim_{h\to 0}\|\bar{y}-\bar{y}_h\|_{H^{1/2}(\Omega_h)}=0 \quad \text{and} \quad \exists C_{\alpha\beta}>0 \text{ such that } \|\bar{y}_h\|_{L^\infty(\Omega_h)}\le C_{\alpha\beta}\ \forall h.$$

Hence, it is easy to prove that $J_h(\bar{u}_h)\to J(\bar{u})$. It remains to prove that $\bar{u}$ is a solution of (P). Let us take any feasible control u for (P), then $u\circ g_h$ is also feasible for (P$_h$). Therefore, since $\bar{u}_h$ is a solution of (P$_h$), we obtain

$$J(u)=\lim_{h\to 0}J_h(u\circ g_h)\ge\lim_{h\to 0}J_h(\bar{u}_h)=J(\bar{u}),$$

which completes the proof. □

THEOREM 4.5. *Let* $\bar{u}$ *be a strict local minimum of* (P)*, then there exists a family* $\{\bar{u}_h\}$ *such that each control* $\bar{u}_h$ *is a local minimum of* (P$_h$) *and* $\bar{u}_h\circ g_h^{-1}\rightharpoonup\bar{u}$ *converges weakly in* $H^{1/2}(\Gamma)$.

Proof. Let $\varepsilon>0$ be such that $\bar{u}$ is the unique global solution of problem

$$(\mathrm{P}_\varepsilon)\begin{cases}\min J(u)=\displaystyle\int_\Omega L(x,y_u(x))\,dx+\frac{N}{2}\int_\Gamma u^2(x)\,d\sigma(x)\\ \text{subject to}\ \ (y_u,u)\in(L^\infty(\Omega)\cap H^{1/2}(\Omega))\times L^2(\Gamma),\\ \alpha\le u(x)\le\beta\quad\text{for a.e.}\ \ x\in\Gamma\ \ \text{and}\ \ \|u-\bar{u}\|_{L^2(\Gamma)}\le\varepsilon.\end{cases}$$

Now, for every h we consider the problems

$$(\mathrm{P}_{h\varepsilon})\begin{cases}\min J_h(u)=\displaystyle\int_{\Omega_h} L(x,y_{h,u}(x))\,dx+\frac{N}{2}\int_{\Gamma_h} u^2(x)\,d\sigma_h(x)\\ \text{subject to}\ \ (y_{h,u},u)\in(L^\infty(\Omega_h)\cap H^{1/2}(\Omega_h))\times L^2(\Gamma_h),\\ \alpha\le u(x)\le\beta\quad\text{for a.e.}\ \ x\in\Gamma_h\ \ \text{and}\ \ \|u\circ g_h^{-1}-\bar{u}\|_{L^2(\Gamma)}\le\varepsilon\end{cases}$$

It is obvious that $\bar{u}\circ g_h$ is a feasible control for each problem (P$_{h\varepsilon}$), therefore there exists at least one solution $u_{h\varepsilon}$ of (P$_{h\varepsilon}$). Let us show that $u_{h\varepsilon}\circ g_h^{-1}\rightharpoonup\bar{u}$ weakly in $H^{1/2}(\Gamma)$ with $h\to 0$.

Since $\{u_{h\varepsilon}\circ g_h^{-1}\}_{h>0}$ is bounded in $L^\infty(\Gamma)$, we can extract a subsequence, still denoted by the same symbol, and an element $\tilde{u}\in L^\infty(\Gamma)$ such that $u_{h\varepsilon}\circ g_h^{-1}\rightharpoonup\tilde{u}$ *weakly in $L^\infty(\Gamma)$ with $h\to 0$. Let us denote by $y_{h\varepsilon}\in H^{1/2}(\Omega_h)\cap L^\infty(\Omega_h)$ the state associated to $u_{h\varepsilon}$ and consider an extension of $y_{h\varepsilon}$ to Ω, still denoted by $y_{h\varepsilon}$, such that

$$\|y_{h\varepsilon}\|_{H^{1/2}(\Omega)}\le C\|y_{h\varepsilon}\|_{H^{1/2}(\Omega_h)}\ \ \text{and}\ \ \|y_{h\varepsilon}\|_{L^\infty(\Omega)}\le C\|y_{h\varepsilon}\|_{L^\infty(\Omega_h)}\ \forall h.$$

The boundedness of $\{u_{h\varepsilon}\circ g_h^{-1}\}$ in $L^\infty(\Gamma)$ implies that of $\{y_{h\varepsilon}\}$ in $H^{1/2}(\Omega)$. Therefore, by taking a subsequence, we can assume that

$$y_{h\varepsilon}\rightharpoonup\tilde{y}\ \text{ in } H^{1/2}(\Omega)\ \text{ and }\ u_{h\varepsilon}\circ g_h^{-1}\rightharpoonup\tilde{u}\ \text{ in } L^2(\Gamma).$$

We are going to prove that $\tilde{y}$ is the state associated to $\tilde{u}$. According to the definition given in §2, we have to prove that the following identity holds

$$\int_\Omega -\tilde{y}\Delta w\,dx+\int_\Omega a(x,\tilde{y})w\,dx=\int_\Gamma\tilde{u}\partial_\nu w\,d\sigma\ \forall w\in H^2(\Omega)\cap H_0^1(\Omega). \tag{4.20}$$

For a given $w \in H^2(\Omega) \cap H^1_0(\Omega)$ we take $w_h \in H^2(\Omega_h) \cap H^1_0(\Omega_h)$, a unique solution of the Dirichlet problem

$$\left\{ \begin{array}{rcll} -\Delta w_h & = & -\Delta w & \text{in } \Omega_h, \\ w_h & = & 0 & \text{on } \Gamma_h. \end{array} \right. \tag{4.21}$$

As in the proof of Theorem 4.2 we have

$$\|w - w_h\|_{H^{3/2}(\Omega_h)} \le C\|w\|_{H^1(\Omega_h)} \le Ch. \tag{4.22}$$

Hence

$$\|\partial_{\nu_h} w - \partial_{\nu_h} w_h\|_{L^2(\Gamma_h)} \le C\left\{\|\Delta(w - w_h)\|_{L^2(\Omega_h)} + \|w - w_h\|_{H^{3/2}(\Omega_h)}\right\}$$

$$= C\|w - w_h\|_{H^{3/2}(\Omega_h)} \le Ch. \tag{4.23}$$

Since $y_{h\varepsilon}$ is the state associated to $u_{h\varepsilon}$ we have

$$\int_{\Omega_h} -y_{h\varepsilon}\Delta w_h\, dx + \int_{\Omega_h} a(x, y_{h\varepsilon}) w_h\, dx = \int_{\Gamma_h} u_{h\varepsilon}\partial_{\nu_h} w_h\, d\sigma_h.$$

In view of (4.21), this identity can be rewritten as follows

$$\int_{\Omega_h} -\Delta w y_{h\varepsilon}\, dx + \int_{\Omega_h} a(x, y_{h\varepsilon}) w_h\, dx = \int_{\Gamma_h} u_{h\varepsilon}\partial_{\nu_h} w_h\, d\sigma_h. \tag{4.24}$$

Now we want to pass to the limit with $h \to 0$ in (4.24). Using the compactness of the imbedding $H^{1/2}(\Omega) \subset L^2(\Omega)$ it is easy to pass to the limit in the first two integrals, which are also the first two integrals of (4.20). Let us consider the right-hand side term of (4.24). Applying (4.23) we get

$$\int_{\Gamma_h} u_{h\varepsilon}\partial_{\nu_h} w_h\, d\sigma_h = \int_{\Gamma_h} u_{h\varepsilon}\partial_{\nu_h} w\, d\sigma_h + O(h). \tag{4.25}$$

Now from Lemma 4.6 below we deduce

$$\int_{\Gamma_h} u_{h\varepsilon}\partial_{\nu_h} w\, d\sigma_h = \int_{\Gamma} (u_{h\varepsilon} \circ g_h^{-1})\partial_\nu w\, d\sigma + O(h). \tag{4.26}$$

Finally, combining (4.25) and (4.26) we get

$$\lim_{h\to 0} \int_{\Gamma_h} u_{h\varepsilon}(x)\partial_{\nu_h} w_h\, d\sigma_h = \int_{\Gamma} \tilde{u}(x)\partial_\nu w(x)\, d\sigma.$$

Thus, we show that (4.20) follows from (4.24) by the limit passage.

Now, using that $u_{h\varepsilon} \circ g_h^{-1} \rightharpoonup \tilde{u}$ weakly in $L^2(\Gamma)$, $y_{h\varepsilon} \to \tilde{y}$ strongly in $L^2(\Omega)$, $\{y_{h\varepsilon}\}_{h>0}$ is bounded in $L^\infty(\Omega)$ and the fact that $u_{h\varepsilon}$ is a solution of $(\mathrm{P}_{h\varepsilon})$ and $\bar{u} \circ g_h^{-1}$ is feasible for problems $(\mathrm{P}_{h\varepsilon})$ we obtain

$$J(\tilde{u}) \le \liminf_{h\to 0} J_h(u_{h\varepsilon}) \le \liminf_{h\to 0} J_h(\bar{u} \circ g_h^{-1}) \le \limsup_{h\to 0} J_h(\bar{u} \circ g_h^{-1}) = J(\bar{u}).$$

Since $\bar{u}$ is the unique solution of (P_ε), the above inequality leads to $\tilde{u} = \bar{u}$ and $J_h(u_{h\varepsilon}) \to J(\bar{u})$, which implies

$$\lim_{h\to 0} \int_{\Gamma_h} u^2_{h\varepsilon}(x)\, d\sigma_h(x) = \int_{\Gamma} \bar{u}^2(x)\, d\sigma(x).$$

Using (3.3)

$$\lim_{h\to 0}\int_\Gamma (u_{h\varepsilon}\circ g_h^{-1})^2(x)\,d\sigma(x) = \int_\Gamma \bar u^2(x)\,d\sigma(x).$$

This identity and the weak convergence imply the strong convergence $u_{h\varepsilon}\circ g_h^{-1}\to\bar u$ in $L^2(\Gamma)$. First consequence of this strong convergence is that the constraint $\|u\circ g_h^{-1}-\bar u\|_{L^2(\Gamma)}\le\varepsilon$ is not active at the controls $u_{h\varepsilon}$ for h small enough. Therefore, $u_{h\varepsilon}$ is a local minimum of problem (P$_h$) for every h small enough. Since $\{\|u_{h\varepsilon}\|_{L^2(\Gamma_h)}\}$ is bounded, then we can argue as in the proof of Theorem 4.4 and conclude that $\{u_{h\varepsilon}\circ g_h^{-1}\}$ is bounded in $H^{1/2}(\Gamma)$ and hence $u_{h\varepsilon}\circ g_h^{-1}\rightharpoonup\bar u$ weakly in $H^{1/2}(\Gamma)$ with $h\to 0$. ◻

Lemma 4.6. *Let $w\in H^2(\Omega)$ and $v\in L^2(\Gamma)$, then there exists a constant $C>0$ independent of w and v such that*

$$\left|\int_\Gamma \partial_\nu w v\,d\sigma - \int_{\Gamma_h}\partial_{\nu_h} w(v\circ g_h)\,d\sigma_h\right| \le Ch\|w\|_{H^2(\Omega)}\|v\|_{L^2(\Gamma)}. \tag{4.27}$$

Proof. First, we observe that (3.3) implies that

$$\left|\int_\Gamma[\nabla w(x)\cdot\nu(x)]v(x)\,d\sigma(x) - \int_{\Gamma_h}[\nabla w(g_h(x))\cdot\nu(g_h(x))]v(g_h(x))\,d\sigma_h(x)\right|$$

$$\le C_\Gamma h^2\int_\Gamma|\nabla w(x)\cdot\nu(x)||v(x)|\,d\sigma(x)$$

$$\le C_\Gamma h^2\|\partial_\nu w\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)} \le Ch^2\|w\|_{H^2(\Omega)}\|v\|_{L^2(\Gamma)}. \tag{4.28}$$

On the other hand,

$$\int_{\Gamma_h}[\nabla w(x)\cdot\nu_h(x)]v(g_h(x))\;d\sigma_h(x) - \int_{\Gamma_h}[\nabla w(g_h(x))\cdot\nu(g_h(x))]v(g_h(x))\,d\sigma_h(x)$$

$$= \int_{\Gamma_h}[\nabla w(x)\cdot(\nu_h(x)-\nu(g_h(x)))]v(g_h(x))\,d\sigma_h(x)$$

$$+\int_{\Gamma_h}[\nabla w(x)-\nabla w(g_h(x))]\cdot\nu(g_h(x))v(g_h(x))\,d\sigma_h(x).$$

From this identity we get, in view of (3.1), (3.2) and (4.6),

$$\left|\int_{\Gamma_h}[\nabla w(x)\cdot\nu_h(x)]v(g_h(x))\;d\sigma_h(x) - \int_{\Gamma_h}[\nabla w(g_h(x))\cdot\nu(g_h(x))]v(g_h(x))\,d\sigma_h(x)\right|$$

$$\le Ch\|w\|_{H^2(\Omega)}\|v\|_{L^2(\Gamma)}. \tag{4.29}$$

Now, (4.28) and (4.29) imply (4.27). ◻

5. Error Estimates.

In this section we assume that $\bar u_h$ is a local minimum of (P_h) such that $\bar u_h \circ g_h^{-1}$ converges weakly in $H^{1/2}(\Gamma)$ to a local minimum $\bar u$ of (P) with $h \to 0$; see Theorems 4.4 and 4.5. The goal of this section is to derive an estimate for $\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)}$, which is established in the following theorem.

Theorem 5.1. *Let $\bar u$ and $\bar u_h$ be as above and let us denote by $\bar y$, $\bar y_h$ and $\bar\varphi$, $\bar\varphi_h$ the states and adjoint states associated to $\bar u$ and $\bar u_h$ respectively. Let us assume that the second order sufficient optimality condition (2.14) is fulfilled for $\bar u$. Then there exists a constant C, independent of h such that the following estimates hold*

$$\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + \|\bar y - \bar y_h\|_{H^{1/2}(\Omega_h)} + \|\bar\varphi - \bar\varphi_h\|_{H^{3/2}(\Omega_h)} \le Ch. \tag{5.1}$$

Before proving this theorem we provide a preliminary result. The proof of Lemma 5.2 is inspired by [5, Lemma 7.2], however there are some important differences.

Lemma 5.2. *Let $\mu > 0$ be taken from Theorem 2.4. Then there exists $h_0 > 0$ such that*

$$\frac{1}{2}\min\{N,\mu\}\|\bar u_h \circ g_h^{-1} - \bar u\|^2_{L^2(\Gamma)} \le (J'(\bar u_h \circ g_h^{-1}) - J'(\bar u))(\bar u_h \circ g_h^{-1} - \bar u). \tag{5.2}$$

Proof. By applying the mean value theorem there is an intermediate element $\hat u_h = \bar u + \theta_h(\bar u_h \circ g_h^{-1} - \bar u)$ such that

$$(J'(\bar u_h \circ g_h^{-1}) - J'(\bar u))(\bar u_h \circ g_h^{-1} - \bar u) = J''(\hat u_h)(\bar u_h \circ g_h^{-1} - \bar u)^2. \tag{5.3}$$

Let us take

$$v_h = \frac{1}{\|\bar u_h \circ g_h^{-1} - \bar u\|_{L^2(\Gamma)}}(\bar u_h \circ g_h^{-1} - \bar u).$$

Taking a subsequence, if necessary, we can assume that $v_h \rightharpoonup v$ weakly in $L^2(\Gamma)$. We show that v belongs to the critical cone $C_{\bar u}$ defined in §2. First of all, observe that v satisfies the sign condition (2.13) since every element v_h satisfies the same condition. Let us prove that $v(x) = 0$ if $N\bar u(x) - \partial_\nu\bar\varphi(x) \neq 0$. To this end it is enough to establish the limit passage

$$\lim_{h\to 0}\int_{\Gamma_h}(N\bar u_h - \partial_{\nu_h}\bar\varphi_h)(v_h \circ g_h)\,d\sigma_h = \int_\Gamma (N\bar u - \partial_\nu\bar\varphi)v\,d\sigma. \tag{5.4}$$

Indeed, from (5.4) we deduce, in view of (3.8), that

$$\int_\Gamma |N\bar u - \partial_\nu\bar\varphi||v|\,d\sigma = \int_\Gamma (N\bar u - \partial_\nu\bar\varphi)v\,d\sigma$$

$$= \lim_{h\to 0}\frac{1}{\|\bar u_h \circ g_h^{-1} - \bar u\|_{L^2(\Gamma)}}\int_{\Gamma_h}(N\bar u_h - \partial_{\nu_h}\bar\varphi_h)(\bar u_h - \bar u \circ g_h)\,d\sigma_h \le 0,$$

which proves the required property. Let us show (5.4). By the strong convergence $\bar u_h \circ g_h^{-1} \to \bar u$ in $L^2(\Gamma)$ combined with (4.14) and (3.2), we have

$$\left|\int_{\Gamma_h}(N\bar u_h - \partial_{\nu_h}\bar\varphi_h)(v_h \circ g_h)\,d\sigma_h - \int_{\Gamma_h}(N\bar u_h - \partial_{\nu_h}\bar\varphi)(v_h \circ g_h)\,d\sigma_h\right|$$

$$\le \|\partial_{\nu_h}\bar\varphi_h - \partial_{\nu_h}\bar\varphi\|_{L^2(\Gamma_h)}\|v_h\circ g_h\|_{L^2(\Gamma_h)}$$

$$\le C_M\left(\|\bar u_h\circ g_h^{-1}-\bar u\|_{L^2(\Gamma)} + h[1+\|u\|_{H^{1/2}(\Gamma)}]\right)\|v_h\|_{L^2(\Gamma)}$$

$$= C_M\left(\|\bar u_h\circ g_h^{-1}-\bar u\|_{L^2(\Gamma)} + h[1+\|u\|_{H^{1/2}(\Gamma)}]\right) \to 0 \ \text{ with } h\to 0. \tag{5.5}$$

On the other hand, from Lemma 4.6 we get

$$\int_{\Gamma_h}\partial_{\nu_h}\bar\varphi(v_h\circ g_h)\,d\sigma_h = \int_\Gamma \partial_\nu\bar\varphi v_h\,d\sigma + O(h) \to \int_\Gamma \partial_\nu\bar\varphi v\,d\sigma \ \text{ with } h\to 0. \tag{5.6}$$

Finally, from (3.3) we obtain

$$\int_{\Gamma_h}\bar u_h(v_h\circ g_h)\,d\sigma_h = \int_\Gamma(\bar u_h\circ g_h^{-1})v_h\,d\sigma + O(h) \to \int_\Gamma \bar u v\,d\sigma \ \text{ with } h\to 0. \tag{5.7}$$

Thus, (5.4) follows from (5.5), (5.6) and (5.7).

Now by the definition of v_h and (2.8), (2.27), we get

$$\lim_{h\to 0} J''(\hat u_h)v_h^2 = \lim_{h\to 0}\left\{\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,y_{\hat u_h}) - \varphi_{\hat u_h}\frac{\partial^2 a}{\partial y^2}(x,y_{\hat u_h})\right] z_{v_h}^2\,dx + N\right\}$$

$$= \int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,\bar y) - \bar\varphi\frac{\partial^2 a}{\partial y^2}(x,\bar y)\right] z_v^2\,dx + N$$

$$= J''(\bar u)v^2 + N(1-\|v\|^2_{L^2(\Gamma)}) \ge N + (\mu - N)\|v\|^2_{L^2(\Gamma)}.$$

Taking into account that $\|v\|_{L^2(\Gamma)}\le 1$, the above inequality leads to

$$\lim_{h\to 0} J''(\hat u_h)v_h^2 \ge \min\{\mu, N\} > 0,$$

which proves the existence of $h_0 > 0$ such that

$$J''(\hat u_h)v_h^2 \ge \frac{1}{2}\min\{\mu, N\} \quad \forall h < h_0.$$

From this inequality, by the definition of v_h and (5.3), we deduce (5.2), which completes the proof. □

Proof of Theorem 5.1. By taking $v = \bar u_h\circ g_h^{-1}$ in (2.12) and $v_h = \bar u\circ g_h$ in (3.8) we get

$$J'(\bar u)(\bar u_h\circ g_h^{-1} - \bar u) = \int_\Gamma (N\bar u - \partial_\nu\bar\varphi)(\bar u_h\circ g_h^{-1} - \bar u)\,d\sigma \ge 0 \tag{5.8}$$

and

$$J_h'(\bar u_h)(\bar u\circ g_h - \bar u_h) = \int_{\Gamma_h}(N\bar u_h - \partial_{\nu_h}\bar\varphi_h)(\bar u\circ g_h - \bar u_h)\,d\sigma_h \ge 0. \tag{5.9}$$

We rewrite inequality (5.9) as follows

$$J'(\bar u_h\circ g_h^{-1})(\bar u - \bar u_h\circ g_h^{-1}) + [J_h'(\bar u_h)(\bar u\circ g_h - \bar u_h) - J'(\bar u_h\circ g_h^{-1})(\bar u - \bar u_h\circ g_h^{-1})] \ge 0. \tag{5.10}$$

From (5.8) and (5.10) we obtain

$$[J'(\bar u_h\circ g_n^{-1})-J'(\bar u)](\bar u_h\circ g_h^{-1}-\bar u)\le J_h'(\bar u_h)(\bar u\circ g_h-\bar u_h)-J'(\bar u_h\circ g_h^{-1})(\bar u-\bar u_h\circ g_h^{-1}).$$

Now, from (5.2) we deduce

$$\frac{1}{2}\min\{N,\mu\}\|\bar u_h\circ g_h^{-1}-\bar u\|^2_{L^2(\Gamma)}\le J_h'(\bar u_h)(\bar u\circ g_h-\bar u_h)-J'(\bar u_h\circ g_h^{-1})(\bar u-\bar u_h\circ g_h^{-1}). \tag{5.11}$$

It remains to derive an estimate for the right-hand side of (5.11). To this end, we introduce $y\in H^1(\Omega)\cap L^\infty(\Omega)$ and $\varphi\in H^2(\Omega)$ as the solutions of the equations

$$\left\{\begin{array}{rcll} -\Delta y+a(x,y) & = & 0 & \text{in } \Omega,\\ y & = & \bar u_h\circ g_h^{-1} & \text{on } \Gamma,\end{array}\right. \tag{5.12}$$

and

$$\left\{\begin{array}{rcll} -\Delta\varphi+\dfrac{\partial a}{\partial y}(x,y)\varphi & = & \dfrac{\partial L}{\partial y}(x,y) & \text{in } \Omega,\\ \varphi & = & 0 & \text{on } \Gamma.\end{array}\right. \tag{5.13}$$

Then we have

$$J'(\bar u_h\circ g_h^{-1})(\bar u-\bar u_h\circ g_h^{-1})=\int_\Gamma(N\bar u_h\circ g_h^{-1}-\partial_\nu\varphi)(\bar u-\bar u_h\circ g_h^{-1})d\sigma. \tag{5.14}$$

From (5.9) we get

$$\begin{aligned}&|J_h'(\bar u_h)(\bar u\circ g_h-\bar u_h)-J'(\bar u_h\circ g_h^{-1})(\bar u-\bar u_h\circ g_h^{-1})|\\ &=\left|\int_{\Gamma_h}(N\bar u_h-\partial_{\nu_h}\bar\varphi_h)(\bar u\circ g_h-\bar u_h)\,d\sigma_h-\int_\Gamma(N\bar u_h\circ g_h^{-1}-\partial_\nu\varphi)(\bar u-\bar u_h\circ g_h^{-1})d\sigma\right|.\end{aligned} \tag{5.15}$$

Using (3.3) we obtain

$$\begin{aligned}&\left|\int_{\Gamma_h}\bar u_h(\bar u\circ g_h-\bar u_h)\,d\sigma_h-\int_\Gamma\bar u_h\circ g_h^{-1}(\bar u-\bar u_h\circ g_h^{-1})\,d\sigma\right|\\ &\le Ch^2\int_\Gamma|\bar u_h\circ g_h^{-1}||\bar u-\bar u_h\circ g_h^{-1}|\,d\sigma\\ &\le Ch^2\|\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}\le Ch^2\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}.\end{aligned} \tag{5.16}$$

On the other hand, from (4.14), Lemma 4.6 and (3.3) we get

$$\begin{aligned}&\left|\int_{\Gamma_h}\partial_{\nu_h}\bar\varphi_h(\bar u\circ g_h-\bar u_h)\,d\sigma_h-\int_\Gamma\partial_\nu\varphi(\bar u-\bar u_h\circ g_h^{-1})d\sigma\right|\\ &\le\|\partial_{\nu_h}\bar\varphi_h-\partial_{\nu_h}\varphi\|_{L^2(\Gamma_h)}\|\bar u\circ g_h^{-1}-\bar u_h\|_{L^2(\Gamma_h)}\\ &+\left|\int_{\Gamma_h}\partial_{\nu_h}\varphi(\bar u\circ g_h-\bar u_h)\,d\sigma_h-\int_\Gamma\partial_\nu\varphi(\bar u-\bar u_h\circ g_h^{-1})d\sigma\right|\end{aligned}$$

$$\leq Ch\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)}. \tag{5.17}$$

Thus, from (5.11), (5.15), (5.16) and (5.17) we conclude

$$\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)} \leq Ch.$$

The remaining estimates of (5.1) follow from the above estimate, (4.2) and (4.9). ☐

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