

# Viability Kernel for Ecosystem Management Models

Eladio OCAÑA ANAYA \*      Michel DE LARA<sup>†</sup>

Ricardo OLIVEROS-RAMOS<sup>‡</sup>      Jorge TAM<sup>‡</sup>

June 18, 2009

## Abstract

We consider sustainable management issues formulated within the framework of control theory. The problem is one of controlling a discrete-time dynamical system (*e. g.* population model) in the presence of state and control constraints, representing conflicting economic and ecological issues for instance. The viability kernel is known to play a basic role for the analysis of such problems and the design of viable control feedbacks, but its computation is not an easy task in general. We study the viability of nonlinear generic ecosystem models under preservation and production constraints. Under simple conditions on the growth rates at the boundary constraints, we provide an explicit description of the viability kernel. A numerical illustration is given for the hake-anchovy couple in the Peruvian upwelling ecosystem.

**Key words:** control theory; state constraints; viability; predator-prey; ecosystem management; Peruvian upwelling ecosystem.

---

\*IMCA, Instituto de Matemática y Ciencias Afines, Universidad Nacional de Ingeniería, Calle los Biólogos 245, La Molina, Lima 12, Perú. eocana@imca.edu.pe, fax +511349-9838

<sup>†</sup>Université Paris-Est, Cermics, 6-8 avenue Blaise Pascal, 77455 Marne la Vallée Cedex 2, France. Corresponding author: delara@cermics.enpc.fr, fax +33164153586

<sup>‡</sup>IMARPE, Instituto del Mar del Perú, Centro de Investigaciones en Modelado Oceanográfico y Biológico Pesquero (CIMOBP), Apartado 22, Callao-Perú. jtam@imarpe.gob.pe, roliveros@imarpe.gob.pe, fax +5114535053

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Discrete-time viability</b>	<b>4</b>
<b>3</b>	<b>Viable control of generic nonlinear ecosystem models</b>	<b>7</b>
3.1	Dynamics and constraints for an ecosystem management model . . . . .	7
3.2	Expression of the viability kernel . . . . .	8
3.3	Viability kernel for a nonlinear predator-prey system . . . . .	11
3.4	Viability kernel for a Lotka-Volterra system . . . . .	12
<b>4</b>	<b>Numerical application to the hake-anchovy couple in the Peruvian upwelling ecosystem</b>	<b>13</b>
<b>5</b>	<b>Conclusion</b>	<b>14</b>

## 1 Introduction

This paper deals with the control of discrete-time dynamical systems of the form  $x(t + 1) = f(x(t), u(t))$ ,  $t \in \mathbb{N}$ , with state  $x(t) \in \mathbb{X}$  and control  $u(t) \in \mathbb{U}$ , in the presence of state and control constraints  $(x(t), u(t)) \in \mathbb{D}$ . The subset  $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$  describes “acceptable configurations of the system”. Such problems of dynamic control under constraints refer to viability (Aubin, 1991) or invariance (Clarke, Ledayev, Stern, and Wolenski, 1995) frameworks. From the mathematical viewpoint, most of viability and weak invariance results are addressed in the continuous time case. However, some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of the continuous dynamics as in (Aubin, 1991; Saint-Pierre, 1994;

Quincampoix and Saint-Pierre, 1995). In the control theory literature, problems of constrained control have also been addressed in the discrete time case (see the survey paper (Blanchini, 1999)); reachability of target sets or tubes for nonlinear discrete time dynamics is examined in (Bertsekas and Rhodes, 1971).

We consider sustainable management issues which can be formulated within such a framework as in (Béné and Doyen, 2000, 2003; Béné, Doyen, and Gabay, 2001; Eisenack, Sheffran, and Kropp, 2006; Martinet and Doyen, 2007; Mullon, Cury, and Shannon, 2004; Rapaport, Terreaux, and Doyen, 2006; De Lara, Doyen, Guilbaud, and Rochet, 2007; De Lara and Doyen, 2008).

The time index  $t$  is an integer and the time period  $[t, t + 1[$  may be a year, a month, etc. The dynamics is generally a population dynamics, with state vector  $x(t)$  being either the biomass of a single species, or a couple of biomasses for a predator–prey system, or a vector of abundances at ages for one or for several species, or abundances at different spatial patches, etc. The control  $u(t)$  may represent catches, harvesting mortality or harvesting effort. The set  $\mathbb{D}$  may include biological, ecological and economic objectives as in (Béné, Doyen, and Gabay, 2001). For instance, if the state  $x$  is a vector of abundances at ages and the control  $u$  is a harvesting effort,  $\mathbb{D} = \{(x, u) \mid B(x) \geq b^b, E(x, u) \geq e^b\}$  represents acceptable or acceptable configurations where conservation is ensured by a biological indicator  $B(x) \geq b^b$  (spawning stock biomass above a reference point, for instance) and economics is taken into account via minimal catches  $E(x, u) \geq e^b$  (catches  $E(x, u)$  above a threshold).

The viability kernel  $\mathbb{V}(f, \mathbb{D})$  associated with the dynamics  $f$  and the acceptable set  $\mathbb{D}$  is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. Unfortunately, its computation is not an easy task in general. In this paper, we focus on the case where the viability kernel is given by a finite algorithm, with emphasis on population dynamics ecosystem models. For nonlinear predator–prey systems, one can find descriptions of viability kernels for Lotka–Volterra systems in (Bonneuil and Müllers, 1997) and population viability analysis in three trophic-level food chains in (Bonneuil and Saint-Pierre, 2005). In these latter references, the control strategies are carried by

the species (“fitness-maximizing” strategies), the constraint set corresponds to having high enough densities, and the models are in continuous time. In our approach, the controls are the harvesting efforts, the constraint set includes both high abundances and high catches, and the model is in discrete time.

The paper is organized as follows. Section 2 is devoted to recalls on discrete–time viability and its possible use for sustainable management. For generic nonlinear ecosystem models, we provide an explicit description of the viability kernel for preservation and production constraints in Section 3, together with viable controls. An illustration in ecosystem management and numerical applications are given for the hake–anchovy couple in the Peruvian upwelling ecosystem in Section 4.

## 2 Discrete–time viability

Let us consider a nonlinear control system described in discrete–time by the difference equation

$$\begin{cases} x(t+1) = f(x(t), u(t)) \text{ for all } t \in \mathbb{N}, \\ x(0) = x_0 \text{ given,} \end{cases} \quad (1)$$

where the *state variable*  $x(t)$  belongs to the finite dimensional state space  $\mathbb{X} = \mathbb{R}^{n_x}$ , the *control variable*  $u(t)$  is an element of the *control set*  $\mathbb{U} = \mathbb{R}^{n_u}$  while the *dynamics*  $f$  maps  $\mathbb{X} \times \mathbb{U}$  into  $\mathbb{X}$ .

A controller or a decision maker describes “acceptable configurations of the system” through a set  $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$  termed the *acceptable set*

$$(x(t), u(t)) \in \mathbb{D} \text{ for all } t \in \mathbb{N}, \quad (2)$$

where  $\mathbb{D}$  includes both system states and controls constraints.

The *state constraints set*  $\mathbb{V}^0$  associated with  $\mathbb{D}$  is obtained by projecting the acceptable set  $\mathbb{D}$  onto the state space  $\mathbb{X}$ :

$$\mathbb{V}^0 := \text{Proj}_{\mathbb{X}}(\mathbb{D}) = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}, (x, u) \in \mathbb{D}\}. \quad (3)$$

Viability is defined as the ability to choose, at each time step  $t \in \mathbb{N}$ , a control  $u(t) \in \mathbb{U}$  such that the system configuration remains acceptable. More precisely, the system is viable if the following feasible set is not empty:

$$\mathbb{V}(f, \mathbb{D}) := \left\{ x_0 \in \mathbb{X} \left| \begin{array}{l} \exists (u(0), u(1), \dots) \text{ and } (x(0), x(1), \dots) \\ \text{satisfying (1) and (2)} \end{array} \right. \right\}. \quad (4)$$

The set  $\mathbb{V}(f, \mathbb{D})$  is called the *viability kernel* (Aubin, 1991) associated with the dynamics  $f$  and the acceptable set  $\mathbb{D}$ . By definition, we have  $\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}^0 = \text{Proj}_{\mathbb{X}}(\mathbb{D})$  but, in general, the inclusion is strict. For a decision maker or control designer, knowing the viability kernel has practical interest since it describes the states from which controls can be found that maintain the system in an acceptable configuration forever. However, computing this kernel is not an easy task in general.

We now focus on the tools to achieve viability. A subset  $\mathbb{V}$  is said to be *weakly invariant* for the dynamics  $f$  in the acceptable set  $\mathbb{D}$ , or a *viability domain* of  $f$  in  $\mathbb{D}$ , if

$$\forall x \in \mathbb{V}, \quad \exists u \in \mathbb{U}, \quad (x, u) \in \mathbb{D} \text{ and } f(x, u) \in \mathbb{V}. \quad (5)$$

That is, if one starts from  $\mathbb{V}$ , an acceptable control may transfer the state in  $\mathbb{V}$ .

Moreover, according to viability theory (Aubin, 1991), the viability kernel  $\mathbb{V}(f, \mathbb{D})$  turns out to be the union of all viability domains, or also the largest viability domain:

$$\mathbb{V}(f, \mathbb{D}) = \bigcup \left\{ \mathbb{V}, \mathbb{V} \subset \mathbb{V}^0, \mathbb{V} \text{ viability domain for } f \text{ in } \mathbb{D} \right\}. \quad (6)$$

A major interest of such a property lies in the fact that any viability domain for the dynamics  $f$  in the acceptable set  $\mathbb{D}$  provides a *lower approximation* of the viability kernel. An *upper approximation*  $\mathbb{V}_k$  of the viability kernel is given by the so called *viability kernel until time  $k$  associated with  $f$  in  $\mathbb{D}$* :

$$\mathbb{V}_k := \left\{ x_0 \in \mathbb{X} \left| \begin{array}{l} \exists (u(0), u(1), \dots, u(k)) \text{ and } (x(0), x(1), \dots, x(k)) \\ \text{satisfying (1) for } t = 0, \dots, k-1 \\ \text{and (2) for } t = 0, \dots, k \end{array} \right. \right\}. \quad (7)$$

We have

$$\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}_{k+1} \subset \mathbb{V}_k \subset \mathbb{V}_0 = \mathbb{V}^0 \text{ for all } k \in \mathbb{N} . \quad (8)$$

It may be seen by induction that the decreasing sequence of viability kernels until time  $k$  satisfies

$$\mathbb{V}_0 = \mathbb{V}^0 \text{ and } \mathbb{V}_{k+1} = \{x \in \mathbb{V}_k \mid \exists u \in \mathbb{U}, (x, u) \in \mathbb{D} \text{ and } f(x, u) \in \mathbb{V}_k\} . \quad (9)$$

By (8), such an algorithm provides approximation from above of the viability kernel as follows:

$$\mathbb{V}(f, \mathbb{D}) \subset \bigcap_{k \in \mathbb{N}} \mathbb{V}_k = \lim_{k \rightarrow +\infty} \downarrow \mathbb{V}_k . \quad (10)$$

Conditions ensuring that equality holds may be found in (Saint-Pierre, 1994). Notice that, when the decreasing sequence  $(\mathbb{V}_k)_{k \in \mathbb{N}}$  of viability kernels up to time  $k$  is stationary, its limit is the viability kernel. Indeed, if  $\mathbb{V}_k = \mathbb{V}_{k+1}$  for some  $k$ , then  $\mathbb{V}_k$  is a viability domain by (9). Now, by (4),  $\mathbb{V}(f, \mathbb{D})$  is the largest of viability domains. As a consequence,  $\mathbb{V}_k = \mathbb{V}(f, \mathbb{D})$  since  $\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}_k$  by (8). We shall use this property in the following Sect. 3.

Once the viability kernel, or any approximation, or a viability domain is known, we have to consider the management or control issue, that is the problem of selecting suitable controls at each time step. For any viability domain  $\mathbb{V}$  and any state  $x \in \mathbb{V}$ , the following subset  $\mathbb{U}_{\mathbb{V}}(x)$  of the decision set  $\mathbb{U}$  is not empty:

$$\mathbb{U}_{\mathbb{V}}(x) := \{u \in \mathbb{U} \mid (x, u) \in \mathbb{D} \text{ and } f(x, u) \in \mathbb{V}\} . \quad (11)$$

Therefore  $\mathbb{U}_{\mathbb{V}(f, \mathbb{D})}(x)$  stands for the largest set of *viable controls associated with*  $x \in \mathbb{X}$ . Then, the decision design consists in the choice of a viable *feedback* control, namely any selection  $\Psi : \mathbb{X} \rightarrow \mathbb{U}$  which associates with each state  $x \in \mathbb{V}(f, \mathbb{D})$  a control  $u = \Psi(x)$  satisfying  $\Psi(x) \in \mathbb{U}_{\mathbb{V}(f, \mathbb{D})}(x)$ .

In the context of sustainable management, viability concepts and methods may help giving a framework for setting decision making. First, one should delineate *perpetual objectives* (the set  $\mathbb{D}$ ). Second, *operational objectives (advice)* can be obtained as viable controls. This

approach is illustrated in (De Lara, Doyen, Guilbaud, and Rochet, 2006), and especially in (De Lara, Doyen, Guilbaud, and Rochet, 2007) for fishery management.

### 3 Viable control of generic nonlinear ecosystem models

For a generic ecosystem model, we provide an explicit description of the viability kernel. Then, we shall specify the results for predator–prey systems, in particular for discrete-time Lotka–Volterra models.

#### 3.1 Dynamics and constraints for an ecosystem management model

For simplicity, we consider a two–dimensional state model. However, the following Proposition 1 may be easily extended to  $n$ –dimensional systems as long as each species is harvested by a specific device: one species, one harvesting effort. The two–dimensional state vector  $(y, z)$  represents biomasses and the two–dimensional control  $(v, w)$  is harvesting effort of each species, respectively. The catches are thus  $vy$  and  $wz$  (measured in biomass).<sup>1</sup>

The dynamics  $f$  in (1) is given by

$$f(y, z, v, w) = \begin{pmatrix} yR_y(y, z, v) \\ zR_z(y, z, w) \end{pmatrix} \quad \text{for all } (y, z, v, w) \in \mathbb{R}^4, \quad (12)$$

where  $R_y : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $R_z : \mathbb{R}^3 \rightarrow \mathbb{R}$  are two functions representing growth coefficients (the growth rates being  $R_y - 1$  and  $R_z - 1$ ). We do not make assumptions on the signs of  $R_y$  and  $R_z$ ; the restrictions on the domain of variation of  $(y, z, v, w)$  will result from the requirement that the trajectories belong to the acceptable set  $\mathbb{D}$ , which will include signs considerations.

---

<sup>1</sup>In fact, any expression of the form  $c(y, v)$  would fit for the catches in the following Proposition 1 as soon as  $v \mapsto c(y, v)$  is strictly increasing and goes from 0 to  $+\infty$  when  $v$  goes from 0 to  $+\infty$ . The same holds for  $d(z, w)$  instead of  $wz$ .

The discrete-time control system (1) with dynamics  $f$  given by (12) now writes:

$$\begin{cases} y(t+1) = y(t)R_y(y(t), z(t), v(t)) \\ z(t+1) = z(t)R_z(y(t), z(t), w(t)) . \end{cases} \quad (13)$$

The implicit assumption in such functional form (12) for the dynamics is that, during one time period, the harvesting effort  $v(t)$  of species  $y$  only affects the same species  $y(t+1)$  and not the other one  $z(t+1)$  (the reverse holds for  $w(t)$  and species  $z$ ). Of course, after two periods,  $z(t+2)$  depends on  $y(t+1)$  which depends on  $v(t)$  so that both efforts affect both species. Thus, the time period is assumed to be short enough for the impact on one species of harvesting the other species to be negligible.

The acceptable set  $\mathbb{D}$  is defined by *minimal biomass thresholds*  $y^b \geq 0, z^b \geq 0$  and *minimal catch thresholds*  $Y^b \geq 0, Z^b \geq 0$ :

$$\mathbb{D} = \{ (y, z, v, w) \in \mathbb{R}^4 \mid y \geq y^b, z \geq z^b, vy \geq Y^b, wz \geq Z^b \} . \quad (14)$$

Such a set includes both *preservation*

$$y(t) \geq y^b, \quad z(t) \geq z^b,$$

and *production* requirements

$$v(t)y(t) \geq Y^b, \quad w(t)z(t) \geq Z^b$$

when thresholds are positive.

### 3.2 Expression of the viability kernel

The following Proposition 1 gives an explicit description of the viability kernel, under some conditions on the minimal thresholds.

**Proposition 1** *Assume that the function  $R_y : \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuously decreasing<sup>2</sup> in the control  $v$  and satisfies  $\lim_{v \rightarrow +\infty} R_y(y, z, v) \leq 0$ , and that  $R_z : \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuously*

---

<sup>2</sup>In all that follows, a mapping  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is said to be increasing if  $x \geq x' \Rightarrow \varphi(x) \geq \varphi(x')$ . The reverse holds for decreasing. Thus, with this definition, a constant mapping is both increasing and decreasing.

decreasing in the control variable  $w$ , and satisfies  $\lim_{w \rightarrow +\infty} R_z(y, z, w) \leq 0$ . If the thresholds in (14) are such that the following growth coefficients are greater than one

$$R_y(y^b, z^b, \frac{Y^b}{y^b}) \geq 1 \quad \text{and} \quad R_z(y^b, z^b, \frac{Z^b}{z^b}) \geq 1, \quad (15)$$

the viability kernel associated with the dynamics  $f$  in (12) and the acceptable set  $\mathbb{D}$  in (14) is given by

$$\mathbb{V}(f, \mathbb{D}) = \left\{ (y, z) \mid y \geq y^b, z \geq z^b, yR_y(y, z, \frac{Y^b}{y}) \geq y^b, zR_z(y, z, \frac{Z^b}{z}) \geq z^b \right\}. \quad (16)$$

Before giving the proof, let us comment the assumptions. That the growth coefficients are decreasing with respect to the harvesting effort is a natural assumption. Conditions (15) mean that, at the point  $(y^b, z^b)$  and applying efforts  $u^b = \frac{Y^b}{y^b}$ ,  $v^b = \frac{Z^b}{z^b}$ , the growth coefficients are greater than one, hence both populations grow; hence, it could be thought that computing the viability kernel is useless since everything looks fine. However, if all is fine at the point  $(y^b, z^b)$ , it is not obvious that this also goes for a larger domain. Indeed, the ecosystem dynamics  $f$  given by (12) has no monotonicity properties that would allow to extend a result valid for a point to a whole domain. What is more, if continuous-time viability results mostly relies upon assumptions at the frontier of the constraints set, this is no longer true for discrete-time viability.

**Proof.** According to induction (9), we have:

$$\begin{aligned}\mathbb{V}_0 &= \{(y, z) \mid y \geq y^b, z \geq z^b\}, \\ \mathbb{V}_1 &= \left\{ (y, z) \left| \begin{array}{l} y \geq y^b, z \geq z^b \text{ and, for some } (v, w) \geq 0, \\ vy \geq Y^b, wz \geq Z^b, yR_y(y, z, v) \geq y^b, zR_z(y, z, w) \geq z^b \end{array} \right. \right\} \\ &= \left\{ (y, z) \left| \begin{array}{l} y \geq y^b, z \geq z^b, yR_y(y, z, \frac{Y^b}{y}) \geq y^b, zR_z(y, z, \frac{Z^b}{z}) \geq z^b \end{array} \right. \right\} \\ &\quad \text{because } v \mapsto R_y(y, z, v) \text{ and } w \mapsto R_z(y, z, w) \text{ are decreasing,} \\ &\quad \text{and thus we may select } v = \frac{Y^b}{y}, w = \frac{Z^b}{z}.\end{aligned}$$

Denoting  $y' = yR_y(y, z, v)$ ,  $z' = zR_z(y, z, w)$ , we obtain,

$$\mathbb{V}_2 = \left\{ (y, z) \left| \begin{array}{l} y \geq y^b, z \geq z^b \text{ and, for some } (v, w) \geq 0, \\ vy \geq Y^b, wz \geq Z^b \\ y' \geq y^b, y'R_y(y', z', \frac{Y^b}{y'}) \geq y^b, z' \geq z^b, z'R_z(y', z', \frac{Z^b}{z'}) \geq z^b \end{array} \right. \right\}.$$

We shall now make use of the property, recalled in Sect. 2, that when the decreasing sequence  $(\mathbb{V}_k)_{k \in \mathbb{N}}$  of viability kernels up to time  $k$  is stationary, its limit is the viability kernel  $\mathbb{V}(f, \mathbb{D})$ . Hence, it suffices to show that  $\mathbb{V}_1 \subset \mathbb{V}_2$  to obtain that  $\mathbb{V}(f, \mathbb{D}) = \mathbb{V}_1$ . Let  $(y, z) \in \mathbb{V}_1$ , so that

$$y \geq y^b, \quad z \geq z^b \quad \text{and} \quad yR_y(y, z, \frac{Y^b}{y}) \geq y^b, \quad zR_z(y, z, \frac{Z^b}{z}) \geq z^b.$$

Since  $R_y : \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuously decreasing in the control variable, with  $\lim_{v \rightarrow +\infty} R_y(y, z, v) \leq 0$ , and since  $yR_y(y, z, \frac{Y^b}{y}) \geq y^b$ , there exists a  $\hat{v} \geq \frac{Y^b}{y}$  (depending on  $y$  and  $z$ ) such that  $y' = yR_y(y, z, \hat{v}) = y^b$ . The same holds for  $R_z : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $z' = zR_z(y, z, \hat{w}) = z^b$ . By (15), we deduce that

$$y'R_y(y', z', \frac{Y^b}{y'}) = y^b R_y(y^b, z^b, \frac{Y^b}{y^b}) \geq y^b \quad \text{and} \quad z'R_z(y', z', \frac{Z^b}{z'}) = z^b R_z(y^b, z^b, \frac{Z^b}{z^b}) \geq z^b.$$

The inclusion  $\mathbb{V}_1 \subset \mathbb{V}_2$  follows. □

As a direct consequence of the proof of Proposition 1, for each  $(y, z) \in \mathbb{V}(f, \mathbb{D})$ , the control  $(\hat{v}, \hat{w})$  belongs to  $\mathbb{U}_{\mathbb{V}(f, \mathbb{D})}(y, z)$ , the set of viable controls defined in (11). More explicitly, we have the following expression of the viable controls set, which results from the above observation and proof.

**Corollary 2** *Suppose that the assumptions of Proposition 1 are satisfied. Denoting*

$$\begin{cases} \hat{v}(y, z) = \max\{v \geq \frac{Y^b}{y} \mid yR_y(y, z, v) = y^b\} \\ \hat{w}(y, z) = \max\{w \geq \frac{Z^b}{z} \mid zR_z(y, z, w) = z^b\}, \end{cases}$$

*the set of viable controls is given by*

$$\mathbb{U}_{\mathbb{V}(f, \mathbb{D})}(y, z) = \left\{ (v, w) \left| \begin{array}{l} \hat{v}(y, z) \geq v \geq \frac{Y^b}{y}, \quad \hat{w}(y, z) \geq w \geq \frac{Z^b}{z}, \\ y'R_y(y', z', \frac{Y^b}{y'}) \geq y^b, \quad z'R_z(y', z', \frac{Z^b}{z'}) \geq z^b \end{array} \right. \right\}$$

*where  $y' = yR_y(y, z, v)$ ,  $z' = zR_z(y, z, w)$ .*

### 3.3 Viability kernel for a nonlinear predator–prey system

Till now, no trophic relationship has been specified between species  $y$  and  $z$ . From now on, we shall focus on nonlinear predator–prey systems. The two–dimensional state vector  $(y, z)$  represents biomasses of preys ( $y$ ) and predators ( $z$ ). This interpretation results from the monotonicity assumptions made on function  $R_z$  in the following Proposition 3, which provides a simpler version of Proposition 1 when the predator  $z$  does not exhibit density–dependence.

**Proposition 3** *Suppose that the dynamics  $f$  is given by*

$$f(y, z, v, w) = \begin{pmatrix} yR_y(y, z, v) \\ zR_z(y, w) \end{pmatrix} \quad \text{for all } (y, z, v, w) \in \mathbb{R}^4. \quad (17)$$

*Assume that the function  $R_y : \mathbb{R}^3 \rightarrow \mathbb{R}$  is decreasing in the control  $v$  and  $R_z : \mathbb{R}^2 \rightarrow \mathbb{R}$  is increasing in the state variable  $y$  and continuously decreasing in the control variable  $w$ , and satisfies  $\lim_{w \rightarrow +\infty} R_z(y, w) \leq 0$ . If*

$$R_y(y^b, z^b, \frac{Y^b}{y^b}) \geq 1 \quad \text{and} \quad R_z(y^b, \frac{Z^b}{z^b}) \geq 1, \quad (18)$$

*the viability kernel associated with the dynamics  $f$  in (17) and the acceptable set  $\mathbb{D}$  in (14) is given by*

$$\mathbb{V}(f, \mathbb{D}) = \left\{ (y, z) \mid y \geq y^b, \quad z \geq z^b, \quad yR_y(y, z, \frac{Y^b}{y}) \geq y^b \right\}. \quad (19)$$

**Proof.** The proof follows the same lines as the one of Proposition 1. The description of the viability kernel is simpler because of the following argument. The monotonicity assumptions on each component of function  $R_z$  and condition  $R_z(y^b, \frac{Z^b}{z^b}) \geq 1$  lead to

$$y \geq y^b \text{ and } z \geq z^b \Rightarrow R_z(y, \frac{Z^b}{z}) \geq R_z(y, \frac{Z^b}{z^b}) \geq R_z(y^b, \frac{Z^b}{z^b}) \geq 1 = \frac{z^b}{z^b} \geq \frac{z^b}{z}.$$

Therefore,  $zR_z(y, \frac{Z^b}{z}) \geq z^b$  for all  $y \geq y^b$  and  $z \geq z^b$ . This explains why (16) reduces to (19).  $\square$

### 3.4 Viability kernel for a Lotka–Volterra system

Consider the following discrete–time Lotka–Volterra system of equations with density–dependence in the prey

$$\begin{cases} y(t+1) = Ry(t) - \frac{R}{\kappa}y^2(t) - \alpha y(t)z(t) - v(t)y(t), \\ z(t+1) = Lz(t) + \beta y(t)z(t) - w(t)z(t), \end{cases} \quad (20)$$

where  $R > 1$ ,  $0 < L < 1$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $\kappa = \frac{R}{R-1}K$ , with  $K > 0$  the carrying capacity for prey. The dynamics  $f$  is as in (17), with  $R_y(y, z, v) = R - \frac{R}{\kappa}y - \alpha z - v$  and  $R_z(y, w) = L + \beta y - w$ . Proposition 3 gives the following Corollary.

**Corollary 4** *Consider the Lotka–Volterra predator–prey model (20). Whenever*

$$y^b \geq \frac{1-L}{\beta} \text{ and } z^b \leq \frac{R-1}{\alpha} - \frac{R(1-L)}{\alpha\beta\kappa},$$

*any minimal catch thresholds  $Y^b$  and  $Z^b$  such that*

$$Y^b \leq Y^{b,*} := y^b(R - \frac{R}{\kappa}y^b - \alpha z^b - 1) \quad (21a)$$

$$Z^b \leq Z^{b,*} := z^b(L + \beta y^b - 1), \quad (21b)$$

*satisfy (18) and the viability kernel associated with the dynamics  $f$  in (20) and the acceptable set  $\mathbb{D}$  in (14) is given by*

$$\mathbb{V}(f, \mathbb{D}) = \left\{ (y, z) \mid y \geq y^b, z^b \leq z \leq \frac{1}{\alpha} \left[ R \left( \frac{\kappa - y}{\kappa} \right) - \frac{Y^b + y^b}{y} \right] \right\}. \quad (22)$$

This Corollary has the following practical consequence. An initial state  $(y, z)$  such that  $y \geq y^b$  and  $z \geq z^b$  belongs to all the viability kernels  $\mathbb{V}(f, \mathbb{D})$  associated to  $Z^b \leq Z^{b,*}$ , by (21b), and to  $Y^b \leq \min \{Y^{b,*}, y(R - Ry/\kappa - \alpha z) - y^b\}$ , by (21a) and (22), as long as this last quantity is nonnegative. In other words, if viably managed, the fishery could produce at least  $\min \{Y^{b,*}, y(R - Ry/\kappa - \alpha z) - y^b\}$  and  $Z^{b,*}$ , while respecting biological thresholds  $y^b$  and  $z^b$ . We shall use this property in the following numerical application.

## 4 Numerical application to the hake–anchovy couple in the Peruvian upwelling ecosystem

We provide a viability analysis of the hake–anchovy Peruvian fisheries between the years 1971 and 1981. For this, we shall consider a discrete-time Lotka–Volterra model for the couple anchovy (prey  $y$ ) and hake (predator  $z$ ), for which the viability kernel has explicit description. The emphasis is not on developing a biological model, but rather on decision-making using such a model.

The period between the years 1971 and 1981 has been chosen because the competition between the fishery and hake was reduced due to low anchovy catches, and because of the absence a strong warm event after El Niño 1972. We have 11 couples of biomasses, and the same for catches. The 5 parameters of the model are estimated minimizing a weighted residual squares sum function using a conjugate gradient method, with central derivatives. Estimated parameters and comparisons of observed and simulated biomasses are shown in Figure 1.

We consider values of  $y^b = 7\,000\,000$  t and  $z^b = 200\,000$  t for minimal biomass thresholds and values of  $Y^b = 2\,000\,000$  t and  $Z^b = 5\,000$  t for minimal catch thresholds (IMARPE, 2000, 2004). Conditions (18) in Proposition 3 are satisfied with these values. Indeed, the expressions in (21a)–(21b) give:

$$\begin{cases} Y^b = 2\,000\,000 \text{ t} \leq Y^{b,*} = 5\,399\,000 \text{ t} \\ Z^b = 5\,000 \text{ t} \leq Z^{b,*} = 56\,800 \text{ t} . \end{cases} \quad (23)$$

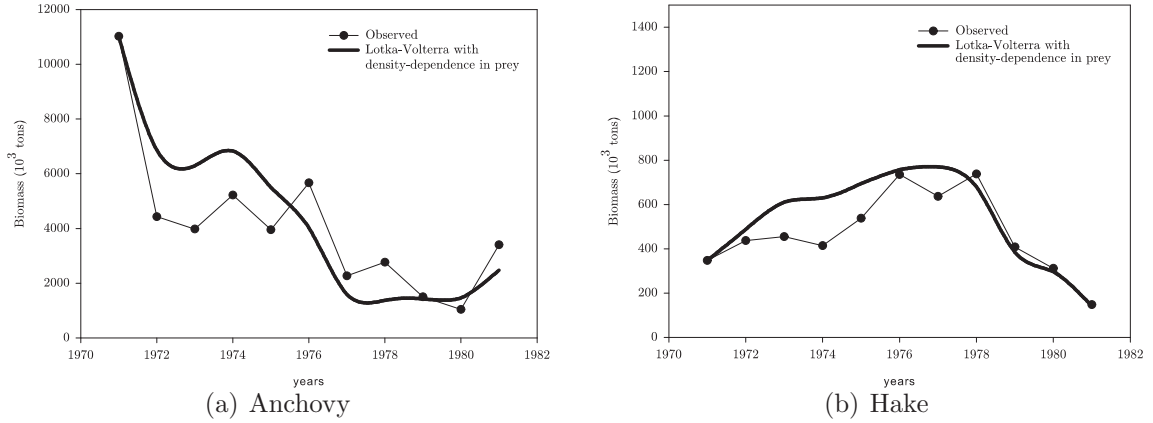


Figure 1: Comparison of observed and simulated biomasses of anchovy and hake using a Lotka–Volterra model with density-dependence in the prey. Model parameters are  $R = 2.25 \text{ year}^{-1}$ ,  $L = 0.945 \text{ year}^{-1}$ ,  $\kappa = 67\,113 \times 10^3 \text{ t}$  ( $K = 37\,285 \times 10^3 \text{ t}$ ),  $\alpha = 1.220 \times 10^{-6} \text{ t}^{-1}$ ,  $\beta = 4.845 \times 10^{-8} \text{ t}^{-1}$ .

The viability kernel is depicted in Figure 2. The unique viable point (within the viability kernel) is the initial point. Thus, based upon this model, the fishery could have been managed – with appropriate viable controls – to produce catches above  $Y^b = 2\,000\,000 \text{ t}$  and  $Z^b = 5\,000 \text{ t}$ , while ensuring biological conservation. What is more, due to the remark following Corollary 4, captures up to  $Y^b = 5\,399\,000 \text{ t}$  and  $Z^b = 56\,800 \text{ t}$  were theoretically achievable in a sustainable way starting from year 1971.

## 5 Conclusion

Motivated by viable management of ecosystems, we have provided a general condition ensuring an explicit construction of viability kernels and have applied this to the viability analysis of generic ecosystem models with harvesting.

Our results have then been applied to a Lotka–Volterra model using the anchovy–hake couple in the Peruvian upwelling ecosystem. We showed that, during the anchovy collapse, theoretically the fishery could have been viably managed to produce catches above the expected threshold levels while ensuring biological conservation.

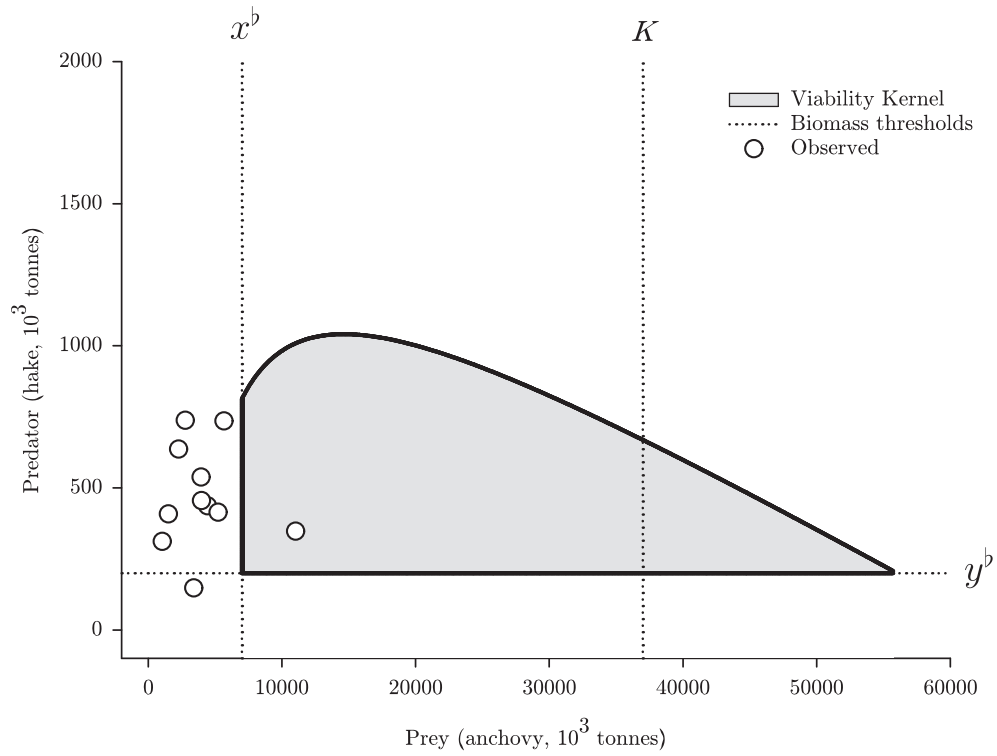


Figure 2: Viability kernel (in grey) for a Lotka–Volterra model with density-dependence in the prey in the predator–prey phase space (with  $y^b = 7\,000\,000\ \tau$ ,  $z^b = 200\,000\ \tau$ ,  $Y^b = 2\,000\,000\ \tau$ ,  $Z^b = 5\,000\ \tau$ ). The unique point within the viability kernel is the initial point, while all subsequent points in the trajectory are outside the state constraint set.

It is interesting to notice that the kind of maximum sustainable yields  $Y^{b,*} = 5\,399\,000$  t and  $Z^{b,*} = 56\,800$  t provided by our approach in (23) are comparable to the 4 250 000 t anchovy yield and the 55 000 t hake yield, respectively, established for the year 2006 (PRODUCE, 2005, 2006b,c), or to the 5 000 000 t anchovy yield and the 35 000 t hake yield, respectively, established for the year 2007 (PRODUCE, 2006a, 2007a,b). So, despite simplicity<sup>3</sup> of the models considered, our approach may provide a mean of designing sustainable yields from an ecosystem point of view.

Thus, control and viability theory methods have allowed us to introduce ecosystem considerations, such as multispecies and multiobjectives, and have contributed to integrate the long term dynamics, which is generally not considered in conventional fishery management. Notice that the World Summit on Sustainable Development (Garcia, Zerbi, Aliaume, Chi, and Lasserre, 2003) encouraged the application of the ecosystem approach by 2010.

**Acknowledgments.** This paper was prepared within the MIFIMA (Mathematics, Informatics and Fisheries Management) international research network. We thank CNRS, INRIA and the French Ministry of Foreign Affairs for their funding and support through the regional cooperation program STIC–AmSud. We thank the staff of the Peruvian Marine Research Institute (IMARPE), especially Erich Diaz and Nathaly Vargas for discussions on anchovy and hake fisheries. We also thank Yboon Garcia (IMCA-Peru and CMM-Chile) for a discussion on the ecosystem model case.

## References

J-P. Aubin. *Viability Theory*. Birkhäuser, Boston, 1991. 542 pp.

---

<sup>3</sup>In addition to hake, there are other important predators of anchovy in the Peruvian upwelling ecosystem, such as mackerel and horse mackerel, seabirds and pinnipeds, which were not considered. Also, anchovy has been an important prey of hake, but other prey species have been found in the opportunistic diet of hake (Tam, Purca, Duarte, Blaskovic, and Espinoza, 2006)

- C. Béné and L. Doyen. Storage and viability of a fishery with resource and market dephased seasonalities. *Environmental Resource Economics*, 15:1–26, 2000.
- C. Béné and L. Doyen. Sustainability of fisheries through marine reserves: a robust modeling analysis. *Journal of Environmental Management*, 69:1–13, 2003.
- C. Béné, L. Doyen, and D. Gabay. A viability analysis for a bio-economic model. *Ecological Economics*, 36:385–396, 2001.
- D. Bertsekas and I. Rhodes. On the minimax reachability of target sets and target tubes. *Automatica*, 7:233–247, 1971. [citeseer.ist.psu.edu/bertsekas71minimax.html](http://citeseer.ist.psu.edu/bertsekas71minimax.html).
- F. Blanchini. Set invariance in control (survey paper). *Automatica*, 35(11):1747–1767, 1999.
- N. Bonneuil and K. Müllers. Viable populations in a prey-predator system. *Journal of Mathematical Biology*, 35(3):261–293, February 1997.
- Noël Bonneuil and Patrick Saint-Pierre. Population viability in three trophic-level food chains. *Applied Mathematics and Computation*, 169(2):1086 – 1105, 2005.
- F. H. Clarke, Y. S. Ledayev, R. J. Stern, and P. R. Wolenski. Qualitative properties of trajectories of control systems: a survey. *Journal of Dynamical Control Systems*, 1:1–48, 1995.
- M. De Lara and L. Doyen. *Sustainable Management of Natural Resources. Mathematical Models and Methods*. Springer-Verlag, Berlin, 2008.
- M. De Lara, L. Doyen, T. Guilbaud, and M.-J. Rochet. Monotonicity properties for the viable control of discrete time systems. *Systems and Control Letters*, 56(4):296–302, 2006.
- M. De Lara, L. Doyen, T. Guilbaud, and M.-J. Rochet. Is a management framework based on spawning-stock biomass indicators sustainable? A viability approach. *ICES J. Mar. Sci.*, 64(4):761–767, 2007.

- K. Eisenack, J. Sheffran, and J. Kropp. The viability analysis of management frameworks for fisheries. *Environmental Modeling and Assessment*, 11(1):69–79, February 2006.
- S. Garcia, A. Zerbi, C. Aliaume, T. Do Chi, and G. Lasserre. The ecosystem approach to fisheries. Issues, terminology, principles, institutional foundations, implementation and outlook. *FAO Fisheries Technical Paper*, 443(71), 2003.
- IMARPE. Trabajos expuestos en el taller internacional sobre la anchoveta peruana (TIAP), 9-12 mayo 2000. *Bol. Inst. Mar Peru*, 19:1–2, 2000.
- IMARPE. Report of the first session of the international panel of experts for assessment of Peruvian hake population. march 2003. *Bol. Inst. Mar Peru*, 21:33–78, 2004.
- V. Martinet and L. Doyen. Sustainable management of an exhaustible resource: a viable control approach. *Journal of Resource and Energy Economics*, 29(1):19–37, 2007.
- C. Mullon, P. Cury, and L. Shannon. Viability model of trophic interactions in marine ecosystems. *Natural Resource Modeling*, 17:27–58, 2004.
- PRODUCE. Establecen régimen provisional de pesca del recurso merluza correspondiente al año 2006. *El Peruano*, page 307804, 30 de diciembre 2005. RM-356-2005-PRODUCE.
- PRODUCE. Establecen régimen provisional de pesca del recurso merluza correspondiente al año 2007. *El Peruano*, page 335485, 27 de diciembre 2006a. RM-357-2006-PRODUCE.
- PRODUCE. Autorizan inicio de actividades pesqueras relativas a los recursos anchoveta y anchoveta blanca en área de dominio marítimo a partir del 2 de mayo de 2006. *El Peruano*, page 317143, 27 de abril 2006b. RM-111-2006-PRODUCE.
- PRODUCE. Autorizan reinicio de actividades pesqueras de los recursos anchoveta y anchoveta blanca en zona del dominio marítimo. *El Peruano*, page 331417, 25 de octubre 2006c. RM-287-2006-PRODUCE.

PRODUCE. Autorizan reinicio de actividades pesqueras de los recursos anchoveta y anchoveta blanca en zona del litoral. *El Peruano*, page 342927, 5 de abril 2007a. RM-095-2007-PRODUCE.

PRODUCE. Autorizan reinicio de actividades pesqueras de los recursos anchoveta y anchoveta blanca en zona del litoral. *El Peruano*, page 356040, 26 de octubre 2007b. RM-327-2007-PRODUCE.

M. Quincampoix and P. Saint-Pierre. An algorithm for viability kernels in Hölderian case: approximation by discrete dynamical systems. *Journal of Mathematical Systems Estimation and Control*, 5:1–13, 1995.

A. Rapaport, J.-P. Terreaux, and L. Doyen. Sustainable management of renewable resource: a viability approach. *Mathematics and Computer Modeling*, 43(5-6):466–484, March 2006.

P. Saint-Pierre. Approximation of viability kernel. *Applied Mathematics and Optimization*, 29:187–209, 1994.

J. Tam, S. Purca, L. O. Duarte, V. Blaskovic, and P. Espinoza. Changes in the diet of hake associated with El Niño 1997-1998 in the Northern Humboldt Current ecosystem. *Advances in Geosciences.*, 6:63–67, 2006.