

# On Decidability Properties of One-Dimensional Cellular Automata

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## Abstract

In a recent paper Sutner proved that the first-order theory of the phase-space  $\mathcal{S}_{\mathcal{A}} = (Q^{\mathbb{Z}}, \longrightarrow)$  of a one-dimensional cellular automaton  $\mathcal{A}$  whose configurations are elements of  $Q^{\mathbb{Z}}$ , for a finite set of states  $Q$ , and where  $\longrightarrow$  is the “next configuration relation”, is decidable [Sut09]. He asked whether this result could be extended to a more expressive logic. We prove in this paper that this is actually the case. We first show that, for each one-dimensional cellular automaton  $\mathcal{A}$ , the phase-space  $\mathcal{S}_{\mathcal{A}}$  is an  $\omega$ -automatic structure. Then, applying recent results of Kuske and Lohrey on  $\omega$ -automatic structures, it follows that the first-order theory, extended with some counting and cardinality quantifiers, of the structure  $\mathcal{S}_{\mathcal{A}}$ , is decidable. We give some examples of new decidable properties for one-dimensional cellular automata. In the case of surjective cellular automata, some more efficient algorithms can be deduced from results of [KL08] on structures of bounded degree. On the other hand we show that the case of cellular automata give new results on automatic graphs.

**Keywords:** One-dimensional cellular automaton; space of configurations;  $\omega$ -automatic structures; first order theory; cardinality quantifiers; decidability properties; surjective cellular automaton; automatic graph; reachability relation.

# 1 Introduction

Some properties of one-dimensional cellular automata, like injectivity or surjectivity, were shown to be decidable by Amoroso and Patt in [AP72]. These two properties are easily expressed as first-order properties of the phase-space  $\mathcal{S}_{\mathcal{A}} = (Q^{\mathbb{Z}}, \longrightarrow)$  of a cellular automaton  $\mathcal{A}$  whose configurations are elements of  $Q^{\mathbb{Z}}$ , for a finite set of states  $Q$ , and where  $\longrightarrow$  is the “next configuration relation”.

It is then very natural to ask whether every first-order property of the structure  $\mathcal{S}_{\mathcal{A}}$  is decidable. Sutner proved recently in [Sut09] that this is actually the case, i.e. that “model-checking for one-dimensional cellular automata in the first order logic  $\mathcal{L}(\longrightarrow)$  with equality is decidable”. He used decidability properties of  $\zeta$ -Büchi automata reading *biinfinite words*.

On the other hand,  $\omega$ -automatic structures are relational structures whose domain and relations are recognizable by Büchi automata reading *infinite words*. The  $\omega$ -automatic structures have very nice decidability and definability properties and have been much studied in the last few years, [Hod83, BG04, KNRS07]. In particular the first-order theory of an  $\omega$ -automatic structure is decidable and the class of  $\omega$ -automatic structures is closed under first-order interpretations.

We show here that, for each one-dimensional cellular automaton  $\mathcal{A}$ , the phase-space  $\mathcal{S}_{\mathcal{A}} = (Q^{\mathbb{Z}}, \longrightarrow)$  is an  $\omega$ -automatic structure. Thus Sutner’s result can be deduced from an earlier result of Hodgson in [Hod83].

But we can now apply to the study of one-dimensional cellular automata some other very recent results on  $\omega$ -automatic structures. Blumensath and Graëdel proved in [BG04] that the first-order theory of an  $\omega$ -automatic structure, extended with the “infinity quantifier”  $\exists^{\infty}$ , is decidable. Kuske and Lohrey proved in [KL08] that the first-order theory, extended with some counting and cardinality quantifiers, of an (injectively)  $\omega$ -automatic structure is decidable. These quantifiers are in the form  $\exists^{\kappa}$ , for any cardinal  $\kappa$ , or are modulo quantifiers in the form  $\exists^{t,k}$ , for integers  $0 \leq t < k$ , whose meaning is “there exist  $t$  modulo  $k$ ”.

This gives an answer to Sutner’s question who asked whether his decidability result could be extended to a more expressive logic.

Sutner’s result implies that one can decide some properties like the following ones : “there exists a 6-cycle” or “there exists exactly five fixed points”. With our new results deduced from properties of  $\omega$ -automatic structures, we can now decide properties like : “there exist uncountably many 3-cycles” or “there exist infinitely many 25-cycles” or “there exist  $t$  modulo  $k$  6-cycles”

or “the set of fixed points is countably infinite”, and so on .

Notice that the algorithms obtained by Sutner in [Sut09] are in general non elementary. This is also the case for the algorithms obtained by Kuske and Lohrey for  $\omega$ -automatic structures. However some more efficient algorithms are obtained in [KL08] in the case of  $\omega$ -automatic structures of bounded degree. In the case of one-dimensional cellular automata this corresponds to the important class of surjective cellular automata. We give the complexities of algorithms obtained from [KL08] in this particular case.

We consider next the set of configurations with finite support of a given cellular automaton, equipped with the next configuration relation. It is an automatic directed graph, i.e. it is a countable structure presentable with automata reading “finite words”. Then we can infer new results on automatic graphs from some results of Sutner [Sut08a]. In particular for each recursively enumerable Turing degree  $\mathbf{d}$  there is an automatic graph in which the reachability relation has exactly the Turing degree  $\mathbf{d}$ . This shows the richness of the class of automatic graphs, from the point of view of Turing degrees.

The paper is organized as follows. In Section 2 we recall definitions of cellular automata and of Büchi automata reading infinite words. We recall in Section 3 the notion of (injectively)  $\omega$ -automatic structure. We get some new decidability properties of one-dimensional cellular automata in Section 4. We show in section 5 that the case of cellular automata give new results on automatic graphs.

## 2 Cellular automata and Büchi automata

When  $\Sigma$  is a finite alphabet, a *non-empty finite word* over  $\Sigma$  is any sequence  $x = a_0.a_1 \dots a_k$ , where  $a_i \in \Sigma$  for  $i = 0, 1, \dots, k$ , and  $k$  is an integer  $\geq 0$ . The *length* of  $x$  is  $k + 1$ , denoted by  $|x|$ . The *empty word* has no letter and is denoted by  $\lambda$ ; its length is 0. For  $x = a_0 \dots a_k$ , we write  $x(i) = a_i$  and  $x[i] = x(0) \dots x(i)$  for  $i \leq k$ .  $\Sigma^*$  is the *set of finite words* (including the empty word) over  $\Sigma$ .

The *first infinite ordinal* is  $\omega$ . An  $\omega$ -word over  $\Sigma$  is an  $\omega$ -sequence  $a_0.a_1 \dots a_n \dots$ , where for all integers  $i \geq 0$ ,  $a_i \in \Sigma$ . When  $\sigma$  is an  $\omega$ -word over  $\Sigma$ , we write  $\sigma = \sigma(0)\sigma(1) \dots \sigma(n) \dots$ , where for all  $i$ ,  $\sigma(i) \in \Sigma$ , and  $\sigma[n] = \sigma(0)\sigma(1) \dots \sigma(n)$  for all  $n \geq 0$ .

The *set of  $\omega$ -words* over the alphabet  $\Sigma$  is denoted by  $\Sigma^\omega$ . An  $\omega$ -language over an alphabet  $\Sigma$  is a subset of  $\Sigma^\omega$ . The complement (in  $\Sigma^\omega$ ) of an  $\omega$ -

language  $V \subseteq \Sigma^\omega$  is  $\Sigma^\omega - V$ , denoted  $V^-$ .

A  $\mathbb{Z}$ -word over the alphabet  $\Sigma$  is a *biinfinite word* :

$$\dots x(-3).x(-2).x(-1).x(0).x(1).x(2).x(3) \dots$$

where for each  $i \in \mathbb{Z}$ , the letter  $x(i)$  is in the alphabet  $\Sigma$ .

The set of  $\mathbb{Z}$ -words over the alphabet  $\Sigma$  is denoted by  $\Sigma^{\mathbb{Z}}$ . It can be identified with the set of functions from  $\mathbb{Z}$  into  $\Sigma$ .

We now define one-dimensional cellular automata.

### Definition 2.1

1. A one-dimensional cellular automaton  $\mathcal{A}$  is a pair  $(Q, \delta)$ , where  $Q$  is a finite set of states and  $\delta$  is a function from  $Q^3$  into  $Q$ , called the “local transition function” of the automaton.
2. A configuration of the cellular automaton  $\mathcal{A}$  is a mapping from  $\mathbb{Z}$  into  $Q$ , i.e. an element of  $Q^{\mathbb{Z}}$ .
3. The global transition function of the cellular automaton  $\mathcal{A}$  is the function  $\Delta$  from  $Q^{\mathbb{Z}}$  into  $Q^{\mathbb{Z}}$  defined by :

$$\forall z \in \mathbb{Z}, \Delta(C)(z) = \delta(C(z-1), C(z), C(z+1))$$

So the cellular automaton may be also denoted by  $(Q^{\mathbb{Z}}, \Delta)$ .

Notice that we have defined the “local transition function” of a one-dimensional cellular automaton as a function from  $Q^3$  into  $Q$ , i.e. the next state of a given cell depends only on the given state of this cell and of its closest neighbours. However all the results in this paper can be easily generalized to the case of a “local transition function” being a function from  $Q^{2r+1}$  into  $Q$ , for an integer  $r \geq 1$ , as in [Sut09].

We recall now the notion of Büchi automaton reading infinite words over a finite alphabet, which can be found for instance in [Tho90, Sta97].

**Definition 2.2** A Büchi automaton is a sextuple  $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$ , where  $K$  is a finite set of states,  $\Sigma$  is a finite alphabet,  $\Delta \subseteq K \times \Sigma \times K$  is the set of transitions,  $q_0 \in K$  is the initial state, and  $F \subseteq K$  is the set of accepting states.

A run  $r$  of the Büchi automaton  $\mathcal{A}$  on an  $\omega$ -word  $x \in \Sigma^\omega$  is an infinite sequence of states  $(q_i)_{i \geq 0}$  such that for each integer  $i \geq 0$   $(q_i, x(i), q_{i+1}) \in \Delta$ .

Notice that the first state of the run is the initial state  $q_0$ .

The run is said to be successful iff there exists a final state  $q_f \in F$  and infinitely many integers  $i \geq 0$  such that  $q_i = q_f$ .

The  $\omega$ -language  $L(\mathcal{A}) \subseteq \Sigma^\omega$  accepted by the Büchi automaton  $\mathcal{A}$  is the set of  $\omega$ -words  $x$  such that there is a successful run  $r$  of  $\mathcal{A}$  on  $x$ .

An  $\omega$ -language  $L \subseteq \Sigma^\omega$  is a regular  $\omega$ -language iff there is a Büchi automaton  $\mathcal{A}$  such that  $L = L(\mathcal{A})$ .

Notice that one can consider a relation  $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega \times \dots \times \Sigma_n^\omega$ , where  $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ , are finite alphabets, as an  $\omega$ -language over the product alphabet  $\Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$ .

We now recall some fundamental closure properties of regular  $\omega$ -languages.

**Theorem 2.3** (see [Tho90, PP04]) *The class of regular  $\omega$ -languages is effectively closed under finite union, finite intersection, and complementation, i.e. we can effectively construct, from two Büchi automata  $\mathcal{A}$  and  $\mathcal{B}$ , some Büchi automata  $\mathcal{C}_1$ ,  $\mathcal{C}_2$ , and  $\mathcal{C}_3$ , such that  $L(\mathcal{C}_1) = L(\mathcal{A}) \cup L(\mathcal{B})$ ,  $L(\mathcal{C}_2) = L(\mathcal{A}) \cap L(\mathcal{B})$ , and  $L(\mathcal{C}_3)$  is the complement of  $L(\mathcal{A})$ .*

### 3 $\omega$ -automatic structures

Let now  $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq n})$  be a relational structure, where  $M$  is the domain and, for each  $i \in [1, n]$ ,  $R_i^M$  is a relation of finite arity  $n_i$  on the domain  $M$ . The structure is said to be  $\omega$ -automatic if the domain and the relations on the domain are accepted by Büchi automata in the following sense.

**Definition 3.1** (see [Hod83, Blu99]) *Let  $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq n})$  be a relational structure, where  $n \geq 1$  is an integer, and each relation  $R_i$  is of finite arity  $n_i$ .*

*An  $\omega$ -automatic presentation of the structure  $\mathcal{M}$  is formed by a tuple of Büchi automata  $(\mathcal{A}, (\mathcal{A}_i)_{1 \leq i \leq n})$ , and a mapping  $h$  from  $L(\mathcal{A})$  onto  $M$ , such that :*

1.  $L(\mathcal{A}) \subseteq \Sigma^\omega$ , for a finite alphabet  $\Sigma$ , and
2. For each  $i \in [1, n]$ , the automaton  $\mathcal{A}_i$  accepts an  $n_i$ -ary relation  $R'_i$  on  $L(\mathcal{A})$ , and
3. The mapping  $h$  is an isomorphism from the structure  $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq n})$  onto the structure  $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq n})$ .

A relational structure is said to be  $\omega$ -automatic if it has an  $\omega$ -automatic presentation.

**Remark 3.2** An  $\omega$ -automatic presentation of a structure as defined above is often said to be an injective  $\omega$ -automatic presentation. It is actually a particular case of the more general notion of (non injective)  $\omega$ -automatic presentation of a structure, see [HKMN08]. However this restricted notion will be sufficient in the sequel so we have only given the definition of an  $\omega$ -automatic presentation of a structure in this particular case. The reader can see [KL08, HKMN08] for more information on this subject.

We recall now two important properties of  $\omega$ -automatic structures.

**Theorem 3.3 (Hodgson [Hod83])** *The first-order theory of an  $\omega$ -automatic structure is decidable.*

**Theorem 3.4 (see [Blu99])** *The class of  $\omega$ -automatic structures is closed under first-order interpretations. If  $\mathcal{M}$  is an  $\omega$ -automatic structure and  $\mathcal{M}'$  is a relational structure which is first-order interpretable in the structure  $\mathcal{M}$ , then the structure  $\mathcal{M}'$  is also  $\omega$ -automatic.*

Some examples of  $\omega$ -automatic structures can be found in [BG04, KL08].

## 4 Decidability properties

We consider a cellular automaton  $\mathcal{A} = (Q^{\mathbb{Z}}, \Delta)$ , where  $Q^{\mathbb{Z}}$  is the set of all possible configurations and  $\Delta$  is the global transition function.

We can replace the function  $\Delta$  by its graph, which is a binary relation  $\longrightarrow$  on the space  $Q^{\mathbb{Z}}$ . The space of configurations of the cellular automaton  $\mathcal{A}$ , equipped with the “next configuration relation”  $\longrightarrow$  will be denoted

$$\mathcal{S}_{\mathcal{A}} = (Q^{\mathbb{Z}}, \longrightarrow).$$

It is a simple relational structure with only one binary relation on the domain  $Q^{\mathbb{Z}}$ . Sutner proved that, for each cellular automaton  $\mathcal{A}$ , the first-order theory of the structure  $\mathcal{S}_{\mathcal{A}}$  is decidable, [Sut09]. We shall see that this result can be deduced from Theorem 3.3 on  $\omega$ -automatic structures.

**Lemma 4.1** *Let  $\mathcal{A} = (Q^{\mathbb{Z}}, \Delta)$  be a one-dimensional cellular automaton. Then the structure  $\mathcal{S}_{\mathcal{A}} = (Q^{\mathbb{Z}}, \longrightarrow)$  is  $\omega$ -automatic.*

**Proof.** We can represent a configuration  $c \in Q^{\mathbb{Z}}$  by an  $\omega$ -word  $x_c$  over the alphabet  $Q \times Q$  which is simply defined by  $x_c(0) = (c(0), c(0))$ , and, for each integer  $n \geq 1$ ,  $x_c(n) = (c(-n), c(n))$ .

The set of  $\omega$ -words in the form  $x_c$  for some configuration  $c \in Q^{\mathbb{Z}}$  is simply the set of  $\omega$ -words  $x$  in  $(Q \times Q)^\omega$  such that  $x(0) = (a, a)$  for some  $a \in Q$ . It is clearly a regular  $\omega$ -language accepted by a Büchi automaton  $\mathcal{B}$  and the function  $h : x_c \rightarrow c$  is a bijection from  $L(\mathcal{B})$  onto  $Q^{\mathbb{Z}}$ .

Due to the **local** definition of the “**local transition function**” of a cellular automaton, it is easy to see that the set of pairs of  $\omega$ -words  $(x_c, x_{c'})$  such that  $c \rightarrow c'$  is accepted by a Büchi automaton  $\mathcal{B}_1$  reading words over the product alphabet  $(Q \times Q) \times (Q \times Q)$ .

In fact the relation  $c \rightarrow c'$  can be checked by a scanner, as described in [PP04, page 283], i.e. “a machine equipped with a finite memory and a sliding window”, here of fixed size 3, to scan the input  $\omega$ -word  $(x_c, x_{c'})$ .  $\square$

Notice that Sutner proved in a similar way that, when representing configurations by *biinfinite words*, the relation  $c \rightarrow c'$  is accepted by a Büchi automaton reading *biinfinite words*, see [PP04, Sut09] for more details. Then he proved that the first-order theory of the structure  $\mathcal{S}_{\mathcal{A}}$  is decidable. We can obtain now this result as a consequence of Lemma 4.1 and Theorem 3.3.

**Corollary 4.2** *Let  $\mathcal{A} = (Q^{\mathbb{Z}}, \Delta)$  be a one-dimensional cellular automaton. Then the first-order theory of the structure  $\mathcal{S}_{\mathcal{A}}$  is decidable.*

**Proof.** It follows directly from Lemma 4.1 and the fact that the first-order theory of an  $\omega$ -automatic structure is decidable, proved by Hodgson in [Hod83].  $\square$

Notice that if  $R_1, R_2, \dots, R_n$  are first-order definable relations over  $Q^{\mathbb{Z}}$  then the expanded structure  $(Q^{\mathbb{Z}}, \rightarrow, R_1, R_2, \dots, R_n)$  is also  $\omega$ -automatic and then has a first-order decidable theory.

We have seen that Corollary 4.2 is an easy consequence of the fact that the structure  $\mathcal{S}_{\mathcal{A}}$  is  $\omega$ -automatic and of Hodgson’s result on the decidability of the first-order theory of an  $\omega$ -automatic structure [Hod83].

On the other hand, an extension of Hodgson’s result to first-order logic extended with the infinity quantifier  $\exists^\infty$  has been proved by Blumensath and Graëdel in [BG04], where  $\exists^\infty$  means “there are infinitely many”. Other extensions of Theorem 3.3 involving other counting and cardinality quantifiers have been proved by Kuske and Lohrey proved in [KL08]. We now recall the

definition of these quantifiers and the results for (injectively)  $\omega$ -automatic structures, [KL08].

The quantifier  $\exists$  is usual in first-order logic. In addition we can consider the infinity quantifier  $\exists^\infty$ , the cardinality quantifiers  $\exists^\kappa$  for any cardinal  $\kappa$ , and the modulo quantifiers  $\exists^{(t,k)}$  where  $t, k$  are integers such that  $0 \leq t < k$ . The set of first-order formulas is denoted by FO. For a class of cardinals  $C$ , the set of formulas using the additional quantifiers  $\exists^\infty$ ,  $\exists^\kappa$  for cardinals  $\kappa \in C$ , and modulo quantifiers  $\exists^{(t,k)}$ , is denoted by  $\text{FO}(\exists^\infty, (\exists^\kappa)_{\kappa \in C}, (\exists^{(t,k)})_{0 \leq t < k})$ . If  $\mathcal{L}$  is a set of formulas, the  $\mathcal{L}$ -theory of a structure  $\mathcal{M}$  is the set of sentences (i.e., formulas without free variables) in  $\mathcal{L}$  that hold in  $\mathcal{M}$ .

If  $\mathcal{M}$  is a relational structure in a signature  $\tau$  which contains only relational symbols and equality, and  $\psi$  is a formula in  $\text{FO}(\exists^\infty, (\exists^\kappa)_{\kappa \in C}, (\exists^{(t,k)})_{0 \leq t < k})$ , then the semantics of the above quantifiers are defined as follows.

- $\mathcal{M} \models \exists^\infty x \psi$  if and only if there are infinitely many  $a \in \mathcal{M}$  such that  $\mathcal{M} \models \psi(a)$ .
- $\mathcal{M} \models \exists^\kappa x \psi$  if and only if the set  $\psi^{\mathcal{M}} = \{a \in \mathcal{M} \mid \mathcal{M} \models \psi(a)\}$  has cardinality  $\kappa$ .
- $\mathcal{M} \models \exists^{(t,k)} x \psi$  if and only if the set  $\psi^{\mathcal{M}} = \{a \in \mathcal{M} \mid \mathcal{M} \models \psi(a)\}$  is finite and its cardinal  $|\psi^{\mathcal{M}}|$  is equal to  $t$  modulo  $k$ .

The  $\text{FO}(\exists^\infty)$ -theory of an (injectively)  $\omega$ -automatic structure has been shown to be decidable by Blumensath and Graëdel in [BG04]. We are now going to recall the more general result proved by Kuske and Lohrey in [KL08].

**Theorem 4.3 ([KL08])** *Let an (injective)  $\omega$ -automatic presentation of a structure  $\mathcal{M}$  formed by a tuple of Büchi automata  $(\mathcal{A}, (\mathcal{A}_i)_{1 \leq i \leq n})$ , and a mapping  $h$ . Let  $C$  be an at most countably infinite set of cardinals and  $\psi(x_1, \dots, x_n)$  be a formula of  $\text{FO}(\exists^\infty, (\exists^\kappa)_{\kappa \in C}, (\exists^{(t,k)})_{0 \leq t < k})$  over the signature of the structure  $\mathcal{M}$ . Then the relation*

$$R = \{(u_1, \dots, u_n) \in L(\mathcal{A})^n \mid \mathcal{M} \models \psi(h(u_1), \dots, h(u_n))\}$$

*is effectively  $\omega$ -automatic, i.e., one can effectively construct a Büchi automaton  $\mathcal{A}_R$  such that  $L(\mathcal{A}_R) = R$ .*

**Corollary 4.4 ([KL08])** *Let  $\mathcal{M}$  be an (injectively)  $\omega$ -automatic structure and let  $C$  be an at most countably infinite set of cardinals. Then the structure  $\mathcal{M}$  has a decidable  $\text{FO}(\exists^\infty, (\exists^\kappa)_{\kappa \in C}, (\exists^{(t,k)})_{0 \leq t < k})$ -theory.*

We can now apply these new results to the study of cellular automata.

**Corollary 4.5** *Let  $\mathcal{A} = (Q^{\mathbb{Z}}, \Delta)$  be a one-dimensional cellular automaton and let  $C$  be an at most countably infinite set of cardinals. Then the structure  $\mathcal{S}_{\mathcal{A}} = (Q^{\mathbb{Z}}, \longrightarrow)$  has a decidable  $\text{FO}(\exists^{\infty}, (\exists^{\kappa})_{\kappa \in C}, (\exists^{(t,k)})_{0 \leq t < k})$ -theory.*

Then Sutner's result is extended to a more powerful logic, containing counting and cardinality quantifiers.

We now give some examples of decidable properties we get for one-dimensional cellular automata.

Applying Theorem 4.3 and its corollaries, we can for example decide whether there are finitely many (respectively, countably many, uncountably many) configurations having finitely many (respectively, countably many, uncountably many) preimages by the global transition function  $\Delta$  of a cellular automaton  $\mathcal{A}$ . Moreover we can construct a Büchi automaton accepting exactly the (set of  $\omega$ -words representing the) configurations having finitely many (respectively, countably many, uncountably many) preimages.

We can also determine exactly the cardinal of the set of fixed points. Moreover if  $R$  is a regular  $\omega$ -language containing  $\omega$ -words representing configurations, then we can determine the cardinal of fixed points  $c \in Q^{\mathbb{Z}}$  such that  $x_c \in R$ . We can also construct a Büchi automaton accepting exactly the (set of  $\omega$ -words  $x_c$  representing the) fixed points  $c$  such that  $x_c \in R$ .

In a similar way we can determine whether there are infinitely many (respectively, countably many, uncountably many) 3-cycles. If there are only finitely many 3-cycles then we can compute the cardinal of this finite set. And we can construct a Büchi automaton accepting exactly  $\omega$ -words  $(x_c, x_{c'}, x_{c''})$  for the 3-cycles  $(c, c', c'')$ .

## 5 More efficient algorithms

A problem in the concrete application of Theorem 4.3 and its corollaries is the complexity of the algorithms we get from them. Sutner noticed in [Sut09] that the algorithm he got for the decidability of the first-order theory of the phase-space  $\mathcal{S}_{\mathcal{A}}$  of a cellular automaton  $\mathcal{A}$  is in general non elementary. This is also the case for the algorithms obtained for (injectively)  $\omega$ -automatic structures, see [KL08].

However some more efficient algorithms can be obtained from results of Kuske and Lohrey on  $\omega$ -automatic structures of bounded degree. A relational structure is of bounded degree iff its Gaifman-graph has bounded degree. The Gaifman-graph of the structure  $\mathcal{S}_{\mathcal{A}} = (Q^{\mathbb{Z}}, \longrightarrow)$  is simply the undirected graph  $(Q^{\mathbb{Z}}, R)$ , where for any configurations  $c, c' \in Q^{\mathbb{Z}}$ ,  $[R(c, c')] \text{ iff } [c \longrightarrow c' \text{ or } c' \longrightarrow c]$ , (see the precise definition of the Gaifman-graph in [KL08]).

Notice first that, for a cellular automaton  $\mathcal{A} = (Q^{\mathbb{Z}}, \Delta)$ , any configuration  $c \in Q^{\mathbb{Z}}$  has a unique successor for the relation  $\longrightarrow$ . Then the phase-space  $\mathcal{S}_{\mathcal{A}}$  is a structure of bounded degree if and only if there is an integer  $k \geq 1$  such that any configuration  $c \in Q^{\mathbb{Z}}$  has at most  $k$  pre-images by the global transition function  $\Delta$  of the cellular automaton  $\mathcal{A}$ .

On the other hand it is well known that this property corresponds to the important class of *surjective* cellular automata, see for example [Kur03, page 217]. As this class is an important one, we are going to recall results of Kuske and Lohrey on  $\omega$ -automatic structures of bounded degree, see [KL08] for more details.

Kuske and Lohrey considered the logic  $\mathcal{L}(\mathcal{Q}_u)$  in a relational signature  $\tau$ . Formulas of the logic  $\mathcal{L}(\mathcal{Q}_u)$  are built from atomic formulas of the form  $R(x_1, \dots, x_k)$  where  $R \in \tau$  is a  $k$ -ary relational symbol and  $x_1, \dots, x_k$  are first-order variables ranging over the universe of the underlying structure, using boolean connectives and quantifications of the form

$$\mathcal{Q}_{\mathcal{C}}y : (\psi_1(\bar{x}, y), \dots, \psi_n(\bar{x}, y)).$$

Above, each  $\psi_i$  is already a formula of  $\mathcal{L}(\mathcal{Q}_u)$ ,  $\bar{x}$  is a sequence of variables, and  $\mathcal{C}$  is an  $n$ -ary relation over cardinals.

We now recall the semantics of a  $\mathcal{Q}_{\mathcal{C}}$  quantifier. Let  $\mathcal{M} = (M, R_1, \dots, R_j)$  be a  $\tau$ -structure where  $M$  is the domain and each  $R_i$  is a  $n_i$ -ary relation on  $M$ . Let  $\bar{u}$  be a tuple of elements of  $M$  of the same length as  $\bar{x}$ , then  $\mathcal{M} \models \mathcal{Q}_{\mathcal{C}}y : (\psi_1(\bar{u}, y), \dots, \psi_n(\bar{u}, y))$  if and only if  $(c_1, \dots, c_n) \in \mathcal{C}$ , where each  $c_i$  is the cardinality of the set  $\{a \in M \mid \mathcal{M} \models \psi_i(\bar{u}, a)\}$ .

The quantifier  $\mathcal{Q}_{\mathcal{C}}$  is called an  $n$ -dimensional counting quantifier.

Notice that all the quantifiers  $\exists, \exists^{\infty}, \exists^{\kappa}$ , for a cardinal  $\kappa$ , and the modulo quantifiers  $\exists^{(t, k)}$ , are particular cases of one-dimensional counting quantifiers. A well known two-dimensional counting quantifier is the Härtig quantifier  $Iy : (\psi_1(\bar{x}, y), \psi_2(\bar{x}, y))$  expressing “the number of  $y$  satisfying  $\psi_1(\bar{x}, y)$  is equal to the number of  $y$  satisfying  $\psi_2(\bar{x}, y)$ ” (the relation  $\mathcal{C}$  is here the identity relation on cardinals).

Let  $\mathbb{C}$  be a class of relations on cardinals. A formula in  $\mathcal{L}(\mathbb{C})$  is a formula of  $\mathcal{L}(\mathcal{Q}_u)$  which uses only quantifiers of the form  $\mathcal{Q}_C$  for  $C \in \mathbb{C}$ . The logic  $\text{FO}(\mathbb{C})$  is the first order logic extended with the quantifiers  $\mathcal{Q}_C$  for  $C \in \mathbb{C}$ . If  $\mathbb{C} = \{C\}$  is a singleton then we write simply  $\text{FO}(C)$  instead of  $\text{FO}(\mathbb{C})$ .

We now recall the decidability result of Kuske and Lohrey.

**Theorem 5.1 (Kuske and Lohrey [KL08])** *Let  $\mathbb{C} = \{C_i \mid i \in \mathbb{N}\}$ , where  $C_i$  is a relation on  $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ . Let  $\mathcal{M}$  be an (injectively)  $\omega$ -automatic structure of bounded degree. Then the  $\mathcal{L}(\mathbb{C})$ -theory of  $\mathcal{M}$  can be decided in triply exponential space by a Turing machine with oracle  $\{(i, \bar{c}) \mid i \in \mathbb{N}, \bar{c} \in C_i\}$ .*

**Corollary 5.2** *Let  $\mathcal{A}$  be a surjective one-dimensional cellular automaton, and let  $\mathbb{C} = \{C_i \mid i \in \mathbb{N}\}$ , where  $C_i$  is a relation on  $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ . Then the  $\mathcal{L}(\mathbb{C})$ -theory of the structure  $\mathcal{S}_{\mathcal{A}}$  can be decided in triply exponential space by a Turing machine with oracle  $\{(i, \bar{c}) \mid i \in \mathbb{N}, \bar{c} \in C_i\}$ .*

Notice that it is proved in [KL08] that even if the continuum hypothesis is not satisfied then allowing cardinals  $\kappa$  with  $\aleph_0 < \kappa < 2^{\aleph_0}$  in the relations  $C_i$  “does not change the results on  $\omega$ -automatic structures”.

Notice that in fact the logic  $\mathcal{L}(\mathcal{Q}_u)$  has a restricted expressive power for structures of bounded degree. This follows from the following result.

**Theorem 5.3** *Let  $\mathcal{M}$  be a  $\tau$ -structure of bounded degree, and let  $\phi(\bar{x})$  be a formula of  $\mathcal{L}(\mathcal{Q}_u)$ . Then there is a first-order formula  $\psi(\bar{x}) \in \text{FO}$  such that  $\mathcal{M} \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ .*

In the case of (injectively)  $\omega$ -automatic structure of bounded degree we can effectively construct the formula  $\psi$ , see [KL08, Corollary 3.13]. We give now directly the important corollary obtained in the case of surjective one-dimensional cellular automata.

**Corollary 5.4** *Let  $\mathcal{A}$  be a surjective one-dimensional cellular automaton, and let  $\mathbb{C} = \{C_i \mid i \in \mathbb{N}\}$ , where  $C_i$  is a relation on  $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ . For a given formula  $\phi(\bar{x}) \in \mathcal{L}(\mathbb{C})$ , one can construct in elementary space (modulo  $\mathbb{C}$ ) a first-order formula  $\psi(\bar{x})$  and a Büchi automaton  $\mathcal{B}_\phi$  such that for any  $\bar{u} = (c_1, \dots, c_k) \in (Q^{\mathbb{Z}})^k$ , (where  $k = |\bar{x}|$ ),*

$$\mathcal{S}_{\mathcal{A}} \models \phi(\bar{u}) \Leftrightarrow \mathcal{S}_{\mathcal{A}} \models \psi(\bar{u}) \Leftrightarrow (x_{c_1}, \dots, x_{c_k}) \in L(\mathcal{B}_\phi).$$

## 6 From cellular automata to automatic structures

We have seen that we can infer new decidability results for cellular automata from recent results about  $\omega$ -automatic structures.

On the other hand we can also obtain new results about automatic structures from results about cellular automata.

If  $\mathcal{A} = (Q, \delta)$  is a one-dimensional cellular automaton with a particular quiescent state  $\# \in Q$ , we consider the set of configurations with finite support  $\mathcal{C}_{fin}$  of  $\mathcal{A}$ . These configurations are in the form  $c \in Q^{\mathbb{Z}}$  with  $c(n) = \#$  for all  $n \leq -k$  or  $n \geq k$  where  $k \geq 0$  is an integer.

We shall consider in this section the notion of automatic structure which is defined as in the case of an  $\omega$ -automatic structure but using automata reading *finite words* instead of *infinite words*. Thus an automatic structure is always a countable structure, see [BG04, KNRS07, Hod83] for more details. We can now state the following result.

**Proposition 6.1** *Let  $\mathcal{A} = (Q^{\mathbb{Z}}, \Delta)$  be a one-dimensional cellular automaton. Then the structure  $\mathcal{S}_{\mathcal{A},fin} = (\mathcal{C}_{fin}, \longrightarrow)$  is automatic.*

Then we have new examples of directed automatic graphs. The directed graph  $\mathcal{S}_{\mathcal{A},fin}$  of the set of configurations with finite support  $\mathcal{C}_{fin}$  of a given one-dimensional cellular automaton  $\mathcal{A}$ , equipped with the “next configuration relation  $\longrightarrow$ ”.

In a directed graph  $(G, \longrightarrow)$ , the reachability relation  $\longrightarrow^*$  is the reflexive and transitive closure of the relation  $\longrightarrow$ . It is clear that in a recursive graph, hence also in an automatic graph, the reachability relation is recursively enumerable. Sutner proved that actually for each recursively enumerable Turing degree  $\mathbf{d}$  there is a cellular automaton  $\mathcal{A}$  such that the reachability relation in the graph  $\mathcal{S}_{\mathcal{A},fin}$  has exactly the Turing degree  $\mathbf{d}$ , see [Sut08a]. So we obtain the following corollary.

**Corollary 6.2** *For each recursively enumerable Turing degree  $\mathbf{d}$  there is an automatic graph  $(G, \longrightarrow)$  in which the reachability relation has exactly the Turing degree  $\mathbf{d}$ .*

Recall that the structure of Turing degrees is very complicated. If  $\mathbf{0}$  denoted the degree of recursive sets and  $\mathbf{0}'$  denotes the degree of complete recursively

enumerable sets, then by the famous result of Friedberg and Muchnik answering Post's question, there is a Turing degree  $\mathbf{d}$  such that  $\mathbf{0} < \mathbf{d} < \mathbf{0}'$ , see [Rog67]. Moreover by Sack's Density Theorem, for two recursively enumerable degrees  $\mathbf{d}_1 < \mathbf{d}_2$  there is a third one  $\mathbf{d}_3$  such that  $\mathbf{d}_1 < \mathbf{d}_3 < \mathbf{d}_2$ .

The above Corollary 6.2 can be refined. In a directed graph  $(G, \longrightarrow)$  the *confluence relation*  $Conf_G$  is defined by: for any  $x, y \in G$ ,  $Conf_G(x, y)$  iff there is some  $z \in G$  such that  $x \xrightarrow{*} z$  and  $y \xrightarrow{*} z$ . Sutner proved that for any recursively enumerable degrees  $\mathbf{d}_1$  and  $\mathbf{d}_2$  there is a cellular automaton  $\mathcal{A}$  such that the reachability relation in the graph  $\mathcal{S}_{\mathcal{A},fin}$  has exactly the Turing degree  $\mathbf{d}_1$  and the confluence relation in the graph  $\mathcal{S}_{\mathcal{A},fin}$  has exactly the Turing degree  $\mathbf{d}_2$ . So we can state the following corollary.

**Corollary 6.3** *For each recursively enumerable Turing degrees  $\mathbf{d}_1$  and  $\mathbf{d}_2$  there is an automatic graph  $(G, \longrightarrow)$  in which the reachability relation has exactly the Turing degree  $\mathbf{d}_1$  and the confluence relation has exactly the Turing degree  $\mathbf{d}_2$ .*

Khoussainov, Nies, Rubin, and Stephan proved in [KNRS07] that the isomorphism problem for automatic graphs is  $\Sigma_1^1$ -complete, hence as complicated as the isomorphism problem for recursive graphs. This showed that the class of automatic graphs is very rich. The above results confirm this from the point of view of Turing degrees.

Notice that if an automatic graph is viewed as the set of configurations of an infinite state transition system, then decision problems naturally appear and are very important in this context. Some decision problems for automatic graphs have been recently studied by Kuske and Lohrey in [KL09]. For instance they proved that the problem of the existence of an Hamiltonian path in a planar automatic graph of bounded degree is  $\Sigma_1^1$ -complete, and that the problem of the existence of an Euler path in an automatic graph is  $\Pi_2^0$ -complete.

We can now infer other results from some results of Sutner on cellular automata. Let  $\mathbb{C}_{\mathbf{d}}$  be the class of cellular automata  $\mathcal{A}$  such that the reachability relation of the graph  $\mathcal{S}_{\mathcal{A},fin}$  has exactly degree  $\mathbf{d}$ . Sutner proved that it is  $\Sigma_3^{\mathbf{d}}$ -complete to decide membership of a given cellular automaton in the class  $\mathbb{C}_{\mathbf{d}}$ , for any recursively enumerable Turing degree  $\mathbf{d}$ . In particular, it is  $\Sigma_3^0$ -complete to decide membership in  $\mathbb{C}_{\mathbf{0}}$  and it is  $\Sigma_4^0$ -complete to decide membership in  $\mathbb{C}_{\mathbf{0}'}$ , see [Sut08a]. We can now state the following result on automatic graphs.

### Corollary 6.4

1. It is  $\Sigma_3^0$ -complete to decide whether the reachability relation of a given automatic graph is recursive.
2. It is  $\Sigma_4^0$ -complete to decide whether the reachability relation of a given automatic graph is  $\Sigma_1^0$ -complete.

**Proof.** The lower bound results follow from the corresponding results for graphs of cellular automata.

The fact that deciding whether the reachability relation of a given automatic graph is recursive is in the class  $\Sigma_3^0$  follow from the known fact that deciding whether a given recursively enumerable set is recursive is in the class  $\Sigma_3^0$ , see [Rog67, page 327]. The fact that deciding whether the reachability relation of a given automatic graph is  $\Sigma_1^0$ -complete is in the class  $\Sigma_4^0$  follow from the known fact that deciding whether a given recursively enumerable set is  $\Sigma_1^0$ -complete is in the class  $\Sigma_4^0$ , see [Rog67, page 330].  $\square$

## 7 Concluding remarks

We have obtained some new decidability results about cellular automata, showing that their graphs are  $\omega$ -automatic and applying recent results of Kuske and Lohrey.

On the other hand we have got new examples of automatic graphs and new results about these graphs, applying recent results of Sutner about cellular automata.

We hope that the interplay of the two domains of cellular automata and of automatic structures will be fruitful and will provide new results in both fields of research.

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