GLOBAL REGULARITY FOR THE NAVIER-STOKES EQUATIONS WITH LARGE, SLOWLY VARYING INITIAL DATA IN THE VERTICAL DIRECTION

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Abstract. In [9] is obtained a class of large initial data generating a global smooth solution to the three dimensional, incompressible Navier-Stokes equations. This data varies slowly in the vertical direction (is a function on ϵx_3) and has a norm which blows up as the small parameter goes to zero. This type of initial data can be seen as the "ill prepared" case (in opposite with the "well prepared" case which was treated in [6]-[8]). In [9] the fluid evolves in a special domain, namely $\Omega = T_h^2 \times \mathbb{R}_v$. The choice of a periodic domain in the horizontal variable plays an important role. The aim of this article is to study the case where the fluid evolves in the full spaces $\mathbb{R}^3$, where we need to overcome the difficulties coming from very low horizontal frequencies. We consider in this paper an intermediate situation between the "well prepared" case and "ill prepared" situation (the norms of the horizontal components of initial data are small but the norm of the vertical component blows up as the small parameter goes to zero). As in [9], the proof uses the analytical-type estimates and the special structure of the nonlinear term of the equation.

1. Introduction

We study in this paper the Navier-Stokes equations with initial data which is slowly varying in the vertical variable. More precisely we consider the system

$$(NS)\begin{cases}\partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \quad \text{in} \quad \mathbb{R}_+\times\Omega\\ \operatorname{div} u = 0\\ u|_{t=0} = u_{0,\varepsilon},\end{cases}$$

where $\Omega = \mathbb{R}^3$ and $u_{0,\varepsilon}$ is a divergence free vector field, whose dependence on the vertical variable x_3 will be chosen to be "slow", meaning that it depends on εx_3 where ε is a small parameter. Our goal is to prove a global existence in time result for the solution generated by this type of initial data, with no smallness assumption on its norm. This type of initial data which is slowly varying in the vertical direction was already studied in [6]-[8]-[9]. We recall that in [8] is studied the case of "well prepared" initial data of the type $(\varepsilon u_0^h(x_h,\varepsilon x_3), u_0^3(x_h,\varepsilon x_3))$ and in [9] is studied the more difficult case of "ill prepared" initial data of the type $(u^h(x_h,\varepsilon x_3), \varepsilon^{-1}u^3(x_h,\varepsilon x_3))$. In this paper, we consider the large initial data between the "well prepared" case treated in [6]-[8] and the "ill prepared" case treated in [9]. More precisely, our initial data is of the form

$$u_{0,\varepsilon} = (\varepsilon^{\frac{1}{2}}u_0^h(x_h,\varepsilon x_3), \varepsilon^{-\frac{1}{2}}u_0^3(x_h,\varepsilon x_3)).$$

Date: 27,Mar.

Key words and phrases. Navier-Stokes equations, global wellposedness.

The mathematical study of the Navier-Stokes equations has a long history. We begin by recalling some important and classical facts about the Navier-Stokes system, focusing on the conditions which imply the global existence of the strong solution.

The first important result about the classical Navier-Stokes system was obtained by J. Leray [16], and asserted that for every finite energy initial data there exists at least one global in time weak solution which verifies the energy estimate. This solution is unique in $\mathbb{R}^2$ but unfortunately the solution is not known to be unique in three dimensional space. The result of J. Leray uses the structure of the nonlinear terms in order to obtain the energy inequality. The question of the uniqueness or of the regularity of the weak solutions is open.

The Fujita-Kato theorem gives a partial response to the construction of global unique solution. Indeed, the theorem of Fujita-Kato [11] allows to construct a unique local in time solution in the homogeneous Sobolev spaces $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, or in the Lebsegue space $L^3(\mathbb{R}^3)$ [14]. If the initial data is small compared to the viscosity $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq c\nu$, then the strong solution exists globally in time. This result was generalized by M. Cannone, Y. Meyer et F. Planchon [2] to Besov spaces of negative index of regularity. More precisely, they proved that, if the initial data belongs to the Besov space $B^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$, and verifies that it is small in the norm of this Besov space, compared to the viscosity, then the solution is global in time. More recently, in [15] is obtained a unique global in time solution for Navier-Stokes equation for small data belonging to a more general space of initial data, which is derivatives of BMO function. Concerning the methods to obtain such results, we recall that proving the existence of a unique, global in time solution to the Navier-Stokes equations is rather standard (it is a consequence of a Banach fixed point theorem) as long as the initial data is chosen small enough in some scale invariant spaces (with invariant norm by the scaling $\lambda u(\lambda^2 t, \lambda x)$) embedded in $\dot{B}^{-1}_{\infty,\infty}$ (the Besov space) , where we recall that

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta} f\|_{L^\infty}.$$

We refer for instance to [2],[11],[15],[24] for a proof in various scale invariant function spaces. These theorems are general results of global existence for small initial data and does not take into account the any particular algebraical properties of the nonlinear terms in the Navier-Stokes equations.

However, proving such a result without any smallness assumption or any geometrical invariance hypothesis, which implies conservation of quantities beyond the scaling, is a challenge. Little progress has been made in that direction: we will not describe all the literature on the question, but refer among others to [18], [2], [4], [5], [6]-[8], [9], [10], [21] and the references therein for more details.

We recall briefly the examples of large initial data which gives global existence of the solution known in the literature. We first notice that for regular axi-symmetric initial data, without swirl, there exists a unique global in time solution for the Navier-Stokes system. This result is based on the conservation of some quantities beyond the scaling regularity level (see [23]).

The case of large initial data (in some sense) for fluids evolving in thin domains was firstly considered by [21]. Roughly speaking, the tridimensional Navier-Stokes system can be seen as a perturbation of the two dimensional Navier-Stokes system if the domain is thin enough in the vertical direction. Generally, if the initial data u_0 can be splitting as $u_0 = v_0 + w_0$, with v_0 a two dimensional free divergence vector

field belonging to $L^2(\mathcal{T}_h^2)$ and $w_0 \in H^{\frac{1}{2}}(\mathcal{T}^3)$, such that

$$\|w_0\|_{H^{\frac{1}{2}}(\mathcal{T}^3)} \exp\left(\frac{\|v_0\|^2_{L^2(\mathcal{T}_h^2)}}{\nu^2}\right) \leq c\nu,$$

then the solution exists globally in time.

The case of initial data with large initial vortex in the vertical direction ($\operatorname{rot} u_0^\varepsilon = \operatorname{rot} u_0 + \epsilon^{-1}(0,0,1)$), or equivalently the case of rotating fluids, was studied in [18] in the case of periodic domains and in [4]-[5] for the case of a rotating fluid in $\mathbb{R}^3$ or in $\mathbb{R}^2 \times (0,1)$. When the rotation is fast enough the fluid tends to have a two-dimensional behavior, far from the boundary of the domain (this is the so called Taylor-Proudman column theorem [20]). For example, in the case where the domain is $\mathbb{R}^3$ the fluctuation of this motion is dispersed to infinity and some Strichartz quantities became small [4] which allow to obtain the global existence of the solution (when ε small enough).

An important issue for the Navier-Stokes equations is to use on maximum the algebrical structure of the nonlinear terms. Some results used in a crucial way this structure and allow to obtain very interesting new results.

The case of the Navier-Stokes equations with vanishing vertical viscosity was firstly studied in [4] where the authors proved local existence for large data in anisotropic Sobolev spaces $H^{0,s}, s > 1/2$, and global existence and uniqueness for small initial data. One of the key observations is that, even if there is no vertical viscosity and thus no smoothing in the vertical variable, the partial derivative ∂_3 is only applied to the component u_3 in the nonlinear term. The divergence-free condition implies that $\partial_3 u_3$ is regular enough to get good estimates of the nonlinear term. In [10] the authors obtained the global existence of the solution for the anisotropic Navier-Stokes system with high oscillatory initial data.

A different idea, but always using the special structure of the Navier-Stokes equations, is used by J.-Y. Chemin and I. Gallagher [6] in order to construct the first example of periodic initial data which is big in C^{-1}, and strongly oscillating in one direction which generates a global solution. Such initial data is given by

$$u_0^N = (N u_h(x_h)\cos(Nx_3), -\operatorname{div}_h u_h(x_h)\sin(Nx_3)),$$

where $\|u_h\|_{L^2(\mathcal{T}_h^2)} \leq C(\ln N)^{\frac{1}{9}}$. This result was generalized to the case of the space $\mathbb{R}^3$ in [7].

In the paper [8], J.-Y. Chemin and I. Gallagher studied the Navier-Stokes equations for initial data which slowly varies in the vertical direction in the well prepared case. The "well prepared" case means that the norm of the initial data is large but does not blow up when the parameter ε converges to zero. We note that important remarks on the pressure term and the bilinear term were used in this paper in order to obtain the global existence for large data.

The case of slowly varying initial data in the vertical direction in the "ill prepared" initial data was recently studied in [9]. We note that the horizontal components has a large norm and the vertical component has a norm which blows up when the parameter goes to zero. After a change of scale of the problem, the system became a Navier-Stokes type equation with an anisotropic viscosity $-\nu\Delta_h u - \nu\epsilon^2\partial_3^2 u$ and anisotropic gradient of the pressure, namely $-(\nabla_h p, \epsilon^2\partial_3 p)$. In this equation we can remark that there is a loos of regularity in the vertical variable in Sobolev estimates. To overcome this difficulty is needed to work with analytical initial data. The most important tool was developed in the paper of J.-Y. Chemin [3] and consisted to make

analytical type estimates, and in the same time to control the size of the analyticity band. This is performed by the control of nonlinear quantities which depend on the solution itself. Even in this situation, it is important to take into account very carefully the special structure of the Navier-Stokes equations. In [9] is obtained in fact a global in time Cauchy-Kowalewskaya type theorem. We recall also that some local in time results for Euler and Prandtl equation with analytic initial data can be found in[22].

In [9] the fluid is supposed to evolve in a special domain $\Omega = T_h^2 \times \mathbb{R}_v$. This choice of domain is justified by the pressure term. Indeed, the pressure verifies the elliptic equation $\Delta_\varepsilon p = \partial_i \partial_j (u^i u^j)$, and consequently, $\nabla_h p = (-\Delta_\varepsilon)^{-1} \nabla_h \partial_i \partial_j (u^i u^j)$. Because we have that Δ_ε^{-1} converges to Δ_h^{-1} it is important to control the low horizontal frequencies. While in the case of the periodic torus in the horizontal variable we have only zero horizontal frequency and high horizontal frequencies.

In this paper our goal is to investigate the case where the fluid evolves in the full space $\mathbb{R}^3$. In that situation, we are able to solve globally in time the equation (conveniently rescaled in ε) for small analytic-type initial data. In the case of the full space $\mathbb{R}^3$ we need to control very precisely the low horizontal frequencies. We also note that we can construct functional spaces where the operator $\Delta_h^{-1} \nabla_h (a \nabla_h b)$ is a bounded operator. However we still need to impose on the initial data more control of the regularity in the low horizontal frequencies (namely we impose that $u_0(\cdot, x_3) \in L^2(\mathbb{R}^2_h) \cap \dot{H}^{-\frac{1}{2}}(\mathbb{R}^2_h)$). In the vertical variable we need to impose analyticity of the data. The method of the proof follows closely the argument of [9], but instead to use pointwise estimates on the fourier variables, we write an equation with a regularizing term in the vertical variable and we use energy estimates on anisotropic Sobolev spaces of the form $H^{0,s}$ respectively $H^{-\frac{1}{2},s}$.

Our main result in the case of the full space $\mathbb{R}^3$ is the following (for the notations see the next section).

Theorem 1.1. Let a be a positive number, $s > \frac{1}{2}$. There exist two positive constants ε_0 and η such that for any divergence free fields v_0 satisfying

$$\|e^{a|D_3|} v_0\|_{H^{0,s}} + \|e^{a|D_3|} v_0\|_{H^{-\frac{1}{2},s}} \leq \eta,$$

and for any $\varepsilon \in (0, \varepsilon_0)$, the Navier-Stokes system (NS) with initial data

$$u_0^\varepsilon = \big(\varepsilon^{\frac{1}{2}} v_0^h(x_h, \varepsilon x_3), \varepsilon^{-\frac{1}{2}} v_0^3(x_h, \varepsilon x_3)\big)$$

has a global smooth solution on $\mathbb{R}^3$.

As we already explain above, in order to prove the main theorem 1.1, we will firstly transform the system using the change of scale

$$u^\varepsilon(t, x_h, x_3) = \big(\varepsilon^{\frac{1}{2}} v^h(t, x_h, \varepsilon x_3), \varepsilon^{-\frac{1}{2}} v^3(t, x_h, \varepsilon x_3)\big)$$

into a system of Navier-Stokes type, with a vertical vanishing viscosity, that is the Laplacian operator became $-\nu \Delta_h v - \epsilon^2 \partial_3 v$ and a changed pressure term became $-(\nabla_h p, \epsilon^2 \partial_3 p)$.

Taking the advantage that we work in the full spaces $\mathbb{R}^3$, we can also consider a different type of initial data, with larger amplitude but strongly oscillating in the horizontal variables, namely, initial data of the form

$$u_0^\varepsilon = \big(\varepsilon^{-\frac{1}{2}} v_0^h(\varepsilon^{-1} x_h, x_3), \varepsilon^{-\frac{3}{2}} v_0^3(\varepsilon^{-1} x_h, x_3)\big).$$

However, this type of initial data has the $\dot{B}^{-1}_{\infty,\infty}$ norm on the same order as the initial data in the previous theorem. In order to solve the Navier-Stokes equations with this new type of initial data, we make the different change of scale,

$$u^\varepsilon(t,x_h,x_3)=\big(\epsilon^{-\frac{1}{2}}v^h(\epsilon^{-2}t,\epsilon^{-1}x_h,x_3),\epsilon^{-\frac{3}{2}}v^3(\epsilon^{-2}t,\epsilon^{-1}x_h,x_3)\big)$$

and we note that the rescaled system that we obtain is exactly the same as that in the proof of theorem 1.1. Consequently, we also obtain the following result.

Theorem 1.2. Let a be a positive number, $s>\frac{1}{2}$. There exist two positive constants ε_0 and η such that for any divergence free fields v_0 satisfying

$$\|e^{a|D_3|}v_0\|_{H^{0,s}}+\|e^{a|D_3|}v_0\|_{H^{-\frac{1}{2},s}}\leq\eta,$$

and for any $\varepsilon\in(0,\varepsilon_0)$, the Navier-Stokes system (NS) with initial data

$$u_0^\varepsilon=\big(\varepsilon^{-\frac{1}{2}}v_0^h(\varepsilon^{-1}x_h,x_3),\varepsilon^{-\frac{3}{2}}v_0^3(\varepsilon^{-1}x_h,x_3)\big)$$

has a global smooth solution on $\mathbb{R}^3$.

2. A simplified model

Let us consider the following equation

$$\partial_t u+\gamma u+a(D)Q(u,u)=0$$

where $a(D)$ is a fourier multiplier of order one and, Q is any quadratic form. Then, if the initial data verifies

$$\|u_0\|_X=\int e^{\delta|\xi|}\hat{u}(\xi)d\xi\leq c\gamma\quad\text{whith}\quad a>0$$

then we have a global solution in the same space. We follow the method introduced in [9] and [3]. The idea of the proof is the following, we want to control the some kind of quantities on the solution, but we must prevent the possible loose of the rayon of the analyticity of the solution. Let us introduce $\theta(t)$ the "loose of analyticity", such that $\dot{\theta}(t)=\int e^{(\delta-\theta(t))|\xi|}|\hat{u}(\xi)|d\xi$, $\theta(0)=0$. We denote by $\Phi=(a-\lambda\theta(t))|\xi|$ and we define

$$\dot{\theta}(t)=\int|\hat{u}_\Phi(\xi)|d\xi=\|u_\Phi\|_X,\quad\theta(0)=0.$$

The computations which follow are performed under the condition $\theta(t)\leq a/\lambda$ (which implies $\Phi\geq 0$). The equation verified by $\hat{u}_\phi$ is the following

$$\partial_t\hat{u}_\Phi+\gamma\hat{u}_\Phi+\lambda\dot{\theta}(t)|\xi|\hat{u}_\Phi+a(\xi)e^\Phi\widehat{(u^2)}=0.$$

As $\dot{\theta}\geq 0$, after integration in ξ, we obtained the following equation

$$\partial_t\|u_\Phi\|_X+\gamma\|u_\Phi\|_X+\lambda\dot{\theta}(t)\int|\xi||\hat{u}_\Phi|d\xi\leq C\int|\xi||\hat{u_\Phi}|\star|\hat{u}_\Phi|(\xi)d\xi.$$

As $|\xi|\leq|\xi-\eta|+|\eta|$, we obtain

$$\int|\xi|\hat{u_\Phi}\star\hat{u}_\Phi(\xi)d\xi\leq 2\bigg(\int|\xi||\hat{u}_\phi|d\xi\bigg)\bigg(\int|\hat{u}_\Phi|d\xi\bigg)=2\dot{\theta}(t)\bigg(\int|\xi||\hat{u}_\phi|d\xi\bigg).$$

So, choosing $\lambda=4C$ we obtain

$$\dot{\theta}(t)=\|u_\Phi(t)\|_X\leq 2\|e^{a|D|}u_0\|_Xe^{-\gamma t}$$

which, for u_0 small enough, gives

$$\theta(t)\leq\gamma^{-1}\|e^{a|D|}u_0\|_X\leq a\lambda^{-1}.$$

This allows to obtain the global in time existence of the solution.

3. Structure of the proof

3.1. **Reduction to a rescaled problem.** We seek the solution of the form

$$u_\varepsilon(t,x) \overset{\text{def}}{=} \Big(\varepsilon^{\frac{1}{2}} v^h(t,x_h,\varepsilon x_3), \varepsilon^{-\frac{1}{2}} v^3(t,x_h,\varepsilon x_3)\Big).$$

This leads to the following rescaled Navier-Stokes system

$$(RNS_\varepsilon) \quad \begin{cases} \partial_t v^h - \Delta_h v^h - \varepsilon^2 \partial_3^2 v^h + \varepsilon^{\frac{1}{2}} v\cdot\nabla v^h = -\nabla^h q, \\ \partial_t v^3 - \Delta_h v^3 - \varepsilon^2 \partial_3^2 v^3 + \varepsilon^{\frac{1}{2}} v\cdot\nabla v^3 = -\varepsilon^2 \partial_3 q, \\ \mathrm{div} v = 0, \\ v(0) = v_0(x), \end{cases} \tag{3.1}$$

where $\Delta_h \overset{\text{def}}{=} \partial_1^2 + \partial_2^2$ and $\nabla_h \overset{\text{def}}{=} (\partial_1, \partial_2)$. As there is no boundary, the rescaled pressure q can be computed with the formula

$$-\Delta_\varepsilon q = \varepsilon^{\frac{1}{2}} \mathrm{div}_h (v\cdot\nabla v), \quad \Delta_\varepsilon = \Delta_h + \varepsilon^2 \partial_3^2. \tag{3.2}$$

When ε tends to zero, Δ_ε^{-1} looks like Δ_h^{-1}. Thus for low horizontal frequencies, an expression of $\nabla_h \Delta_h^{-1}$ cannot be estimated in L^2. This is one reason why the authors in [9] work in $\mathbb{T}^2 \times \mathbb{R}$. To obtain a similar result in $\mathbb{R}^3$, we need to introduce the following anisotropic Sobolev space.

Definition 3.1. Let $s, \sigma \in \mathbb{R}$, $\sigma < 1$. The anisotropic Sobolev space $H^{\sigma,s}$ is defined by

$$H^{\sigma,s} = \{f \in \mathcal{S}'(\mathbb{R}^3); \|f\|_{H^{\sigma,s}} < \infty\},$$

where

$$\|f\|_{H^{\sigma,s}}^2 \overset{\text{def}}{=} \int_{\mathbb{R}^3} |\xi_h|^{2\sigma} (1+|\xi_3|^2)^s |\hat{f}(\xi)|^2 d\xi, \quad \xi = (\xi_h, \xi_3).$$

For any $f, g \in H^{\sigma,s}$, we denote

$$(f,g)_{H^{\sigma,s}} \overset{\text{def}}{=} (|D_h|^\sigma \langle D_3\rangle^s f, |D_h|^\sigma \langle D_3\rangle^s g)_{L^2}, \quad \langle D_3\rangle = (1+|D_3|^2)^{\frac{1}{2}}.$$

We prove that

Theorem 3.2. Let a be a positive number, $s > \frac{1}{2}$. There exist two positive constants ε_0 and η such that for any divergence free fields v_0 satisfying

$$\|e^{a|D_3|} v_0\|_{H^{0,s}} + \|e^{a|D_3|} v_0\|_{H^{-\frac{1}{2},s}} \le \eta,$$

and for any $\varepsilon \in (0, \varepsilon_0)$, (RNS_ε) has a global smooth solution on $\mathbb{R}^3$.

3.2. **Definition of the functional setting.** As in [9], the proof relies on exponential decay estimates for the Fourier transform of the solution. Thus, for any locally bounded function Ψ on $\mathbb{R}^+ \times \mathbb{R}^3$ and for any function f, continuous in time and compactly supported in Fourier space, we define

$$f_\Psi(t) \overset{\text{def}}{=} \mathcal{F}^{-1}\big(e^{\Psi(t,\cdot)} \widehat{f}(t,\cdot)\big).$$

Now we introduce two key quantities we want to control in order to prove the theorem. We define the function $\theta(t)$ by

$$\dot{\theta}(t) \overset{\text{def}}{=} \varepsilon \|v_\Phi^h(t)\|_{H^{\frac{1}{2},s}}^2 + \|v_\Phi^3(t)\|_{H^{\frac{1}{2},s}}^2 \quad \text{and} \quad \theta(0) = 0, \tag{3.3}$$

and we also define

$$\Psi(t) \stackrel{\text{def}}{=} \|v_\Phi(t)\|^2_{H^{0,s}} + \int_0^t \|\nabla_h v_\Phi(\tau)\|^2_{H^{0,s}} d\tau, \tag{3.4}$$

where

$$\Phi(t,\xi) \stackrel{\text{def}}{=} (a - \lambda\theta(t))|\xi_3| \tag{3.5}$$

for some λ that will be chosen later on.

3.3. **Main steps of the proof.**

Proposition 3.3. A constant C_0 exists such that, for any positive λ and for any t satisfying $\theta(t) \leq a/\lambda$, we have

$$\theta(t) \leq \exp\big(C_0\Psi(t)\big)\Big[\|e^{a|D_3|}v_0\|^2_{H^{-\frac{1}{2},s}} + C_0\int_0^t \dot\theta(\tau)\Psi(\tau)d\tau\Big].$$

Proposition 3.4. There exist C_1 and λ_0 such that for $\lambda \geq \lambda_0$ and for any t satisfying $\theta(t) \leq a/\lambda$, we have

$$\Psi(t) \leq \|e^{a|D_3|}v_0\|^2_{H^{0,s}} \exp\big(C_1\Psi(t)\big).$$

The proof of Proposition 3.3 and 3.4 will be presented in section 4 and section 5 respectively. For the moment, let us assume that they are true and conclude the proof of Theorem 3.2. As in [9], we use a continuation argument. For any $\lambda \geq \lambda_0$ and η, let us define

$$\mathcal{T}_\lambda \stackrel{\text{def}}{=} \{T : \theta(T) \leq 4\eta^2, \Psi(T) \leq 2\eta^2\}.$$

Similar to the argument in [9], $\mathcal{T}_\lambda$ is of the form $[0, T^*)$ for some positive T^*. Thus, it suffices to prove that $T^* = +\infty$. In order to use Proposition 3.3 and 3.4, we need to assume that $\theta(T) \leq \frac{a}{\lambda}$, which leads to the condition

$$4\eta^2 \leq \frac{a}{\lambda}.$$

From Proposition 3.3 and 3.4, it follows that for all $T \in \mathcal{T}_\lambda$,

$$\begin{aligned} &\theta(T) \leq \exp(2C_0\eta^2)(\eta^2 + 2C_0\eta^2\theta(T)),\\ &\Psi(T) \leq \eta^2\exp(2C_1\eta^2) \end{aligned} \tag{3.6}$$

Now we choose η such that

$$\exp(2C_0\eta^2) < 2, \quad \exp(2C_1\eta^2) < 2, \quad 4C_0\eta^2 < \frac{1}{2}.$$

With this choice of η, then we infer from (3.6) that

$$\theta(T) < 4\eta^2, \quad \Psi(T) < 2\eta^2, \tag{3.7}$$

which ensures that $T^* = +\infty$, thus we conclude the proof of Theorem 3.2. □

4. The action of subadditive phases on products

For any function f, we denote by f^+ the inverse Fourier transform of $|\widehat{f}|$. Let us notice that the map $f \mapsto f^+$ preserves the norm of all $H^{\sigma,s}$ spaces. Throughout this section, Ψ will denote a locally bounded function on $\mathbb{R}^+ \times \mathbb{R}^3$ which satisfies the following inequality

$$\Psi(t,\xi) \leq \Psi(t,\xi-\eta) + \Psi(t,\eta). \tag{4.1}$$

Before presenting the product estimates, let us recall the Littlewood-Paley decomposition. Choose two nonnegative even functions χ, $\varphi \in \mathcal{S}(\mathbb{R})$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R},\ |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R},\ \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \text{for} \quad \xi \in \mathbb{R},$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \text{for} \quad \xi \in \mathbb{R}\setminus\{0\}.$$

The frequency localization operators Δ_j^v and S_j^v in the vertical direction are defined by

$$\Delta_j^v f = \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi_3|)\widehat{f}\big) \quad \text{for}\, j \geq 0, \quad S_j^v f = \mathcal{F}^{-1}\big(\chi(2^{-j}|\xi_3|)\widehat{f}\big) = \sum_{j'\leq j-1} \Delta_{j'}^v f,$$
$$\Delta_{-1}^v f = S_0^v f, \quad \Delta_j^v f = 0 \quad \text{for}\, j \leq -2.$$

And the frequency localization operators $\dot{\Delta}_j^h$ and S_j^h in the horizontal direction are defined by

$$\dot{\Delta}_j^h f = \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi_h|)\widehat{f}\big), \quad S_j^h f = \sum_{j'\leq j-1} \dot{\Delta}_{j'}^h f, \quad \text{for}\ j \in \mathbb{Z}.$$

It is easy to verify that

$$\|f\|_{H^{\sigma,s}}^2 \approx \sum_{j,k\in\mathbb{Z}} 2^{2js}2^{2k\sigma}\|\Delta_j^v\dot{\Delta}_k^h f\|_{L^2}^2. \tag{4.2}$$

In the sequel, we will constantly use the Bony's decomposition from [1] that

$$fg = T_f^v g + R_f^v g, \tag{4.3}$$

with

$$T_f^v g = \sum_j S_{j-1}^v f \Delta_j^v g, \quad R_f g = \sum_j S_{j+2}^v f \Delta_j^v g.$$

We also use the Bony's decomposition in the horizontal direction

$$fg = T_f^h g + T_f^h g + R^h(f,g), \tag{4.4}$$

with

$$T_f^h g = \sum_j S_{j-1}^h f \dot{\Delta}_j^h g, \quad R^h(f,g) = \sum_{|j'-j|\leq 1} \dot{\Delta}_j^h f \dot{\Delta}_{j'}^h g.$$

Lemma 4.1. (Bernstein's inequality) Let $1 \leq p \leq q \leq \infty$. Assume that $f \in L^p(\mathbb{R}^d)$, then there exists a constant C independent of f, j such that

$$\mathrm{supp}\hat{f} \subset \{|\xi| \leq C2^j\} \Rightarrow \|\partial^\alpha f\|_{L^q} \leq C2^{j|\alpha|+dj(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$
$$\mathrm{supp}\hat{f} \subset \{\frac{1}{C}2^j \leq |\xi| \leq C2^j\} \Rightarrow \|f\|_{L^p} \leq C2^{-j|\alpha|}\sup_{|\beta|=|\alpha|}\|\partial^\beta f\|_{L^p}.$$

Lemma 4.2. Let $s > \frac{1}{2}, \sigma_1, \sigma_2 < 1$ and $\sigma_1 + \sigma_2 > 0$. Assume that $a_\Psi \in H^{\sigma_1,s}$ and $b_\Psi \in H^{\sigma_2,s}$. Then there holds

$$\|\big[\Delta_j^v \dot{\Delta}_k^h (T_a^v b)\big]_\Psi\|_{L^2} + \|\big[\Delta_j^v \dot{\Delta}_k^h (R_a^v b)\big]_\Psi\|_{L^2} \leq C c_{j,k} 2^{(1-\sigma_1-\sigma_2)k} 2^{-js} \|a_\Psi\|_{H^{\sigma_1,s}} \|b_\Psi\|_{H^{\sigma_2,s}},$$

with the sequence $(c_{j,k})_{j,k\in\mathbb{Z}}$ satisfying $\sum\limits_{j,k} c_{j,k} \leq 1$.

Proof. Let us firstly prove the case when the function Ψ is identically 0. Below we only present the proof of $R_a b$, the proof for $T_a b$ is very similar. Using Bony's decomposition (4.4) in the horizontal direction, we write

$$\begin{aligned}
\Delta_j \dot{\Delta}_k^h (R_a^v b) &= \sum_{j'} \Delta_j^v \dot{\Delta}_k^h (S_{j'+2}^v a \Delta_{j'}^v b) \\
&= \sum_{j'} \Delta_j^v \dot{\Delta}_k^h \big(T^h_{S_{j'+2}^v a} \Delta_{j'}^v b + T^h_{\Delta_{j'}^v b} S_{j'+2}^v a + R^h(S_{j'+2}^v a, \Delta_{j'}^v b)\big) \\
&:= I + II + III.
\end{aligned}$$

Considering the support of the Fourier transform of $T^h_{S_{j'+2}^v a} \Delta_{j'}^v b$, we have

$$I = \sum_{j'\geq j-4} \sum_{|k'-k|\leq 4} \Delta_j^v \dot{\Delta}_k^h \big(S_{j'+2}^v S_{k'-1}^h a \Delta_{j'}^v \dot{\Delta}_{k'}^h b\big).$$

Then we get by Lemma 4.1 that

$$\begin{aligned}
\|I\|_{L^2} &\leq C \sum_{j'\geq j-4} \sum_{|k'-k|\leq 4} \|S_{j'+2}^v S_{k'-1}^h a \Delta_{j'} \dot{\Delta}_{k'}^h b\|_{L^2} \\
&\leq C \sum_{j'\geq j-4} \sum_{|k'-k|\leq 4} \|S_{j'+2}^v S_{k'-1}^h a\|_{L^\infty} \|\Delta_{j'}^v \dot{\Delta}_{k'}^h b\|_{L^2}.
\end{aligned}$$

We use Lemma 4.1 again to get

$$\begin{aligned}
\|S_{j'+2}^v S_{k'-1}^h a\|_{L^\infty} &\leq \sum_{j''\leq j'+1} \sum_{k''\leq k'-2} \|\Delta_{j''}^v \dot{\Delta}_{k''}^h a\|_{L^\infty} \\
&\leq C \sum_{j''\leq j'+1} \sum_{k''\leq k'-2} 2^{k''} \|\Delta_{j''}^v \dot{\Delta}_{k''}^h a\|_{L^\infty_{x_3} L^2_{x_h}} \\
&\leq C \sum_{j''\leq j'+1} \sum_{k''\leq k'-2} 2^{\frac{j''}{2}} 2^{k''} \|\Delta_{j''}^v \dot{\Delta}_{k''}^h a\|_{L^2} \\
&\leq C 2^{(1-\sigma_1)k} \|a\|_{H^{\sigma_1,s}},
\end{aligned}$$

from which, it follows that

$$\begin{aligned}
\|I\|_{L^2} &\leq C 2^{(1-\sigma_1)k} \|a\|_{H^{\sigma_1,s}} \sum_{j'\geq j-4} \sum_{|k'-k|\leq 4} \|\Delta_{j'}^v \dot{\Delta}_{k'}^h b\|_{L^2} \\
&\leq C c_{j,k} 2^{-js} 2^{(1-\sigma_1-\sigma_2)k} \|a\|_{H^{\sigma_1,s}} \|b\|_{H^{\sigma_2,s}}. \qquad (4.5)
\end{aligned}$$

Similarly, we have

$$II = \sum_{j'\geq j-4} \sum_{|k'-k|\leq 4} \Delta_j^v \dot{\Delta}_k^h (\Delta_{j'}^v S_{k'-1}^h b S_{j'+2}^v \dot{\Delta}_{k'}^h a).$$

Then we get by Lemma 4.1 that

$$\begin{aligned}\|II\|_{L^2} &\leq C\sum_{j'\geq j-4}\sum_{|k'-k|\leq 4}\|\Delta_{j'}^v S_{k'-1}^h b\|_{L^2_{x_3}L^\infty_{x_h}}\|S_{j'+2}^v\dot{\Delta}_{k'}^h a\|_{L^2_{x_h}L^\infty_{x_3}}\\ &\leq C2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|_{H^{\sigma_1,s}}\|b\|_{H^{\sigma_2,s}}\sum_{j'\geq j-4}\sum_{|k'-k|\leq 4}2^{-(j'-j)s}c_{k'}c_{j'}\\ &\leq Cc_{j,k}2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|_{H^{\sigma_1,s}}\|b\|_{H^{\sigma_2,s}}. \qquad (4.6)\end{aligned}$$

Now, let us turn to III. We have

$$III=\sum_{j'\geq j-4}\sum_{k',k''\geq k-2;|k'-k''|\leq 1}\Delta_j^v\dot{\Delta}_k^h(S_{j'+2}^v\dot{\Delta}_{k'}^h a\Delta_{j'}^v\dot{\Delta}_{k''}^h b).$$

So, we have by Lemma 4.1 that

$$\begin{aligned}&\|III\|_{L^2}\\ &\leq C\sum_{j'\geq j-4}\sum_{k',k''\geq k-2;|k'-k''|\leq 1}2^k\|S_{j'+2}^v\dot{\Delta}_{k'}^h a\Delta_{j'}^v\dot{\Delta}_{k''}^h b\|_{L^2_{x_3}L^1_{x_h}}\\ &\leq C\sum_{j'\geq j-4}\sum_{k',k''\geq k-2;|k'-k''|\leq 1}2^k\|S_{j'+2}^v\dot{\Delta}_{k'}^h a\|_{L^\infty_{x_3}L^2_{x_h}}\|\Delta_{j'}^v\dot{\Delta}_{k''}^h b\|_{L^2}\\ &\leq C2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|_{H^{\sigma_1,s}}\|b\|_{H^{\sigma_2,s}}\sum_{j'\geq j-4}\sum_{k'\geq k-2}2^{-(\sigma_1+\sigma_2)(k'-k)}2^{-(j'-j)s}c_{k'}c_{j'}\\ &\leq Cc_{j,k}2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|_{H^{\sigma_1,s}}\|b\|_{H^{\sigma_2,s}}. \qquad (4.7)\end{aligned}$$

Summing up (4.5)-(4.7), we obtain

$$\|\Delta_j^v\dot{\Delta}_k^h(R_a b)\|_{L^2}\leq Cc_{j,k}2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|_{H^{\sigma_1,s}}\|b\|_{H^{\sigma_2,s}}.$$

The lemma is proved in the case when the function Ψ is identically 0. In order to treat the general case, we only need to notice the fact that

$$|\mathcal{F}\big[\Delta_j\dot{\Delta}_k^h(R_a b)\big]_\Psi(\xi)|\leq\mathcal{F}\big[\Delta_j\dot{\Delta}_k^h(R_{a_\Psi^+}b_\Psi^+)\big](\xi).$$

This finishes the proof of Lemma 4.2. □

As a consequence of Lemma 4.2 and (4.2), we have

Lemma 4.3. Let $s>\frac{1}{2},\sigma_1,\sigma_2<1$ and $\sigma_1+\sigma_2>0$. Assume that $a_\Psi\in H^{\sigma_1,s}$ and $b_\Psi\in H^{\sigma_2,s}$. Then there holds

$$\|(ab)_\Psi\|_{H^{\sigma_1+\sigma_2-1,s}}\leq C\|a_\Psi\|_{H^{\sigma_1,s}}\|b_\Psi\|_{H^{\sigma_2,s}}.$$

5. Classical analytical-type estimates

In this section, we prove Proposition 3.3. In this part, we don't need to use any regularizing effect from the analyticity, but only the fact that the $e^{\Phi(t,\xi_3)}$ is a sublinear function.

Notice that $\partial_t v_\Phi+\lambda\dot{\theta}(t)|D_3|v_\Phi=(\partial_t v)_\Phi$, we find from (3.1) that

$$\left\{\begin{array}{l}\partial_t v_\Phi^h+\lambda\dot{\theta}(t)|D_3|v_\Phi^h-\Delta_h v_\Phi^h-\varepsilon^2\partial_3^2 v_\Phi^h+\varepsilon^{\frac{1}{2}}(v\cdot\nabla v^h)_\Phi=-\nabla_h q_\Phi,\\ \partial_t v_\Phi^3+\lambda\dot{\theta}(t)|D_3|v_\Phi^3-\Delta_h v_\Phi^3-\varepsilon^2\partial_3^2 v_\Phi^3+\varepsilon^{\frac{1}{2}}(v\cdot\nabla v^3)_\Phi=-\varepsilon^2\partial_3 q_\Phi,\\ \mathrm{div}\, v_\Phi=0,\\ v_\Phi(0)=e^{a|D_3|}v_0(x).\end{array}\right. \qquad (5.1)$$

Step 1. Estimates on the vertical component v_Φ^3

Note that $\dot{\theta}(t) \geq 0$, we get from the second equation of (5.1) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v^3_\Phi(t)\|^2_{H^{-\frac{1}{2},s}} + \|\nabla_h v^3_\Phi(t)\|^2_{H^{-\frac{1}{2},s}} + \|\varepsilon\partial_3 v^3_\Phi(t)\|^2_{H^{-\frac{1}{2},s}} \\
&\leq -\varepsilon^{\frac{1}{2}}\big((v^h\cdot\nabla_h v^3)_\Phi, v^3_\Phi\big)_{H^{-\frac{1}{2},s}} + \varepsilon^{\frac{1}{2}}\big((v^3\mathrm{div}_h v^h)_\Phi, v^3_\Phi\big)_{H^{-\frac{1}{2},s}} - \varepsilon^2\big(\partial_3 q_\Phi, v^3_\Phi\big)_{H^{-\frac{1}{2},s}} \\
&\overset{\mathrm{def}}{=} I + II + III. \qquad (5.2)
\end{aligned}$$

Here we used the fact that $\mathrm{div}\, v = 0$ such that

$$v\cdot\nabla v^3 = v^h\cdot\nabla_h v^3 - v^3\mathrm{div}_h v^h.$$

For II, Lemma 4.3 applied gives

$$\begin{aligned}
|II| &\leq \varepsilon^{\frac{1}{2}}\|(v^3\mathrm{div}_h v^h)_\Phi\|_{H^{-\frac{1}{2},s}}\|v^3_\Phi\|_{H^{-\frac{1}{2},s}} \\
&\leq C\varepsilon^{\frac{1}{2}}\|v^3_\Phi\|_{H^{\frac{1}{2},s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|v^3_\Phi\|_{H^{-\frac{1}{2},s}} \\
&\leq \frac{1}{100}\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}} + C\varepsilon\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|v^3_\Phi\|^2_{H^{-\frac{1}{2},s}}. \qquad (5.3)
\end{aligned}$$

For I, we get by integration by parts that

$$I = \varepsilon^{\frac{1}{2}}\big((\mathrm{div}_h v^h v^3)_\Phi, v^3_\Phi\big)_{H^{-\frac{1}{2},s}} + \varepsilon^{\frac{1}{2}}\big((v^h v^3)_\Phi, \nabla_h v^3_\Phi\big)_{H^{-\frac{1}{2},s}} \overset{\mathrm{def}}{=} I_1 + I_2.$$

As in (5.3), we have

$$|I_1| \leq \frac{1}{100}\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}} + C\varepsilon\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|v^3_\Phi\|^2_{H^{-\frac{1}{2},s}}, \qquad (5.4)$$

and by Lemma 4.3,

$$\begin{aligned}
|I_2| &\leq \varepsilon^{\frac{1}{2}}\|(v^3 v^h)_\Phi\|_{H^{-\frac{1}{2},s}}\|\nabla_h v^3_\Phi\|_{H^{-\frac{1}{2},s}} \\
&\leq C\varepsilon^{\frac{1}{2}}\|v^h_\Phi\|_{H^{0,s}}\|v^3_\Phi\|_{H^{\frac{1}{2},s}}\|\nabla_h v^3_\Phi\|_{H^{-\frac{1}{2},s}} \\
&\leq C\varepsilon\|v^h_\Phi\|^2_{H^{0,s}}\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}} + \frac{1}{100}\|\nabla_h v^3_\Phi\|^2_{H^{-\frac{1}{2},s}}. \qquad (5.5)
\end{aligned}$$

Now, we turn to the estimates of the pressure. Recall that the pressure verifies

$$-\Delta_\varepsilon p = \varepsilon^{\frac{1}{2}}\big[\partial_i\partial_j(v^i v^j) + \partial_i\partial_3(v^i v^3) - 2\partial_3(v^3\mathrm{div}_h v^h)\big].$$

Here and in what follows the index i, j run from 1 to 2. Thus, we can write $p = p^1 + p^2 + p^3$ with

$$\begin{aligned}
p^1 &= \varepsilon^{\frac{1}{2}}(-\Delta_\varepsilon)^{-1}\partial_i\partial_j(v^i v^j), \\
p^2 &= \varepsilon^{\frac{1}{2}}(-\Delta_\varepsilon)^{-1}\partial_i\partial_3(v^i v^3), \qquad (5.6) \\
p^3 &= -2\varepsilon^{\frac{1}{2}}(-\Delta_\varepsilon)^{-1}\partial_3(v^3\mathrm{div}_h v^h).
\end{aligned}$$

We get by integration by parts that

$$\varepsilon^2(\partial_3 p^1_\Phi, v^3_\Phi)_{H^{-\frac{1}{2},s}} = -\varepsilon(p^1_\Phi, \varepsilon\partial_3 v^3_\Phi)_{H^{-\frac{1}{2},s}} \leq C\varepsilon^2\|p^1_\Phi\|^2_{H^{-\frac{1}{2},s}} + \frac{1}{100}\|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{-\frac{1}{2},s}},$$

which together with the fact that the operator $\partial_i\partial_j(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$ and Lemma 4.3 implies that

$$\begin{aligned}
\varepsilon^2(\partial_3 p^1_\Phi, v^3_\Phi)_{H^{-\frac{1}{2},s}} &\leq C\varepsilon^3\|(v^h\otimes v^h)_\Phi\|^2_{H^{-\frac{1}{2},s}} + \frac{1}{100}\|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{-\frac{1}{2},s}} \\
&\leq C\varepsilon^2\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|^2_{H^{\frac{1}{2},s}}\|v^h_\Phi\|^2_{H^{0,s}} + \frac{1}{100}\|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{-\frac{1}{2},s}}. \qquad (5.7)
\end{aligned}$$

For the term containing p_2, we get by integration by parts that

$$\varepsilon^2(\partial_3 p^2_\Phi, v^3_\Phi)_{H^{-\frac12,s}} = -\varepsilon^{\frac12}(\varepsilon^2\partial_3^2(-\Delta_\varepsilon)^{-1}(v^i v^3)_\Phi, \partial_i v^3_\Phi)_{H^{-\frac12,s}},$$

then using the fact that $(\varepsilon\partial_3)^2(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$ and Lemma 4.3, we have

$$\begin{aligned}\varepsilon^2(\partial_3 p^2_\Phi, v^3_\Phi)_{H^{-\frac12,s}} &\le C\varepsilon^{\frac12}\|(v^3v^h)_\Phi\|_{H^{-\frac12,s}}\|\nabla_h v^3_\Phi\|_{H^{-\frac12,s}}\\ &\le C\varepsilon\|v^3_\Phi\|^2_{H^{\frac12,s}}\|v^h_\Phi\|^2_{H^{0,s}} + \frac{1}{100}\|\nabla_h v^3_\Phi\|^2_{H^{-\frac12,s}}. \end{aligned}\tag{5.8}$$

For the last term coming from p_3, we use again the fact that $(\varepsilon\partial_3)^2(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$ and obtain

$$\begin{aligned}\varepsilon^2(\partial_3 p^3_\Phi, v^3_\Phi)_{H^{-\frac12,s}} &\le C\varepsilon^{\frac12}\|(v^3\mathrm{div}\, v^h)_\Phi\|_{H^{-\frac12,s}}\|v^3_\Phi\|_{H^{-\frac12,s}}\\ &\le C\varepsilon^{\frac12}\|v^3_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|v^3_\Phi\|_{H^{-\frac12,s}}\\ &\le C\varepsilon\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|v^3_\Phi\|^2_{H^{-\frac12,s}} + \frac{1}{100}\|v^3_\Phi\|^2_{H^{\frac12,s}}. \end{aligned}\tag{5.9}$$

Summing up (5.2)-(5.5) and (5.7)-(5.9), we obtain

$$\begin{aligned}&\frac{d}{dt}\|v^3_\Phi(t)\|^2_{H^{-\frac12,s}} + \|v^3_\Phi(t)\|^2_{H^{\frac12,s}}\\ &\le C\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|v^3_\Phi\|^2_{H^{-\frac12,s}} + C(\|v^3_\Phi\|^2_{H^{\frac12,s}} + \|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{\frac12,s}})\|v^h_\Phi\|^2_{H^{0,s}}. \end{aligned}\tag{5.10}$$

Here we used the fact that

$$\|\nabla_h v^3_\Phi\|^2_{H^{-\frac12,s}} \approx \|v^3_\Phi\|^2_{H^{\frac12,s}}.$$

Step 2. Estimates on the horizontal component v^h_Φ

From the first equation of (5.1), we infer that

$$\begin{aligned}&\frac12\frac{d}{dt}\|\varepsilon^{\frac12}v^h_\Phi(t)\|^2_{H^{-\frac12,s}} + \|\varepsilon^{\frac12}\nabla_h v^h_\Phi(t)\|^2_{H^{-\frac12,s}} + \varepsilon\|\varepsilon\partial_3 v^h_\Phi(t)\|^2_{H^{-\frac12,s}}\\ &\quad\le -\varepsilon\big((v\cdot\nabla v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}} - \varepsilon\big(\nabla_h q_\Phi, v^h_\Phi\big)_{H^{-\frac12,s}} \overset{\text{def}}{=} I + II. \end{aligned}\tag{5.11}$$

We rewrite I as

$$I = -\varepsilon\big((v^h\cdot\nabla_h v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}} - \varepsilon\big((v^3\partial_3 v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}} \overset{\text{def}}{=} I_1 + I_2.$$

Lemma 4.3 applied gives

$$\begin{aligned}|I_1| &\le \varepsilon\|(v^h\nabla_h v^h)_\Phi\|_{H^{-\frac12,s}}\|\varepsilon^{\frac12}v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le C\varepsilon\|v^h_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|\varepsilon^{\frac12}v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le \frac{1}{100}\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{\frac12,s}} + C\varepsilon\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}}. \end{aligned}\tag{5.12}$$

For I_2, we use integration by parts and $\mathrm{div}\, v = 0$ to get

$$\begin{aligned}I_2 &= -\varepsilon\big((\mathrm{div}_h v^h v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}} + \big((v^h v^3)_\Phi, \varepsilon^{\frac12}\varepsilon\partial_3 v^h_\Phi\big)_{H^{-\frac12,s}},\\ &\overset{\text{def}}{=} I_{21} + I_{22}.\end{aligned}$$

As in (5.12), we have

$$|I_{21}| \le \frac{1}{100}\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{\frac12,s}} + C\varepsilon\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}},\tag{5.13}$$

and by Lemma 4.3,

$$\begin{aligned}|I_{22}| &\leq \|(v^3v^h)_\Phi\|_{H^{-\frac{1}{2},s}}\varepsilon^{\frac{1}{2}}\|\varepsilon\partial_3 v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq C\|v^h_\Phi\|_{H^{0,s}}\|v^3_\Phi\|_{H^{\frac{1}{2},s}}\varepsilon^{\frac{1}{2}}\|\varepsilon\partial_3 v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq \frac{1}{100}\varepsilon\|\varepsilon\partial_3 v^h_\Phi\|^2_{H^{\frac{1}{2},s}} + C\|v^h_\Phi\|^2_{H^{0,s}}\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}}. \end{aligned} \tag{5.14}$$

In order to deal with the pressure, we write $p = p^1 + p^2 + p^3$ with p^1, p^2, p^3 defined by (5.6). Using the fact that the operator $\partial_i\partial_j(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$ and Lemma 4.3, we have

$$\begin{aligned}\varepsilon(\nabla_h p^1_\Phi, v^h_\Phi)_{H^{-\frac{1}{2},s}} &= -\varepsilon((-\Delta_\varepsilon)^{-1}\partial_i\partial_j(v^iv^j)_\Phi, \varepsilon^{\frac{1}{2}}\mathrm{div}_h v^h_\Phi)_{H^{-\frac{1}{2},s}}\\ &\leq C\varepsilon\|(v^h\otimes v^h)_\Phi\|_{H^{-\frac{1}{2},s}}\|\varepsilon^{\frac{1}{2}}\nabla_h v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq C\varepsilon^{\frac{1}{2}}\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|_{H^{\frac{1}{2},s}}\|v^h_\Phi\|_{H^{0,s}}\|\varepsilon^{\frac{1}{2}}\nabla_h v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq \frac{1}{100}\|\varepsilon^{\frac{1}{2}}\nabla_h v^h_\Phi\|^2_{H^{-\frac{1}{2},s}} + C\varepsilon\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|^2_{H^{\frac{1}{2},s}}\|v^h_\Phi\|^2_{H^{0,s}}. \end{aligned} \tag{5.15}$$

For the term coming from p_2, we integrate by parts to get

$$\varepsilon(\nabla_h p^2_\Phi, v^h_\Phi)_{H^{-\frac{1}{2},s}} = -(\varepsilon\partial_i\partial_3(-\Delta_\varepsilon)^{-1}(v^iv^3)_\Phi, \varepsilon^{\frac{1}{2}}\mathrm{div}_h v^h_\Phi)_{H^{-\frac{1}{2},s}},$$

then note that $\varepsilon\partial_3\partial_i(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$, we get by Lemma 4.3 that

$$\begin{aligned}\varepsilon(\nabla_h p^2_\Phi, v^h_\Phi)_{H^{-\frac{1}{2},s}} &\leq C\|(v^3v^h)_\Phi\|_{H^{-\frac{1}{2},s}}\|\varepsilon^{\frac{1}{2}}\nabla_h v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq C\|v^3_\Phi\|_{H^{\frac{1}{2},s}}\|v^h_\Phi\|_{H^{0,s}}\|\varepsilon^{\frac{1}{2}}\nabla_h v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq C\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}}\|v^h_\Phi\|^2_{H^{0,s}} + \frac{1}{100}\|\varepsilon^{\frac{1}{2}}\nabla_h v^h_\Phi\|^2_{H^{-\frac{1}{2},s}}. \end{aligned} \tag{5.16}$$

Similarly, we have

$$\begin{aligned}\varepsilon(\nabla_h p^3_\Phi, v^h_\Phi)_{H^{-\frac{1}{2},s}} &\leq C\|(v^3\mathrm{div}_h v^h)_\Phi\|_{H^{-\frac{1}{2},s}}\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq C\|v^3_\Phi\|_{H^{\frac{1}{2},s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|_{H^{-\frac{1}{2},s}}\\ &\leq \frac{1}{100}\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}} + C\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|^2_{H^{-\frac{1}{2},s}}. \end{aligned} \tag{5.17}$$

Summing up (5.11)-(5.17), we obtain

$$\begin{aligned}&\frac{d}{dt}\|\varepsilon^{\frac{1}{2}}v^h_\Phi(t)\|^2_{H^{-\frac{1}{2},s}} + \|\varepsilon^{\frac{1}{2}}v^h_\Phi(t)\|^2_{H^{\frac{1}{2},s}} - \frac{1}{50}\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}}\\ &\leq C\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|^2_{H^{-\frac{1}{2},s}} + C\big(\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}} + \|\varepsilon^{\frac{1}{2}}v^h_\Phi\|^2_{H^{\frac{1}{2},s}}\big)\|v^h_\Phi\|^2_{H^{0,s}}. \end{aligned} \tag{5.18}$$

Step 3. Estimate on the function $\theta(t)$

Combining (5.10) with (5.18), we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\|\varepsilon^{\frac{1}{2}}v^h_\Phi(t)\|^2_{H^{-\frac{1}{2},s}} + \|v^3_\Phi(t)\|^2_{H^{-\frac{1}{2},s}}\big) + \big(\|\varepsilon^{\frac{1}{2}}v^h_\Phi(t)\|^2_{H^{\frac{1}{2},s}} + \|v^3_\Phi(t)\|^2_{H^{\frac{1}{2},s}}\big)\\ &\quad\leq C\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\big(\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|^2_{H^{-\frac{1}{2},s}} + \|v^3_\Phi\|^2_{H^{-\frac{1}{2},s}}\big)\\ &\qquad + C\big(\|\varepsilon^{\frac{1}{2}}v^h_\Phi\|^2_{H^{\frac{1}{2},s}} + \|v^3_\Phi\|^2_{H^{\frac{1}{2},s}}\big)\|v^h_\Phi\|^2_{H^{0,s}}, \end{aligned}$$

from which and Gronwall's inequality, it follows that

$$\begin{aligned}
&\|\varepsilon^{\frac12}v^h_\Phi(t)\|^2_{H^{-\frac12,s}}+\|v^3_\Phi(t)\|^2_{H^{-\frac12,s}}+\int_0^t\big(\|\varepsilon^{\frac12}v^h_\Phi(\tau)\|^2_{H^{\frac12,s}}+\|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}\big)d\tau\\
&\leq\exp\big(C\int_0^t\|\nabla_h v^h_\Phi(\tau)\|^2_{H^{0,s}}d\tau\big)\Big[\|e^{a|D_3|}v_0\|^2_{H^{-\frac12,s}}+\\
&\qquad+C\int_0^t\big(\|\varepsilon^{\frac12}v^h_\Phi(\tau)\|^2_{H^{\frac12,s}}+\|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}\big)\|v^h_\Phi(\tau)\|^2_{H^{0,s}}d\tau\Big].
\end{aligned}$$

In particular, we have

$$\theta(t)\leq\exp\big(C\Psi(t)\big)\Big[\|e^{a|D_3|}v_0\|^2_{H^{-\frac12,s}}+C\int_0^t\dot\theta(\tau)\Psi(\tau)d\tau\Big].$$

This finishes the proof of Proposition 3.3. □

6. Regularizing effect du analyticity

Let's now prove Proposition 3.4. Here we will encounter two kinds of bad terms, where we lose a vertical derivative. The first one is $(v^3\partial_3 v^h)_\Phi$. We will see that in an energy estimate, we have no loss of vertical derivative in this term (by integrating by parts, using commutators and of course $\partial_3 v^3=-\mathrm{div}_h v^h$). In the term ∇p, we really lose a vertical derivative.

Step 1. Estimates on the horizontal component v^h_Φ

Let us recall that v^h_Φ verifies the equations

$$\partial_t v^h_\Phi+\lambda\dot\theta(t)|D_3|v^h_\Phi-\Delta_h v^h_\Phi-\varepsilon^2\partial_3^2 v^h_\Phi+\varepsilon^{\frac12}(v\cdot\nabla v^h)_\Phi=-\nabla_h q_\Phi.$$

Note that $\dot\theta\geq0$, we perform an energy estimate in $H^{0,s}$ to obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|v^h_\Phi\|^2_{H^{0,s}}+\lambda\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s+1/2}}+\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\|\varepsilon\partial_3 v^h_\Phi\|^2_{H^{0,s}}\\
&\leq\varepsilon^{\frac12}((v^h\otimes v^h)_\Phi,\nabla_h v^h_\Phi)_{H^{0,s}}-\varepsilon^{\frac12}(\partial_3(v^3v^h)_\Phi,v^h_\Phi)_{H^{0,s}}-(\nabla_h p_\Phi,v^h_\Phi)_{H^{0,s}}\\
&\overset{\mathrm{def}}{=}I+II+III.
\end{aligned}\tag{6.1}$$

We get by Lemma 4.3 and the interpolation that

$$\begin{aligned}
|I|&\leq C\varepsilon^{\frac12}\|(v_h\otimes v^h)_\Phi\|_{H^{0,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\\
&\leq C\varepsilon^{\frac12}\|v^h_\Phi\|_{H^{\frac12,s}}\|v^h_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\\
&\leq C\varepsilon^{\frac12}\|v^h_\Phi\|_{H^{0,s}}\|v^h_\Phi\|_{H^{1,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\\
&\leq C\varepsilon\|v^h_\Phi\|^2_{H^{0,s}}\|v^h_\Phi\|^2_{H^{1,s}}+\frac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}.
\end{aligned}\tag{6.2}$$

In order to estimate II, we use Bony's decomposition (4.3) to rewrite it as

$$II=-\varepsilon^{\frac12}(\partial_3(T^v_{v^h}v_3)_\Phi,v^h_\Phi)_{H^{0,s}}-\varepsilon^{\frac12}(\partial_3(R^v_{v_3}v^h)_\Phi,v^h_\Phi)_{H^{0,s}}\overset{\mathrm{def}}{=}II_1+II_2.$$

From the proof of Lemma 4.2, it is easy to find that

$$\begin{aligned}
|II_2|&\leq C\||D_3|^{\frac12}(R^v_{v^3}v^h)_\Phi\|_{H^{-\frac12,s}}\||\varepsilon D_3|^{\frac12}|D_h|^{\frac12}v^h_\Phi\|_{H^{0,s}}\\
&\leq C\|v^3_\Phi\|_{H^{\frac12,s}}\|v^h_\Phi\|_{H^{0,s+\frac12}}\|\nabla_\varepsilon v^h\|_{H^{0,s}}\\
&\leq C\|v^3_\Phi\|^2_{H^{\frac12,s}}\|v^h_\Phi\|^2_{H^{0,s+\frac12}}+\frac{1}{100}\|\nabla_\varepsilon v^h\|^2_{H^{0,s}}.
\end{aligned}\tag{6.3}$$

Due to $\operatorname{div} v = 0$, we rewrite II_1 as

$$II_1 = \varepsilon^{\frac{1}{2}}((T^v_{v^h}\operatorname{div} v^h)_\Phi, v^h_\Phi)_{H^{0,s}} - \varepsilon^{\frac{1}{2}}((T^v_{\partial_3 v^h} v^3)_\Phi, v^h_\Phi)_{H^{0,s}} \overset{\text{def}}{=} II_{11} + II_{12}.$$

Using Lemma 4.2, we have

$$\begin{aligned} |II_{11}| &\leq \varepsilon^{\frac{1}{2}}\|(T^v_{v^h}\operatorname{div} v^h)_\Phi\|_{H^{-\frac{1}{2},s}}\|v^h_\Phi\|_{H^{\frac{1}{2},s}} \\ &\leq C\varepsilon^{\frac{1}{2}}\|v^h_\Phi\|_{H^{\frac{1}{2},s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|v^h_\Phi\|_{H^{\frac{1}{2},s}} \\ &\leq C\varepsilon\|v^h_\Phi\|^2_{H^{0,s}}\|v^h_\Phi\|^2_{H^{1,s}} + \frac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}. \end{aligned} \tag{6.4}$$

From the proof of Lemma 4.2, we can conclude that

$$\begin{aligned} |II_{12}| &\leq C\|v^3_\Phi\|_{H^{\frac{1}{2},s}}\|v^h_\Phi\|_{H^{0,s+\frac{1}{2}}}\|\nabla_\varepsilon v^h\|_{H^{0,s}} \\ &\leq C\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}}\|v^h_\Phi\|^2_{H^{0,s+\frac{1}{2}}} + \frac{1}{100}\|\nabla_\varepsilon v^h\|^2_{H^{0,s}}. \end{aligned} \tag{6.5}$$

We next turn to the estimate of the pressure. Recall that $p = p^1 + p^2 + p^3$ with p^1, p^2, p^3 defined by (5.6). Using the fact that $(-\Delta_\varepsilon)^{-1}\partial_i\partial_j$ is bounded on $H^{\sigma,s}$ and Lemma 4.3, we get

$$\begin{aligned} (\nabla_h p^1_\Phi, v^h_\Phi)_{H^{0,s}} &= -\varepsilon^{\frac{1}{2}}((-\Delta_\varepsilon)^{-1}\partial_i\partial_j(v^i v^j)_\Phi, \operatorname{div} v^h_\Phi)_{H^{0,s}} \\ &\leq C\varepsilon^{\frac{1}{2}}\|(v^h \otimes v^h)_\Phi\|_{H^{0,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}} \\ &\leq C\varepsilon^{\frac{1}{2}}\|v^h_\Phi\|_{H^{\frac{1}{2},s}}\|v^h_\Phi\|_{H^{\frac{1}{2},s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}} \\ &\leq C\varepsilon\|v^h_\Phi\|^2_{H^{0,s}}\|v^h_\Phi\|^2_{H^{1,s}} + \frac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}. \end{aligned} \tag{6.6}$$

Notice that $\partial_i\partial_j(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$, then exactly as in the estimate of II, we can obtain

$$\begin{aligned} &(\nabla_h p^2_\Phi, v^h_\Phi)_{H^{0,s}} \\ &\leq C\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}}\|v^h_\Phi\|^2_{H^{0,s+\frac{1}{2}}} + C\varepsilon\|v^h_\Phi\|^2_{H^{0,s}}\|v^h_\Phi\|^2_{H^{1,s}} + \frac{1}{100}\|\nabla_\varepsilon v^h\|^2_{H^{0,s}}. \end{aligned} \tag{6.7}$$

We write

$$\nabla_h p_3 = -2\partial_3|D_3|^{-\frac{1}{2}}\big(\nabla_h|D_h|^{\frac{1}{2}}|\varepsilon D_3|^{1/2}(-\Delta_\epsilon)^{-1}\big)|D_h|^{-\frac{1}{2}}(v^3\operatorname{div}_h v^h)$$

thus,

$$(\nabla_h p^3_\Phi, v^h_\Phi)_{H^{0,s}} = -2(\big(\nabla_h|D_h|^{\frac{1}{2}}|\varepsilon D_3|^{\frac{1}{2}}(-\Delta_\epsilon)^{-1}\big)|D_h|^{-\frac{1}{2}}(v^3\operatorname{div}_h v^h), \partial_3|D_3|^{-\frac{1}{2}}v^h_\Phi)_{H^{0,s}}.$$

Note that $\nabla_h|D_h|^{\frac{1}{2}}|\varepsilon D_3|^{\frac{1}{2}}(-\Delta_\epsilon)^{-1}$ is a bounded operator on $H^{\sigma,s}$, we get by Lemma 4.3 that

$$\begin{aligned} (\nabla_h p^3_\Phi, v^h_\Phi)_{H^{0,s}} &\leq C\||D_h|^{-1/2}(v^3\operatorname{div}_h v^h)\|_{H^{0,s}}\|\partial_3|D_3|^{-\frac{1}{2}}v^h_\Phi\|_{H^{0,s}} \\ &\leq C\|v^3_\Phi\|_{H^{\frac{1}{2},s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|v^h\|_{H^{0,s+1/2}} \\ &\leq C\|v^3_\Phi\|^2_{H^{\frac{1}{2},s}}\|v^h_\Phi\|^2_{H^{0,s+1/2}} + \frac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}. \end{aligned} \tag{6.8}$$

Summing up (6.1)-(6.8), we get by taking λ big enough that

$$\frac{d}{dt}\|v^h_\Phi(t)\|^2_{H^{0,s}} + \|\nabla_h v^h_\Phi(t)\|^2_{H^{0,s}} \leq C\|v^h_\Phi\|^2_{H^{0,s}}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}. \tag{6.9}$$

Step 2. Estimates on the vertical component v^3_Φ

Recall that v^3_Φ verifies the equation

$$\partial_t v^3_\Phi + \lambda\dot\theta(t)|D_3|v^3_\Phi - \Delta_h v^3_\Phi - \varepsilon^2\partial_3^2 v^3_\Phi + \varepsilon^{\frac12}(v\cdot\nabla v^3)_\Phi = -\varepsilon^2\partial_3 q_\Phi.$$

We perform an energy estimate in $H^{0,s}$ to obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|v^3_\Phi\|^2_{H^{0,s}} + \|\nabla_h v^3_\Phi\|^2_{H^{0,s}} + \|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{0,s}}\\
&\le -\varepsilon^{\frac12}((v^h\cdot\nabla_h v^3)_\Phi, v^3_\Phi)_{H^{0,s}} + \varepsilon^{\frac12}((v^3\mathrm{div}_h v^h)_\Phi, v^3_\Phi)_{H^{0,s}} - \varepsilon^2(\partial_3 p_\Phi, v^3_\Phi)_{H^{0,s}}\\
&\overset{\text{def}}{=} I + II + III. \qquad (6.10)
\end{aligned}$$

Using Lemma 4.3 and the inerpolation, we have

$$\begin{aligned}
|I| &\le C\varepsilon^{\frac12}\|v^h_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^3_\Phi\|_{H^{0,s}}\|v^3_\Phi\|_{H^{\frac12,s}}\\
&\le C\varepsilon\big(\|v^h_\Phi\|^2_{H^{0,s}}\|v^h_\Phi\|^2_{H^{1,s}} + \|v^3_\Phi\|^2_{H^{0,s}}\|v^3_\Phi\|^2_{H^{1,s}}\big) + \frac{1}{100}\|\nabla_h v^3_\Phi\|^2_{H^{0,s}}, \qquad (6.11)
\end{aligned}$$

and

$$\begin{aligned}
|II| &\le C\varepsilon^{\frac12}\|v^3_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|v^3_\Phi\|_{H^{\frac12,s}}\\
&\le C\varepsilon\|v^3_\Phi\|^2_{H^{0,s}}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}} + \frac{1}{100}\|\nabla_h v^3_\Phi\|^2_{H^{0,s}}. \qquad (6.12)
\end{aligned}$$

Using the decomposition (5.6), we can similarly obtain

$$|III| \le C\varepsilon\|v_\Phi\|^2_{H^{0,s}}\|\nabla_h v_\Phi\|^2_{H^{0,s}} + \frac{1}{100}\|\nabla_h v^3_\Phi\|^2_{H^{0,s}}. \qquad (6.13)$$

Summing up (6.10)-(6.13), we obtain

$$\frac{d}{dt}\|v^3_\Phi\|^2_{H^{0,s}} + \|\nabla_h v^3_\Phi\|^2_{H^{0,s}} \le C\varepsilon\|v_\Phi\|^2_{H^{0,s}}\|\nabla_h v_\Phi\|^2_{H^{0,s}}. \qquad (6.14)$$

Combining (6.9) with (6.14), we get

$$\frac{d}{dt}\|v_\Phi\|^2_{H^{0,s}} + \|\nabla_h v_\Phi\|^2_{H^{0,s}} \le C\|v_\Phi\|^2_{H^{0,s}}\|\nabla_h v_\Phi\|^2_{H^{0,s}},$$

from which and Gronwall's inequality, we infer that

$$\|v_\Phi(t)\|^2_{H^{0,s}} + \int_0^t \|\nabla_h v_\Phi(\tau)\|^2_{H^{0,s}}d\tau \le \|e^{a|D_3|}v_0\|^2_{H^{0,s}}\exp\big(C\int_0^t\|\nabla_h v_\Phi(\tau)\|^2_{H^{0,s}}d\tau\big),$$

that is,

$$\Psi(t) \le \|e^{a|D_3|}v_0\|^2_{H^{0,s}}\exp\big(C\Psi(t)\big).$$

This finishes the proof of Proposition 3.4. □

ACKNOWLEDGEMENT

This work was partly done when Zhifei Zhang was visiting Department of Mathematics of Paris-Sud University as a Postdoctor Fellowship. He would like to thank the hospitality and support of the Department.

References

[1] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. École Norm. Sup., **14** (1981), pages 209-246.

[2] M. Cannone, Y. Meyer and F. Planchon, Solutions autosimilaires des équations de Navier-Stokes, Séminaire "Équations aux Dérivées Partielles" de l'École polytechnique, Exposé VIII, 1993–1994.

[3] J.-Y. Chemin, Le système de Navier-Stokes incompressible soixante dix ans après Jean Leray, Séminaire et Congrès, **9**, 2004, pages 99–123.

[4] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, *Modélisation Mathématique et Analyse Numérique*, **34**, 2000, pages 315–335.

[5] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Mathematical Geophysics: An introduction to rotating fluids and to the Navier-Stokes equations, *Oxford University Press*, 2006.

[6] J.-Y. Chemin and I. Gallagher, On the global wellposedness of the 3-D Navier-Stokes equations with large initial data, *Annales de l'École Normale Supérieure*, **39**, 2006, pages 679–698.

[7] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in $\mathbf{R}^3$ *to appear in Annales de l'Institut Henri Poincaré, Analyse Non Linéaire.*

[8] J.-Y. Chemin and I. Gallagher, Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, *to appear in Transactions of the Americal Mathematical Society.*

[9] J.-Y. Chemin, I. Gallagher and M. Paicu, Global regularity for some classes of large solutions to the Navier-Stokes equations, 2008.

[10] J.-Y. Chemin and P. Zhang, On the global wellposedness of the 3-D incompressible anisotropic Navier-Stokes equations, *Communications in Mathematical Physics*, **272**, 2007, pages 529–566.

[11] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archive for Rational Mechanics and Analysis*, **16**, 1964, pages 269–315.

[12] D. Iftimie, A uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity, *SIAM Journal of Mathematical Analysis*, **33**, 2002), pages 1483–1493.

[13] D. Iftimie, G. Raugel and G.R. Sell, Navier-Stokes equations in thin 3D domains with Navier boundary conditions, *Indiana University Mathematical Journal*, **56**, 2007, pages 1083–1156.

[14] T. Kato, Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions, *Mathematische Zeitschrift*, **187**, 1984, pages 471-480 .

[15] H. Koch and D. Tataru, Well–posedness for the Navier–Stokes equations, *Advances in Mathematics*, **157**, 2001, pages 22–35.

[16] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63**, 1933, pages 193–248.

[17] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *Journal de Mathématiques Pures et Appliquées*, **12**, 1933, pages 1–82.

[18] A. Mahalov and B. Nicolaenko, Global solvability of three-dimensional Navier-Stokes equations with uniformly high initial vorticity, (Russian. Russian summary) *Uspekhi Mat. Nauk* **58**, 2003, pages 79–110; translation in *Russian Math. Surveys* **58**, 2003, pages 287–318.

[19] M.Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Revista Rev. Matemtica Iberoamericana*, **21**, 2005, page 179–235.

[20] J. Pedlovsky, *Geophysical fluid dynamics*, Springer-Verlag, (1979).

[21] G. Raugel and G.R. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions, *Journal of the American Mathematical Society*, **6**, 1993, pages 503–568.

[22] M. Sammartino and R. E. Caflisch, Zero Viscosity Limit for Analytic Solutions, of the Navier-Stokes Equation on a Half-Space. I. Existence for Euler and Prandtl Equations, *Communications in Mathematical Physics*, **192**, 1998, pages 433–461.

[23] M. Ukhovskii and V. Iudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space. *Prikl. Mat. Meh.* 32 59–69 (Russian); translated as *Journal of Applied Mathematics and Mechanics*, **32**, 1968, pages 52–61.

[24] F. Weissler, The Navier-Stokes Initial Value Problem in L^p, *Archiv for Rational Mechanics and Analysis*, **74**, 1980, pages 219-230.

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