Solving BSDE with adaptive control variate. A note on the rate of convergence of the operator $\mathcal{P}^k$

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This note is a complement of the paper "Solving BSDE with adaptive control variate" [1]. It deals with the convergence of the approximating operator $\mathcal{P}$, based on a non parametric regression technique called local averaging, and defined in Definition 1.1.
Although the computations are quite standard (see [3], [2]), the specificities of the paper are the following

- the support of the variables is unbounded;
- the error has to be measured using specific L_2-norms;
- errors on the gradient are provided.

1 Definitions

Let us first introduce some notations

- Let $C_b^{k,l}$ be the set of continuously differentiable functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^d$ with continuous and uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l).
- C_p^k denotes the set of C^{k-1} functions whose k-th derivative is piecewise continuous.
- Constants $c_{i,j}(\cdot)$ and $C(d)$. For any function ϕ in $C_b^{i,j}$, $c_{i,j}(\phi)$ denotes $\sum_{k,l=0}^{i,j} |\partial_t^k \partial_x^l \phi|_\infty$. For $i = j = 0$, we set $c_0(\phi) := c_{0,0}(\phi)$. $C(d)$ denotes a constant depending only on d.
- Functions $K(T)$. $K(\cdot)$ denotes a generic function non decreasing in T which may depend on d, μ, β, on the coefficients b and σ (through σ_0, σ_1, $c_{1,3}(\sigma)$, $c_{0,1}(\partial_t\sigma)$, $c_{1,3}(b)$) and on other constants appearing in [1, Appendix A]. The parameter β is defined in [1, Section 2.1], μ is defined in [1, Section 3.2], σ_0 and σ_1 are defined in [1, Hypothesis 1]. .
- Functions $K_0(T)$. $K_0(T)$ are analogous to $K(T)$ except that they may also depend on the operator $\mathcal{P}$ (through $c_1(K_t)$ and $c_2(K_x)$, defined in Section [1, Section 7].

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Definition 1.1. We approximate a function $v(t,x)$ by

$$\mathcal{P}^k v(t,x) = \frac{r_n^k(t,x)}{f_n^k(t,x)} g(2^{d+1}T\lambda(B) f_n^k(t,x)), \tag{1.1}$$

where

- $r_n^k(t,x) = \frac{1}{nh_t h_x^d} \sum_{i=1}^n K_t(\frac{t-T_i^k}{h_t}) K_x(\frac{x-X_i^k}{h_x}) v(T_i^k, X_i^k)$;
- $f_n^k(t,x) = \frac{1}{nh_t h_x^d} \sum_{i=1}^n K_t(\frac{t-T_i^k}{h_t}) K_x(\frac{x-X_i^k}{h_x})$;
- the points $(T_i^k, X_i^k)_{1\leq i\leq n}$ are uniformly distributed on $[0,T]\times B$ where $B := B_\infty(0,a) = [-a,a]^d$, and $\mathcal{A}_k$ denotes the set of points $(T_i^k, X_i^k)_{1\leq i\leq n}$;
- $\lambda(B) = (2a)^d$;
- and g is such that

$$g(y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } y > 1, \\ -y^4 + 2y^2 & \text{if } y \in [0,1]; \end{cases} \tag{1.2}$$

- The kernel function K_t is defined on the compact support $[-1,1]$, bounded, even, non-negative, C_p^2 and $\int_{\mathbb{R}} K_t(u)du = 1$;
- The kernel function K_x is defined on the compact support $[-1,1]^d$, bounded, and such that $\forall y = (y_1, \cdots, y_d) \in \mathbb{R}^d, \;\; K_x(y) = \Pi_{j=1}^d K_x^j(y_j)$, where for $j = 1, \cdots, d$ $K_x^j : \mathbb{R} \to \mathbb{R}$ is an even non-negative C_p^2 function and $\int_{\mathbb{R}} K_x^j(u)du = 1$;
- δ_n denotes $\frac{1}{nh_t h_x^d}$, and $T\lambda(B)\delta_n \ll 1$;
- $h_x \ll a$ and $h_t \ll \frac{T}{2}$. Since we study the convergence when h_t and h_x tend to 0, we assume in the following that $h_t \leq 1$ and $h_x \leq 1$.

Remark 1.2. We give some useful bounds for g and its first derivative. The function $G : x \longmapsto \frac{g(x)}{x}$ is bounded by 2, g' is bounded by 2, $x \longmapsto \frac{g'(x)}{x}$ is bounded by 4 and $x \longmapsto \frac{g(x)}{x^2}$ is bounded by 2. Then, G' is bounded by 6.

Remark 1.3. This choice for the operator $\mathcal{P}^k$ is not harmless. $\mathcal{P}^k$ should be continuous and differentiable. That's why we multiply $\frac{r_n^k}{f_n^k}$ by a regularising function g at the point $2^{d+1}T\lambda(B)f_n^k$. Since the function $f_n^k(t,x)$ converges to $\frac{1}{T\lambda(B)}$ when n goes to infinity for $t \in]0,T[$ and $|x_i| < a$, $i = 1, \cdots, d$, $g(2^{d+1}T\lambda(B)f_n^k(t,x))$ converges to 1 when n goes to ∞. Hence, if $f_n^k \sim \frac{1}{T\lambda(B)}$, $\mathcal{P}^k v(t,x) = \frac{r_n^k(t,x)}{f_n^k(t,x)}$, which is a standard estimator. The function g has an impact on $\mathcal{P}^k$ only when f_n^k is strictly positive and small (compared to $\frac{1}{T\lambda(B)}$).

We also introduce the space $H_{\beta,X}^{m,\mu}$:

Definition 1.4 (Space $H^{m,\mu}_{\beta,X}$)**.** Let X denote the $\mathbb{R}^d$-valued process solution of

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{1.3}$$

where W is a q-dimensional standard Brownian motion, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times q}$. For any $m \leq 2, \beta > 0, \mu > 0$, let $H^{m,\mu}_{\beta,X}$ define the space of functions $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\|v\|^2_{H^{m,\mu}_{\beta,X}} = \int_0^T e^{\beta s} \int_{\mathbb{R}^d} e^{-\mu|x|} \sum_{k\leq m} \mathbb{E}|\partial^k_x v(s, X^x_s)|^2 dxds < \infty.$$

Definition 1.5 (Function ν^t_μ)**.** For any $s, t \in [0,T]$ and any $x, y \in \mathbb{R}^d$ such that $t < s$ we define $\nu^t_\mu(s, y) := \int_{\mathbb{R}^d} e^{-\mu|x|}p(t, x; s, y)dx$, where μ is a positive constant and p is the transition density function of the process X defined by (1.3).

Remark 1.6. Using the definition of ν, we also get $\|v\|^2_{H^\mu_{\beta,X}} = \int_0^T e^{\beta s} \int_{\mathbb{R}^d} dy\nu^0_\mu(s, y)|v(s, y)|^2$.

Hypothesis 1. We assume that the coefficients σ and b are Lipschitz and bounded measurable functions on $[0,T] \times \mathbb{R}^d$. We also assume that σ satisfies the ellipticity condition.

2 Main results

We aim at proving the following Propositions, which correspond to [1, Theorem 7.4].

Proposition 2.1. *Assume Hypothesis 1. We also assume that v is a $C^{1,2}([0,T]\times\mathbb{R}^d)$ function and v and $\partial_x v$ are bounded by $c_{0,1}(v)$ and v satisfies $\forall t, t' \in [0,T], \forall x \in \mathbb{R}^d$, $|\partial_x v(t, x) - \partial_x v(t', x)| \leq c_{1/2}(v)\sqrt{|t' - t|}$, where $c_{1/2}(v)$ is a positive constant. Then,*

$$\begin{aligned}\mathbb{E}\|\mathcal{P}^k v - v\|^2_{H^\mu_{\beta,X}} + \mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|^2_{H^\mu_{\beta,X}} \leq &\epsilon_1(\mathcal{P})(\mathbb{E}\|v\|^2_{H^{2,\mu}_{\beta,X}} + \mathbb{E}\|\partial_t v\|^2_{H^\mu_{\beta,X}}) \\ &+ e_2(\mathcal{P})(c^2_{1/2}(v) + c^2_{0,1}(v)),\end{aligned}$$

where $\epsilon_1(\mathcal{P}) = K_0(T)(h^2_t + h^2_x + \frac{T\lambda(B)\delta_n}{h^2_x})$, $e_2(\mathcal{P}) = K_0(T)(h_t + e^{-\mu a}\frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}} + \frac{T\lambda(B)\delta_n}{h^2_x})$. Moreover, if v is a $C^{1,2}_b$ function, we get $\mathbb{E}\|\mathcal{P}^k v - v\|^2_{H^\mu_{\beta,X}} + \mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|^2_{H^\mu_{\beta,X}} \leq (\epsilon_1(\mathcal{P}) + \epsilon_2(\mathcal{P}))(c^2_{1/2}(v) + c^2_{1,2}(v))$.

The proof of Proposition 2.1 is done in Sections 4 and 5. Section 4 (resp. Section 5) deals with the bound for $\mathbb{E}\|\mathcal{P}^k v - v\|^2_{H^\mu_{\beta,X}}$ (resp. $\mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|^2_{H^\mu_{\beta,X}}$).

Proposition 2.2. *Under Hypothesis 1, for any random function v from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}$ independent of $\mathcal{A}_k$, one has*

$$\mathbb{E}\|\mathcal{P}^k v\|^2_{H^\mu_{\beta,X}} + \mathbb{E}\|\partial_x(\mathcal{P}^k v)\|^2_{H^\mu_{\beta,X}} \leq c_4(\mathcal{P})\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}, \text{ where } c_4(\mathcal{P}) = \frac{K_0(T)}{h^2_x}.$$

If $\mathbb{E}(v(t, x)) = 0$, one has $\mathbb{E}\|\mathcal{P}^k v\|^2_{H^\mu_{\beta,X}} + \mathbb{E}\|\partial_x(\mathcal{P}^k v)\|^2_{H^\mu_{\beta,X}} \leq \epsilon_4(\mathcal{P})\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}$, where $\epsilon_4(\mathcal{P}) = K_0(T)\frac{T\lambda(B)\delta_n}{h^2_x}$.

The proof of Proposition 2.1 is done in Section 6.

Remark 2.3. For the sake of clearness, we omit the superscript k in the definition of r^k_n and f^k_n. From now on, r_n (resp. f_n) denotes r^k_n (resp. f^k_n).

3 Properties on f_n, r_n and other useful results

In this Section, we only recall some technical results on f_n and r_n proved in [4].

Lemma 3.1 (Lemma 12.13, [4]). *For all $(s,y) \in [0,T] \times \mathbb{R}^d$,*

$$\mathbb{E}[f_n(s,y)] = \frac{1}{T\lambda(B)} \int_{-1\vee\frac{-s}{h_t}}^{1\wedge\frac{T-s}{h_t}} K_t(r)dr\Pi_{j=1}^d \int_{-1\vee\frac{-a-y_j}{h_x}}^{1\wedge\frac{a-y_j}{h_x}} K_x^j(x_j)dx_j, \tag{3.1}$$

and $\mathbb{E}[f_n(s,y)] \le \frac{1}{T\lambda(B)}$. Moreover, for $(s,y) \in [0,T] \times B$, $\mathbb{E}[f_n(s,y)] \ge \frac{1}{T\lambda(B)2^{d+1}}$, and for $(s,y) \in [h_t, T-h_t] \times B_\infty(0,a-h_x)$, $\mathbb{E}[f_n(s,y)] = \frac{1}{T\lambda(B)} = \overline{f}(s,y)$.

Proposition 3.2 (Proposition 12.20, [4]). *Assume $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a bounded function. Then, for all $(s,y) \in [0,T] \times \mathbb{R}^d$,*

$$(\mathbb{E}[r_n(s,y)])^2 \le \frac{c_0^2(v)}{(T\lambda(B))^2}\mathbf{1}_{\{y\in B_\infty(0,a+h_x)\}},$$

$$Var(r_n(s,y)) \le 2^{d+1}\frac{c_0^2(K_t)c_0^2(K_x)c_0^2(v)\delta_n}{T\lambda(B)}\mathbf{1}_{\{y\in B_\infty(0,a+h_x)\}}.$$

Using $T\lambda(B)\delta_n << 1$ yields $\mathbb{E}(r_n^2(s,y)) \le 2^{d+2}\frac{c_0^2(K_t)c_0^2(K_x)c_0^2(v)}{(T\lambda(B))^2}\mathbf{1}_{\{y\in B_\infty(0,a+h_x)\}}$.
Assume $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^+$ is a continuous function. Then,

$$\mathbb{E}(r_n(s,y))^2 \le \frac{2^{d+1}}{h_t h_x^d (T\lambda(B))^2} \int_0^T drK_t^2\left(\frac{s-r}{h_t}\right)\int_B dzK_x^2\left(\frac{y-z}{h_x}\right)v^2(r,z),$$

$$Var(r_n(s,y)) \le \frac{1}{nh_t^2h_x^{2d}T\lambda(B)} \int_0^T drK_t^2\left(\frac{s-r}{h_t}\right)\int_B dzK_x^2\left(\frac{y-z}{h_x}\right)v^2(r,z).$$

Using $T\lambda(B)\delta_n << 1$ yields

$$\mathbb{E}(r_n^2(s,y)) \le \frac{2^{d+2}}{h_t h_x^d (T\lambda(B))^2} \int_0^T drK_t^2\left(\frac{s-r}{h_t}\right)\int_B dzK_x^2\left(\frac{y-z}{h_x}\right)v^2(r,z).$$

Lemma 3.3 (Lemma 12.17, [4]). *For all $(s,y) \in [0,T] \times \mathbb{R}^d$, for all $i \in \{1,\cdots,d\}$, the following assertion holds*

$$\mathbb{E}(\partial_{x_i} f_n(s,y))^2 \le 2^{2d+3}\frac{c_0^2(K_t)c_{0,1}^2(K_x)}{h_x^2(T\lambda(B))^2}\mathbf{1}_{\{y\in B_\infty(0,a+h_x)\}}.$$

Proposition 3.4 (Proposition 12.19, [4]). *Under Hypothesis 1, one has*

$$\int_0^T dse^{\beta s}\int_B dy\nu^0(s,y)|\mathbb{E}(\partial_{x_i} f_n(s,y))|^2 \le \frac{K_0(T)}{h_x(T\lambda(B))^2}e^{-\mu a}a^{d-1}.$$

Lemma 3.5 (Lemma 12.10, [4]). *Assume Hypothesis 1 and let f be a function from $[0,T]\times\mathbb{R}^d$ into $\mathbb{R}^+$, g_t a positive bounded function with compact support in $[-1,1]$ and g_x a positive bounded function with compact support in $[-1,1]^d$. Then,*

$$\int_0^T dse^{\beta s}\int_{\mathbb{R}^d} dy\nu^0(s,y)\int_0^T drg_t\left(\frac{s-r}{h_t}\right)\int_{\mathbb{R}^d} dzg_x\left(\frac{y-z}{h_x}\right)f(r,z) \le$$
$$K(T)c_0(g_t)c_0(g_x)h_th_x^d\int_0^T dre^{\beta r}\int_{\mathbb{R}^d}dz\nu^0(r,z)f(r,z).$$

4 Proof of Proposition 2.1: term $\mathbb{E}\|\mathcal{P}^k v - v\|^2_{H^\mu_{\beta,X}}$

The study of $\mathbb{E}\|\mathcal{P}^k v - v\|^2_{H^\mu_{\beta,X}}$ will be done in two steps. To do so, we add and substract

$$C_n(s,y) := \frac{r_n(s,y)}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}} \tag{4.1}$$

to the term $v - \mathcal{P}^k v$. C_n approximates well v inside the domain $[0,T]\times B$. We get $\mathbb{E}\|\mathcal{P}^k v - v\|^2_{H^\mu_{\beta,X}} \leq 2\mathbb{E}\|\mathcal{P}^k v - C_n\|^2_{H^\mu_{\beta,X}} + 2\mathbb{E}\|C_n - v\|^2_{H^\mu_{\beta,X}}$. The two following sections are devoted to the study of $\mathbb{E}\|C_n - v\|^2_{H^\mu_{\beta,X}}$ and $\mathbb{E}\|\mathcal{P}^k v - C_n\|^2_{H^\mu_{\beta,X}}$.

4.1 Study of $\mathbb{E}\|C_n - v\|^2_{H^\mu_{\beta,X}}$

Using the definition of C_n, we get

$$C_n(s,y) - v(s,y) = \frac{r_n(s,y) - v(s,y)\mathbb{E}[f_n(s,y)]}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}} - v(s,y)\mathbf{1}_{\{s\notin[0,T]\cup y\notin B\}}. \tag{4.2}$$

Then, we split $\mathbb{E}\|C_n - v\|^2_{H^\mu_{\beta,X}}$ in two terms, by using the bias-variance decomposition: $\mathbb{E}\|C_n - v\|^2_{H^\mu_{\beta,X}} = \|\mathbb{E}(C_n - v)\|^2_{H^\mu_{\beta,X}} + \|\mathrm{Std}(C_n - v)\|^2_{H^\mu_{\beta,X}}$, where $\mathrm{Std}(Y(s,y)) = \sqrt{\mathrm{Var}(Y(s,y))}$.

4.1.1 Study of $\|\mathbb{E}(C_n - v)\|^2_{H^\mu_{\beta,X}}$

We have

$$\mathbb{E}(C_n - v)(s,y) = \frac{\mathbb{E}[r_n(s,y)] - v(s,y)\mathbb{E}[f_n(s,y)]}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}} - v(s,y)\mathbf{1}_{\{s\notin[0,T]\cup y\notin B\}}.$$

Since $\mathbb{E}[r_n(s,y)] = \frac{1}{T\lambda(B)h_t h_x^d}\int_B dz \int_0^T dr v(r,z) K_x(\frac{y-z}{h_x})K_t(\frac{s-r}{h_t})$, we obtain

$$\mathbb{E}[r_n(s,y)] - v(s,y)\mathbb{E}[f_n(s,y)] = \frac{1}{T\lambda(B)h_t h_x^d}\int_B dz \int_0^T dr K_x(\frac{y-z}{h_x})K_t(\frac{s-r}{h_t})(v(r,z) - v(s,y)).$$

We use the second property of Lemma 3.1 and the equality $v(r,z) - v(s,y) = v(r,z) - v(s,z) + v(s,z) - v(s,y)$ to bound $|\mathbb{E}(C_n - v)(s,y)|$ by $|A_1(s,y)| + |A_2(s,y)| + |A_3(s,y)|$, where

$$\begin{aligned}
A_1(s,y) &:= \frac{2^{d+1}}{h_t h_x^d}\int_B dz \int_0^T dr K_x\left(\frac{y-z}{h_x}\right)K_t\left(\frac{s-r}{h_t}\right)(v(r,z) - v(s,z)),\\
A_2(s,y) &:= \frac{2^{d+1}}{h_t h_x^d}\int_B dz \int_0^T dr K_x\left(\frac{y-z}{h_x}\right)K_t\left(\frac{s-r}{h_t}\right)(v(s,z) - v(s,y)),\\
A_3(s,y) &:= v(s,y)\mathbf{1}_{\{s\notin[0,T]\cup y\notin B\}}.
\end{aligned}$$

We analyze each term in the following three Lemmas.

Lemma 4.1. *Let us assume Hypothesis 1 and v is a function C^1 in time. Then,*

$$\|A_1(s,y)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)h_t^2\|\partial_t v\|^2_{H^\mu_{\beta,X}},$$

If $\partial_t v$ is bounded, we get $\|A_1(s,y)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)c_{1,0}^2(v)h_t^2$.

This Lemma ensues from [4, Lemma 12.36].

Lemma 4.2. *Let us assume Hypothesis 1 and v is a function C^1 in space. There exists a function $K_0(T)$ such that*

$$\|A_2(s,y)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)h_x^2\|\partial_x v\|^2_{H^\mu_{\beta,X}}.$$

In particular, if $\partial_x v$ is bounded, we get $\|A_2(s,y)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)c^2_{0,1}(v)h_x^2$.

Proof. The proof of this Lemma is the same as the one of Lemma 4.1, except that we split the difference $v(s,z)-v(s,y)$ as a sum of d terms: $v(s,z)-v(s,y)=\sum_{i=1}^d v(s,\overline{z}_i)-v(s,\overline{z}_{i-1})$, where $\overline{z}_i=(z_1,z_2,\cdots,z_i,y_{i+1},\cdots,y_d), \forall i\in\{1,\cdots,d\}$, and $\overline{z}_0=y$. For all $i\in\{1,\cdots,d\}$, we get $v(s,\overline{z}_i)-v(s,\overline{z}_{i-1})=\int_{y_i}^{z_i} dl\partial_{x_i}v(s,\overline{z}_i^l)$, where $\overline{z}_i^l=(z_1,\cdots,z_{i-1},l,y_{i+1},\cdots,y_d)$. ■

Lemma 4.3. *Assume Hypothesis 1 and v is bounded. Then, $\|A_3(s,y)\|^2_{H^\mu_{\beta,X}} \leq K(T)c_0^2(v)e^{-\frac{\mu a}{\sqrt{d}}}$.*

Proof. Since v is bounded, we get $\|A_3(s,y)\|^2_{H^\mu_{\beta,X}} \leq c_0^2(v)\int_0^T dre^{\beta r}\int_{B^c} dy\nu^0_\mu(r,y)$. To conclude, we use $\nu^0_\mu(r,y)\leq 2^dKe^{c_2 r}e^{-\mu|y|}$ (see the proof of [1, Proposition 3.8]). ■

Combining Lemmas 4.1, 4.2, 4.3 yields to the following Proposition.

Proposition 4.4. *Let us assume Hypothesis 1 and v is a bounded $C^{1,1}$ function. Then,*

$$\|\mathbb{E}(C_n-v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)(h_t^2\|\partial_t v\|^2_{H^\mu_{\beta,X}}+h_x^2\|\partial_x v\|^2_{H^\mu_{\beta,X}})+c_0^2(v)K(T)e^{-\frac{\mu a}{\sqrt{d}}}.$$

Moreover, if $\partial_t v$ and $\partial_x v$ are bounded, we get $\|\mathbb{E}(C_n-v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)c^2_{1,1}(v)(h_t^2+h_x^2+e^{-\frac{\mu a}{\sqrt{d}}})$.

4.1.2 Study of $\|\mathbf{Std}(C_n-v)\|^2_{H^\mu_{\beta,X}}$

Let us study $\|\mathrm{Std}(C_n-v)\|^2_{H^\mu_{\beta,X}}=\int_0^T dse^{\beta s}\int_{\mathbb{R}^d} dy\nu^0_\mu(s,y)\mathrm{Var}(C_n-v)(s,y)$.

Proposition 4.5. *Let us assume Hypothesis 1. Then,*

$$\|Std(C_n-v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)T\lambda(B)\delta_n\|v\|^2_{H^\mu_{\beta,X}}.$$

If v is bounded, we get $\|Std(C_n-v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)c_0^2(v)T\lambda(B)\delta_n$.

Proof. Using (4.2) leads to $\|\mathrm{Std}(C_n-v)\|^2_{H^\mu_{\beta,X}}=\int_0^T dse^{\beta s}\int_B dy\nu^0_\mu(s,y)\mathrm{Var}(r_n(s,y))\frac{1}{\mathbb{E}[f_n(s,y)]^2}$. We use Lemma 3.1 to get

$$\|\mathrm{Std}(C_n-v)\|^2_{H^\mu_{\beta,X}} \leq 2^{2d+2}(T\lambda(B))^2\int_0^T dse^{\beta s}\int_B dy\nu^0_\mu(s,y)\mathrm{Var}(r_n(s,y)).$$

The end of the proof is similar to the one of [4, Proposition 12.34]. ■

4.1.3 Conclusion

We combine Propositions 4.4 and 4.5 to get the following result

Proposition 4.6. *Let us assume Hypothesis 1 and v is a bounded $C^{1,1}$ function. Then,* $\mathbb{E}\|C_n - v\|^2_{H^\mu_{\beta,X}} \leq K_0(T)(T\lambda(B)\delta_n\|v\|^2_{H^\mu_{\beta,X}} + h_t^2\|\partial_t v\|^2_{H^\mu_{\beta,X}} + h_x^2\|\partial_x v\|^2_{H^\mu_{\beta,X}}) + c_0^2(v)K(T)e^{-\frac{\mu a}{\sqrt{d}}}$. *Moreover, if $\partial_t v$ and $\partial_x v$ are bounded, we get* $\|\mathbb{E}(C_n - v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)c^2_{1,1}(v)(T\lambda(B)\delta_n + h_t^2 + h_x^2 + e^{-\frac{\mu a}{\sqrt{d}}})$.

4.2 Study of $\mathbb{E}\|\mathcal{P}^k v - C_n\|^2_{H^\mu_{\beta,X}}$

By using the definition of $\mathcal{P}^k v(s,y)$ and $C_n(s,y)$, we write

$$\mathcal{P}^k v(s,y) - C_n(s,y) = r_n(s,y)\left[\frac{1}{f_n(s,y)}g(2^{d+1}T\lambda(B)f_n(s,y)) - \frac{1}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}\right].$$

If $y \notin B_\infty(0, a+h_x)$, $\mathcal{P}^k v(s,y) - C_n(s,y) = 0$. If $y \in B_\infty(0,a+h_x) \setminus B$, $\mathcal{P}^k v(s,y) - C_n(s,y) = \frac{r_n(s,y)}{f_n(s,y)}g(2^{d+1}T\lambda(B)f_n(s,y))$. Since g is bounded by 1 and $|r_n(s,y)| \leq f_n(s,y)\sup_{(s,y)\in[0,T]\times B_\infty(0,a+h_x)\setminus B}|v(s,y)|$, we get $|\mathcal{P}^k v(s,y) - C_n(s,y)| \leq c_0(v)$. If $y \in B$, $\mathcal{P}^k v(s,y) - C_n(s,y) = \frac{r_n(s,y)}{f_n(s,y)}[g(2^{d+1}T\lambda(B)f_n(s,y)) - \frac{f_n(s,y)}{\mathbb{E}[f_n(s,y)]}]$. Let us give two upper bounds for $\mathcal{P}^k v(s,y) - C_n(s,y)$ when $y \in B$.

Lemma 4.7. *For $y \in B$, the two following assertions hold*

$$|\mathcal{P}^k v(s,y) - C_n(s,y)| \leq 2^{d+3}T\lambda(B)\frac{r_n(s,y)}{f_n(s,y)}|f_n(s,y) - \mathbb{E}[f_n(s,y)]|,$$
$$|\mathcal{P}^k v(s,y) - C_n(s,y)| \leq 2^{d+3}(T\lambda(B))^2|r_n(s,y)||f_n(s,y) - \mathbb{E}[f_n(s,y)]|.$$

Proof. Let $\tilde{g}(x) := g(2^{d+1}T\lambda(B)x) - \frac{x}{\mathbb{E}[f_n(s,y)]}$. Then, we use the second property of Lemma 3.1 to get $\tilde{g}(\mathbb{E}[f_n(s,y)]) = 0$, and $\mathcal{P}^k v(s,y) - C_n(s,y) = \frac{r_n(s,y)}{f_n(s,y)}(\tilde{g}(f_n(s,y)) - \tilde{g}(\mathbb{E}[f_n(s,y)]))$. Moreover, Remark 1.2 leads to $|\tilde{g}(f_n(s,y)) - \tilde{g}(\mathbb{E}[f_n(s,y)])| \leq 2^{d+3}T\lambda(B)|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$. The first result follows. To get the second one, we introduce $\overline{g}(x) := \frac{g(2^{d+1}T\lambda(B)x)}{x}$. We have $\overline{g}(\mathbb{E}[f_n(s,y)]) = \frac{1}{\mathbb{E}[f_n(s,y)]}$ and $|\overline{g}(f_n(s,y)) - \overline{g}(\mathbb{E}[f_n(s,y)])| \leq 2^{2d+3}(T\lambda(B))^2|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$. ■

Proposition 4.8. *Assume Hypothesis 1 and v is bounded. Then, $\forall \epsilon \geq 0$ such that $\epsilon^2 \leq (T\lambda(B))^{-2}$, one has*

$$\mathbb{E}\|\mathcal{P}^k v - C_n\|^2_{H^\mu_{\beta,X}} \leq K_0(T)\epsilon^2(T\lambda(B))^2\|v\|^2_{H^\mu_{\beta,X}} + K_0(T)c_0^2(v)(T\lambda(B))^2(\epsilon^2 + \frac{\delta_n}{T\lambda(B)})\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right).$$

Proof. Using Lemma 4.7, we split $\mathcal{P}^k v(s,y) - C_n(s,y)$ in two terms, depending on the value of $|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$ w.r.t. a constant ϵ. When $|f_n(s,y) - \mathbb{E}[f_n(s,y)]| \leq \epsilon$, we use the second inequality of Lemma 4.7, otherwise we use the first one. Since $r_n(s,y) \leq c_0(v)f_n(s,y)$, we use [4, Proposition 12.16] to get $\mathbb{E}|\mathcal{P}^k v(s,y) - C_n(s,y)|^2 \leq K(T)\epsilon^2(T\lambda(B))^4\mathbb{E}[r_n^2(s,y)] + K(T)c_0^2(v)(\epsilon^2 + \frac{\delta_n}{T\lambda(B)})\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right)$. We apply Proposition 3.2 and Lemma 3.5 to conclude. ■

4.3 Conclusion

To conclude, we combine Propositions 4.6 and 4.8 (with $\epsilon^2 = \frac{\delta_n}{T\lambda(B)}$). We obtain

Proposition 4.9. *We assume Hypothesis 1 and v is a bounded $C^{1,1}$ function. Then,*

$$\mathbb{E}\|\mathcal{P}^k v - v\|^2_{H^\mu_{\beta,X}} \leq K_0(T)(T\lambda(B)\delta_n\|v\|^2_{H^\mu_{\beta,X}} + h_t^2\|\partial_t v\|^2_{H^\mu_{\beta,X}} + h_x^2\|\partial_x v\|^2_{H^\mu_{\beta,X}}) + c_0^2(v)K(T)(e^{-\frac{\mu a}{\sqrt{d}}} + T\lambda(B)\delta_n).$$

Moreover, if $\partial_t v$ and $\partial_x v$ are bounded, we get $\|\mathbb{E}(\mathcal{P}^k v - v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)c_{1,1}^2(v)(T\lambda(B)\delta_n + h_t^2 + h_x^2 + e^{-\frac{\mu a}{\sqrt{d}}})$.

5 Proof of Proposition 2.1: term $\mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|^2_{H^\mu_{\beta,X}}$

We study $\mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|^2_{H^\mu_{\beta,X}}$ componentwise, then we deal with $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} v\|^2_{H^\mu_{\beta,X}}$, for $1 \leq i \leq d$. The study of this term will be done in two steps. To do so, we add and substract the term $\partial_{x_i} C_n(s,y)$ (see (4.1) for the definition of C_n) to $\partial_{x_i}(\mathcal{P}^k v)(s,y) - \partial_{x_i} v(s,y)$.

$$\partial_{x_i} C_n(s,y) = \left(\frac{\partial_{x_i} r_n(s,y)}{\mathbb{E}[f_n(s,y)]} - r_n(s,y)\frac{\mathbb{E}[\partial_{x_i} f_n(s,y)]}{(\mathbb{E}[f_n(s,y)])^2}\right)\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}} \tag{5.1}$$

We get $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} v\|^2_{H^\mu_{\beta,X}} \leq 2\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|^2_{H^\mu_{\beta,X}} + 2\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|^2_{H^\mu_{\beta,X}}$. The two following sections are devoted to the study of $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|^2_{H^\mu_{\beta,X}}$ and $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|^2_{H^\mu_{\beta,X}}$.

5.1 Study of $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|^2_{H^\mu_{\beta,X}}$

Using the definition of $\partial_{x_i} C_n(s,y)$, we get

$$\begin{aligned}\partial_{x_i} C_n(s,y) - \partial_{x_i} v(s,y) = -\partial_{x_i} v(s,y)\mathbf{1}_{\{s\notin[0,T]\cup y\notin B\}} \\ + \left(\frac{\partial_{x_i} r_n(s,y) - \partial_{x_i} v(s,y)\mathbb{E}[f_n(s,y)]}{\mathbb{E}[f_n(s,y)]} - r_n(s,y)\frac{\mathbb{E}[\partial_{x_i} f_n(s,y)]}{(\mathbb{E}[f_n(s,y)])^2}\right)\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}.\end{aligned} \tag{5.2}$$

Then, we split $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|^2_{H^\mu_{\beta,X}}$ in two terms, by using the bias-variance decomposition: $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|^2_{H^\mu_{\beta,X}} = \|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|^2_{H^\mu_{\beta,X}} + \|\mathrm{Std}(\partial_{x_i} C_n - \partial_{x_i} v)\|^2_{H^\mu_{\beta,X}}$.

5.1.1 Study of $\|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|^2_{H^\mu_{\beta,X}}$

By using (5.2), we split $\mathbb{E}(\partial_{x_i} C_n(s,y) - \partial_{x_i} v(s,y))$ in three terms :

$$\begin{aligned}B_1(s,y) &= \frac{\mathbb{E}[\partial_{x_i} r_n(s,y)] - \partial_{x_i} v(s,y)\mathbb{E}[f_n(s,y)]}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}, \\ B_2(s,y) &= -\mathbb{E}[r_n(s,y)]\frac{\mathbb{E}[\partial_{x_i} f_n(s,y)]}{(\mathbb{E}[f_n(s,y)])^2}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}, \\ B_3(s,y) &= -\partial_{x_i} v(s,y)\mathbf{1}_{\{s\notin[0,T]\cup y\notin B\}}\end{aligned}$$

such that $\mathbb{E}(\partial_{x_i} C_n(s,y) - \partial_{x_i} v(s,y)) = B_1(s,y) + B_2(s,y) + B_3(s,y)$.
We analyze each term in the three following Lemmas.

Lemma 5.1. *Let us assume Hypothesis 1. We also assume v is a bounded $C^{0,2}$ function which satisfies $\forall t, t' \in [0,T], \forall x \in \mathbb{R}^d$, $|\partial_x v(t,x) - \partial_x v(t',x)| \le c_{1/2}(v)\sqrt{|t'-t|}$. Then,*

$$\|B_1(s,y)\|^2_{H^\mu_{\beta,X}} \le K_0(T) h_x^2 \|\partial_x^2 v\|^2_{H^\mu_{\beta,X}} + K_0(T)(c^2_{1/2}(v) h_t + c_0^2(v) e^{-\mu a}\frac{a^{d-1}}{h_x}).$$

If $\partial_x^2 v$ is bounded, we get $\|B_1(s,y)\|^2_{H^\mu_{\beta,X}} \le K_0(T)(c^2_{0,2}(v)h_x^2 + c^2_{1/2}(v)h_t + c_0^2(v)e^{-\mu a}\frac{a^{d-1}}{h_x})$.

Proof. Let us recall $B_1(s,y) = \frac{\mathbb{E}[\partial_{x_i} r_n(s,y)] - \partial_{x_i} v(s,y)\mathbb{E}[f_n(s,y)]}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}$. We have $\mathbb{E}(\partial_{x_i} r_n(s,y)) = \frac{1}{h_t h_x^{d+1}}\frac{1}{T\lambda(B)}\int_0^T dr K_t\left(\frac{s-r}{h_t}\right)\int_B dz \partial_{x_i} K_x\left(\frac{y-z}{h_x}\right) v(r,z)$. We integrate by parts $\frac{1}{h_x}\int_{-a}^a dz_i (K_x^i)'\left(\frac{y_i-z_i}{h_x}\right)v(r,z)$ and we get $\frac{1}{h_x}\int_{-a}^a dz_i (K_x^i)'\left(\frac{y_i-z_i}{h_x}\right)v(r,z) = -K_x^i\left(\frac{y_i-a}{h_x}\right)v(r,z_a^i) + K_x^i\left(\frac{y_i+a}{h_x}\right)v(r,z_{-a}^i) + \int_{-a}^a dz_i \partial_{x_i} v(r,z) K_x^i\left(\frac{y_i-z_i}{h_x}\right)$, where z_y^i denotes the vector $(z_1,\cdots,z_{i-1},y,z_{i+1},\cdots,z_d)$. Then, $\mathbb{E}[\partial_{x_i} r_n(s,y)] - \partial_{x_i} v(s,y)\mathbb{E}[f_n(s,y)] = \frac{1}{T\lambda(B)h_t h_x^d}\int_0^T dr K_t\left(\frac{s-r}{h_t}\right)\int_B dz K_x\left(\frac{y-z}{h_x}\right)[\partial_{x_i} v(r,z) - \partial_{x_i} v(s,y)] + \frac{1}{T\lambda(B)h_t h_x^d}\int_0^T dr K_t\left(\frac{s-r}{h_t}\right)\int_{[-a,a]^{d-1}} dz^i \prod_{j=1,j\neq i}^d K_x^j(\frac{y_j-z_j}{h_x})[-K_x^i\left(\frac{y_i-a}{h_x}\right)v(r,z_a^i) + K_x^i\left(\frac{y_i+a}{h_x}\right)v(r,z_{-a}^i)]$. Combining this with the bound $\mathbb{E}[f_n(s,y)] \ge \frac{1}{2^{d+1}T\lambda(B)}$ (see Lemma 3.1) leads to the following upper bound for $B_1(s,y)$: $|B_1(s,y)| \le B_{11}(s,y) + B_{12}(s,y) + B_{13}(s,y)$, where

$$B_{11}(s,y) = \frac{2^{d+1}}{h_t h_x^d}\int_0^T dr K_t\left(\frac{s-r}{h_t}\right)\int_B dz K_x\left(\frac{y-z}{h_x}\right)[\partial_{x_i} v(r,z) - \partial_{x_i} v(s,z)]\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}},$$
$$B_{12}(s,y) = \frac{2^{d+1}}{h_t h_x^d}\int_0^T dr K_t\left(\frac{s-r}{h_t}\right)\int_B dz K_x\left(\frac{y-z}{h_x}\right)[\partial_{x_i} v(s,z) - \partial_{x_i} v(s,y)]\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}},$$
$$B_{13}(s,y) = \frac{2^{d+1}}{h_t h_x} c_0(v) c_0(K_x)\int_0^T dr K_t\left(\frac{s-r}{h_t}\right)(\mathbf{1}_{\{|y+a|\le h_x\}} + \mathbf{1}_{\{|y-a|\le h_x\}})\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}.$$

By using [4, Lemmas 12.53 and 12.54], we get $\|B_{11}(s,y)\|^2_{H^\mu_{\beta,X}} \le c^2_{1/2}(v)K(T)h_t$, $\|B_{12}(s,y)\|^2_{H^\mu_{\beta,X}} \le K_0(T)h_x^2\|\partial_x^2 v\|^2_{H^\mu_{\beta,X}}$. It remains to bound $\|B_{13}(s,y)\|^2_{H^\mu_{\beta,X}}$. We obtain $\|B_{13}(s,y)\|^2_{H^\mu_{\beta,X}} \le \frac{K_0(T)c_0^2(v)}{h_x^2}\int_0^T ds e^{\beta s}\int_{B\setminus B(0,a-h_x)} dy \nu_\mu^0(s,y)$. Since $\int_{B\setminus B(0,a-h_x)} dy \nu_\mu^0(s,y) \le K(T)e^{-\mu(a-h_x)}a^{d-1}h_x$ (see [4, Equation (12.5), page 132]), we get $\|B_{13}(s,y)\|^2_{H^\mu_{\beta,X}} \le \frac{K_0(T)c_0^2(v)}{h_x}e^{-\mu a}a^{d-1}$.

■

Lemma 5.2. *Assume Hypothesis 1 and v is bounded. Then, $\|B_2(s,y)\|^2_{H^\mu_{\beta,X}} \le K_0(T)c_0^2(v)e^{-\mu a}\frac{a^{d-1}}{h_x}$.*

Proof. Since v is bounded, we get $\mathbb{E}[r_n(s,y)] \le c_0(v)\mathbb{E}[f_n(s,y)]$. Hence, $|B_2(s,y)| \le c_0(v)\frac{\mathbb{E}[\partial_{x_i} f_n(s,y)]}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}$. To conclude, we apply Lemma 3.1 and Proposition 3.4. ■

Lemma 5.3. *Assume Hypothesis 1 and $\partial_x v$ is bounded. Then,* $\|B_3(s,y)\|^2_{H^\mu_{\beta,X}} \leq K(T)c^2_{0,1}(v)e^{-\frac{\mu a}{\sqrt{d}}}$.

Proof. We refer to the proof of Lemma 4.3. ■

We combine Lemmas 5.1, 5.2 and 5.3 to get the following Proposition.

Proposition 5.4. *Assume Hypothesis 1 and v is a $C^{0,2}-C^{0,1}_b$ function which satisfies $\forall t,t' \in [0,T], \forall x \in \mathbb{R}^d$, $|\partial_x v(t,x) - \partial_x v(t',x)| \leq c_{1/2}(v)\sqrt{|t'-t|}$. Then,*

$$\|\mathbb{E}(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)h_x^2\|\partial_x^2 v\|^2_{H^\mu_{\beta,X}} + K_0(T)(c^2_{1/2}(v)h_t + c_0^2(v)e^{-\mu a}\frac{a^{d-1}}{h_x} + c^2_{0,1}(v)e^{-\frac{\mu a}{\sqrt{d}}}).$$

Moreover, if $\partial_x^2 v$ is bounded, we get $\|\mathbb{E}(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)(c^2_{0,2}(v)+c^2_{1/2}(v))(h_x^2 + h_t + e^{-\mu a}\frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}})$.

5.1.2 Study of $\|\mathbf{Std}(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}}$

Let us study $\|\mathrm{Std}(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}} = \int_0^T dse^{\beta s}\int_{\mathbb{R}^d} dy\nu^0_\mu(s,y)\mathrm{Var}(\partial_{x_i}C_n - \partial_{x_i}v)(s,y)$.

Proposition 5.5. *Assume Hypothesis 1. It holds*

$$\|Std(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)\frac{T\lambda(B)\delta_n}{h_x^2}\|v\|^2_{H^\mu_{\beta,X}}.$$

If v is bounded, we get $\|Std(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)c_0^2(v)\frac{T\lambda(B)\delta_n}{h_x^2}$.

Proof. We have $\|\mathrm{Std}(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}} = \int_0^T dse^{\beta s}\int_B dy\nu^0_\mu(s,y)\mathrm{Var}(\partial_{x_i}C_n(s,y))$. Using (5.1) leads to $\mathrm{Var}(\partial_{x_i}C_n(s,y)) \leq 2\left(\frac{1}{(\mathbb{E}[f_n(s,y)])^2}\mathrm{Var}(\partial_{x_i}r_n(s,y)) + \mathrm{Var}(r_n(s,y))\frac{(\mathbb{E}[\partial_{x_i}f_n(s,y)])^2}{(\mathbb{E}[f_n(s,y)])^4}\right)\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}$. Then, we use [4, Proposition 12.51] and Lemma 3.1 to get $\int_0^T dse^{\beta s}\int_B dy\nu^0_\mu(s,y)\frac{\mathrm{Var}(\partial_{x_i}r_n(s,y))}{(\mathbb{E}[f_n(s,y)])^2} \leq K_0(T)\frac{T\lambda(B)\delta_n}{h_x^2}\|v\|^2_{H^\mu_{\beta,X}}$. It remains to bound $\int_0^T dse^{\beta s}\int_B dy\nu^0_\mu(s,y)\mathrm{Var}(r_n(s,y))\frac{(\mathbb{E}[\partial_{x_i}f_n(s,y)])^2}{(\mathbb{E}[f_n(s,y)])^4}$. To do it, we use Lemmas 3.1, 3.3 and the proof of Proposition 4.5. ■

5.1.3 Conclusion

We combine Propositions 5.4 and 5.5 to get the following result

Proposition 5.6. *Assume Hypothesis 1 and v is a $C^{1,2}-C^{0,1}_b$ function which satisfies $\forall t,t' \in [0,T], \forall x \in \mathbb{R}^d$, $|\partial_x v(t,x) - \partial_x v(t',x)| \leq c_{1/2}(v)\sqrt{|t'-t|}$. Then,*

$$\begin{aligned}\mathbb{E}\|\partial_{x_i}C_n - \partial_{x_i}v\|^2_{H^\mu_{\beta,X}} \leq & K_0(T)\left(h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2}\right)\|v\|^2_{H^{2,\mu}_{\beta,X}} \\ & + K_0(T)\left(c^2_{1/2}(v)h_t + c_0^2(v)e^{-\mu a}\frac{a^{d-1}}{h_x} + c^2_{0,1}(v)e^{-\frac{\mu a}{\sqrt{d}}}\right).\end{aligned}$$

Moreover, if $\partial_x^2 v$ is bounded, we get $\|\mathbb{E}(\partial_{x_i}C_n - \partial_{x_i}v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)(c^2_{0,2}(v)+c^2_{1/2}(v))(h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2} + h_t + e^{-\mu a}\frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}})$.

5.2 Study of $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|^2_{H^\mu_{\beta,X}}$

We have $\partial_{x_i}(\mathcal{P}^k v)(s,y) = \partial_{x_i} r_n(s,y)\frac{g(2^{d+1}T\lambda(B)f_n(s,y))}{f_n(s,y)} + 2^{2d+2}(T\lambda(B))^2 r_n(s,y)G'(2^{d+1}T\lambda(B)f_n(s,y))\partial_{x_i} f_n(s,y)$, where G has been introduced in Remark 1.2. Using the definition of $\partial_{x_i} C_n$ (see (5.1)) yields

$$\partial_{x_i}(\mathcal{P}^k v)(s,y) - \partial_{x_i} C_n(s,y) = \partial_{x_i} r_n(s,y)\left(\frac{g(2^{d+1}T\lambda(B)f_n(s,y))}{f_n(s,y)} - \frac{1}{\mathbb{E}[f_n(s,y)]}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}\right)$$
$$+ r_n(s,y)\left(2^{d+2}(T\lambda(B))^2\partial_{x_i} f_n(s,y)G'(2^{d+1}T\lambda(B)f_n(s,y)) - \frac{\mathbb{E}[\partial_{x_i} f_n(s,y)]}{(\mathbb{E}[f_n(s,y)])^2}\mathbf{1}_{\{s\in[0,T]\}}\mathbf{1}_{\{y\in B\}}\right) \tag{5.3}$$

Let us study $\partial_{x_i}(\mathcal{P}^k v)(s,y) - \partial_{x_i} C_n(s,y)$ w.r.t. the value of y. The first Lemma ensues from the Definition of f_n.

Lemma 5.7. *If $y \notin B_\infty(0, a+h_x)$, $\partial_{x_i}(\mathcal{P}^k v)(s,y) - \partial_{x_i} C_n(s,y) = 0$.*

Lemma 5.8. *If $y \in B_\infty(0,a+h_x)\setminus B$ and v is bounded, for all $i \in \{1,\cdots,d\}$,*

$$\mathbb{E}|\partial_{x_i}(\mathcal{P}^k v)(s,y) - \partial_{x_i} C_n(s,y)|^2 \le \frac{K_0(T)c_0^2(v)}{h_x^2}.$$

Proof. Let us introduce $\overline{f}^i_n(s,y) = \frac{1}{nh_t h_x^{d+1}}\sum_{i=1}^n K_t\left(\frac{s-T_i}{h_t}\right)|\partial_{x_i} K_x|\left(\frac{y-X_i}{h_x}\right)$. The indicators in (5.3) are null. Since $\frac{g(y)}{y}$ is bounded by 2 (see Remark 1.2 for the bounds for g and its first derivative), $\frac{|\partial_{x_i} r_n(s,y)|}{f_n(s,y)} g(2^{d+1}T\lambda(B)f_n(s,y)) \le 2^{d+2}T\lambda(B)c_0(v)\overline{f}^i_n(s,y)$. It remains to bound the term containing G'. To do so, we write $G'(y) = \frac{g'(y)}{y} - \frac{g(y)}{y^2}$. Since $|G(y)| \le 2$ and $|r_n(s,y)| \le c_0(v)f_n(s,y)$, $|r_n(s,y)|\frac{g(2^{d+1}T\lambda(B)f_n(s,y))}{f_n^2(s,y)}|\partial_{x_i} f_n(s,y)| \le 2^{d+2}T\lambda(B)c_0(v)|\partial_{x_i} f_n(s,y)|$. Since g' is bounded by 2, $2^{d+1}T\lambda(B)|r_n(s,y)|\frac{g'(2^{d+1}T\lambda(B)f_n(s,y))}{f_n(s,y)}|\partial_{x_i} f_n(s,y)|$ is bounded by $2^{d+2}T\lambda(B)c_0(v)|\partial_{x_i} f_n(s,y)|$. To conclude, we use $|\partial_{x_i} f_n(s,y)| \le \overline{f}^i_n(s,y)$ and [4, Lemma 12.8], which states $\mathbb{E}(\overline{f}^i_n(s,y))^2 \le \frac{K_0(T)}{h_x^2(T\lambda(B))^2}$. ∎

Lemma 5.9. *If $y \in B$, $\partial_{x_i}(\mathcal{P}^k v)(s,y) - \partial_{x_i} C_n(s,y) = A(s,y) + B(s,y) + C(s,y)$ where*

$$A(s,y) = \partial_{x_i} r_n(s,y)\left(\frac{g(2^{d+1}T\lambda(B)f_n(s,y))}{f_n(s,y)} - \frac{1}{\mathbb{E}[f_n(s,y)]}\right),$$
$$B(s,y) = 2^{2d+2}(T\lambda(B))^2 r_n(s,y)\partial_{x_i} f_n(s,y)[G'(2^{d+1}T\lambda(B)f_n(s,y)) - G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)])],$$
$$C(s,y) = 2^{2d+2}(T\lambda(B))^2 r_n(s,y)G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)])[\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)]].$$

Proof. We add and substract $= 2^{2d+2}(T\lambda(B))^2 r_n(s,y)\partial_{x_i} f_n(s,y)G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)])$ in (5.3) and we use $G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)]) = -\frac{1}{2^{2d+2}(T\lambda(B))^2\mathbb{E}[f_n(s,y)]^2}$ (since $2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)] \ge 1$). ∎

5.2.1 Bound for $\mathbb{E}[A^2(s,y)]$

Lemma 5.10. *If v is bounded, $\forall \epsilon \geq 0$ such that $\epsilon^2 \leq (T\lambda(B))^{-2}$, we have*

$$\mathbb{E}[A^2(s,y)] \leq K(T)(T\lambda(B))^4\epsilon^2\mathbb{E}[\partial_{x_i} r_n(s,y)]^2 \\ + K_0(T)c_0^2(v)\frac{(T\lambda(B))^2}{h_x^2}\left[\left(\epsilon^2 + \frac{\delta_n}{T\lambda(B)}\right)\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{T\lambda(B)}\exp\left(-\frac{c}{T\lambda(B)\delta_n}\right)\right].$$

Proof. Studying $A(s,y)$ boils down to study $\mathcal{P}^k v - C_n$ where r_n is replaced by $\partial_{x_i} r_n$. First, the second inequality of Lemma 4.7 gives us

$$|A(s,y)| \leq 2^{d+3}(T\lambda(B))^2|\partial_{x_i} r_n(s,y)||f_n(s,y) - \mathbb{E}[f_n(s,y)]|,$$

and since $|\partial_{x_i} r_n(s,y)| \leq c_0(v)\overline{f}_n^i(s,y)$ ($\overline{f}_n^i(s,y)$ has been introduced in the proof of Lemma 5.8), we also have $|A(s,y)| \leq 2^{d+3}(T\lambda(B))^2 c_0(v)\overline{f}_n^i(s,y)|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$. As in the proof of Proposition 4.8, we split $A(s,y)$ in two terms, depending on the value of $|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$ w.r.t. a constant ϵ_0. When $|f_n(s,y) - \mathbb{E}[f_n(s,y)]| \leq \epsilon_0$, we use the first inequality, otherwise we use the second one. We get

$$|A(s,y)| \leq 2^{d+3}(T\lambda(B))^2\epsilon_0|\partial_{x_i} r_n(s,y)| \\ + 2^{d+3}(T\lambda(B))^2 c_0(v)\overline{f}_n^i(s,y)|f_n(s,y) - \mathbb{E}[f_n(s,y)]|\mathbf{1}_{\{|f_n(s,y)-\mathbb{E}[f_n(s,y)]|\geq\epsilon_0\}}.$$

We split again the second right hand term of the above inequality by introducing $\pm\mathbb{E}[\overline{f}_n^i(s,y)]$. We get

$$|A(s,y)| \leq 2^{d+3}(T\lambda(B))^2\epsilon_0|\partial_{x_i} r_n(s,y)| \\ + 2^{d+3}(T\lambda(B))^2 c_0(v)|\overline{f}_n^i(s,y) - \mathbb{E}[\overline{f}_n^i(s,y)]||f_n(s,y) - \mathbb{E}[f_n(s,y)]|\mathbf{1}_{\{|f_n(s,y)-\mathbb{E}[f_n(s,y)]|\geq\epsilon_0\}} \\ + 2^{d+3}(T\lambda(B))c_0(v)\frac{K_0(T)}{h_x}|f_n(s,y) - \mathbb{E}[f_n(s,y)]|\mathbf{1}_{\{|f_n(s,y)-\mathbb{E}[f_n(s,y)]|\geq\epsilon_0\}},$$

where we have used $\mathbb{E}(\overline{f}_n^i(s,y))^2 \leq \frac{K_0(T)}{h_x^2(T\lambda(B))^2}$. Then, we split the second term of the r.h.s. in two terms, depending on the value of $|\overline{f}_n^i(s,y) - \mathbb{E}[\overline{f}_n^i(s,y)]|$ w.r.t. a constant ϵ_1. We obtain

$$|A(s,y)|^2 \leq K(T)(T\lambda(B))^4\epsilon_0^2|\partial_{x_i} r_n(s,y)|^2 \\ + K(T)(T\lambda(B))^4 c_0^2(v)\overline{E}^2(s,y)E^2(s,y)\mathbf{1}_{\{E(s,y)\geq\epsilon_0\}}\mathbf{1}_{\{\overline{E}(s,y)\geq\epsilon_1\}} \\ + K_0(T)(T\lambda(B))^2 c_0^2(v)\left(\frac{1}{h_x^2} + (T\lambda(B))^2\epsilon_1^2\right)E^2(s,y)\mathbf{1}_{\{E(s,y)\geq\epsilon_0\}}.$$

where $E(s,y) := |f_n(s,y) - \mathbb{E}[f_n(s,y)]|$ and $\overline{E}(s,y) := |\overline{f}_n^i(s,y) - \mathbb{E}[\overline{f}_n^i(s,y)]|$. To conclude, it remains to apply [4, Propositions 12.16 and 12.18] (since $\overline{f}_n^i(s,y)$ is almost $\partial_{x_i} f_n$), Cauchy Schwarz inequality, to choose $\epsilon_1 = \frac{\epsilon_0}{h_x}$ and to use $\epsilon_0^2 \leq (T\lambda(B))^{-2}$. We get

$$\mathbb{E}[A^2(s,y)] \leq K(T)(T\lambda(B))^4\epsilon_0^2\mathbb{E}[\partial_{x_i} r_n(s,y)]^2 \\ + K_0(T)c_0^2(v)\frac{(T\lambda(B))^2}{h_x^2}(\epsilon_0^2 + \frac{\delta_n}{T\lambda(B)})\exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right) \\ + K_0(T)c_0^2(v)\frac{(T\lambda(B))^4}{h_x^2}\left[\left(\epsilon_0^2 + \frac{\delta_n}{T\lambda(B)}\right)^2\exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))^3}\exp\left(-\frac{c}{T\lambda(B)\delta_n}\right)\right].$$

Since $\epsilon_0^2 \le (T\lambda(B))^{-2}$ and $T\lambda(B)\delta_n \ll 1$, we obtain $(T\lambda(B))^2(\epsilon_0^2 + \frac{\delta_n}{T\lambda(B)})^2 \le \epsilon_0^2 + \frac{\delta_n}{T\lambda(B)}$, and the result follows. ■

5.2.2 Bound for $\mathbb{E}[B^2(s,y)]$

Lemma 5.11. *If v is bounded, $\forall \epsilon \ge 0$ such that $\epsilon^2 \le (T\lambda(B))^{-2}$, we have*

$$\mathbb{E}[B^2(s,y)] \le C(d)(T\lambda(B))^4 \frac{\epsilon^2}{h_x^2}\mathbb{E}[r_n(s,y)]^2$$
$$+ K_0(T)c_0^2(v)\frac{(T\lambda(B))^2}{h_x^2}\left[\left(\epsilon^2 + \frac{\delta_n}{T\lambda(B)}\right)\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))}\exp\left(-\frac{c}{T\lambda(B)\delta_n}\right)\right].$$

Proof. First, we split $B(s,y)$ in two terms $B_1 + B_2$ by introducing $\pm\mathbb{E}[\partial_{x_i} f_n(s,y)]$. We get

$$B_1(s,y) = 2^{2d+2}(T\lambda(B))^2 r_n(s,y)(\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)])\Delta G,$$
$$B_2(s,y) = 2^{2d+2}(T\lambda(B))^2 r_n(s,y)\mathbb{E}[\partial_{x_i} f_n(s,y)]\Delta G,$$

where $\Delta G := G'(2^{d+1}T\lambda(B)f_n(s,y)) - G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)])$.
Bound for B_2. First, we use Lemma 3.3 to bound $\mathbb{E}[\partial_{x_i} f_n(s,y)]$: $\mathbb{E}[\partial_{x_i} f_n(s,y)] \le \frac{K_0(T)}{T\lambda(B)hx}$. Then, we give two bounds for B_2. The first one uses that G' is a Lipschitz function on $[0,\infty[$. We get $G'(2^{d+1}T\lambda(B)f_n(s,y)) - G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)]) \le C(d)T\lambda(B)|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$. Thus, $|B_2(s,y)| \le C(d)\frac{(T\lambda(B))^2}{h_x}r_n(s,y)|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$. The second bound relies on the inequality $r_n(s,y) \le c_0(v)f_n(s,y)$ and the fact that the function $\tilde{g}(x) := xG'(2^{d+1}T\lambda(B)x) - xG'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)])$ satisfies $\tilde{g}(\mathbb{E}[f_n(s,y)]) = 0$ and is a Lipschitz function. We get $|B_2(s,y)| \le C(d)c_0(v)\frac{T\lambda(B)}{h_x}|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$. Once again, we split $B_2(s,y)$ in two terms, depending on the value of $|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$ w.r.t. a constant ϵ. When $|f_n(s,y) - \mathbb{E}[f_n(s,y)]| \le \epsilon$, we use the first inequality, otherwise we use the second one. By using [4, Proposition 12.16], we get

$$\mathbb{E}[B_2(s,y)]^2 \le \epsilon^2\frac{(T\lambda(B))^4}{h_x^2}\mathbb{E}[r_n(s,y)^2] + K_0(T)c_0^2(v)\frac{(T\lambda(B))^2}{h_x^2}(\epsilon^2 + \frac{\delta_n}{T\lambda(B)})\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right).$$

Bound for B_1. As for B_2, we give two bounds for B_1: $|B_1(s,y)| \le C(d)(T\lambda(B))^3 r_n(s,y)|\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)]||f_n(s,y) - \mathbb{E}[f_n(s,y)]|$ and $|B_1(s,y)| \le C(d)c_0(v)(T\lambda(B))^2|\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)]||f_n(s,y) - \mathbb{E}[f_n(s,y)]|$. Then, we split B_1 in four terms, depending on the value of $|f_n(s,y) - \mathbb{E}[f_n(s,y)]|$ w.r.t. a constant ϵ_0 and on the value of $|\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)]|$ w.r.t. a constant ϵ_1. We introduce $E'(s,y) := |\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)]|$. Then, we get

$$|B_1(s,y)| \le |B_1(s,y)|\mathbf{1}_{\{E' \le \epsilon_1\}}\mathbf{1}_{\{E \le \epsilon_0\}} + |B_1(s,y)|\mathbf{1}_{\{E' \le \epsilon_1\}}\mathbf{1}_{\{E > \epsilon_0\}}$$
$$+ |B_1(s,y)|\mathbf{1}_{\{E' > \epsilon_1\}}\mathbf{1}_{\{E \le \epsilon_0\}} + |B_1(s,y)|\mathbf{1}_{\{E' > \epsilon_1\}}\mathbf{1}_{\{E > \epsilon_0\}}.$$

We bound the first term (resp. the three other terms) by using the first (resp. second) bound for B_1. Applying Cauchy-Schwarz inequality, [4, Propositions 12.16 and 12.18] and choosing

$\epsilon_1 = \frac{\epsilon_0}{h_x}$ yield

$$\mathbb{E}|B_1(s,y)|^2 \leq K(T)(T\lambda(B))^6 \frac{\epsilon_0^4}{h_x^2}\mathbb{E}[r_n(s,y)]^2$$
$$+ K(T)c_0^2(v)\frac{(T\lambda(B))^4\epsilon_0^2}{h_x^2}(\epsilon_0^2 + \frac{\delta_n}{T\lambda(B)})\exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right)$$
$$+ K(T)c_0^2(v)\frac{(T\lambda(B))^4}{h_x^2}\left[\left(\epsilon_0^2 + \frac{\delta_n}{T\lambda(B)}\right)^2 \exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))^3}\exp\left(-\frac{c}{T\lambda(B)\delta_n}\right)\right].$$

For $\epsilon_0^2 \leq (T\lambda(B))^{-2}$, we get $(T\lambda(B))^6 \frac{\epsilon_0^4}{h_x^2} \leq (T\lambda(B))^4 \frac{\epsilon_0^2}{h_x^2} \leq \frac{(T\lambda(B))^2}{h_x^2}$. Hence, the first two terms of the bound for $\mathbb{E}|B_1(s,y)|^2$ are smaller than the terms bounding $\mathbb{E}|B_2(s,y)|^2$. We end the proof as in Lemma 5.10. ■

5.2.3 Bound for $\mathbb{E}[C^2(s,y)]$

Lemma 5.12. *If v is bounded, $\forall \epsilon \geq 0$ such that $\epsilon^2 \leq (T\lambda(B))^{-2}$, we have*

$$\mathbb{E}[C^2(s,y)] \leq K(T)(T\lambda(B))^4 \frac{\epsilon^2}{h_x^2}\mathbb{E}[r_n(s,y)]^2$$
$$+ K_0(T)c_0^2(v)\frac{(T\lambda(B))^2}{h_x^2}\left[\left(\epsilon^2 + \frac{\delta_n}{T\lambda(B)}\right)\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))}\exp\left(-\frac{c}{T\lambda(B)\delta_n}\right)\right].$$

Proof. We recall $C(s,y) = 2^{2d+2}(T\lambda(B))^2 r_n(s,y)G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s,y)])[\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)]]$. We use that G' is bounded and we split $C(s,y)$ in two terms depending on the value of $E' = |\partial_{x_i} f_n(s,y) - \mathbb{E}[\partial_{x_i} f_n(s,y)]|$ w.r.t. a constant ϵ_1. We get

$$|C(s,y)|^2 \leq C(d)(T\lambda(B))^4\epsilon_1^2|r_n(s,y)|^2 + C(d)(T\lambda(B))^4 c_0^2(v)|f_n(s,y)|^2(E')^2\mathbf{1}_{\{E'>\epsilon_1\}},$$

where we have used $r_n(s,y) \leq c_0(v)f_n(s,y)$. Then, we split the second term of the r.h.s. of the above inequality by introducing $\pm\mathbb{E}[f_n(s,y)]$. Since $\mathbb{E}[f_n(s,y)] \leq \frac{1}{T\lambda(B)}$, we obtain

$$\begin{aligned}|C(s,y)|^2 \leq & K(T)(T\lambda(B))^4\epsilon_1^2|r_n(s,y)|^2 + K(T)(T\lambda(B))^2 c_0^2(v)(E')^2\mathbf{1}_{\{E'>\epsilon_1\}} \\ & + K(T)(T\lambda(B))^4 c_0^2(v)(E')^2\mathbf{1}_{\{E'>\epsilon_1\}}|f_n(s,y) - \mathbb{E}[f_n(s,y)]|^2.\end{aligned}$$

Finally, we split the last term of the above inequality in two terms depending on the value of $E = |f_n(s,y) - \mathbb{E}[f_n(s,y)]|$ w.r.t. a constant ϵ_0. We get

$$\begin{aligned}|C(s,y)|^2 \leq & K(T)(T\lambda(B))^4\epsilon_1^2|r_n(s,y)|^2 + C(d)(T\lambda(B))^4\left(\frac{1}{(T\lambda(B))^2} + \epsilon_0^2\right)c_0^2(v)(E')^2\mathbf{1}_{\{E'>\epsilon_1\}} \\ & + K(T)(T\lambda(B))^4 c_0^2(v)(E')^2\mathbf{1}_{\{E'>\epsilon_1\}}E^2\mathbf{1}_{\{E>\epsilon_0\}}.\end{aligned}$$

Combining Cauchy Schwarz inequality,[4, Propositions 12.16 and 12.18], choosing $\epsilon_1 = \frac{\epsilon_0}{h_x}$ and using $\epsilon_0 \leq (T\lambda(B))^{-1}$ and $T\lambda(B)\delta_n \ll 1$ lead to the result. ■

5.2.4 Conclusion

Combining Lemmas 5.10, 5.11 and 5.12 leads to the following Proposition.

Proposition 5.13. *If v is bounded, $\forall \epsilon \geq 0$ such that $\epsilon^2 \leq (T\lambda(B))^{-2}$ and $y \in B$, we have*

$$\mathbb{E}[|\partial_{x_i}(\mathcal{P}^k v)(s,y) - \partial_{x_i} C_n(s,y)|^2] \leq C(d)(T\lambda(B))^4\epsilon^2\left(\mathbb{E}[\partial_{x_i} r_n(s,y)]^2 + \frac{1}{h_x^2}\mathbb{E}[r_n(s,y)]^2\right)$$
$$+ K_0(T)c_0^2(v)\frac{(T\lambda(B))^2}{h_x^2}\left[\left(\epsilon^2 + \frac{\delta_n}{T\lambda(B)}\right)\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))}\exp\left(-\frac{c}{T\lambda(B)\delta_n}\right)\right].$$

Combining Lemmas 5.7, 5.8, Proposition 5.13 and following the same proof as [4, Theorem 12.50] yields

Proposition 5.14. *Assume Hypothesis 1 and v is bounded. Then, $\forall \epsilon \geq 0$ such that $\epsilon^2 \leq (T\lambda(B))^{-2}$, we have*

$$\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|^2_{H^\mu_{\beta,X}} \leq K_0(T)(T\lambda(B))^2\epsilon^2(\|\partial_x v\|^2_{H^\mu_{\beta,X}} + \frac{1}{h_x^2}\|v\|^2_{H^\mu_{\beta,X}}) + c_0^2(v)\frac{K_0(T)}{h_x}e^{-\mu a}a^{d-1}$$
$$+ K_0(T)c_0^2(v)\frac{(T\lambda(B))^2}{h_x^2}\left[\left(\epsilon^2 + \frac{\delta_n}{T\lambda(B)}\right)\exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{T\lambda(B)}\exp\left(-\frac{c}{T\lambda(B)\delta_n}\right)\right].$$

5.3 Conclusion

We combine Propositions 5.14 and 5.6 with $\epsilon^2 = \frac{\delta_n}{T\lambda(B)}$ to get the following result

Proposition 5.15. *Assume Hypothesis 1 and v is a $C^{1,2} - C_b^{0,1}$ function satisfying $\forall t, t' \in [0,T], \forall x \in \mathbb{R}^d$, $|\partial_x v(t,x) - \partial_x v(t',x)| \leq c_{1/2}(v)\sqrt{|t'-t|}$. Then,*

$$\mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|^2_{H^\mu_{\beta,X}} \leq K_0(T)\left(h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2}\right)\|v\|^2_{H^{2,\mu}_{\beta,X}}$$
$$+ K_0(T)\left(c_{1/2}^2(v)h_t + c_0^2(v)e^{-\mu a}\frac{a^{d-1}}{h_x} + c_{0,1}^2(v)e^{-\frac{\mu a}{\sqrt{d}}} + c_0^2(v)\frac{T\lambda(B)\delta_n}{h_x^2}\right).$$

Moreover, if $\partial_x^2 v$ is bounded, we get $\|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|^2_{H^\mu_{\beta,X}} \leq K_0(T)(c_{0,2}^2(v) + c_{1/2}^2(v))(h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2} + h_t + e^{-\mu a}\frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}})$.

6 Proof of Proposition 2.2

6.1 Bound for $\mathbb{E}\|\mathcal{P}^k v\|^2_{H^\mu_{\beta,X}}$

From the definition of $\mathcal{P}^k$ and since $\frac{g(x)}{x}$ is bounded by 2, we deduce $|\mathcal{P}^k v(s,y)|^2 \leq 2^{2d+3}(T\lambda(B))^2|r_n|^2(s,y)$. Then, Proposition 3.2 gives

$$\begin{aligned}\mathbb{E}(|\mathcal{P}^k v(s,y)|^2) &\leq 2^{2d+3}(T\lambda(B))^2\mathbb{E}(|r_n|^2(s,y)),\\ &\leq \frac{K_0(T)}{h_t h_x^d}\int_0^T dr K_t^2(\frac{s-r}{h_t})\int_B dz K_x^2(\frac{y-z}{h_x})\mathbb{E}(v^2(r,z)).\end{aligned}$$

Using the definition of $\|\cdot\|_{H^\mu_{\beta,X}}$ and Lemma 3.5 yields $\mathbb{E}\|\mathcal{P}^k v\|^2_{H^\mu_{\beta,X}} \leq K_0(T)\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}$. If v is unbiased, $\mathbb{E}(|r_n|^2(s,y)) = \mathrm{Var}(r_n(s,y))$. Proposition 3.2 gives

$$\begin{aligned}\mathbb{E}(|\mathcal{P}^k v(s,y)|^2) &\leq 2^{2d+3}(T\lambda(B))^2\mathrm{Var}(r_n(s,y)),\\ &\leq \frac{K_0(T)T\lambda(B)}{nh_t^2h_x^{2d}}\int_0^T drK_t^2(\frac{s-r}{h_t})\int_B dzK_x^2(\frac{y-z}{h_x})\mathbb{E}(v^2(r,z)),\end{aligned}$$

and we get $\mathbb{E}\|\mathcal{P}^k v\|^2_{H^\mu_{\beta,X}} \leq K_0(T)T\lambda(B)\delta_n\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}$.

6.2 Bound for $\mathbb{E}\|\partial_x(\mathcal{P}^k v)(t,x)\|^2_{H^\mu_{\beta,X}}$

We have $\partial_{x_i}\mathcal{P}^k v(s,y) = 2^{d+1}T\lambda(B)\partial_{x_i}r_n(s,y)G(2^{d+1}T\lambda(B)f_n(s,y)) + (2^{d+1}T\lambda(B))^2 r_n(s,y)\partial_{x_i}f_n(s,y)G'(2^{d+1}T\lambda(B)f_n(s,y))$ where G has been introduced in Remark 1.2. Using the bounds for G and G', we obtain $|\partial_{x_i}\mathcal{P}^k v(s,y)| \leq 2^{d+2}T\lambda(B)|\partial_{x_i}r_n(s,y)| + 6*2^{2d+2}(T\lambda(B))^2|r_n(s,y)||\partial_{x_i}f_n(s,y)|$, and

$$\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v)\|^2_{H^\mu_{\beta,X}} \leq C(d)(T\lambda(B))^2(\mathbb{E}\|\partial_{x_i}r_n\|^2_{H^\mu_{\beta,X}} + (T\lambda(B))^2\mathbb{E}\|(r_n\partial_{x_i}f_n)\|^2_{H^\mu_{\beta,X}}).$$

6.2.1 Bound for $(T\lambda(B))^2\mathbb{E}\|\partial_{x_i}r_n\|^2_{H^\mu_{\beta,X}}$

We write $\mathbb{E}(|\partial_{x_i}r_n(s,y)|^2) = (\mathbb{E}(\partial_{x_i}r_n(s,y)))^2 + \mathrm{Var}(\partial_{x_i}r_n(s,y))$. As in Proposition 3.2, we get $(\mathbb{E}(\partial_{x_i}r_n(s,y)))^2 \leq \frac{1}{h_th_x^{d+2}}\frac{1}{(T\lambda(B))^2}\int_0^T drK_t^2\left(\frac{s-r}{h_t}\right)\int_B dz(\partial_{x_i}K_x)^2\left(\frac{y-z}{h_x}\right)\mathbb{E}(v^2(r,z))$, $\mathrm{Var}((\partial_{x_i}r_n(s,y))) \leq \frac{1}{nh_th_x^{d+2}T\lambda(B)}\int_0^T drK_t^2\left(\frac{s-r}{h_t}\right)\int_B dz(\partial_{x_i}K_x)^2\left(\frac{y-z}{h_x}\right)\mathbb{E}(v^2(r,z))$. Since we assume $T\lambda(B)\delta_n << 1$, Lemma 3.5 yields $(T\lambda(B))^2\mathbb{E}\|\partial_{x_i}r_n\|^2_{H^\mu_{\beta,X}} \leq \frac{K_0(T)}{h_x^2}\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}$. If v is unbiased, we get $(T\lambda(B))^2\mathbb{E}\|\partial_{x_i}r_n\|^2_{H^\mu_{\beta,X}} \leq K_0(T)\frac{T\lambda(B)\delta_n}{h_x^2}\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}$.

6.2.2 Bound for $(T\lambda(B))^4\mathbb{E}\|r_n\partial_{x_i}f_n\|^2_{H^\mu_{\beta,X}}$

First, we develop the product $r_n^2(s,y)(\partial_{x_i}f_n)^2(s,y)$ by using the following formulae $(\partial_{x_i}f_n(s,y))^2 = \frac{1}{n^2h_t^2h_x^{2d+2}}\left(\sum_{j=1}^n K_t^2(j)(\partial_{x_i}K_x)^2(j) + \sum_{i,j=1,i\neq j}^n K_t(i)K_t(j)(\partial_{x_i}K_x)(i)(\partial_{x_i}K_x)(j)\right) := A+B$, $r_n^2(s,y) = \frac{1}{n^2h_t^2h_x^{2d}}\left(\sum_{k=1}^n K_t^2(k)K_x^2(k)v^2(k) + \sum_{k,l=1,k\neq l}^n K_t(k)K_t(l)K_x(k)K_x(l)v(k)v(l)\right) := C+D$, where $K_t(j) := K_t\left(\frac{s-T_j}{h_t}\right)$, $K_x(j) := K_x\left(\frac{y-X_j}{h_x}\right)$, $(\partial_{x_i}K_x)(j) := (\partial_{x_i}K_x)\left(\frac{y-X_j}{h_x}\right)$ and $v(k) := v(T_k,X_k)$. Developing $A\times C$ leads to

$$\begin{aligned}\mathbb{E}(A\times C) = \frac{1}{n^4h_t^4h_x^{4d+2}}&\left(n\mathbb{E}(K_t^4(1)(\partial_{x_i}K_x)^2(1)K_x^2(1)v^2(1))\right.\\ &\left.+n(n-1)\mathbb{E}(K_t^2(1)(\partial_{x_i}K_x)^2(1))\mathbb{E}(K_t^2(1)K_x^2(1)v^2(1))\right).\end{aligned}$$

Since $\mathbb{E}(K_t^2(1)(\partial_{x_i}K_x)^2(1))$ is bounded by $K_0(T)\frac{h_th_x^d}{T\lambda(B)}$, we get $\mathbb{E}(A\times C) \leq \frac{\delta_n^3}{h_th_x^{d+2}}\mathbb{E}(K_t^4(1)(\partial_{x_i}K_x)^2(1)K_x^2(1)v^2(1)) + \frac{\delta_n^2}{T\lambda(B)h_th_x^{d+2}}\mathbb{E}(K_t^2(1)K_x^2(1)v^2(1))$. We write terms of type $\mathbb{E}(g_t(1)g_x(1)v^2(1))$ as $\frac{1}{T\lambda(B)}\int_0^T dr g_t\left(\frac{s-r}{h_t}\right)\int_B dz g_x\left(\frac{y-z}{h_x}\right)\mathbb{E}(v^2(r,z))$, where g_t (resp g_x) represents a function depending on K_t (resp. on K_x and $\partial_x K_x$). Finally, by using the same procedure for the other terms, we obtain $\mathbb{E}((\partial_{x_i} f_n(s,y))^2(r_n^2(s,y))) \leq \frac{K_0(T)}{(T\lambda(B))^4 h_t h_x^{d+2}}\int_0^T dr g_t\left(\frac{s-r}{h_t}\right)\int_B dz g_x\left(\frac{y-z}{h_x}\right)\mathbb{E}(v^2(r,z))$. Applying Lemma 3.5 yields $(T\lambda(B))^4\mathbb{E}\|r_n\partial_{x_i} f_n\|^2_{H^\mu_{\beta,X}} \leq \frac{K_0(T)}{h_x^2}\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}$.

If v is unbiased, terms like $(\mathbb{E}(K_t(1)K_x(1)v(1)))^2$ are null and this leads to $(T\lambda(B))^4\mathbb{E}\|r_n\partial_{x_i} f_n\|^2_{H^\mu_{\beta,X}} \leq \frac{K_0(T)T\lambda(B)\delta_n}{h_x^2}\mathbb{E}\|v\|^2_{H^\mu_{\beta,X}}$.

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