

On FO^2 quantifier alternation over words^{*}

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Abstract. We show that each level of the quantifier alternation hierarchy within $\text{FO}^2[<]$ on words is a variety of languages. We use the notion of condensed rankers, a refinement of the rankers defined by Weis and Immerman, to produce a decidable hierarchy of varieties which is interwoven with the quantifier alternation hierarchy – and conjecturally equal to it. It follows that the latter hierarchy is decidable within one unit, a much more precise result than what is known about the quantifier alternation hierarchy within $\text{FO}[<]$, where no decidability result is known beyond the very first levels.

First-order logic is an important object of study in connection with computer science and language theory, not least because many important and natural problems are first-order definable: our understanding of the expressive power of this logic and the efficiency of the solution of related algorithmic problems are of direct interest in such fields as verification. Here, by first-order logic, we mean the first-order logic of the linear order, $\text{FO}[<]$, interpreted on finite words.

In this context, there has been continued interest in fragments of first-order logic, defined by the limitation of certain resources, e.g. the quantifier alternation hierarchy (which is closely related with the dot-depth hierarchy of star-free languages). It is still an open problem whether each level of this hierarchy is decidable.⁴ Another natural restriction concerns the number of variables used (and re-used!) in a formula. It is interesting, notably because the trade-off between formula size and number of variables is known to be related with the trade-off between parallel time and number of processes, see [18, 5, 1, 4].

In this paper, we concentrate on $\text{FO}^2[<]$, the 2-variable fragment of $\text{FO}[<]$. It is well-known that every $\text{FO}[<]$ -formula is logically equivalent with a formula using only 3 variables, but that $\text{FO}^2[<]$ is properly less expressive than $\text{FO}[<]$. The expressive power of $\text{FO}^2[<]$ was characterized in many interesting fashions (see [12, 14, 15, 3]), and in particular, we know how to decide whether an $\text{FO}[<]$ -formula is equivalent to one in $\text{FO}^2[<]$.

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⁴ On the other hand, the quantifier alternation hierarchy collapses at level 2 for the first-order logic of the successor $\text{FO}[S]$ [16, 9].

A recent result of Weis and Immerman refined a result of Schwentick, Thérien and Vollmer [12] to give a combinatorial description of the $\text{FO}_m^2[\prec]$ -definable languages (those that can be defined by an $\text{FO}^2[\prec]$ -formula with quantifier alternation bounded above by m), using the notion of rankers. Rankers are finite sequences of instructions of the form *go to the next a -position to the right* (resp. *left*) of the current position.

Our first set of results shows that \mathcal{FO}_m^2 (the $\text{FO}_m^2[\prec]$ -definable languages), and the classes of languages defined by rankers having m alternations of directions (right *vs.* left), are varieties of languages. This means that membership of a language L in these classes depends only on the syntactic monoid of L , which justifies an algebraic approach of decidability.

Our investigation shows that rankers are actually better suited to characterize a natural hierarchy within unary temporal logic, and we introduce the new notion of a condensed ranker, that is more adapted to discuss the quantifier alternation hierarchy within $\text{FO}^2[\prec]$. There again, the alternation of directions in rankers defines hierarchies of varieties of languages \mathcal{R}_m and \mathcal{L}_m , with particularly interesting properties. Indeed, we show that these varieties are decidable, that they admit a neat characterization in terms of closure under deterministic and co-deterministic products, and that $\mathcal{R}_m \cup \mathcal{L}_m \subseteq \mathcal{FO}_m^2 \subseteq \mathcal{R}_{m+1} \cap \mathcal{L}_{m+1}$. The latter containments show that we can effectively compute, given a language $L \in \mathcal{FO}^2$, an integer m such that L is in \mathcal{FO}_{m+1}^2 , possibly in \mathcal{FO}_m^2 , but not in \mathcal{FO}_{m-1}^2 . This is much more precise than the current level of knowledge on the general quantifier alternation hierarchy in $\text{FO}[\prec]$.⁵

1 An algebraic approach to study FO_m^2

If $u \in A^+$ is a non-empty word, we denote by $u[i]$ the letter of u in position i ($1 \leq i \leq |u|$), and by $u[i, j]$ be the factor $u[i] \cdots u[j]$ of u ($1 \leq i \leq j \leq |u|$). Then we identify the word u with the logical structure $(\{1, \dots, |u|\}, (\mathbf{a})_{a \in A})$, where \mathbf{a} denotes the set of integers i such that $u[i] = a$.

Let $\text{FO}[\prec]$ (resp. $\text{FO}^k[\prec]$, $k \geq 0$) denote the set of first-order formulas using the unary predicates \mathbf{a} ($a \in A$) and the binary predicate $<$ (resp. and at most k variable symbols). It is well-known that $\text{FO}^3[\prec]$ is as expressive as $\text{FO}[\prec]$ and that $\text{FO}^2[\prec]$ is properly less expressive.

In the sequel, we omit specifying the predicate $<$ and we write simply FO or FO^k . The classes of FO - and FO^2 -definable languages have well-known beautiful characterizations [12, 14, 15, 3]. Two are of particular interest in this paper.

- The algebraic characterization in terms of recognizing monoids: a language is FO -definable if and only if it is recognized by a finite aperiodic monoid, i.e., one in which $x^n = x^{n+1}$ for each element x and for all n large enough (Schützenberger and McNaughton-Ladner, see [13]); and a language is FO^2 -definable if and only if it is recognized by a finite monoid in \mathbf{DA} (see [14]), a class of monoids with

⁵ Unfortunately, it does not help with the general problem since a language L is $\text{FO}^2[\prec]$ -definable if and only if L and its complement are Σ_2 -definable [11].

many interesting characterizations, which will be discussed later. These algebraic characterizations prove the decidability of the corresponding classes of languages: L is FO (resp. FO²) definable if and only if the (effectively computable) syntactic monoid of L is in the (decidable) class of aperiodic monoids (resp. in DA).

- The language-theoretic characterization: a language is in FO-definable if and only if it is star-free, i.e., it can be obtained from singletons using Boolean operations and concatenation products (Schützenberger, see [8]); a language is FO²-definable if and only if it can be written as the disjoint union of unambiguous products of the form $B_0^* a_1 B_1^* \cdots a_k B_k^*$, where $k \geq 0$, the a_i are letters and the B_i are subsets of the alphabet. Such a product is called *unambiguous* if each word $u \in B_0^* a_1 B_1^* \cdots a_k B_k^*$ admits a unique factorization in the form $u = u_0 a_1 u_1 \cdots a_k u_k$ such that $u_i \in B_i^*$ for each i .

We now concentrate on FO²-formulas and we define two important parameters concerning such formulas. To simplify matters, we consider only formulas where negation is used only on atomic formulas so that, in particular, no quantifier is negated. This is naturally possible up to logical equivalence. Now, with each formula $\varphi \in \text{FO}^2$, we associate in the natural way a parsing tree: each occurrence of a quantification, $\exists x$ or $\forall x$, yields a unary node, each occurrence of \vee or \wedge yields a binary node, and the leaves are labeled with atomic or negated atomic formulas. Each path from root to leaf in this parsing tree has a *quantifier label*, which is the sequence of quantifier node labels (\exists or \forall) encountered along this path. A *block* in this quantifier label is a maximal factor consisting only of \exists or only of \forall . The *quantifier depth* of φ is the maximum length of the quantifier label of a path in the parsing tree of φ , and the *number of blocks* of φ is the maximum number of blocks in the quantifier label of a path in its parsing tree.

We let $\text{FO}_{m,n}^2$ denote the set of first-order formulas with quantifier depth at most n and with at most m blocks and let FO_m^2 denote the union of the $\text{FO}_{m,n}^2$ for all n . We also denote by \mathcal{FO}^2 (\mathcal{FO}_m^2) the class of FO² (FO_m^2)-definable languages. Weis and Immerman's characterization of the expressive power of $\text{FO}_{m,n}^2[<]$ [18] in terms of rankers, see Theorem 1.2 below, forms the basis of our own results.

1.1 Rankers and logic

A *ranker* [18] is a non-empty word on the alphabet $\{X_a, Y_a \mid a \in A\}$.⁶ Rankers may define positions in words: given a word $u \in A^+$ and a letter $a \in A$, we denote by $X_a(u)$ (resp. $Y_a(u)$) the least (resp. greatest) integer $1 \leq i \leq |u|$ such that $u[i] = a$. If a does not occur in u , we say that $Y_a(u)$ and $X_a(u)$ are not defined. If in addition q is an integer such that $1 \leq q \leq |u|$, we let

$$\begin{aligned} X_a(u, q) &= X_a(u[q+1, |u|]) \\ Y_a(u, q) &= Y_a(u[1, q-1]). \end{aligned}$$

⁶ Weis and Immerman write \triangleright_a and \triangleleft_a instead of X_a and Y_a . We rather follow the notation in [3], where X and Y refer to the future and past operators of LTL.

These definitions are extended to all rankers: if r' is a ranker, $Z \in \{X_a, Y_a \mid a \in A\}$ and $r = r'Z$, we let $r(u, q) = Z(u, r'(u, q))$ if $r'(u, q)$ and $Z(u, r'(u, q))$ are defined, and we say that $r(u, q)$ is undefined otherwise.

Finally, if r starts with an X- (resp. Y-) letter, we say that r defines the position $r(u) = r(u, 0)$ (resp. $r(u) = r(u, |u| + 1)$), or that it is undefined on u if this position does not exist. Then $L(r)$ is the language of all words on which r is defined. We say that the words u and v agree on a class R of rankers if exactly the same rankers from R are defined on u and v .

The *depth* of a ranker r is defined to be its length (as a word). A *block* in r is a maximal factor in $\{X_a \mid a \in A\}^+$ (an X-block) or in $\{Y_a \mid a \in A\}^+$ (a Y-block). If $n \geq m$, we denote by $R_{m,n}^X$ (resp. $R_{m,n}^Y$) the set of m -block, depth n rankers, starting with an X- (resp. Y-) block, and we let $R_{m,n} = R_{m,n}^X \cup R_{m,n}^Y$ and $\underline{R}_{m,n}^X = \bigcup_{n' \leq n} R_{m,n'}^X \cup \bigcup_{m' < m, n' < n} R_{m',n'}$. We define $\underline{R}_{m,n}^Y$ dually and we let $\underline{R}_m^X = \bigcup_{n \geq m} \underline{R}_{m,n}^X$, $\underline{R}_m^Y = \bigcup_{n \geq m} \underline{R}_{m,n}^Y$ and $\underline{R}_m = \underline{R}_m^X \cup \underline{R}_m^Y$.

Rankers and temporal logic Let us depart for a moment from the consideration of FO^2 -formulas, to observe that rankers are naturally suited to describe the different levels of a natural class of temporal logic. The symbols X_a and Y_a ($a \in A$) can be seen as modal (temporal) operators, with the *future* and *past* semantics respectively. We denote the resulting temporal logic (known as *unary temporal logic*) by TL: its only atomic formula is \top , the other formulas are built using Boolean connectives and modal operators. Let $u \in A^+$ and let $0 \leq i \leq |u| + 1$. We say that \top holds at every position i , $(u, i) \models \top$; Boolean connectives are interpreted as usual; and $(u, i) \models X_a \varphi$ (resp. $Y_a \varphi$) if and only if $(u, j) \models \varphi$, where j is the least a -position such that $i < j$ (resp. the greatest a -position such that $j < i$). We also say that $u \models X_a \varphi$ (resp. $Y_a \varphi$) if $(u, 0) \models X_a \varphi$ (resp. $(u, 1 + |u|) \models Y_a \varphi$).

TL is a fragment of *propositional temporal logic* PTL; the latter is expressively equivalent to FO and TL is expressively equivalent to FO^2 , see [14].

As in the case of FO^2 -formulas, one may consider the parsing tree of a TL-formula and define inductively its depth and number of alternations (between past and future operators). If $n \geq m$, the fragment $\text{TL}_{m,n}^X$ (resp. $\text{TL}_{m,n}^Y$) consists of the TL-formulas with depth n and with m alternations, in which every branch (of the parsing tree) with exactly m alternations starts with future (resp. past) operators. The fragments $\text{TL}_{m,n}$, $\underline{\text{TL}}_{m,n}^X$, $\underline{\text{TL}}_{m,n}^Y$, $\underline{\text{TL}}_m^X$, $\underline{\text{TL}}_m^Y$ and $\underline{\text{TL}}_m$ are defined according to the same pattern as in the definition of $R_{m,n}$, $\underline{R}_{m,n}^X$, $\underline{R}_{m,n}^Y$, \underline{R}_m^X , \underline{R}_m^Y and \underline{R}_m . We also denote by $\mathcal{TL}_{m,n}^X$ (\mathcal{TL}_m^X , $\underline{\mathcal{TL}}_m^X$, etc) the class of $\text{TL}_{m,n}^X$ (TL_m^X , $\underline{\text{TL}}_m^X$, etc)-definable languages. The following result is elementary.

Proposition 1.1. *Let $1 \leq m \leq n$. Two words satisfy the same $\text{TL}_{m,n}^X$ formulas if and only if they agree on rankers from $R_{m,n}^X$. A language is in $\mathcal{TL}_{m,n}^X$ if and only if it is a Boolean combination of languages of the form $L(r)$, $r \in R_{m,n}^X$.*

Similar statements hold for $\text{TL}_{m,n}^Y$, $\text{TL}_{m,n}$, $\underline{\text{TL}}_{m,n}^X$, $\underline{\text{TL}}_{m,n}^Y$, $\underline{\text{TL}}_m^X$, $\underline{\text{TL}}_m^Y$ and $\underline{\text{TL}}_m$, relative to the corresponding classes of rankers.

Rankers and FO^2 The connection established by Weis and Immerman [18] between rankers and formulas in $\text{FO}_{m,n}^2$, Theorem 1.2 below, is deeper. If x, y are integers, we let $\text{ord}(x, y)$, the *order type* of x and y , be one of the symbols $<, >$ or $=$, depending on whether $x < y$, $x > y$ or $x = y$.

Theorem 1.2. *Let $u, v \in A^*$ and let $1 \leq m \leq n$. Then u and v satisfy the same formulas in $\text{FO}_{m,n}^2$ if and only if*

- (WI 1) *u and v agree on rankers from $R_{m,n}$,*
- (WI 2) *if the rankers $r \in \underline{R}_{m,n}$ and $r' \in \underline{R}_{m-1,n-1}$ are defined on u and v , then $\text{ord}(r(u), r'(u)) = \text{ord}(r(v), r'(v))$.*
- (WI 3) *if $r \in \underline{R}_{m,n}$ and $r' \in \underline{R}_{m,n-1}$ are defined on u and v and end with different direction letters, then $\text{ord}(r(u), r'(u)) = \text{ord}(r(v), r'(v))$.*

Corollary 1.3. *For each $n \geq m \geq 1$, $\underline{\mathcal{TL}}_{m,n} \subseteq \mathcal{FO}_{m,n}^2$ and $\underline{\mathcal{TL}}_m \subseteq \mathcal{FO}_m^2$.*

FO_m^2 and $\underline{\mathcal{TL}}_m$ -definable languages form varieties Our first result is the following. We refer the reader to [8] and to Section 1.2 below for background and discussion on varieties of languages.

Proposition 1.4. *For each $n \geq m \geq 1$, the classes $\underline{\mathcal{TL}}_{m,n}^X, \underline{\mathcal{TL}}_{m,n}^Y, \underline{\mathcal{TL}}_m^Y, \underline{\mathcal{TL}}_m^Y, \underline{\mathcal{TL}}_{m,n}, \underline{\mathcal{TL}}_m, \mathcal{FO}_{m,n}^2$ and \mathcal{FO}_m^2 are varieties of languages.*

Sketch of proof. Let $\rho_{m,n}$ be the relation for two words to agree on $\underline{\mathcal{TL}}_{m,n}^X$ -formulas. Using Proposition 1.1, one verifies that $\rho_{m,n}$ is a finite index congruence. Then a language is $\underline{\mathcal{TL}}_{m,n}^X$ -definable if and only if it is a union of $\rho_{m,n}$ -classes, if and only if it is recognized by the finite monoid $A^*/\rho_{m,n}$. It follows that these languages are exactly those accepted by the monoids in the pseudovariety generated by the $A^*/\rho_{m,n}$, for all finite alphabets A , and hence they form a variety of languages.

The proof for the other fragments of \mathcal{TL} is similar. For the fragments of FO^2 , we use Theorem 1.2 instead of Proposition 1.1. \square

This result shows that, for a given regular language L , $\underline{\mathcal{TL}}_m^X$ - (resp. $\underline{\mathcal{TL}}_m^-$, FO_m^2 -, etc) definability is characterized algebraically, that is, it depends only on the syntactic monoid of L . This justifies using the algebraic path to tackle decidability of these definability problems. Eilenberg's theory of varieties provides the mathematical framework.

1.2 A short survey on varieties and pseudovarieties

We summarize in this section the information on monoid and variety theory that will be relevant for our purpose, see [8, 2, 14, 15] for more details.

A language $L \subseteq A^*$ is *recognized* by a monoid M if there exists a morphism $\varphi: A^* \rightarrow M$ such that $L = \varphi^{-1}(\varphi(L))$. For instance, if $u \in A^*$ and $B \subseteq A$, let $\text{alph}(u) = \{a \in A \mid u = vaw \text{ for some } v, w \in A^*\}$ and $[B] = \{u \in A^* \mid \text{alph}(u) =$

$B\}$. Then $[B]$ is recognized by the direct product of $|B|$ copies of the 2-element monoid $\{0, 1\}$ (multiplicative).

A *pseudovariety* of monoids is a class of finite monoids closed under taking direct products, homomorphic images and submonoids. Pseudovarieties of subsemigroups are defined similarly. A *class of languages* \mathcal{V} is a collection $\mathcal{V} = (\mathcal{V}(A))_A$, indexed by all finite alphabets A , such that $\mathcal{V}(A)$ is a set of languages in A^* . If \mathbf{V} is a pseudovariety of monoids, we let $\mathcal{V}(A)$ be the set of languages of A^* recognized by a monoid in \mathbf{V} . The class \mathcal{V} is closed under Boolean operations, residuals and inverse homomorphic images. Classes of recognizable languages with these properties are called *varieties* of languages, and Eilenberg's theorem (see [8]) states that the correspondence $\mathbf{V} \mapsto \mathcal{V}$, from pseudovarieties of monoids to varieties of languages, is one-to-one and onto. Moreover, the decidability of membership in the pseudovariety \mathbf{V} , implies the decidability of the variety \mathcal{V} : indeed, a language is in \mathcal{V} if and only if its (effectively computable) syntactic monoid is in \mathbf{V} .

For every finite semigroup S and $s \in S$, we denote by s^ω the unique power of s which is idempotent. The *Green relations* are another important concept to describe monoids: if S is a monoid and $s, t \in S$, we say that $s \leq_{\mathcal{J}} t$ (resp. $s \leq_{\mathcal{R}} t$, $s \leq_{\mathcal{L}} t$) if $s = utv$ (resp. $s = tv$, $s = ut$) for some $u, v \in S$. We also say that $s \mathcal{J} t$ is $s \leq_{\mathcal{J}} t$ and $t \leq_{\mathcal{J}} s$. The relations \mathcal{R} and \mathcal{L} are defined similarly.

Pseudovarieties that will be important in this paper are the following.

- \mathbf{J}_1 , the pseudovariety of idempotent and commutative monoids, whose corresponding variety of languages consists of the Boolean combinations of languages of the form $[B]$.

- \mathbf{R} , \mathbf{L} and \mathbf{J} , the pseudovarieties of \mathcal{R} -, \mathcal{L} - and \mathcal{J} -trivial monoids; a monoid is, say, \mathcal{R} -trivial if each of its \mathcal{R} -classes is a singleton.

- \mathbf{DA} , the pseudovariety of all monoids in which $(xy)^\omega x(xy)^\omega = (xy)^\omega$ for all x, y ; \mathbf{DA} has a great many characterizations in combinatorial, algebraic and logical terms [2, 11, 12, 14, 15].

- \mathbf{K} (resp. \mathbf{D} , \mathbf{LI}) is the pseudovariety of semigroups in which $x^\omega y = x^\omega$ (resp. $yx^\omega = x^\omega$, $x^\omega yx^\omega = x^\omega$) for all x, y .

Finally, if \mathbf{V} is a pseudovariety of semigroups and \mathbf{W} is a pseudovariety of monoids, we say that a finite monoid M lies in the *Mal'cev product* $\mathbf{W} \overset{m}{\circ} \mathbf{V}$ if there exists a finite monoid T and onto morphisms $\alpha: T \rightarrow M$ and $\beta: T \rightarrow N$ such that $N \in \mathbf{W}$ and $\beta^{-1}(e) \in \mathbf{V}$ for each idempotent e of N . Then $\mathbf{W} \overset{m}{\circ} \mathbf{V}$ is a pseudovariety of monoids and we have in particular [8, 2, 10]:

$$\mathbf{K} \overset{m}{\circ} \mathbf{J}_1 = \mathbf{K} \overset{m}{\circ} \mathbf{J} = \mathbf{R}, \quad \mathbf{D} \overset{m}{\circ} \mathbf{J}_1 = \mathbf{D} \overset{m}{\circ} \mathbf{J} = \mathbf{L}, \quad \mathbf{LI} \overset{m}{\circ} \mathbf{J}_1 = \mathbf{LI} \overset{m}{\circ} \mathbf{J} = \mathbf{DA}.$$

We denote by $\mathbf{TL}_{m,n}^X$, $\mathbf{TL}_{m,n}^Y$, \mathbf{TL}_m^Y , \mathbf{TL}_m^X , $\mathbf{TL}_{m,n}$, \mathbf{TL}_m , $\mathbf{FO}_{m,n}^2$ and \mathbf{FO}_m^2 the pseudovarieties corresponding to the language varieties discovered in Proposition 1.4.

2 Main results

Our main tool to approach the decidability of \mathbf{FO}_m^2 -definability lies in a variant of rankers, which we borrow from a proof in Weis and Immerman's paper [18]. As

in the turtle language of [12], a ranker can be seen as a sequence of instructions: go to the next a to the right, go to the next b to the left, etc. We say that a ranker r is *condensed on u* if it is defined on u , and if the sequence of positions visited *zooms in* on $r(u)$, never crossing over a position already visited. Formally, $r = Z_1 \cdots Z_n$ is condensed on u if there exists a chain of open intervals

$$(0, |u| + 1) = (i_0, j_0) \supset (i_1, j_1) \supset \cdots \supset (i_{n-1}, j_{n-1}) \ni r(u)$$

such that for all $1 \leq \ell \leq n - 1$ the following properties are satisfied:

- If $Z_\ell Z_{\ell+1} = X_a X_b$ then $(i_\ell, j_\ell) = (X_a(u, i_{\ell-1}), j_{\ell-1})$.
- If $Z_\ell Z_{\ell+1} = Y_a Y_b$ then $(i_\ell, j_\ell) = (i_{\ell-1}, Y_a(u, j_{\ell-1}))$.
- If $Z_\ell Z_{\ell+1} = X_a Y_b$ then $(i_\ell, j_\ell) = (i_{\ell-1}, X_a(u, i_{\ell-1}))$.
- If $Z_\ell Z_{\ell+1} = Y_a X_b$ then $(i_\ell, j_\ell) = (Y_a(u, j_{\ell-1}), j_{\ell-1})$.

For instance, the ranker $X_a Y_b X_c$ is defined on the words bac and bca , but it is condensed only on bca . Rankers in \underline{R}_1 , or of the form $X_a Y_{b_1} \cdots Y_{b_k}$ or $Y_a X_{b_1} \cdots X_{b_k}$, are condensed on all words on which they are defined. We denote by $L_c(r)$ the set of all words on which r is condensed.

Condensed rankers form a natural notion, which is equally well-suited to the task of describing FO_m^2 -definability (see Theorem 2.4 below). With respect to TL, for which Proposition 1.1 shows a perfect match with the notion of rankers, they can be interpreted as adding a strong notion of unambiguity, see Section 3 below and the work of Lodaya, Pandya and Shah [7] on unambiguous interval temporal logic.

2.1 Condensed rankers determine a hierarchy of pseudovarieties

Let us say that two words u and v *agree on condensed rankers from a set R* of rankers, if the same rankers are condensed on u and v . We write $u \triangleright_{m,n} v$ (resp. $u \triangleleft_{m,n} v$) if u and v agree on condensed rankers in $\underline{R}_{m,n}^X$ (resp. $\underline{R}_{m,n}^Y$).

These relations turn out to have a very nice recursive characterization. For each word $u \in A^*$ and letter a occurring in u , the *a -left* (resp. *a -right*) *factorization* of u is the factorization that isolates the leftmost (resp. rightmost) occurrence of a in u ; that is, the factorization $u = u_- a u_+$ such that a does not occur in u_- (resp. u_+). We say that the word $a_1 \cdots a_r$ is a *subword* of u if u can be factored as $u = u_0 a_1 u_1 \cdots a_r u_r$, with the $u_i \in A^*$.

Proposition 2.1. *The relations $\triangleright_{m,n}$ and $\triangleleft_{m,n}$ ($n \geq m \geq 1$) are uniquely determined by the following properties.*

- $u \triangleright_{1,n} v$ if and only if $u \triangleleft_{1,n} v$, if and only if u and v have the same subwords of length at most n .
- If $m \geq 1$, then $u \triangleright_{m,n} v$ if and only if $\text{alph}(u) = \text{alph}(v)$, $u \triangleleft_{m-1,n-1} v$ and for each letter $a \in \text{alph}(u)$, the a -left factorizations $u = u_- a u_+$ and $v = v_- a v_+$ satisfy $u_- \triangleleft_{m-1,n-1} v_-$ and $u_+ \triangleright_{m,n-1} v_+$.
- If $m \geq 1$, then $u \triangleleft_{m,n} v$ if and only if $\text{alph}(u) = \text{alph}(v)$, $u \triangleright_{m-1,n-1} v$ and for each letter $a \in \text{alph}(u)$, the a -right factorizations $u = u_- a u_+$ and $v = v_- a v_+$ satisfy $u_+ \triangleright_{m-1,n-1} v_+$ and $u_- \triangleleft_{m,n-1} v_-$.

Corollary 2.2. *The relations $\triangleright_{m,n}$ and $\triangleleft_{m,n}$ are finite-index congruences.*

For each $m \geq 1$, let us denote by \mathbf{R}_m (resp. \mathbf{L}_m) the pseudovariety generated by the quotients $A^*/\triangleright_{m,n}$ (resp. $A^*/\triangleleft_{m,n}$), where $n \geq m$ and A is a finite alphabet. Corollary 2.2 shows that a language L is in the corresponding variety \mathcal{R}_m (resp. \mathcal{L}_m) if and only if L is a Boolean combination of languages of the form $L_c(r)$, with $r \in \underline{R}_m^X$ (resp. \underline{R}_m^Y).

By definition, for all $m \geq 1$, \mathbf{R}_m and \mathbf{L}_m are contained in both \mathbf{R}_{m+1} and \mathbf{L}_{m+1} . According to the first statement of Proposition 2.1, $\triangleright_{1,n} = \triangleleft_{1,n}$ is the congruence defining the piecewise n -testable languages studied by Simon in the early 1970s, and that, in consequence, $\mathbf{R}_1 = \mathbf{L}_1 = \mathbf{J}$, the pseudovariety of \mathcal{J} -trivial monoids [8].

In addition, one can show that if a position in a word u is defined by a ranker $r \in \underline{R}_{m,n}^X$ (resp. $\underline{R}_{m,n}^Y$), then the same position is defined by a ranker $s \in \underline{R}_{m,n}^X$ (resp. $\underline{R}_{m,n}^Y$) which is condensed on u . This leads to the following result.

Proposition 2.3. *Let $n \geq m \geq 1$. If the words u and v agree on condensed rankers in $\underline{R}_{m,n}^X$ (resp. $\underline{R}_{m,n}^Y$), then they agree on rankers from the same class. In particular, $\underline{\mathbf{TL}}_m^X \subseteq \mathbf{R}_m$ and $\underline{\mathbf{TL}}_m^Y \subseteq \mathbf{L}_m$*

As indicated above, condensed rankers allow for a description of FO_m^2 -definability, as neat as with ordinary rankers: more precisely, we show that the statement of Weis and Immerman's theorem can be modified to used condensed rankers instead.

Theorem 2.4. *Let $u, v \in A^*$ and let $1 \leq m \leq n$. Then u and v satisfy the same formulas in $\text{FO}_{m,n}^2$ if and only if*

- (WI 1c) *u and v agree on condensed rankers from $R_{m,n}$,*
- (WI 2c) *if the rankers $r \in \underline{R}_{m,n}$ and $r' \in \underline{R}_{m-1,n-1}$ are condensed on u and v , then $\text{ord}(r(u), r'(u)) = \text{ord}(r(v), r'(v))$.*
- (WI 3c) *if $r \in \underline{R}_{m,n}$ and $r' \in \underline{R}_{m,n-1}$ are condensed on u and v and end with different direction letters, then $\text{ord}(r(u), r'(u)) = \text{ord}(r(v), r'(v))$.*

Thus there is a connection between \mathcal{FO}_m^2 and the varieties \mathcal{R}_m and \mathcal{L}_m . But much more can be said about the latter varieties.

2.2 Language hierarchies

Proposition 2.1 also leads to a description of the language varieties \mathcal{R}_m and \mathcal{L}_m in terms of deterministic and co-deterministic products. Recall that a product of languages $L = L_0 a_1 L_1 \cdots a_k L_k$ ($k \geq 1$, $a_i \in A$, $L_i \subseteq A^*$) is said to be *deterministic* if, for $0 \leq i \leq k$, each word $u \in L$ has a unique prefix in $L_0 a_1 L_1 \cdots L_{i-1} a_i$. If for each i , the letter a_i does not occur in L_{i-1} , the product $L_0 a_1 L_1 \cdots a_k L_k$ is called *visibly deterministic*: this is obviously a particular case of a deterministic product.

The definition of a *co-deterministic* or *visibly co-deterministic* product is dual, in terms of suffixes instead of prefixes. If \mathcal{V} is a class of languages and A is

a finite alphabet, let $\mathcal{V}^{det}(A)$ (resp. $\mathcal{V}^{vdet}(A)$, $\mathcal{V}^{codet}(A)$, $\mathcal{V}^{vcodet}(A)$) be the set of all Boolean combinations of languages of $\mathcal{V}(A)$ and of deterministic (resp. visibly deterministic, co-deterministic, visibly co-deterministic) products of languages of $\mathcal{V}(A)$. Schützenberger gave algebraic characterizations of the closure operations $\mathcal{V} \mapsto \mathcal{V}^{det}$ and $\mathcal{V} \mapsto \mathcal{V}^{codet}$, see [8]: if \mathcal{V} is a variety of languages and if \mathbf{V} is the corresponding pseudovariety of monoids, then \mathcal{V}^{det} and \mathcal{V}^{codet} are varieties of languages and the corresponding pseudovarieties are, respectively, $\mathbf{K} \widehat{\circledast} \mathbf{V}$ and $\mathbf{D} \widehat{\circledast} \mathbf{V}$. Then we show the following.

Proposition 2.5. *For each $m \geq 1$, we have $\mathcal{R}_{m+1} = \mathcal{L}_m^{vdet} = \mathcal{L}_m^{det}$, $\mathbf{R}_{m+1} = \mathbf{K} \widehat{\circledast} \mathbf{L}_m$, $\mathcal{L}_{m+1} = \mathcal{R}_m^{vcodet} = \mathcal{R}_m^{codet}$ and $\mathbf{L}_{m+1} = \mathbf{D} \widehat{\circledast} \mathbf{R}_m$. In particular, $\mathbf{R}_2 = \mathbf{R}$ and $\mathbf{L}_2 = \mathbf{L}$.*

Sketch of proof. Proposition 2.1 shows that $\mathcal{R}_{m+1} \subseteq \mathcal{L}_m^{vdet}$, which is trivially contained in \mathcal{L}_m^{det} . The last containment is proved algebraically, by showing that if $\gamma: A^* \rightarrow M$ is an onto morphism, and $M \in \mathbf{K} \widehat{\circledast} \mathbf{L}_m$, then for some large enough n , $u \triangleright_{m+1,n} v$ implies $\gamma(u) = \gamma(v)$: thus M is a quotient of $A^*/\triangleright_{m+1,n}$ and hence, $M \in \mathbf{R}_{m+1}$. This proof relies on a technical property of semigroups in **DA**: if $a \in A$ occurs in $\text{alph}(v)$ and $\gamma(u) \mathcal{R} \gamma(uv)$, then $\gamma(uva) \mathcal{R} \gamma(u)$. \square

It turns out that the \mathbf{R}_m and the \mathbf{L}_m were studied in the semigroup-theoretic literature (Kufleitner, Trotter and Weil, [17, 6]). In [6], it is defined as the hierarchy of pseudovarieties obtained from \mathbf{J} by repeated applications of the operations $\mathbf{X} \mapsto \mathbf{K} \widehat{\circledast} \mathbf{X}$ and $\mathbf{X} \mapsto \mathbf{D} \widehat{\circledast} \mathbf{X}$. Proposition 2.5 shows that it is the same hierarchy as that considered in this paper⁷. The following results are proved in [6, Section 4].

Proposition 2.6. *The hierarchies $(\mathbf{R}_m)_m$ and $(\mathbf{L}_m)_m$ are infinite chains of decidable pseudovarieties, and their unions are equal to **DA**. Moreover, every m -generated monoid in **DA** lies in $\mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$.*

The decidability statement in Proposition 2.6 is in fact a consequence of a more precise statement (see [17, 6]) which gives defining pseudoidentities for the \mathbf{R}_m and \mathbf{L}_m . Let x_1, x_2, \dots be a sequence of variables. If u is a word on that alphabet, we let \bar{u} be the mirror image of u , that is, the word obtained from reading u from right to left. We let

$$\begin{aligned} G_2 &= x_2 x_1, & I_2 &= x_2 x_1 x_2, \\ \text{for } n > 2, & G_n &= x_n \overline{G_{n-1}}, & I_n &= G_n x_n \overline{I_{n-1}}, \\ & \varphi(x_1) &= (x_1^\omega x_2^\omega x_1^\omega)^\omega, & \varphi(x_2) &= x_2^\omega, \\ \text{and, for } n > 2, & \varphi(x_n) &= (x_n^\omega \varphi(\overline{G_{n-1}} G_{n-1})^\omega x_n^\omega)^\omega. \end{aligned}$$

Then we have [6]:

⁷ More precisely, the pseudovarieties \mathbf{R}_m and \mathbf{L}_m in [6] are pseudovarieties of semigroups, and the \mathbf{R}_m and \mathbf{L}_m considered in this paper are the classes of monoids in these pseudovarieties.

Proposition 2.7. For each $m \geq 2$, $\mathbf{R}_m = \mathbf{DA} \cap [\varphi(G_m) = \varphi(I_m)]$ and $\mathbf{L}_m = \mathbf{DA} \cap [\varphi(\overline{G_m}) = \varphi(\overline{I_m})]$.

Example 2.8. For \mathbf{R}_2 , this yields the pseudo-identity $x_2^\omega (x_1^\omega x_2^\omega x_1^\omega)^\omega = x_2^\omega (x_1^\omega x_2^\omega x_1^\omega)^\omega x_2^\omega$. One can verify that, together with the pseudo-identity defining \mathbf{DA} , this is equivalent to the usual pseudo-identity describing $\mathbf{R} = \mathbf{R}_2$, namely $(st)^\omega s = (st)^\omega$.

For $\mathbf{R}_3 = \mathbf{K} \circledast \mathbf{L}$, no pseudo-identity was known in the literature. We get

$$\begin{aligned}\varphi(G_3) &= (x_3^\omega ((x_1^\omega x_2^\omega x_1^\omega)^\omega x_2^\omega (x_1^\omega x_2^\omega x_1^\omega)^\omega)^\omega x_3^\omega)^\omega \\ \varphi(I_3) &= (x_3^\omega ((x_1^\omega x_2^\omega x_1^\omega)^\omega x_2^\omega (x_1^\omega x_2^\omega x_1^\omega)^\omega)^\omega x_3^\omega)^\omega \\ &\quad (x_3^\omega ((x_1^\omega x_2^\omega x_1^\omega)^\omega x_2^\omega (x_1^\omega x_2^\omega x_1^\omega)^\omega)^\omega x_3^\omega)^\omega x_2^\omega (x_1^\omega x_2^\omega x_1^\omega)^\omega x_2^\omega.\end{aligned}$$

2.3 Connection with the \mathbf{TL}_m and the \mathbf{FO}_m^2 hierarchies

Proposition 2.3 established a containment between the \mathbf{R}_m (resp. \mathbf{L}_m) and the \mathbf{TL}_m hierarchies. A technical analysis allows us to prove a containment in the other direction, but one that is not very tight – showing the difference between the consideration of condensed rankers and that of ordinary rankers.

Proposition 2.9. $\mathbf{R}_2 = \mathbf{TL}_2^X$ and $\mathbf{L}_2 \subseteq \mathbf{TL}_2^Y$. If $m \geq 3$ and if two words agree on rankers in $\underline{\mathbf{R}}_{\lfloor 3m/2 \rfloor}^X$ (resp. $\underline{\mathbf{R}}_{\lfloor 3m/2 \rfloor}^Y$), then they agree on condensed rankers in $\underline{\mathbf{R}}_m^X$ (resp. $\underline{\mathbf{R}}_m^Y$). In particular $\mathbf{R}_m \subseteq \mathbf{TL}_{\lfloor 3m/2 \rfloor}^X$ and $\mathbf{L}_m \subseteq \mathbf{TL}_{\lfloor 3m/2 \rfloor}^Y$.

Example 2.10. The language $L_c(X_a Y_b X_c)$ is in \mathcal{R}_3 and not in $\underline{\mathcal{L}}_3^X$.

The connection between the \mathbf{R}_m , \mathbf{L}_m and \mathbf{FO}_m^2 hierarchies is tighter.

Theorem 2.11. Let $m \geq 1$. Every language in \mathcal{R}_m or \mathcal{L}_m is \mathbf{FO}_m^2 -definable, and every \mathbf{FO}_m^2 -definable language is in $\mathcal{R}_{m+1} \cap \mathcal{L}_{m+1}$. Equivalently, we have

$$\mathbf{R}_m \vee \mathbf{L}_m \subseteq \mathbf{FO}_m^2 \subseteq \mathbf{R}_{m+1} \cap \mathbf{L}_{m+1},$$

where $\mathbf{V} \vee \mathbf{W}$ denotes the least pseudovariety containing \mathbf{V} and \mathbf{W} .

Sketch of proof. The containment $\mathbf{R}_m \vee \mathbf{L}_m \subseteq \mathbf{FO}_m^2$ follows directly from Property (WI 1c) in Theorem 2.4. The proof of the converse containment also relies on that theorem. We show that if $u \triangleright_{m+1, 2n}$ or $u \triangleleft_{m+1, 2n}$, then Properties (WI 1c), (WI 2c) and (WI 3c) hold for m, n . This is done by a complex and quite technical induction. \square

If $m = 1$, we know that $\mathbf{R}_2 \cap \mathbf{L}_2 = \mathbf{R} \cap \mathbf{L} = \mathbf{J} = \mathbf{R}_1 \vee \mathbf{L}_1$: this reflects the elementary observation that \mathbf{FO}_1^2 -definable languages, like \mathbf{FO}_1 -definable languages, are the piecewise testable languages. For $m \geq 2$, we conjecture that $\mathbf{R}_m \vee \mathbf{L}_m$ is properly contained in $\mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$. The following shows it holds for $m = 2$.

Example 2.12. $L = \{b, c\}^*ca\{a, b\}^*$ is \mathbf{FO}_2^2 -definable, by the following formula:

$$\begin{aligned} & \exists i (\mathbf{c}(i) \wedge (\forall j (j < i \rightarrow \neg \mathbf{a}(j))) \wedge (\forall j (j > i \rightarrow \neg \mathbf{c}(j)))) \\ & \wedge \exists i (\mathbf{a}(i) \wedge (\forall j (j < i \rightarrow \neg \mathbf{a}(j))) \wedge (\forall j (j > i \rightarrow \neg \mathbf{c}(j)))) \\ & \wedge \forall i (\mathbf{b}(i) \rightarrow (\exists j (j < i \wedge \mathbf{a}(j)) \vee (\exists j (j > i \wedge \mathbf{c}(j)))). \end{aligned}$$

The words $u_n = (bc)^n(ab)^n$ are in L , while the words $v_n = (bc)^nb(ca)^n$ are not. Almeida and Azevedo showed that $\mathbf{R}_2 \vee \mathbf{L}_2$ is defined by the pseudo-identity $(bc)^\omega(ab)^\omega = (bc)^\omega b(ab)^\omega$ [2, Theorem 9.2.13 and Exercise 9.2.15]. In particular, for each language K recognized by a monoid in $\mathbf{R}_2 \vee \mathbf{L}_2$, the words u_n and v_n (for n large enough) are all in K , or all in the complement of K . Therefore L is not recognized by such a monoid, which proves that $\mathbf{R}_2 \vee \mathbf{L}_2$ is strictly contained in \mathbf{FO}_2^2 , and hence also in $\mathbf{R}_3 \cap \mathbf{L}_3$. It also shows that $\underline{\mathcal{TL}}_2$ is properly contained in \mathcal{FO}_2^2 .

Finally, we formulate the following conjecture.

Conjecture 2.13. For each $m \geq 1$, $\mathbf{FO}_m^2 = \mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$.

3 Consequences

The main consequence we draw of Theorem 2.11 and of the decidability of the pseudovarieties \mathbf{R}_m and \mathbf{L}_m is summarized in the next statement.

Theorem 3.1. *Given an \mathbf{FO}^2 -definable language L , one can compute an integer m such that L is \mathbf{FO}_{m+1}^2 -definable but not \mathbf{FO}_{m-1}^2 -definable. That is: we can decide the quantifier alternation level of L within one unit.*

Sketch of proof. If $M \in \mathbf{DA}$, we can compute the largest m such that $M \notin \mathbf{R}_m \cap \mathbf{L}_m$ (Proposition 2.6). Then $M \notin \mathbf{FO}_{m+1}^2 \setminus \mathbf{FO}_{m-1}^2$ by Theorem 2.11. \square

The fact that the \mathbf{R}_m and \mathbf{L}_m form strict hierarchies (Proposition 2.6), together with Theorem 2.11, proves that the \mathcal{FO}_m^2 hierarchy is infinite. Weis and Immerman had already proved this result by combinatorial means [18], whereas our proof is algebraic. From that result on the \mathcal{FO}_m^2 , it is also possible to recover the strict hierarchy result on the \mathbf{R}_m and \mathbf{L}_m and the fact that their union is equal to \mathbf{DA} .

By the same token, Propositions 2.3 and 2.9 show that the $\underline{\mathcal{TL}}_m$ (resp. \mathbf{TL}_m) hierarchy is infinite and that its union is all of \mathcal{FO}^2 (resp. \mathbf{DA}).

Similarly, the fact that an m -generated element of \mathbf{DA} lies in $\mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$ (Proposition 2.6), shows that an \mathbf{FO}^2 -definable language in A^* lies in $\mathcal{R}_{|A|+1} \cap \mathcal{L}_{|A|+1}$, and hence in \mathcal{FO}_{m+1}^2 – a fact that was already established by combinatorial means by Weis and Immerman [18, Theorem 4.6]. It also shows that such a language is in $\underline{\mathcal{TL}}_{\frac{3}{2}(|A|+1)}$ by Proposition 2.9.

Finally we note the following refinement on [6, Proposition 4.6]. It was mentioned in the introduction that the languages in \mathcal{FO}^2 are disjoint unions of unambiguous products of the form $B_0^*a_1B_1^* \cdots a_kB_k^*$, where each B_i is a subset of A . Propositions 2.5 and 2.6 imply the following statement.

Proposition 3.2. *The least variety of languages containing the languages of the form B^* ($B \subseteq A$) and closed under visibly deterministic and visibly co-deterministic products, is \mathcal{FO}^2 .*

Every unambiguous product of languages of the form $B_0^ a_1 B_1^* \cdots a_k B_k^*$ (with each $B_i \subseteq A$), can be expressed in terms of the B_i^* and the a_i using only Boolean operations and at most $|A| + 1$ applications of visibly deterministic and visibly co-deterministic products, starting with a visibly deterministic (resp. co-deterministic) product.*

The weaker statement with the word *visibly* deleted was proved by the authors in [6], as well as by Lodaya, Pandya and Shah [7].

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