Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate fully non-linear elliptic equations

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Abstract. In this paper, we study fully non-linear elliptic equations in non-divergence form which can be degenerate when "the gradient is small". Typical examples are either equations involving the m-Laplace operator or Bellman-Isaacs equations from stochastic control problems. We establish an Alexandroff-Bakelman-Pucci estimate and we prove a Harnack inequality for viscosity solutions of such degenerate elliptic equations.

Keywords: Degenerate fully nonlinear elliptic equation, singular elliptic equation, non-divergence form, Alexandroff-Bakelman-Pucci estimate, weak Harnack inequality, local maximum principle, Harnack inequality, Hölder regularity, viscosity solutions

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1 Introduction

Following the original strategy of Krylov and Safonov [23, 24], Delarue [13] proved by probabilistic methods a Harnack inequality for quasi-linear elliptic equations of the form

$$-\operatorname{Tr}(A(x,u,Du)D^2u) + H(x,u,Du) = 0\,, \quad x \in \Omega \tag{1}$$

(where Ω is a domain of $\mathbb{R}^n$) in the case where the $n \times n$ matrix $A(x,p)$ can degenerate. Precisely, he assumes

$$\Lambda^{-1}\lambda(p)I \le A(x,u,p) \le \Lambda\lambda(p)I \tag{2}$$

$$H(x,u,p) \le \Lambda(1+\lambda(p))(1+|p|) \tag{3}$$

where $\Lambda \ge 1$, $\lambda : \mathbb{R}^n \to \mathbb{R}^+$ is continuous and such that $\lambda(p) \ge \lambda_F$ if $|p| \ge M_F$. In (2), I denotes the identity matrix and the inequalities are understood in the sense of the usual partial order on the set of real symmetric matrices. The model example of (1) is the m-Laplace equation where $A(x,p) = |p|^{m-2}$ for some $m > 2$. An important application of the Harnack inequality is the derivation of an Hölder estimate for the solution of (1).

In this paper, we generalize this result to the case of fully non-linear elliptic equations in non-divergence form

$$F(x, u, Du, D^2u) = 0\,, \quad x \in \Omega \tag{4}$$

which can be either degenerate or singular. We do so by proving first an Alexandroff-Bakelman-Pucci (ABP for short) estimate. This is the first main difference with [13] and the first main contribution of this paper. Important examples of (4) which are out of the scope of [13] are Bellman-Isaacs equations appearing in the context of stochastic control problems. We also generalize and/or recover results from [11, 3] where an ABP estimate and a Harnack inequality respectively are obtained for

$$-F_0(Du, D^2u) - b(x) \cdot Du|Du|^\alpha - cu|u|^\alpha + f_0(x) = 0\,, \quad x \in \Omega \tag{5}$$

where F positively homogeneous of order $\alpha \in (-1, 1)$ (see Section 6 for precise assumptions). If $\alpha \in [0, 1)$, the equation is degenerate. If $\alpha \in (-1, 0]$, the equation is singular. Even if this equation does not formally enter into our general framework, we will explain how the results of [11, 3] can be derived from ours.

Known results. Krylov and Safonov [23, 24] first proved a Harnack inequality for second order elliptic equations in non-divergence form with measurable coefficients. This result is often presented as the counterpart of the De Giorgi and Nash estimates [12, 26] for divergence form equations.

As far as degenerate elliptic equations are concerned, De Giorgi and Nash estimates were obtained for equations in divergence form and for degeneracies of p-Laplace type. See for instance [28, 25].

Krylov and Safonov estimates were obtained by Caffarelli [5] for fully non-linear elliptic equations of the form $F(x, D^2u) = 0$ (see also [30, 17]). As explained in [17], a fondamental tool in this approach is the Alexandroff-Bakelman-Pucci estimate. Many authors extended these results since then; see for instance [15, 21, 7, 27] and references therein.

To the best of our knowledge and as far as degenerate elliptic equations in non-divergence form are concerned, the Krylov and Safonov estimates obtained by Delarue [13] are the first ones.

After this work was completed, Birindelli and Demengel [3] obtained a Harnack inequality for degenerate elliptic equations of the form (5) with $\alpha \in [0, 1)$ in dimension 2. Reading their interesting paper, we understood that we could recover (and in fact extend) their results and deal with singular equations (see Section 6). Their work aims at generalizing the results of Dávila, Felmer and Quaas [10] where the same elliptic equation is considered but with $\alpha \in (-1, 0]$ so that the equation is singular. We also mention that an ABP estimate is proved in [11] for degenerate and singular equations. We will explain that it can be also derived from ours (see Section 6).

Main results. Let us now describe a bit more precisely our main results. We use the techniques developped by Caffarelli [5] (see also [6]) instead of probability arguments to get, apart from the Alexandroff-Bakelman-Pucci estimate, a weak Harnack inequality and a local maximum principle. It is then easy to derive a Harnack inequality and a Hölder estimate of a solution of (4).

First and foremost, we mention that, as in [5, 13], we use the notion of viscosity solution [9] since the equation is fully non-linear. We recall that if singular equations of the form (5) are considered, the classical notion of viscosity solutions must be adapted; see [2].

We next make precise the standing assumptions that the non-linearity F must satisfy. Throughout the paper, $\mathcal{S}_n$ denotes the space of real symmetric $n \times n$ matrices and B_R denotes the open ball of radius $R \geq 0$.

Assumption (A).

- F is *continuous* on $\Omega \times \mathbb{R} \times \mathbb{R}^n \setminus B_{M_F} \times \mathcal{S}_n$ for some $M_F \geq 0$;
- F is *(degenerate) elliptic, i.e.* for all $x \in \Omega$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$ ($p \neq 0$ for singular equation) and $X, Y \in \mathcal{S}_n$,
$$X \leq Y \Rightarrow F(x,r,p,Y) \leq F(x,r,p,X)\,.$$
- F is *proper* i.e. it is non-decreasing with respect to its r variable.

Our first main result (Theorem 1) is an ABP estimate for lower semi-continuous super-solutions of (4) on a ball B_d where F is *strictly elliptic for "large gradients"*

$$\left.\begin{array}{r} X \geq 0 \\ |p| \geq M_F \\ F(x,r,p,X) \geq 0 \end{array}\right\} \Rightarrow -\lambda_F \operatorname{tr}(X) + \sigma(x)|p| + g(x,u) \geq 0 \tag{6}$$

for some continuous functions g and σ and some constants $M_F \geq 0$, $\lambda_F > 0$. This condition holds true if F satisfies (7) but it is more general. An ABP estimate permits to control $\sup_{B_d} u^-$ in terms of $M_\partial = \sup_{\partial B_d} u^-$ and the L^n-norms of $g(x, M_\partial)$ and σ appearing in (6). In order to get such an estimate, we use the techniques that we introduced in [18]. As we already mentioned it in [18], the ABP estimate that we are able to obtain differs slightly from classical ones in the sense that we can prove it under a weaker condition than (7); moreover, the super-solution is only lower semi-continuous. We recall that this is an apriori estimate: structure conditions ensuring the uniqueness of the solution are not required. We finally mention that when the equation is strictly elliptic ($M_F = 0$), we recover the classical ABP estimate.

Our second main result (Corollary 1) is a Harnack inequality for (4). This inequality is a consequence of a weak Harnack inequality and a local maximum principle proved by generalizing in an appropriate way (2) and (3). In view of

(2), one can consider the quasilinear equation (1) where A and H are replaced with

$$\tilde{A}(x,u,Du) = \frac{1}{\lambda(Du)} A(x,u,Du) \quad \text{and} \quad \tilde{H}(x,u,Du) = \frac{1}{\lambda(Du)} A(x,u,Du)\,.$$

Hence, the new quasi-linear equation is uniformly elliptic. However, the first order term is, in this case, eventually singular and (2) can be seen as an assumption concerning the first order term. In the case of the m-Laplace equation, $\lambda(p) = |z|^{m-2}$ and H has therefore a polynomial growth of order $m-1$. Assumptions (2), (3) are replaced with

$$\left.\begin{array}{r} |p| \geq M_F \\ F(x,u,p,X) \geq 0 \end{array}\right\} \Rightarrow \mathcal{M}^+(X) + \sigma(x)|p| + \gamma_F u + f(x) \geq 0\,, \tag{7}$$

$$\left.\begin{array}{r} |p| \geq M_F \\ F(x,u,p,X) \leq 0 \end{array}\right\} \Rightarrow \mathcal{M}^-(X) - \sigma(x)|p| + \gamma_F u - f(x) \leq 0 \tag{8}$$

where $\sigma, f : \overline{B} \to \mathbb{R}$ are continuous and M_F and γ_F are non-negative constants. It is important to remark that if F satisfies (7), (8), then it can be degenerate or singular and it can have a superlinear growth in p.

An important consequence of the Harnack inequality is the Hölder regularity of solutions of (4) (see Theorem 2). As far as the regularity of solutions of (4) is concerned, we notice that by assuming (7) and (8), we cannot expect more than Lipschitz continuity. Indeed, by making such an assumption, we somehow forget about all small gradients and we cannot expect these small gradients to be regular. We also point out that it is easier to prove the uniqueness of a Hölder continuous function than to prove a strong comparison result between discontinuous viscosity sub- and super-solutions (which is the classical way to get uniqueness of viscosity solutions [9]). To finish with, we shed light on the fact that, as for the ABP estimate, we recover the Harnack inequality of [5] in the strictly elliptic case ($M_F = 0$).

Extensions. We will explain how to deal with non-linearities, after redefining them if necessary, growing quadratically with respect to the gradient. Precisely, (7) and (8) are replaced with

$$\left.\begin{array}{r} |p| \geq M_F \\ F(x,u,p,X) \geq 0 \end{array}\right\} \Rightarrow \mathcal{M}^+(X) + \sigma(x)|p| + \sigma_2|p|^2 + \gamma_F u + f(x) \geq 0\,, \tag{9}$$

$$\left.\begin{array}{r} |p| \geq M_F \\ F(x,u,p,X) \leq 0 \end{array}\right\} \Rightarrow \mathcal{M}^-(X) - \sigma(x)|p| - \sigma_2|p|^2 + \gamma_F u - f(x) \leq 0 \tag{10}$$

where $\sigma, f : \overline{B} \to \mathbb{R}$ are continuous and M_F, σ_2 and γ_F are non-negative constants. In this case, it is known [30, 22] that it is not possible to get a weak Harnack inequality which does not depend on the L^∞-norm of the solution. See Section 5 for more details and comments.

As far as extensions of these results are concerned, we would like to mention next that we could have used L^p-viscosity solutions [4] instead of classical continuous viscosity solutions in order to be able to deal with discontinuous coefficients. We chose not to do so in order to avoid technicalities but we think that this can be done. We also mention that it is sometimes more difficult to get a classical ABP estimate when using this notion of solution; for instance in [21], the ABP estimate does not involve the contact set of the function.

We also mention that the parabolic case will be addressed in a future work.

Additional comments. Assumption (6) permits to take into account non-linearity growing linearly with respect to the gradient. Such an assumption appears in [29] where Trudinger proved that strong solutions satisfy a weak Harnack inequality for such non-linearities if σ is sufficiently integrable. This result has been generalized to viscosity solutions since then; see for instance [16, 20].

We recall that it is possible to use the techniques introduced in [19] in order to prove the Hölder regularity of viscosity solutions much more easily. But the estimate of the Hölder constant depends in this case on the modulus of continuity of the coefficients of the equation.

Organization of the article. The paper is organized as follows. In Section 2, we construct a barrier function that will be used when proving the Harnack inequality. We also recall the definition of two Pucci operators. In Section 3, we establish an ABP estimate. In Section 4, we successively prove a weak Harnack inequality and a local maximum principle. We also derive from these two results a Harnack inequality. In Section 5, we explain how to deal with elliptic equations with quadratic dependence on the gradient. As applications of our results, we generalize and/or recover some results from [3, 11] in Section 6. Section 7 is dedicated to proofs of our main results. Appendices A and B are added for the sake of completeness of proofs and for the reader's convenience. In particular, we give in Appendix A detailed proofs of results which can be easily derived from classical ones. In Appendix B, we explain how to adapt proofs of Section 7 in order to derive Theorem 7 in Section 6.

Notation. A ball of radius r centered at x is denoted by $B(x,r)$ or $B_r(x)$. If $x = 0$, we simply write B_r. ω_n denotes the volume of the unit ball. The hypercube $\Pi_{i=1}^n(x_i - r/2, x_i + r/2)$ is denoted by $Q_r(x)$. If $x = 0$, we simply write Q_r.

Given a vector $a \neq 0$, $\hat{a}$ denotes $a/|a|$. Given two vectors $a, b \in \mathbb{R}^n$. I denotes the identity matrix. The set of real symmetric $n \times n$ matrices is denoted by $\mathcal{S}_n$.

A constant is universal if it only depends on n (dimension), q (constant greater than n fixed in all the paper), λ_F and Λ_F (constants appearing in the maximal Pucci operator).

Given a lower semi-continuous function u, $D^{2,-}u(x)$ (resp. $D^{2,-}u(x)$) denotes the set of all subjets (resp. limiting subjets) of u at point x. See [9] for definitions.

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2 Preliminaries

Pucci operators. We recall the definition of two important second order non-linear elliptic operators. For all $M \in \mathcal{S}_n$, we define

$$\begin{aligned} \mathcal{M}^+(M) &= \sup_{A \in \mathcal{A}_{\lambda_F,\Lambda_F}} (-\mathrm{Tr}\,(AM)) \\ \mathcal{M}^-(M) &= \inf_{A \in \mathcal{A}_{\lambda_F,\Lambda_F}} (-\mathrm{Tr}\,(AM)) \end{aligned}$$

where $\mathcal{A}_{\lambda_F,\Lambda_F} = \{A \in \mathcal{S}_n : \lambda_F I \le A \le \Lambda_F I\}$. We will refer to these operators as the *maximal* and *minimal Pucci operators*. Remark that $\mathcal{M}^+$ is subadditive. More precisely, it is the support function of the set $-\mathcal{A}_{\lambda_F,\Lambda_F}$. We will also use the fact that $\mathcal{M}^-(M) = -\mathcal{M}^+(-M)$.

Construction of a barrier. We now construct a barrier that will be used when proving the (weak) Harnack inequality.

Lemma 1 (Construction of a barrier). *Given a constant $\varepsilon_0 > 0$, there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$, a universal constant $M_{\mathrm{B}} > 1$ and constants $C_{\mathrm{B}} > 0$, $R, r > 0$ (with $R \ge (3r/2)\sqrt{n}$) depending only on the dimension n, λ_F, Λ_F and ε_0, such that*

$$\varphi \ge 0 \quad \text{in } \mathbb{R}^n \setminus B_R \tag{11}$$

$$\varphi \le -2 \quad \text{in } Q_{3r} \tag{12}$$

$$\varphi \ge -M_{\mathrm{B}} \quad \text{in } \mathbb{R}^n \tag{13}$$

$$|D\varphi| \le \varepsilon_0 \quad \text{in } \mathbb{R}^n \tag{14}$$

$$\mathcal{M}^-\varphi + C_{\mathrm{B}}\xi \ge 0 \quad \text{in } \mathbb{R}^n \tag{15}$$

where $\xi : \mathbb{R}^n \to [0,1]$ is a continuous function supported in $\bar{Q}_r$.

Remark 1. We recall that this barrier function will be used to prove the weak Harnack inequality. At first glance, it is not clear why we need to construct a function φ such that $\mathcal{M}^-\varphi \ge 0$ on Q_r and $\varphi \le -2$ on Q_{3r}. This will be clearer when applying the cube decomposition in order to estimate the volume of all the level sets (and not only one) of a super-solution. And we choose $R \ge (3r/2)\sqrt{n}$ in order that $Q_{3r} \subset B_R$.

Proof. We follow [6] by choosing φ under the following form for $x \notin B_r$

$$\varphi(x) = M_1 - M_2 |x|^{-\alpha}$$

where $\alpha > 0$ will be chosen later and $M_1, M_2 > 0$ have to be chosen such that (11), (12) and (14) hold true (with $R \geq (3r/2)\sqrt{n}$). It is enough to impose

$$\begin{array}{rcl} M_2 & \leq & M_1 R^\alpha \,, \\ ((3r/2)\sqrt{n})^\alpha (M_1 + 2) & \leq & M_2 \,, \\ M_2 & \leq & \varepsilon_0 \dfrac{r^{\alpha+1}}{\alpha} \,. \end{array}$$

After elementary computations, it is equivalent to

$$((3r/2)\sqrt{n})^\alpha (M_1 + 2) \leq M_2 \leq \min(M_1 R^\alpha, \varepsilon_0 r^{\alpha+1}/\alpha) \,.$$

One can choose M_2 and M_1 so that they satisfy the previous condition if and only if

$$2 \frac{((3r/2)\sqrt{n})^\alpha}{R^\alpha - ((3r/2)\sqrt{n})^\alpha} \leq M_1 \leq \frac{\varepsilon_0}{\alpha((3/2)\sqrt{n})^\alpha} r - 2 \,.$$

Hence, we choose $R = q(3r/2)\sqrt{n}$ with $q > 1$ and $r > 0$ satisfying

$$\frac{2}{q^\alpha - 1} \leq \frac{\varepsilon_0}{\alpha((3/2)\sqrt{n})^\alpha} r - 2 \,.$$

It is now enough to choose $q > 1$ such that $\frac{2}{q^\alpha - 1} \leq 1$ and r such that

$$\frac{\varepsilon_0}{\alpha((3/2)\sqrt{n})^\alpha} r \geq 3 \,.$$

We now choose $\alpha > 0$ so that (15) holds true. If $x \notin B_r$, we have

$$\mathcal{M}^-(D^2\varphi(x)) \quad = \quad -\alpha M_2 |x|^{-(\alpha+2)} (\Lambda_F (n-1) - \lambda_F (\alpha + 1)) \,.$$

Hence it is enough to choose $\alpha > \max(0, \frac{\Lambda_F}{\lambda_F}(n-1) - 1)$ to conclude.

It is next easy to extend φ on $\mathbb{R}^n$ such that (12) and (14) remain true and (13) is satisfied too for some universal constant $M_B > 1$. Indeed, we have outside B_r

$$\varphi \geq M_1 - M_2 r^{-\alpha} \geq 2 \frac{1}{q^\alpha - 1} - \frac{\varepsilon_0 r}{\alpha} \,.$$

It is now enough to remark that q and $\varepsilon_0 r$ can be choosen universal and we also saw above that α can be chosen universal too. Hence M_B can be chosen universal. $\square$

Rescaling solutions. We will have to rescale sub- or super-solutions several times. We need to know how non-linearities are rescaled in order, for instance, to determine if these new F's satisfy assumptions.

Lemma 2 (Rescaling solutions)**.** *Given* $R_0 > 0$*,* $t_0 > 0$ *and* $x_0 \in \mathbb{R}^n$*, let* u *be a super-solution of* F *on* $Q_{t_0 R_0}(x_0)$*. Consider the linear map* $T : Q_{R_0} \to Q_{t_0 R_0}(x_0)$ *defined by* $T(y) = x_0 + t_0 y$*. Then the scaled solution* $u_s(y) = \frac{1}{M_0} u(T(y))$ *is a super-solution of* $F_s = 0$ *in* Q_{R_0} *with*

$$F_s(y, v, q, Y) = \frac{t_0^2}{M_0} F(x_0 + t_0 y, M_0 v, t_0^{-1} M_0 q, t_0^{-2} M_0 Y)\,.$$

If F *satisfies* (7) *(resp.* (8)*), then* F_s *satisfies* (7) *(resp.* (8)*) with constants* M_s, γ_s *and functions* σ_s *and* f_s

$$M_s = \frac{t_0 M_F}{M_0}, \quad \gamma_s = t_0^2 \gamma_F, \quad \sigma_s = t_0 . \sigma \circ T, \quad f_s = \frac{t_0^2}{M_0} f \circ T\,.$$

In particular,

$$\|f_s\|_{L^n(Q_{R_0})} = \frac{t_0}{M_0} \|f\|_{L^n(Q_{t_0 R_0(x_0)})}\,, \qquad \|\sigma_s\|_{L^q(Q_{R_0})} = t_0^{1-\frac{n}{q}} \|\sigma\|_{L^q(Q_{t_0 R_0}(x_0))}\,.$$

3 An ABP estimate

As explained in the introduction, we can prove an ABP estimate as soon as the non-linearity F satisfies a strict ellipticity condition "for large gradients". We must also prescribe a growth condition with respect to first order terms. We thus assume that F satisfies (6). Our first main result is the following theorem.

Theorem 1 (ABP estimate)**.** *Consider a non-linearity* F *which satisfies* (A) *and* (6)*. Let* u *be a (lsc) super-solution of* (4) *in* B_d*. Then*

$$\sup_{B_d} u^- \leq \sup_{\partial B_d} u^- + Cd \left(M_F + \left(\int_{B_d \cap \{u + M_\partial = \Gamma(u)\}} (f^+)^n \right)^{1/n} \right) \tag{16}$$

where $M_\partial = \sup_{\partial B_d} u^-$*,* $\Gamma(u)$ *is the convex hull of* $\min(u + M_\partial, 0)$ *extended by* 0 *on* B_{2d}*,* $f(x) = g(x, -M_\partial)$ *and* C *is a constant (only) depending on* $\|\sigma\|_{L^n(B_d)}$*,* n *and* λ_F*.*

Remark 2*.* Remark that when the equation is not degenerate ($M_F = 0$), Eq. (16) corresponds to the classical ABP estimate.

Remark 3*.* The constant C equals $3e^{C_{\mathrm{ABP}}(1+||\sigma||^n_{L^n(B_d)})}$ where $C_{\mathrm{ABP}} = \frac{2^{n-2}}{\omega_n \lambda_F^n}$.

Sketch of proof. The proof follows the ideas of [6, 18]. The key lemma is the following one.

Lemma 3. *The function* $\Gamma(u)$ *is* $C^{1,1}$ *on* $\mathcal{B} = \{x \in B_d : |D\Gamma(u)(x)| \geq M_F\}$*.*

Remark 4. Remark that before knowing that $\Gamma(u)$ is $C^{1,1}$, $D\Gamma(u)$ is not uniquely determined. Hence $\mathcal{B}$ should be first defined as follows

$$\mathcal{B} = \{x \in B_d : \forall (p,A) \in D^{2,-}\Gamma(u)(x), |p| \geq M_F\}\,.$$

Lemma 3 is proved together with

Lemma 4. *The Hessian of* $\Gamma(u)$ *satisfies on* $\mathcal{B}$ *the following properties*

1. $D^2\Gamma(u) = 0$ *a.e. in* $\mathcal{B} \setminus \{u + M_\partial = \Gamma(u)\}$ *;*
2. $D^2\Gamma(u)(x) \leq \lambda_F^{-1}\big\{\sigma(x)|D\Gamma(u)(x)| + f^+(x)\big\}I$ *a.e. in* $\mathcal{B} \cap \{u + M_\partial = \Gamma(u)\}$.

Proofs of these two lemmata can be adapted from the classical one by remarking that points x_i called by $x \in \mathcal{B}$ when computing the convex hull $\Gamma(u)$ (see Proposition 1 in Appendix A) satisfy $D\Gamma(u)(x_i) = D\Gamma(u)(x)$. In particular, $x_i \in \mathcal{B}$, *i.e.* $|D\Gamma(u)(x_i)| \geq M_F$ and consequently (6) can be used. The reader is referred to Appendix A where detailed proofs are given for his convenience.

Lemma 5. *The following inclusion holds true*

$$B_{M/(3d)}(0) \setminus B_{M_F}(0) \subset D\Gamma(u)(\mathcal{B})\,. \tag{17}$$

where M *denotes* $\sup_{B_d} u^- - \sup_{\partial B_d} u^-$ *and* $\mathcal{B} = \{x \in B_d : |D\Gamma(u)(x)| \geq M_F\}$.

Proof. This lemma is a consequence of the classical fact

$$B_{M/(3d)}(0) \subset D\Gamma(u)(B_d)\,.$$

□

From now on, we assume without loss of generality that $M/(3d) \geq M_F$. We then use Lemma 3 in order to apply the area formula (see [14, Theorem 3.2.5] and Remark 6 below) to the Lipschitz map $D\Gamma(u) : \mathcal{B} \to \mathbb{R}^n$ and to the function $g(p) = (|p|^{n/(n-1)} + \mu^{n/(n-1)})^{(1-n)}$ for some positive real number μ to be fixed later.

$$\int_{\mathcal{B}} g(p)dp = \int_{\mathcal{B}} g(D\Gamma(u))\,\det D^2\Gamma(u)\,.$$

On one hand, we can use Lemmata 4 and 5 in order to get

$$\begin{aligned}
\int_{B_{M/(3d)}(0)\setminus B_{M_F}(0)} g(p)dp &\leq \int_{D\Gamma(u)(\mathcal{B})} g(p)dp \\
&\leq \int_{\mathcal{B}} g(D\Gamma(u))\,\det D^2\Gamma(u) \\
&\leq \frac{1}{\lambda_F^n}\int_{\mathcal{B}\cap\{u+M_\partial=\Gamma(u)\}} g(D\Gamma(u))(\sigma|D\Gamma(u)| + f^+)^n \\
&\leq \frac{1}{\lambda_F^n}\int_{\mathcal{B}\cap\{u+M_\partial=\Gamma(u)\}} (|\sigma|^n + \mu^{-n}(f^+)^n)\,.
\end{aligned}$$

If now one chooses μ such that $\mu^n = \int_{\mathcal{B}\cap\{u+M_\partial=\Gamma(u)\}}(f^+)^n$, we obtain from the inequality $g(p) \geq 2^{2-n}(|p|^n + \mu^n)^{-1}$ the following estimate

$$\begin{aligned} \frac{2^{2-n}}{n}\omega_n \ln\frac{(M/(3d))^n+\mu^n}{(M_F)^n+\mu^n} &= 2^{2-n}\omega_n\int_{M_F}^{M/d}\frac{r^{n-1}dr}{r^n+\mu^n} \\ &\leq \int_{B_{M/d}(0)\setminus B_{M_F}(0)} g(p)dp \\ &\leq \lambda_F^{-n}(1+||\sigma||_n^n) \end{aligned}$$

where ω_n denotes the volume of the unit ball. It is now easy to get (16). $\square$

Remark 5. We see from the previous proof that Assumptions (A) and (6) on F are important in order to get the following property

$$(18) \qquad \forall(p,A)\in\mathcal{J}^{2,-}u(x) \,:\, \left.\begin{array}{r} A\geq 0 \\ |p|\geq M_F \end{array}\right\} \Rightarrow \lambda_F \operatorname{Tr} A \leq \sigma(x)|p| + f(x)\,.$$

As a matter of fact, the previous piece of information is the relevant one in order to get (16). Indeed, in Lemma 4, the second estimate can be rewritten as follows

$$\lambda_F D^2\Gamma(u)(x) \leq \{\sigma(x)|D\Gamma(u)| + f(x)\}I\,.$$

Remark 6. The area formula in [14] is stated for maps $G : \mathbb{R}^n \to \mathbb{R}^n$ that are Lipschitz continuous on $\mathbb{R}^n$ (in our case). However, the result still holds true if G is only Lipschitz continuous on $\mathcal{B}$ since it is always possible to extend it in a Lipschitz map $\tilde{G}$ on $\mathbb{R}^n$ with $G = \tilde{G}$ on $\mathcal{B}$.

4 Harnack inequality

In this section, we explain how to derive a Harnack inequality from the ABP estimate. As usual, we obtain it by deriving on one hand a weak Harnack inequality and on the other hand a local maximum principle for the fully nonlinear equation (4).

In order to get a weak Harnack inequality and local maximum principle respectively, Condition (6) is strengthened by assuming (7) and (8) respectively.

The Harnack inequality is obtained as a combination of a weak Harnack inequality and a local maximum principle. Here are precise statements.

Theorem 2 (Weak Harnack inequality)**.** *Given $q > n$ and a non-linearity F satisfying* (A) *and* (7) *for some continuous functions f and σ in Q_1, consider a non-negative super-solution u of* (4) *in Q_1. Then*

$$(19) \qquad \|u\|_{L^{p_0}(Q_{1/4})} \leq C(\inf_{Q_{1/2}} u + \max(M_F, \|f\|_{L^n(Q_1)}))$$

where $p_0 > 0$ is universal and C (only) depends on n, q, λ_F, Λ_F, γ_F and $\|\sigma\|_{L^q(Q_1)}$.

Theorem 3 (Local maximum principle)**.** *Given $q > n$ and a non-linearity F satisfying* (A) *and* (8) *for some continuous functions f and σ on Q_1, consider a sub-solution u of* (4) *in Q_1. Then for any $p > 0$,*

$$\sup_{Q_{1/4}} u \le C(p)(\|u^+\|_{L^p(Q_{1/2})} + \max(M_F, \|f\|_{L^n(Q_1)})) \tag{20}$$

where $C(p)$ is a constant (only) depending on n, q, λ_F, Λ_F, γ_F, $\|\sigma\|_{L^q(Q_1)}$ and p.

Combining these two results, we obtain the second main result of this paper.

Corollary 1 (Harnack inequality)**.** *Given $q > n$ and a non-linearity F satisfying* (A), (7) *and* (8) *for some continuous functions f and σ on Q_1, consider a non-negative solution u of* (4) *in Q_1. Then*

$$\sup_{Q_{1/2}} u \le C(\inf_{Q_{1/2}} u + \max(M_F, \|f\|_{L^n(Q_1)})) \tag{21}$$

where C is a constant (only) depending on n, q, $\lambda_F, \Lambda_F, \gamma_F$ and $\|\sigma\|_{L^q(Q_1)}$.

An important consequence of Corollary 1 is the following regularity result.

Corollary 2 (Interior Hölder regularity)**.** *Given $q > n$ and a non-linearity F satisfying* (A), (7) *and* (8) *for some continuous functions f and σ on Q_1, consider a solution u of* (4) *in Q_1. Then u is α-Hölder continuous on $Q_{\frac{1}{2}}$ and*

$$\sup_{\substack{x,y \in Q_{\frac{1}{2}} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le C_\alpha(\|u\|_{L^\infty(Q_1)} + \max(M_F, \|f\|_{L^n(Q_1)} + \gamma_F \|u\|_{L^\infty(Q_1)})) \tag{22}$$

where α and C_α depend (only) on n, q, λ_F, Λ_F, γ_F and $\|\sigma\|_{L^q(Q_1)}$.

5 Quadratic growth in Du

In this section, we extend the results of the previous section to elliptic equations with a first order term (after changing the original equation if necessary; see the Introduction) which can grow quadratically with respect to the gradient. Precisely, (7) and (8) are replaced with (9) and (10).

Through a Cole-Hopf transform, an immediate consequence of Theorems 2 and 3 are the following results.

Theorem 4 (Weak Harnack inequality)**.** *Given $q > n$ and a non-linearity F satisfying* (A) *and* (9) *for some continuous functions f and σ in Q_1, consider a non-negative super-solution u of* (4) *in Q_1. Then*

$$\|u\|_{L^{p_0}(Q_{1/4})} \le C(\inf_{Q_{1/2}} u + \max(M_F, \|f\|_{L^n(Q_1)})) \tag{23}$$

where $p_0 > 0$ is universal and C (only) depends on $\|u\|_{L^\infty(Q_1)}$, n, q, λ_F, Λ_F, γ_F and $\|\sigma\|_{L^q(Q_1)}$.

Remark 7. As explained in [30, 22], one cannot expect to get weak Harnack inequality for such non-linearities with a constant $C > 0$ which does not depend on a bound on u.

Remark 8. The constant C can be written

$$C = C_0 \frac{\frac{\sigma_2 \|u\|_{L^\infty(Q_1)}}{\lambda_F}}{1 - e^{-\frac{\sigma_2 \|u\|_{L^\infty(Q_1)}}{\lambda_F}}}$$

where C_0 (only) depends on n, q, λ_F, Λ_F, γ_F and $\|\sigma\|_{L^q(Q_1)}$.

Theorem 5 (Local maximum principle)**.** *Given $q > n$ and a non-linearity F satisfying* (A) *and* (10) *for some continuous functions f and σ on Q_1, consider a sub-solution u of* (4) *in Q_1. Then for any $p > 0$,*

$$\sup_{Q_{1/4}} u \leq C(\|u^+\|_{L^p(Q_{1/2})} + \max(M_F, \|f\|_{L^n(Q_1)})) \tag{24}$$

where C (only) depends on $\|u\|_{L^\infty(Q_1)}$, n, q, λ_F, Λ_F, γ_F, $\|\sigma\|_{L^q(Q_1)}$ and p.

Remark 9. The constant C can be written

$$C = C_0 \frac{\frac{\sigma_2 \|u\|_{L^\infty(Q_1)}}{\lambda_F}}{1 - e^{-\frac{\sigma_2 \|u\|_{L^\infty(Q_1)}}{\lambda_F}}}$$

where C_0 (only) depends on n, q, λ_F, Λ_F, γ_F, $\|\sigma\|_{L^q(Q_1)}$ and p.

It is now easy to derive a Harnack inequality and an interior Hölder estimate.

Corollary 3 (Harnack inequality)**.** *Given $q > n$ and a non-linearity F satisfying* (A), (9) *and* (10) *for some continuous functions f and σ on Q_1, consider a non-negative solution u of* (4) *in Q_1. Then*

$$\sup_{Q_{1/2}} u \leq C(\inf_{Q_{1/2}} u + \max(M_F, \|f\|_{L^n(Q_1)})) \tag{25}$$

where C (only) depends on $\|u\|_{L^\infty(Q_1)}$, n, q, λ_F, Λ_F,γ_F and $\|\sigma\|_{L^q(Q_1)}$.

Corollary 4 (Interior Hölder regularity)**.** *Given $q > n$ and a non-linearity F satisfying* (A), (9) *and* (10) *for some continuous functions f and σ on Q_1, consider a solution u of* (4) *in Q_1. Then u is α-Hölder continuous on $Q_{\frac{1}{2}}$ and*

$$\sup_{\substack{x,y \in Q_{\frac{1}{2}} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C_\alpha(\|u\|_{L^\infty(Q_1)} + \max(M_F, \|f\|_{L^n(Q_1)} + \gamma_F \|u\|_{L^\infty(Q_1)})) \tag{26}$$

where α and C_α depend (only) on $\|u\|_{L^\infty(Q_1)}$, n, q, λ_F, Λ_F, γ_F and $\|\sigma\|_{L^q(Q_1)}$.

6 Applications: results of [3, 11]

In [3, 11], Eq. (5) is considered. In [3], α lies in $[0, 1)$ and in [11], α lies in $(-1, 1)$. They assume

Assumption (H)

- (H1) $F(tp, \mu X) = |t|^\alpha \mu F(p, X)$ for $t \neq 0$ and $\mu \geq 0$;
- (H2) $|p|^\alpha \mathcal{M}^-(N) \leq F(p, M+N) - F(p, M) \leq |p|^\alpha \mathcal{M}^+(N)$ for some $\alpha \in (-1, 1)$.

The ABP estimate obtained in [11] is the following one

Theorem 6 ([11, Theorem 1])**.** *Under Assumption (H),* $\alpha \in (-1,1)$ *and* $c \leq 0$*, any super-solution of* (5) *satisfies*

$$\sup_{B_d} u^- \leq \sup_{\partial B_d} u^- + Cd \left(\left(\int_{B_d \cap \{u + M_\partial = \Gamma(u)\}} (f_0^+)^n \right)^{1/n} \right)^{\frac{1}{1+\alpha}} \tag{27}$$

where $M_\partial = \sup_{\partial B_d} u^-$*,* $\Gamma(u)$ *is the convex hull of* $\min(u + M_\partial, 0)$ *extended by* 0 *on* B_{2d}*,* $f(x) = g(x, -M_\partial)$ *and* C *is a constant (only) depending on* $\|\sigma\|_{L^n(B_d)}$*,* n*,* α *and* $\|c\|_\infty$*.*

Dávila, Felmer and Quaas pointed out (in the case $\alpha \geq 0$) to us that it can be obtained from ours. As we shall see, we can also do it when $\alpha < 0$. See below.

The Harnack inequality obtained in [3] is the following one

Theorem 7 ([3, Theorems 3.1 and 3.2])**.** *Under Assumption (H) with* $\alpha \in (-1,1)$*, any non-negative solution of* (5) *satisfies*

$$\sup_B u \leq C(\inf_B u + \|f_0\|_{L^N(B)}^{\frac{1}{1+\alpha}}) \,. \tag{28}$$

where C *is a constant (only) depending on* n*,* q*,* λ_F, Λ_F*,* $\|c\|_\infty$*,* α *and* $\|\sigma\|_{L^q(Q_1)}$*.*

Remark 10. This result is proved in [3] only in dimension 2. Moreover, it is slightly more precise since it depends on q and $\|\sigma\|_{L^q(Q_1)}$ instead of $\|\sigma\|_{L^\infty(Q_1)}$.

Their results are not included in ours but they can be derived with little additional work. We mention that Birindelli and Demengel do not derive this Harnack inequality from an ABP estimate.

Proofs of Theorems 6 and 7. Dávila, Felmer and Quaas kindly explained to us the link between their result and ours in the case $\alpha \geq 0$ in a simple enlightening case.

We first remark that Assumption (H2) implies $|p|^\alpha \mathcal{M}^-(X) \leq F_0(p, X) \leq |p|^\alpha \mathcal{M}^+(X)$.

- If $\alpha \geq 0$, (7) and (8) are satisfied for any $M_F > 0$ with $\sigma = |b|$, $f = \frac{-f_0}{M_F^\alpha}$ and $\gamma_F u$ is replaced with $cu|u|^\alpha$. Moreover, (6) is satisfied for any $M_F > 0$ and with $\sigma = |b|$ and $g(x,u) = \frac{f_0(x) - cu|u|^\alpha}{M_F^\alpha}$. In particular, $g(x, -M_\partial) \le \frac{f_0(x)}{M_F^\alpha}$ since $c \le 0$. Hence, our result gives

$$\sup_{B_d} u^- \le \sup_{\partial B_d} u^- + Cd \left(M_F + \frac{1}{M_F^\alpha} \left(\int_{B_d \cap \{u + M_\partial = \Gamma(u)\}} (f_0^+)^n \right)^{1/n} \right).$$

Optimizing with respect to $M_F > 0$ gives (27).

- If $\alpha = -\beta < 0$, then $F(x,u,p,X) \ge 0$ implies

  $$\mathcal{M}^+(X) + |b(x)||p| + (f_0 - cu|u|^{-\beta})_+ |p|^\beta \ge 0\,.$$

  Now using $ab \le a^r/r + b^q/q$ with $r = 1/\beta$ and $q = \frac{1}{1-\beta}$, we obtain (6) with $\sigma = |b| + \frac{1}{\beta}$ and $g(x,u) = (1-\beta)(f_0 - cu|u|^{-\beta})_+^{\frac{1}{1-\beta}}$. In particular, $g(x, -M_\partial) \le (1-\beta)(f_0)_+^{\frac{1}{1-\beta}}$. The result then follows in this case too.

The Harnack inequality of [3] when $c = 0$ can be easily obtained from ours in any dimension (but not when $c \neq 0$). The case $c \neq 0$ can also be treated but it requires to modify proofs a bit more. In particular, the barrier constructed in Section 2 must be adapted. Notice that the singular ($\alpha \le 0$) and degenerate ($\alpha \ge 0$) can be treated with the same techniques. Main differences are explained in Appendix for the reader's convenience (see Section B).

7 Proofs

7.1 Proof of the weak Harnack inequality

Proof of the weak Harnack inequality (Theorem 2). The proof of the weak Harnack inequality is performed in four steps. First, the problem is reduced to the case of a cube Q of universal side-length (Lemma 6), then it is proved that non-negative super-solutions can be bounded from above on Q by a universal constant on a set of universal positive measure (Lemma 7). Next, the measures of all level sets of super-solutions (restricted to Q) are (universally) estimated from above. Finally, we prove the weak Harnack inequality in Q.

Step 1. As explained above, we first reduce the problem to a simpler one.

Lemma 6 (Reduction of the problem)**.** *Consider a non-negative super-solution u of (4) in Q_{2R}. Then there exist universal constants p_0, ε_0 and C satisfying*

(29)
$$\left. \begin{aligned} \inf\nolimits_{Q_{3r}} u \le 1 \\ \max(M_F, \gamma_F, \|f\|_{L^n(Q_R)}, \|\sigma\|_{L^q(Q_R)}) \le \varepsilon_0 \end{aligned} \right\} \Rightarrow \|u\|_{L^{p_0}(Q_r)} \le C\,.$$

We now explain how to derive the weak Harnack inequality from it. Let v be a super-solution of (4) in $Q_{R/t}$ for some $t > 0$. We then define a function $v_s(y) = \frac{v(ty)}{V}$ with $V > 0$ and $t \in (0,1)$ to be chosen later. Thanks to Lemma 2 with $x_0 = 0$, $M_0 = V$, $R_0 = R/t$, the new function v_s satisfies $F_s = 0$ in Q_R for a non-linearity F_s satisfying (A) and (7) with

$$M_s = \frac{tM_F}{V}, \quad \gamma_s = \gamma_F t^2, \quad \sigma_s(y) = t\sigma(ty), \quad f_s(y) = \frac{f(ty)}{V}\,.$$

Hence, if one chooses

$$\begin{aligned} V &= \inf_{Q_{3r/t}} v + \delta + \varepsilon_0^{-1} \max(M_F, \|f\|_{L^n(Q_{R/t})}) \\ t &= \left(\left(\frac{\|\sigma\|_{L^q(Q_{R/t})}}{\varepsilon_0} \right)^{q/(q-n)} + \left(\frac{\gamma_F}{\varepsilon^0} \right)^{1/2} + 1 \right)^{-1} \end{aligned}$$

we obtain that v satisfies

$$\begin{aligned} \inf_{Q_{3r}} v_s &\le 1 \\ \max(M_s, \gamma_s, \|f_s\|_{L^n(Q_R)}, \|\sigma_s\|_{L^q(Q_R)}) &\le \varepsilon_0\,. \end{aligned}$$

We thus can apply Lemma 6 and we obtain from (29) the following estimate (after letting $\delta \to 0$)

$$\|u\|_{L^{p_0}(Q_{r/t})} \le C(\inf_{Q_{3r/t}} u + \max(M_F, \|f\|_{L^n(Q_{R/t})}))\,. \tag{30}$$

A standard covering procedure permits to get (19).

Step 2. In this step, we obtain a (universal) upper bound M for non-negative super-solutions in Q_R on a set of (universal) positive measure μ if the L^n-norm of f on Q_R, the L^q-norm of σ on Q_R, M_F and γ_F are (universally) small.

Lemma 7 (Upper bound on a subset of positive measure)**.** *There exist universal constants* $r, R > 0$, $\varepsilon_0 > 0$, $\mu \in (0,1)$ *and* $M_{\mathrm{B}} > 0$ *such that for any non-negative super-solution* u *of* (4) *in* Q_R*, we have*

$$\left. \begin{aligned} \inf_{Q_{3r}} u \le 1 \\ \max(M_F, \gamma_F, \|f\|_{L^n(Q_R)}, \|\sigma\|_{L^q(Q_R)}) \le \varepsilon_0 \end{aligned} \right\} \Rightarrow |\{u \le M_{\mathrm{B}}\} \cap Q_r| \ge \mu |Q_r|\,.$$

The proof of this lemma relies on the barrier function φ that we constructed in the preliminary section and on the ABP estimate applied to $w = u + \varphi$.

Proof of Lemma 7. Given $\varepsilon_0 > 0$ to be fixed later, we consider φ from Lemma 1 and define $w = u + \varphi$. We want to apply the ABP estimate (Theorem 1) to the function w on the ball B_R.

- First, $u \ge 0$ and $\varphi \ge 0$ on ∂B_R hence $M_\partial = \sup_{\partial B_R} w^- = 0$.

- Since $\inf_{Q_{3r}} u \leq 1$ and $\varphi \leq -2$ in Q_{3r}, we conclude that $\inf_{Q_{3r}} w \leq -1$; in other words, we have $\sup_{Q_{3r}} w^- \geq 1$.
- We also claim that w is a super-solution of an appropriate equation. More precisely, we claim that w satisfies (18) in $\{w \leq 0\} \cap B_R$ for some appropriate continuous functions $\overline{f}$ and $\overline{\sigma}$.

Let us justify the last assertion and make precise what $\overline{f}$ and $\overline{\sigma}$ are. We write

$$\begin{aligned} 0 &\leq F(x, u, Du, D^2 u) \\ &= F(x, w - \varphi, Dw - D\varphi, D^2 w - D^2 \varphi) \\ &\leq F(x, w + M_{\mathrm{B}}, Dw - D\varphi, D^2 w - D^2 \varphi)\,. \end{aligned}$$

Assume next that $|Dw| \geq M_F + \varepsilon_0 =: \overline{M}_F$, $D^2 w \geq 0$ (in the viscosity sense) and $w \leq 0$. Then $|Dw - D\varphi| \geq M_F$ and we obtain from (7) the following inequality

$$0 \leq \mathcal{M}^+(D^2 w) - \mathcal{M}^-(D^2 \varphi) + \sigma |Dw| + \gamma_F M_B + \sigma \varepsilon_0 + f$$

(we used the fact that $\mathcal{M}^+$ is subadditive and the relation between the two Pucci operators). Use next that $D^2 w \geq 0$ and φ satisfies (15)

$$\lambda_F \Delta w \leq \sigma |Dw| + C_{\mathrm{B}} \xi + \gamma_F M_B + \sigma \varepsilon_0 + f\,.$$

Hence (18) holds true with

$$\overline{\sigma} = \sigma \quad \text{ and } \quad \overline{f}(x) = C_{\mathrm{B}} \xi + \gamma_F M_B + \sigma \varepsilon_0 + f\,.$$

By using the ABP estimate for w and the properties listed above satisfied by this function, we obtain

$$1 \leq \sup_{B_R} w^- \leq 3 e^{C_{\mathrm{ABP}}(1 + \|\sigma\|^n_{L^n(B_R)})} R \left(\bar{M}_F + \left(\int_{\{\Gamma(w) = w\} \cap B_R} (\overline{f}^+)^n \right)^{1/n} \right)$$

where $\Gamma(w)$ is the convex hull of $\min(w, 0)$ after extending w to B_{2R} by setting $w \equiv 0$ outside B_R. We now use the fact that

$$\max(M_F, \gamma_F, \|f\|_{L^n(Q_R)}, \|\sigma\|_{L^q(Q_R)}) \leq \varepsilon_0\,,$$

together with definitions of $\overline{f}$, $\overline{M}_F$ and the fact that $\operatorname{supp} \xi \subset Q_r$ in order to get

$$1 \leq 3 e^{C_{\mathrm{ABP}}(1 + R^{1 - \frac{n}{q}} \varepsilon_0^n)} R (3 \varepsilon_0 + R^{\frac{1}{n} - \frac{1}{q}} \varepsilon_0^2 + \varepsilon_0 M_B + C_{\mathrm{B}} |\{\Gamma(w) = w\} \cap Q_r|)\,.$$

It is now enough to remark that

$$\{\Gamma(w) = w\} \subset \{w \leq 0\} \subset \{u \leq M_{\mathrm{B}}\}$$

and to choose $\varepsilon_0 \in (0,1)$ such that

$$3 e^{C_{\mathrm{ABP}}(1 + R^{1 - \frac{n}{q}} \varepsilon_0^n)} R (3 \varepsilon_0 + R^{\frac{1}{n} - \frac{1}{q}} \varepsilon_0^2 + \varepsilon_0 M_B) \leq \frac{1}{2}$$

to conclude. We used here that M_B is universal; in particular, it does not depend on ε_0. $\square$

Step 3. We derive from the previous lemma (Lemma 7) an estimate of any level set of super-solutions u under consideration. Precisely, we use Lemma 2 together with the Calderón-Zygmund cube decomposition lemma (see Lemma 15 in Appendix A) in order to get the following result.

Lemma 8 (Estimate of the measure of level sets in Q_r)**.** *Let u be as in Lemma 7. Then there exist universal constants $\varepsilon > 0$ and $d > 0$ such that for all $t > 0$,*

$$|\{u \geq t\} \cap Q_r| \leq d t^{-\varepsilon} \, . \tag{31}$$

The proof of Lemma 4.6 in [6] can be easily adapted (with minor changes). For the reader's convenience, a detailed proof is given in Appendix A.

Step 4. We finally explain how to derive Lemma 6. We first recall the following useful fact: if u is a non-negative function, then

$$\int_{Q_r} u^{p_0} = p_0 \int_0^{+\infty} t^{p_0 - 1} |\{u \geq t\} \cap Q_r| dt \, .$$

We can use the results of Lemmata 7 and 8: we thus choose $p_0 = \varepsilon/2$ where ε appears (31) in order to get

$$\frac{1}{p_0} \int_{Q_r} u^{p_0} \leq \int_0^1 t^{\varepsilon/2-1} |Q_r| dt + \int_1^{+\infty} t^{\varepsilon/2-1} t^{-\varepsilon} dt =: C \, .$$

This achieves the proof of Lemma 6 and the proof of Theorem 2. □

7.2 Proof of the local maximum principle

The proof of the local maximum principle is easily adapted from [6]. However, we give a detailed proof for the sake of completeness.

Proof of Theorem 3. The proof is divided in two steps. First, the problem is reduced to the case where the L^ε-norm of u is small; it is to be proven that u is bounded by a universal constant (Step 1). Then we explain how to get the universal bound (Steps 2 and 3).

Step 1. We state the lemma to be proven in Step 2.

Lemma 9. *Consider a sub-solution u of* (4) *in Q_R. Then there exists a universal constant $C > 0$ such that*

$$\left. \begin{array}{r} \|u^+\|_{L^\varepsilon(Q_r)} \leq d^{1/\varepsilon} \\ \max(M_F, \gamma_F, \|f\|_{L^n(Q_R)}, \|\sigma\|_{L^q(Q_R)}) \leq \varepsilon_0 \end{array} \right\} \Rightarrow \sup_{Q_{\frac{r}{4}}} u \leq C$$

where ε and d appears in Lemma 8.

We now explain how to derive Theorem 3 from this lemma. First, it is enough to get (20) for a particular p since the full result can be obtained by interpolation. In view of the previous lemma, we consider $p = \varepsilon$. By scaling u and by using a covering argument, we obtain the desired result.

Step 2. We remark that the assumption $\|u^+\|_{L^\varepsilon(Q_r)} \leq d^{1/\varepsilon}$ implies

$$|\{u \geq t\} \cap Q_r| \leq t^{-\varepsilon} \int_{Q_r} (u^+)^\varepsilon \leq d t^{-\varepsilon} \,.$$

Remark that this estimate already appeared in the proof of the weak Harnack inequality; see (31) above. We next prove the following lemma.

Lemma 10. *Consider a sub-solution u of* (4) *in Q_R satisfying* (31) *and F be such that*

$$\max(M_F, \gamma_F, \|f\|_{L^n(Q_R)}, \|\sigma\|_{L^q(Q_R)}) \leq \varepsilon_0 \,.$$

Then there exists universal constants $M_0 > 1$ and $\Sigma > 0$ such that

$$\left.\begin{array}{r} x_0 \in Q_{\frac{r}{2}}, j \in \mathbb{N} \\ u(x_0) \geq \nu^{j-1} M_0 \end{array}\right\} \Rightarrow \sup_{Q_{l_j}(x_0)} u > \nu^j M_0$$

where $l_j = \Sigma \frac{M_0^{-\varepsilon/n}}{\nu^{\varepsilon j/n}} < \frac{r}{2}$ and $\nu = M_0/(M_0 - 1/2) > 1$.

Proof of Lemma 10. We first choose Σ and M_0 such that

$$\Sigma M_0^{-\varepsilon/n} \leq \frac{r}{2}$$

so that $l_j < \frac{r}{2}$ and $Q_{l_j}(x_0) \subset Q_r$. We now argue by contradiction by assuming that $\sup_{Q_{l_j}(x_0)} u < \nu^j M_0$. We have to exhibit a contradiction.

On one hand, we have from (31) and the fact that $r < R$ and $l_j < r/2$

$$|\{u \geq \nu^j \frac{M_0}{2}\} \cap Q_{\frac{l_j r}{R}}(x_0)| \leq d \nu^{-j\varepsilon} \left(\frac{M_0}{2}\right)^{-\varepsilon} \,. \tag{32}$$

On the other hand, since we have $\sup_{Q_{l_j}(x_0)} u \leq \nu^j M_0$ by assumption, we can consider the following transformation

$$T(y) = x_0 + \frac{l_j}{R} y$$

which defines a bijection between Q_R and $Q_{l_j}(x_0)$. The function v defined on Q_R as follows

$$v(y) = \frac{\nu M_0 - \frac{u(T(y))}{\nu^{j-1}}}{(\nu - 1) M_0} \geq 0$$

thus satisfies $F_s(y, v, Dv, D^2 v) = 0$ in Q_R with F_s satisfying (A) and (7) with

$$M_s = \frac{t}{\nu^{j-1}(\nu - 1) M_0} M_F, \qquad \sigma_s(y) = t\sigma(x_0 + ty),$$
$$\gamma_s = t^2 \gamma_F, \qquad f_s(y) = \frac{t}{\nu^{j-1}(\nu - 1) M_0} t f(x_0 + ty)$$

where $t = \frac{l_j}{R} < \frac{1}{4}$. It is clear that $\gamma_s \le \gamma_F \le \varepsilon_0$. Notice that

$$(\nu - 1)M_0 = \frac{M_0}{2M_0 - 1} > \frac{1}{2} > t$$

hence $M_s \le M_F \le \varepsilon_0$ and $f_s(y) \le t f(x_0 + ty)$. We also have

$$\begin{aligned} \|\sigma_s\|_{L^q(Q_R)} &\le t^{1-\frac{n}{q}} \|\sigma\|_{L^q(Q_R)} \le \varepsilon_0 \\ \|f_s\|_{L^n(Q_R)} &\le \|f\|_{L^n(Q_R)} \le \varepsilon_0 \,. \end{aligned}$$

Moreover, $v(0) = \frac{\nu M_0 - \frac{u(x_0)}{\nu^{j-1}}}{(\nu-1)M_0} \le 1$ by assumption on u; thus $\inf_{Q_{3r}} v \le 1$. Hence, v satisfies the assumptions of Lemma 7 and we therefore obtain from (31) the following estimate

$$|\{v \ge M_0\} \cap Q_r| \le d M_0^{-\varepsilon} \,.$$

We thus obtain

$$|\{u \le \nu^j \frac{M_0}{2}\} \cap Q_{\frac{l_j r}{R}}(x_0)| \le \left(\frac{l_j}{R}\right)^n d M_0^{-\varepsilon} \,. \tag{33}$$

Combining (32) and (33), we thus obtain

$$\left(\frac{l_j r}{R}\right)^n \le d \nu^{-j\varepsilon} \left(\frac{M_0}{2}\right)^{-\varepsilon} + \left(\frac{l_j}{R}\right)^n d M_0^{-\varepsilon} \,.$$

We also choose M_0 such that $d M_0^{-\varepsilon} \le \frac{r^n}{2}$, and we obtain

$$\frac{1}{2}\left(\frac{l_j r}{R}\right)^n \le d \nu^{-j\varepsilon} \left(\frac{M_0}{2}\right)^{-\varepsilon} \,.$$

Use now the definition of l_j and get

$$\frac{1}{2}\left(\frac{\Sigma r}{R}\right)^n \le d 2^{\varepsilon} \,.$$

We next choose $\Sigma > d^{\frac{1}{n}} 2^{\frac{\varepsilon+1}{n}} \frac{R}{r}$ in order to get a contradiction. □

Step 3. We prove Lemma 9. By Step 2, we know that the sub-solution u satisfies the conclusion of Lemma 10. In particular, the series $\sum_j l_j$ converges and we can find a universal integer $j_0 \ge 1$ such that $\sum_{j \ge j_0} l_j \le \frac{r}{8}$.

We now claim that $\sup_{Q_{\frac{r}{4}}} u \le \nu^{j_0 - 1} M_0$. We argue by contradiction by assuming that this is not true and by exhibiting a contradiction. Let us assume that there exists $x_{j_0} \in Q_{\frac{r}{4}}$ and $u(x_{j_0}) \ge \nu^{j_0-1} M_0$. Hence, we can apply Lemma 10 and we get a point x_{j_0+1} such that $|x_{j_0+1} - x_{j_0}|_\infty \le l_{j_0}/2$ and $u(x_{j_0+1}) \ge \nu^{j_0} M_0$. By induction, we construct a sequence $(x_j)_{j \ge j_0}$ such that

$|x_{j+1} - x_j| \leq l_j/2$ and $u(x_{j+1}) \geq \nu^j M_0$ as long as $x_j \in Q_{\frac{r}{2}}$. This is always the case since

$$|x_j|_\infty \leq |x_{j_0}|_\infty + \sum_{k=j_0}^{j-1} |x_{k+1} - x_k| \leq \frac{r}{8} + \frac{r}{8} \leq \frac{r}{4}\,.$$

We now get a contradiction since u is upper semi-continuous; indeed, it is bounded from above on $Q_{\frac{r}{2}}$ so it cannot satisfy $u(x_{j+1}) \geq \nu^j M_0$ for all $j \geq j_0$. The proof is now complete. □

7.3 Proofs of Theorems 4 and 5

Proofs of Theorems 4 and 5. Both proofs rely on a transform of Cole-Hopf type in order to remove quadratic terms.

In order to understand why the exponential change of variables is the right one, we consider $v = h^{-1}(u)$ for some increasing function h and we remark that v satisfies

$$\mathcal{M}^+(D^2 v) + \sigma(x)|Dv| + \frac{f^+(x)}{h'(v)} \geq 0$$

as soon as h satisfies $\lambda_F h'' - \sigma_2 (h')^2 = 0$. We thus choose

$$h(t) = \frac{\lambda_F}{\sigma_2} \ln \left(1 - \frac{\sigma_2 t}{\lambda_F}\right)^{-1}.$$

We thus derive (23) from (19) by remarking that

$$\frac{1 - e^{-\frac{\sigma_2 \|u\|_{L^\infty(Q_1)}}{\lambda_F}}}{\frac{\sigma_2 \|u\|_{L^\infty(Q_1)}}{\lambda_F}} u \leq v \leq u$$

and $\frac{1}{h'(t)} = 1 - \frac{\sigma_2 t}{\lambda_F} \leq 1$.

We proceed in the same way in order to prove Theorem 5. Remark that we can assume without loss of generality that the solution is non-negative. □

A Additional proofs

A.1 Proofs of Lemmata 3 and 4

In this paragraph, we explain how to prove Lemmata 3 and 4 by adapting the techniques of [18].

We first recall useful facts from convex analysis. The first one deals with the convex hull U^{**} of a function U.

Proposition 1. *Let Ω be a bounded convex open set and $U : \overline{\Omega} \to \mathbb{R}$ be lsc. For $x \in \overline{\Omega}$, consider $(p, A) \in D^{2,-}U^{**}(x)$. There then exist $x_1, \ldots, x_q \in \overline{\Omega}$, $q \leq n$, $\lambda_1, \ldots, \lambda_q \in (0, 1]$, $\sum_{i=1}^q \lambda_i = 1$ such that*

$$\left\{ \begin{array}{l} x = \sum_{i=1}^q \lambda_i x_i \\ U^{**}(x) = \sum_{i=1}^q \lambda_i U(x_i)\,. \end{array} \right. \tag{34}$$

Moreover U^{**} *is linear on the convex hull of* $\{x_1,\dots,x_q\}$. *In particular,* $A\le 0$ *for a.e.* $x\in\{U=U^{**}\}$.

We next recall a result from [18] (see also [1]) about the subjet of the convex hull U^{**} of a function U.

Proposition 2 ([18, Proposition 3])**.** *Let* Ω *be a bounded convex open set and* $U:\overline{\Omega}\to\mathbb{R}$ *be lower semi-continuous. For* $x\in\overline{\Omega}$, *consider* $(p,A)\in D^{2,-}U^{**}(x)$. *Consider* x_i *and* λ_i *such that* (34) *hold true. Then for every* $\varepsilon>0$, *there are* $A_i\in\mathbb{S}^{N-1}$, $i=1,\dots,q$, *such that*

$$\begin{cases} (p,A_i)\in \overline{D}^{2,-}U(x_i), \\ A_\varepsilon \le \Box_{i=1}^{q}\left(\lambda_i^{-1}A_i\right) \end{cases} \tag{35}$$

where $\Box$ *denotes the parallel sum of matrices. We recall that*

$$(A\Box B)\xi\cdot\xi = \inf_{\zeta\in\mathbb{R}^n}\{A(\xi-\zeta)\cdot(\xi-\zeta)+B\zeta\cdot\zeta\}\,.$$

We next recall a (necessary and) sufficient condition for a function to be semi-concave.

Lemma 11 ([1, Lemma 1])**.** *Consider a bounded convex open set* Ω *and* $U:\Omega\to\mathbb{R}$ *a lower semi-continuous function. Assume that there exists* $C>0$ *such that for all* $x\in\Omega$ *and all* $(p,A)\in D^{2,+}U(x)$, $A\le CI$. *Then* $U-C|\cdot|^2/2$ *is concave.*

We finally recall a useful approximation lemma from [1].

Lemma 12 ([1])**.** *Consider a convex set* Ω *and a convex function* $V:\Omega\to\mathbb{R}$. *For all* $(p,A)\in D^{2,-}V(x)$, *there exists* $(x_n)_n$ *and* $(p_n,A_n)\in D^{2,-}V(x_n)$ *such that* $x_n\to x$, $A_n\ge 0$ *and* $A\le A_n+\frac{1}{n}$.

We now turn to the proofs of the two lemmata.

Proofs of Lemmata 3 and 4. The function $v=u+M_\partial$ is a super-solution of

$$G(x,v,Dv,D^2v)=0$$

with $G(x,r,p,X)=F(x,r+M_\partial,p,X)$. Then $\Gamma(u)$ is the convex hull of the function $\min(v,0)$.

We first reduce the problem to the study of subjet of the function $\Gamma(u)$.

Lemma 13. *Assume that* $\Gamma(u)$ *satisfies the following properties*

$$\exists C>0 \;/\; \forall x\in\mathcal{B}, \forall (p,A)\in D^{2,-}\Gamma(u)(x), A\le CI\,, \tag{36}$$

$$\begin{cases} \forall x\in\mathcal{B}\cap\{\Gamma(u)=u+M_\partial\}, \forall(p,A)\in D^{2,-}\Gamma(u)(x), \\ \qquad\qquad A\le \lambda_F^{-1}(\sigma(x)|p|+f^+(x))I\,, \end{cases} \tag{37}$$

$$\Gamma(u) \text{ is linear on } \mathcal{B}\setminus\{x\in B_d:\Gamma(u)=u+M_\partial\}\,. \tag{38}$$

Then $\Gamma(u)$ *satisfies conclusions of Lemmata 3 and 4.*

Proof. Thanks to Lemma 11, Eq. (36) implies that $\Gamma(u)$ is semi-concave in $\mathcal{B}$. Since $\Gamma(u)$ is convex, this implies that $\Gamma(u)$ is $C^{1,1}$ in $\mathcal{B}$. Hence Lemma 3 is proved. We next remark that (38) implies Point 1 in Lemma 4. Eventually, (37) together with Alexandroff theorem permits to get Point 2. Le recall that Alexandroff theorem implies that a convex function is almost every twice differentiable. Hence the proof of Lemma 13 is now complete. $\Box$

We now prove the following lemma in order to achieve the proof of Lemmata 3 and 4.

Lemma 14. *The function* $\Gamma(u)$ *satisfies* (36), (37) *and* (38).

Proof. We first remark that (38) is a consequence of Proposition 1 and of Alexandroff theorem.

We now turn to the proof of (36) and (37). Consider next $x \in \mathcal{B}$ and $(p, A) \in D^{2,-}\Gamma(u)(x)$. Notice that we cannot just prove (36) for a.e. $x \in \mathcal{B}$. In view of the definition of $\mathcal{B}$ (see also Remark 4), we know that $|p| \geq M_F$. Thanks to Lemma 12, we can assume without loss of generality that $A \geq 0$. We now distinguish two cases.

Case 1: $x \in \mathcal{B} \cap \{\Gamma(u) = u + M_\partial\}$**.** In such a case, $(p, A) \in D^{2,-}\Gamma(u)(x) = D^{2,-}u(x)$, and since $|p| \geq M_F$, we have $F(x, u(x), p, A) \geq 0$. Now (6) yields

$$-\lambda_F \mathrm{Tr} A + \sigma(x)|p| + f^+(x) \geq 0$$

and since $A \geq 0$, we conclude that (37) holds true and the right hand side is bounded in B_d since $\Gamma(u)$ is Lipschitz continuous and σ and f^+ are continuous.

Remark that the previous inequality also holds true for A such that $(p, A) \in \bar{D}^{2,+}\Gamma(u)(x)$, $A \geq 0$, since the equation is also satisfied for limiting semi-jets.

Case 2: $x \in \mathcal{B} \setminus \{\Gamma(u) = u + M_\partial\}$**.** There then exist $x_i \in \bar{B}_d$ and $\lambda_i \in (0, 1]$, $i = 1, \ldots, q$, such that (34) holds true (where $U = u + M_\partial$). We know that there is at most one point x_i on ∂B_{2d} and the others are in B_d; if not, $\Gamma(u) \equiv 0$ and there is nothing to prove. Moreover, $x_i \in \mathcal{B}$ for $i = 1, \ldots, q$.

By Proposition 2, for any $\varepsilon > 0$, there exist q matrices $\lambda_i^{-1} A_i \geq A_\varepsilon \geq 0$ such that $\Box_{i=1}^{q} \lambda_i^{-1} A_i \geq A_\varepsilon$ and $(p, A_i) \in \bar{D}^{2,+}\Gamma(u)(x_i) = \bar{D}^{2,+}u(x_i)$.

If there are no points on ∂B_{2d}, we deduce from Case 1 that for all i, $A_i \leq CI$ and $A_\varepsilon \leq CI$ follows.

If $x_p \in \partial B_{2d}$, say, then we deduce from (34) that $\lambda_p \leq 2/3$; hence, there exists $i \in \{1, \ldots, p-1\}$ such that $\lambda_i \geq 1/3n$. For instance $i = 1$. Then we conclude that

$$A_\varepsilon \leq \frac{1}{\lambda_1} A_1 \leq 3nCI.$$

Passing to the limit on ε, we obtain $A \leq CI$ (for some new constant C). $\Box$

$\Box$

A.2 Proof of Lemma 8

In order to prove Lemma 8, we need the Calderón-Zygmund cube decomposition such as stated in [6]. We thus first recall it. We use notation from [6]. Given $r > 0$, the cube Q_r is split in 2^n cubes of half side-length. We do the same with all the new cubes and we iterate the process. The cubes obtained in this way are called *dyadic cubes*. If Q is a dyadic cube of Q_r, $\tilde{Q}$ denotes a dyadic cube such that Q is one of 2^n cubes obtained from $\tilde{Q}$.

Lemma 15 (Cube decomposition)**.** *Consider $r > 0$ and two measurable subsets $A \subset B \subset Q_r$. Consider $\delta \in (0,1)$ such that*

- $|A| \leq \delta |Q_r|$ *;*
- *if Q is a dyadic cube of Q_r such that $|A \cap Q| > \delta |Q|$, then $\tilde{Q} \subset B$.*

Then $|A| \leq \delta |B|$.

As far as the proof of this lemma is concerned, the reader is referred to [6]. We now turn to the proof of Lemma 8.

Proof of Lemma 8. We are going to prove the following estimate

$$|\{u \geq (M_B)^k\} \cap Q_r| \leq (1-\mu)^k |Q_r| \tag{39}$$

where M_B and μ are given by Lemma 7. The reader can check that (31) derives from (39) with $d = (1-\mu)^{-1}$ and $\varepsilon = -\ln(1-\mu)/\ln M_B$.

We prove (39) by induction. Lemma 7 implies that (39) holds for $k = 1$. We now consider $k \geq 2$, we assume that (39) holds for $k-1$ and we prove it for k. To do so, we are going to apply Lemma 15 with the two following sets $A \subset B \subset Q_r$

$$\begin{aligned} A &= \{u > (M_B)^k\} \cap Q_r\,, \\ B &= \{u > (M_B)^{k-1}\} \cap Q_r \end{aligned}$$

and with $\delta = 1 - \mu$. Remark that $A \subset \{u > M_B\} \cap Q_r$; hence $|A| \leq (1-\mu)|Q_r|$. It thus remains to prove that if Q is a dyadic cube of Q_r such that

$$|A \cap Q| > (1-\mu)|Q| \tag{40}$$

then the predecessor $\tilde{Q}$ of Q satisfies $\tilde{Q} \subset B$. Consider such a dyadic cube $Q = Q_{\frac{r}{2^i}}(x_0)$ and suppose that $\tilde{Q}$ is not contained in B. Then there exists $\tilde{x} \in \tilde{Q}$ such that $u(\tilde{x}) \leq (M_B)^{k-1}$. We now use Lemma 2 with $R_0 = R$, $t_0 = \frac{1}{2^i}$ and $M_0 = (M_B)^{k-1}$ to get a rescaled function u_s satisfying $F_s = 0$ with F_s such that (7) holds with constants $M_s \leq M_F$, $\gamma_s \leq \gamma_F$ and functions f_s, σ_s satisfying $\|f_s\|_{L^n(Q_R)} \leq \|f\|_{L^n(Q_R)}$ and $\|\sigma_s\|_{L^q(Q_R)} \leq \|\sigma\|_{L^q(Q_R)}$. We thus can apply Lemma 7 if $\inf_{Q_{3r}} u_s \leq 1$. This is indeed the case

$$\inf_{Q_{3r}} u_s \leq \frac{u(\tilde{x})}{(M_B)^{k-1}} \leq 1\,.$$

Hence, $|Q \setminus A| > (1-\mu)|Q|$ which contradicts (40). $\square$

A.3 Proof of Corollary 2

Proof of Corollary 2. We use the notation of [6]: for all $r \in (0,1)$, $m_r = \inf_{Q_r} u$, $M_r = \sup_{Q_r} u$, $o_r = M_r - m_r = \mathrm{osc}_{Q_r} u$. The non-negative functions $u - m_1$ and $M_1 - u$ satisfy equations $F_- = 0$, $F^+ = 0$ respectively for some non-linearities F_- and F^+ satisfying (7), (8) with f replaced with $f + \gamma_F M_1$. Hence, we can apply the Harnack inequality two $M_1 - u$ and $u - m_1$ and get

$$\begin{aligned} M_{1/2} - m_1 &\le C(m_{1/2} - m_1 + \max(M_F, \|f\|_{L^n(Q_1)} + \gamma_F |m_1|))\,, \\ M_1 - m_{1/2} &\le C(M_1 - M_{1/2} + \max(M_F, \|f\|_{L^n(Q_1)} + \gamma_F |M_1|)) \end{aligned}$$

where we can assume without loss of generality that $C > 1$. Adding these two inequalities and rearranging terms, we obtain

$$\mathrm{osc}_{Q_{1/2}} u \le \frac{C-1}{C+1} \mathrm{osc}_{Q_1} u + 2 \max(M_F, \|f\|_{L^n(Q_1)} + \gamma_F \|u\|_{L^\infty(Q_1)})\,.$$

We now use Lemma 8.3 in [17] in order to get (22). $\square$

B How to get Theorem 7

In this section, we explain how to adapt the proofs of Theorems 2, 3 in order to get Theorem 7. We must be especially careful when rescaling solutions.

- First, the barrier from Lemma 1 must be adapted in order to ensure $M_B \le 1/2$. To do so, we simply replace (12) with $\varphi \le -1/3$. The reader can check that our proof works with minor modifications.
- In Lemma 2, $\gamma_s = t_0^2 \gamma_F$ is replaced with $\gamma_s = t_0^2 M_0^\alpha \gamma_F$ with $\gamma_F = \|c\|_\infty$.
- In Lemmata 6 and 7, $\inf_{Q_{3r}} u \le 1$ is replaced with $\inf_{Q_{3r}} u \le 1/4$. In the proof of Lemma 6, V and t should be chosen properly (since the choice is quite obvious, we do not give it). In the proof of Lemma 7, $M_B < 1$ permits to replace $M_B^{1+\alpha}$ with 1 in computations.
- In the proof of Lemma 8, Lemma 2 is used with $M_0 = 4(M_B)^{k-1}$ and gives $\gamma_s = \frac{4^\alpha M_B^{\alpha(k-1)}}{4^i} \gamma_F \le \gamma_F$.

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