# Geometric Invariant Theory and Generalized Eigenvalue Problem II

N. Ressayre

March 6, 2009

**Abstract**

Let $G$ be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. Fix maximal tori and Borel subgroups of $G$ and $\hat{G}$. Consider the cone $\mathcal{LR}^\circ(\hat{G}, G)$ generated by the pairs $(\nu, \hat{\nu})$ of strictly dominant characters such that $V_\nu$ is a submodule of $V_{\hat{\nu}}$. The main result of this article is a bijective parametrisation of the faces of $\mathcal{LR}^\circ(\hat{G}, G)$. We also explain when such a face is contained in another one.

In way, we obtain results about the faces of the Dolgachev-Hu's $G$-ample cone. We also apply our results to reprove known results about the moment polytopes.

## 1 Introduction

Let $G$ be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. Fix maximal tori and Borel subgroups of $G$ and $\hat{G}$. Consider the cone $\mathcal{LR}^\circ(\hat{G}, G)$ generated by the pairs $(\nu, \hat{\nu})$ of strictly dominant characters such that $V_\nu$ is a submodule of $V_{\hat{\nu}}$. This work is a continuation of [Res07]. We obtain results about general GIT-cones and apply it to obtain a bijective parametrisation of the faces of the cone $\mathcal{LR}^\circ(\hat{G}, G)$.

Consider a connected reductive group $G$ acting on a projective variety $X$. To any $G$-linearized line bundle $\mathcal{L}$ on $X$ we associate the following open subset $X^{\rm ss}(\mathcal{L})$ of $X$:

$$X^{\rm ss}(\mathcal{L}) = \left\{ x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0 \right\}.$$

The points of $X^{\rm ss}(\mathcal{L})$ are said to be *semistable* for $\mathcal{L}$. Note that if $\mathcal{L}$ is not ample, this notion of semistability is not the standard one. In particular, the quotient $\pi_{\mathcal{L}} \,:\, X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$ is a good quotient, if $\mathcal{L}$ is ample. In this context, we ask for:

What are the $\mathcal{L}$'s with non empty set $X^{\rm ss}(\mathcal{L})$ ?

Let us fix a freely finitely generated subgroup $\Lambda$ of the group $\rm Pic^G(X)$ of $G$-linearized line bundles on $X$. Let $\Lambda_\mathbb{Q}$ denote the $\mathbb{Q}$-vector space containing $\Lambda$ as a lattice. Consider the convex cones $\mathcal{TC}^G_\Lambda(X)$ (resp. $\mathcal{AC}^G_\Lambda(X)$) generated in $\Lambda_\mathbb{Q}$ by the $\mathcal{L}$'s (resp. the ample $\mathcal{L}$'s) in $\Lambda$ which have non zero $G$-invariant sections. By [DH98] (see also [Res00]), $\mathcal{AC}^G_\Lambda(X)$ is a closed convex rational polyhedral cone in the dominant cone of $\Lambda_\mathbb{Q}$. We are interested in the faces of $\mathcal{AC}^G_\Lambda(X)$ and $\mathcal{TC}^G_\Lambda(X)$.

We need to introduce a definition due to D. Luna. Assume that $X$ is smooth. Let $\mathcal{O}$ be an orbit of $G$ in $X$. For $x\in\mathcal{O}$, we consider the action of the isotropy $G_x$ on the normal space $N_x$ of $\mathcal{O}$ in $X$ at $x$. The pair $(G_x,N_x)$ is called the *type* of the orbit $\mathcal{O}$ and is defined up to conjugacy by $G$. The main part of Theorem 3 is:

**Theorem A** *We assume that $X$ is smooth. Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$.*

*Then, the type of the closed orbit in $\pi_\mathcal{L}^{-1}(\xi)$ for $\xi\in X^{\rm ss}(\mathcal{L})/\!/G$ general does not depends on the choice of an ample $G$-linearized line bundle $\mathcal{L}$ in the relative interior of $\mathcal{F}$.*

*We will call this type the type of $\mathcal{F}$.*

Let $\mathcal{F}$ be a face of $\mathcal{AC}^G(X)$. Let $\mathcal{L}_0$ be any point in the relative interior of $\mathcal{F}$. The local geometry $\mathcal{AC}^G(X)$ around $\mathcal{F}$ is described by the convex cone $\mathcal{C}_\mathcal{F}$ generated by the vectors $p-\mathcal{L}_0$ for $p\in\mathcal{AC}^G(X)$.

We now introduce some notation to describe this cone. Consider the quotient $\pi\,:\,X^{\rm ss}(\mathcal{L})\longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$. Let $x$ be any point in $X^{\rm ss}(\mathcal{L})$ with closed orbit in $X^{\rm ss}(\mathcal{L})$, and so, reductive isotropy $G_x$. Then, the fiber $\pi^{-1}(\pi(x))$ is isomorphic to a fiber product $G\times_{G_x}L$, for an affine $G_x$-variety $L$ with a fixed point as unique closed orbit. Let $X(G_x)$ denote the group of characters of $G_x$. Consider the semicone $\mathcal{C}_x$ in $X(G_x)\otimes\mathbb{Q}$ generated by the weights of $G_x$ on the set of regular functions on $L$. Finally, we consider is the linear map $\mu\,:\,\Lambda_\mathbb{Q}\longrightarrow X(G_x)\otimes\mathbb{Q}$ obtained by considering the action of $G_x$ on the fibers $\mathcal{L}_x$ in $\mathcal{L}\in\Lambda$ over $x$.

**Theorem B** *With above notation, we have:*

$$\mathcal{C}_\mathcal{F}=\mu^{-1}(\mathcal{C}_x).$$

In the symplectic setting, S. Sjamaar obtained a description of the local structure of the moment polytope (see [Sja98]) which is closed from Theorem B. In Sjamaar's situation, $G_x^\circ$ is a torus which simplifies a little bit.

Now, assume that the variety $X$ equals $Y \times G/B$, for a $G$-variety $Y$. Let $\mathcal{L}$ be an ample $G$-linearized line bundle on $Y$. Let $\Lambda$ be the subgroup of $\mathrm{Pic^G(X)}$ generated by the pullback of $\mathcal{L}$ and the pullbacks of the $G$-linearized line bundles on $G/B$. Then, $\mathcal{TC}^G_\Lambda(X)$ is a cone over the moment polytope $P(Y,\mathcal{L})$ defined in [Bri99]; in particular, the faces of $\mathcal{TC}^G_\Lambda(X)$ correspond bijectively to the faces of $P(Y,\mathcal{L})$.

Following [Bri99], we show in Proposition 7 below, that any moment polytope $P(Y,\mathcal{L})$ can be describe in terms of one which intersects the interior of the dominant chamber. We now assume that $P(Y,\mathcal{L})$ intersects the interior of the dominant chamber and that $Y$ is smooth. In Proposition 8 below, we associate to each face of $P(Y,\mathcal{L})$ which intersects the interior of the dominant chamber a well $B$-covering pair (see Definition 7.2) of $Y$ improving (with stronger assumptions) [Bri99, Theorem 1 and 2].

Now, $G$ is assumed to be embedded in another connected reductive group $\hat{G}$. We fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of $G$ and $\hat{G}$. Consider the diagonal action of $G$ on $\hat{G}/\hat{B} \times G/B$ and the associated GIT-cone $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$. Actually, $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ identifies with the cone generated by pairs $(\nu, \hat{\nu})$ of strictly dominant character of $T \times \hat{T}$ such that the dual of the $G$-module associated to $\nu$ can be $G$-equivariantly embedded in the $\hat{G}$-module associated to $\hat{\nu}$. The interior of $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ is non empty if and only if no non trivial connected normal subgroup of $G$ is normal in $\hat{G}$: we assume, from now on that $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ has non empty interior. Theorem 5 below gives a bijective parametrisation of the faces of $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$. Moreover, we can read very easily the inclusions between faces using this parametrisation. To avoid too many notation, in this introduction, we will only state our results in the case when $\hat{G} = G^2$.

For any standard parabolic subgroup $P$ of $G$, we consider the cohomology group $\mathrm{H}^*(G/P, \mathbb{Z})$ and its basis consisting in classes of Schubert varieties. We consider on this group the Belkale-Kumar product $\odot_0$ defined in [BK06]. The coefficient-structure of this product in this basis are either 0 or the coefficient-structure of the usual cup product. These coefficients are parametrized by the triple of Schubert classes.

**Theorem C** *The group $G$ is assumed to be semi-simple. The set of faces of $\mathcal{AC}^G((G/B)^3)$ correspond bijectively to the set of structure coefficient of $(\mathrm{H}^*(G/P, \mathbb{Z}), \odot_0)$ equal to one, for the various standard parabolic subgroups $P$ of $G$.*

We will now explain how to read the inclusion off this parametrization. Let $P$ and $P'$ be two standard parabolic subgroups. Let $\Lambda_1, \Lambda_2$ and $\Lambda_3$

(resp. $\Lambda'_1, \Lambda'_2$ and $\Lambda'_3$) three Schubert varieties in $G/P$ (resp. $G/P'$) such the corresponding coefficients structure for $\odot_0$ equal to one. Let $\mathcal{F}$ and $\mathcal{F}'$ denote the corresponding faces of $\mathcal{AC}^G((G/B)^3)$.

**Theorem D** *Let $\mathcal{F}$ and $\mathcal{F}'$ be two faces of $\mathcal{AC}^G((G/B)^3)$. The following are equivalent:*

*(i)* $\mathcal{F} \subset \mathcal{F}'$*;*

*(ii)* $P \subset P'$ *and* $\pi(\Lambda_i) = \Lambda'_i$ *for* $i = 1, 2$ *and* $3$*, where* $\pi : G/P \longrightarrow G/P'$ *is the* $G$*-equivariant morphism mapping* $P/P$ *on* $P'/P'$*.*

**Convention.** The ground field $\mathbb{K}$ is assumed to be algebraically closed of characteristic zero. The notation introduced in the environments "**Notation.**" are fixed for all the sequence of the article.

## 2 An example of GIT-cone

Let us fix a connected reductive group $G$ acting on an irreducible projective algebraic variety $X$.

### 2.1 An Ad Hoc notion of semistability

As in the introduction, for any $G$-linearized line bundle $\mathcal{L}$ on $X$, we consider the following set of *semistable points*:

$$X^{\rm ss}(\mathcal{L}) = \left\{ x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0 \right\}.$$

To precise the acting group, we sometimes denote $X^{\rm ss}(\mathcal{L})$ by $X^{\rm ss}(\mathcal{L}, G)$.

The subset $X^{\rm ss}(\mathcal{L})$ is open and stable by $G$. Note that this definition of $X^{\rm ss}(\mathcal{L})$ is the standard one only when $\mathcal{L}$ is ample. Indeed, one usually imposes that the open subset defined by the non vanishing of $\sigma$ to be affine.

If $\mathcal{L}$ is ample, there exists a categorical quotient:

$$\pi \,:\, X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G,$$

such that $X^{\rm ss}(\mathcal{L})/\!/G$ is a projective variety and $\pi$ is affine. A point $x \in X^{\rm ss}(\mathcal{L})$ is said to be *stable* if $G_x$ is finite and $G.x$ is closed in $X^{\rm ss}(\mathcal{L})$. Then, for all stable point $x$ we have $\pi^{-1}(\pi(x)) = G.x$; and the set $X^{\rm s}(\mathcal{L})$ of stable points is open in $X$.

## 2.2 Definitions

Let us recall from the introduction that $\Lambda$ is a freely finitely generated subgroup of $\mathrm{Pic^G(X)}$ and $\Lambda_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space containing $\Lambda$ as a lattice. Since $X^{\rm ss}(\mathcal{L}) = X^{\rm ss}(\mathcal{L}^{\otimes n})$, for any $G$-linearized line bundle and any positive integer $n$, we can define $X^{\rm ss}(\mathcal{L})$ for any $\mathcal{L} \in \Lambda_{\mathbb{Q}}$. We consider the following *total $G$-cone*:

$$\mathcal{TC}^G_\Lambda(X) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} \ : \ X^{\rm ss}(\mathcal{L}) \text{ is not empty}\}.$$

Since the tensor product of two non zero $G$-invariant sections is a non zero $G$-invariant section, $\mathcal{TC}^G_\Lambda(X)$ is a convex cone.

Consider the convex cones $\Lambda^+_{\mathbb{Q}}$ and $\Lambda^{++}_{\mathbb{Q}}$ generated respectively by the semiample and ample elements of $\Lambda$. For all $\mathcal{L} \in \Lambda^+_{\mathbb{Q}}$ (resp. $\Lambda^{++}_{\mathbb{Q}}$), there exists a positive integer $n$ such that $\mathcal{L}^{\otimes n}$ is a semiample (resp. ample) $G$-linearized line bundle on $X$ in $\Lambda$. So, any set of semistable points associated to a point in $\Lambda^+_{\mathbb{Q}}$ (resp. $\Lambda^{++}_{\mathbb{Q}}$) is in fact a set of semistable point associated to a semiample (resp. ample) $G$-linearized line bundle. We consider the following *semiample and ample $G$-cones*:

$$\mathcal{SAC}^G_\Lambda(X) = \mathcal{TC}^G_\Lambda(X) \cap \Lambda^+_{\mathbb{Q}} \quad \text{and} \quad \mathcal{AC}^G_\Lambda(X) = \mathcal{TC}^G_\Lambda(X) \cap \Lambda^{++}_{\mathbb{Q}}.$$

By [DH98] (see also [Res00]), $\mathcal{AC}^G_\Lambda(X)$ is a closed convex rational polyhedral cone in $\Lambda^{++}_{\mathbb{Q}}$. This cone is the central object of this article.

Two points $\mathcal{L}$ and $\mathcal{L}'$ in $\mathcal{AC}^G_\Lambda(X)$ are said to be *GIT-equivalent* if $X^{\rm ss}(\mathcal{L}) = X^{\rm ss}(\mathcal{L}')$. An equivalence class is simply called a *GIT-class*.

For $x \in X$, the *stability set of $x$* is the set of $\mathcal{L} \in \Lambda^{++}_{\mathbb{Q}}$ such that $X^{\rm ss}(\mathcal{L})$ contains $x$; it is denoted by $\Omega_\Lambda(x)$ or $\Omega_\Lambda(G.x)$. In [Res00], we have studied the geometry of the GIT-classes and the stability sets with lightly different assumptions (no $\Lambda$ for example). However all the results and proofs of [Res00] remain valuable here. In particular, there are only finitely many GIT-classes; and each GIT-class is the relative interior of a closed convex polyhedral cone of $\Lambda^{++}_{\mathbb{Q}}$.

## 2.3 An example of $G$-ample cone

**Notation.** If $\Gamma$ is an affine algebraic group, $[\Gamma, \Gamma]$ will denote its derived subgroup and $X(\Gamma)$ will denote the character group of $\Gamma$.

For later use, we consider here a $G$-ample cone for the action of $G$ over an affine variety. More precisely, let $V$ be an affine $G$-variety containing a fix point $O$ as unique closed orbit. The action of $G$ over the fiber gives a morphism $\mu^\bullet(O,G)\ :\ \mathrm{Pic^G(V)}\longrightarrow \mathrm{X(G)}$. By [Res00, Lemma 7], $\mu^\bullet(O,G)$ is an isomorphism. We denote by $V^{\rm ss}(\chi)$ the set of semistable points for the $G$-linearized line bundle $\mathcal{L}_\chi$ associated to $\chi\in X(G)$; that is, the trivial line bundle on $V$ linearized by $\chi$. As in the projective case, we consider the $G$-ample cone $\mathcal{AC}^G(V)$ in $X(G)\otimes\mathbb{Q}$.

For any $\chi\in X(G)$, we have:

$$\mathrm{H}^0(V,\mathcal{L}_\chi)^G=\{f\in\mathbb{K}[V]\ :\ \forall x\in V\quad (g.f)(x)=\chi(g)f(x)\}=\mathbb{K}[V]_\chi.$$

Note that $\mathrm{H}^0(V,\mathcal{L}_\chi)^G$ is contained in $\mathbb{K}[V]^{[G,G]}$. Set

$$S=\{\chi\in X(G)\ :\ \mathrm{H}^0(V,\mathcal{L}_\chi)^G\quad\text{is non trivial}\}.$$

It is the set of weights of $G/[G,G]$ in $\mathbb{K}[V]^{[G,G]}$. We have:

**Lemma 1** *We assume that $V$ is irreducible. The set $S$ is a finitely generated semigroup in $X(G)$. Moreover, $\mathcal{AC}^G(V)$ is the convex cone generated by $S$; it is strictly convex.*

**Proof.** Since $\mathbb{K}[V]^{[G,G]}$ is a finitely generated algebra, $S$ is a finitely generated semigroup. The fact that $\mathcal{AC}^G(V)$ is generated by $S$ is obvious. Finally, $\mathcal{AC}^G(V)$ is strictly convex since $\mathrm{H}^0(V,\mathcal{L}_0)^G=\mathbb{K}$. □

# 3 Slice Etale Theorem

In this section, we recall some very useful results of D. Luna. We fix an ample $G$-linearized line bundle $\mathcal{L}$ on the irreducible projective $G$-variety $X$.

## 3.1 Closed orbits in general position

**Notation.** If $H$ is a subgroup of $G$, $N_G(H)$ denotes the normalizer of $H$ in $G$. Consider the quotient $\pi\,:\,X^{\rm ss}(\mathcal{L})\longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$. For all $\xi\in X^{\rm ss}(\mathcal{L})/\!/G$, we denote by $T(\xi)$ the unique closed orbit of $G$ in $\pi^{-1}(\xi)$. We denote by $(X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}$ the set of $\xi$ such that there exists an open neighborhood $\Omega$ of $\xi$ in $X^{\rm ss}(\mathcal{L})/\!/G$ such that the orbits $T(\xi')$ are isomorphic to $T(\xi)$, for all $\xi'\in\Omega$.

Since $\pi$ is a gluing of affine quotients, some results on the actions of $G$ on affine variety remains true for $X^{\rm ss}(\mathcal{L})$. For example, the following theorem is a result of Luna and Richardson (see [LR79, Section 3] and [Lun75, Corollary 4] or [PV91, Section 7]):

**Theorem 1** *With above notation, if $X$ is normal, we have:*

*(i)* *The set $(X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$ is a non empty open subset of $X^{\mathrm{ss}}(\mathcal{L})/\!/G$. Let $H$ be the isotropy of a point in $T(\xi_0)$ with $\xi_0 \in (X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$. The group $H$ has fixed points in $T(\xi)$ for any $\xi \in X^{\mathrm{ss}}(\mathcal{L})/\!/G$.*

*(ii)* *Let $Y$ be the closure of $\pi^{-1}\left((X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}\right)^H$ in $X$. It is the union of some components of $X^H$. Then, $H$ acts trivially on some positive power $\mathcal{L}_{|Y}^{\otimes n}$ of $\mathcal{L}_{|Y}$. Moreover, the natural map*

$$Y^{\mathrm{ss}}(\mathcal{L}_{|Y}^{\otimes n})/\!/(N_G(H)/H) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G$$

*is an isomorphism. Moreover, $Y$ contains stable points for the action of $N_G(H)/H$ and the line bundle $\mathcal{L}_{|Y}^{\otimes n}$.*

A subgroup $H$ as in Theorem 1 will be called *a generic closed isotropy of $X^{\mathrm{ss}}(\mathcal{L})$*. The conjugacy class of $H$ which is obviously unique is called *the generic closed isotropy of $X^{\mathrm{ss}}(\mathcal{L})$*.

## 3.2 The principal Luna stratum

When $X$ is smooth, the open subset $(X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$ is called the *principal Lana stratum* and has very useful properties (see [Lun73] or [PV91]):

**Theorem 2 (Luna)** *We assume that $X$ is smooth. Let $H$ be a generic closed isotropy of $X^{\mathrm{ss}}(\mathcal{L})$.*

*Then, there exists a $H$-module $L$ such that for any $\xi \in (X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$, there exist points $x$ in $T(\xi)$ satisfying:*

*(i)* *$G_x = H$;*

*(ii)* *the $H$-module $T_xX/T_x(G.x)$ is isomorphic to the sum of $L$ and its fix points, in particular, it is independent of $\xi$ and $x$;*

*(iii)* *for any $v \in L$, $0$ belongs to the closure of $H.v$;*

*(iv)* *the fiber $\pi^{-1}(\xi)$ is isomorphic to $G \times_H L$.*

## 3.3 The fibers of quotient morphisms

Another useful consequence of Luna's Slice Etale Theorem is (see [Lun73] or [PV91]):

**Proposition 1** *Let $x$ be a semistable point for $\mathcal{L}$ whose the orbit is closed in $X^{\rm ss}(\mathcal{L})$. Then, there exists an affine $G_x$-variety $V$ containing a unique closed orbit which is a fixed point and such that $\pi^{-1}(\pi(x))$ is isomorphic to $G\times_{G_x} V$.*

# 4 About faces of the $G$-ample cone

## 4.1 Isotropy subgroups associated to faces of $\mathcal{AC}^G_\Lambda(X)$

Let $\varphi$ be a linear form on $\Lambda_\mathbb{Q}$ which is non negative on $\mathcal{AC}^G_\Lambda(X)$. If the set of $\mathcal{L}\in\mathcal{AC}^G_\Lambda(X)$ such that $\varphi(\mathcal{L})=0$ is non empty it will be called *a face of* $\mathcal{AC}^G_\Lambda(X)$. Now, we associate two invariants to a face $\mathcal{F}$ of $\mathcal{AC}^G_\Lambda(X)$.

**Theorem 3** *Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$. Let $\mathcal{L}$ be a point in the relative interior of $\mathcal{F}$. Then, we have:*

*(i) The generic closed isotropy of $X^{\rm ss}(\mathcal{L})$ does not depend on the point $\mathcal{L}$ in the relative interior of $\mathcal{F}$, but only in $\mathcal{F}$. We call this isotropy the generic closed isotropy of $\mathcal{F}$.*

*Let us fix a generic closed isotropy $H$ of $\mathcal{F}$.*

*(ii) For any $\mathcal{M}\in\mathcal{F}$, $H$ fixes points in any closed orbit of $G$ in $X^{\rm ss}(\mathcal{M})$.*

*(iii) The closure $Y$ of $\left(\pi_{\mathcal{L}}^{-1}\left((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}\right)\right)^H$ in $X$ does not depends on a choice of $\mathcal{L}$. Let $Y_\mathcal{F}$ denote this subvariety of $X^H$; it is the union of some components of $X^H$.*

*(iv) The group $H$ acts trivially on some positive power $\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}}$ of $\mathcal{L}_{|Y_\mathcal{F}}$. Moreover, the natural map*

$$Y^{\rm ss}_\mathcal{F}(\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}})/\!/(N_G(H)/H)\longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$$

*is an isomorphism. Moreover, $Y_\mathcal{F}$ contains stable points for the action of $N_G(H)/H$ and the line bundle $\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}}$.*

*(v) Set $Y^+_\mathcal{F}:=\{x\in X\ :\ \overline{H.x}\cap Y_\mathcal{F}\neq\emptyset\}$. Then $G.Y^+_\mathcal{F}$ contains an open subset of $X$.*

**Proof.** Let $\mathcal{L}_1$, $\mathcal{L}_2\in\mathcal{AC}^G_\Lambda(X)$. By an easy argument of convexity, to prove Assertion (i) it is sufficient to prove that the generic closed isotropy of $X^{\rm ss}(\mathcal{L})$ does not depend on $\mathcal{L}$ in the open segment $]\mathcal{L}_1,\mathcal{L}_2[$. Let us fix $\mathcal{L}$, $\mathcal{L}'\in$

$]\mathcal{L}_1, \mathcal{L}_2[$. Let $x \in X$ which maps in $(X^{\rm ss}(\mathcal{M})/\!/G)_{\rm pr}$, for $\mathcal{M} = \mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$ and $\mathcal{L}'$ by the quotient maps.

Recall that $\Omega_\Lambda(x)$ is a polyhedral convex cone. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to $\Omega_\Lambda(x)$, $\mathcal{L}$ and $\mathcal{L}'$ belong to the relative interior of the same face of $\Omega_\Lambda(x)$. By [Res00, Proposition 6, Assertion (iii)], there exists $x' \in \overline{G.x}$ such that this face is $\Omega_\Lambda(x')$. But, [Res00, Proposition 6, Assertion (i)] shows that the closed orbits in $X^{\rm ss}(\mathcal{L}) \cap \overline{G.x'}$ and $X^{\rm ss}(\mathcal{L}') \cap \overline{G.x'}$ are equal. Now, our choice of the point $x$ implies that the generic closed isotropies of $X^{\rm ss}(\mathcal{L})$ and $X^{\rm ss}(\mathcal{L}')$ are equal.

Let $H$ be a generic closed isotropy of $X^{\rm ss}(\mathcal{L})$. Let $Y$ be the closure of $X^H \cap \pi_{\mathcal{L}}^{-1}((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr})$. By Theorem 1, $N_G(H)$ acts transitively on the set of irreducible components of $Y$. Let $Y_1$ be such a component of $X^H$. Again by Theorem 1, $\pi_{\mathcal{L}}(Y_1 \cap X^{\rm ss}(\mathcal{L})) = X^{\rm ss}(\mathcal{L})/\!/G$; that is, any closed $G$-orbit in $X^{\rm ss}(\mathcal{L})$ intersects $Y_1$. Finally, $Y$ is the union of irreducible components of $X^H$ which intersect a general closed $G$-orbit in $X^{\rm ss}(\mathcal{L})$. But, the above proof of Assertion (i) shows that a general closed orbit in $X^{\rm ss}(\mathcal{L})$ is also a closed orbit in $X^{\rm ss}(\mathcal{L}')$. In particular, $Y$ is the closure of $X^H \cap \pi_{\mathcal{L}'}^{-1}((X^{\rm ss}(\mathcal{L}')/\!/G)_{\rm pr})$. Assertion (iii) follows.

Let us now fix a generic closed isotropy $H$ of $\mathcal{F}$. Let $\mathcal{M}_1 \in \mathcal{F}$. By Assertion (i) of Theorem 1, to prove the second assertion, it is sufficient to prove that the generic closed isotropy of $X^{\rm ss}(\mathcal{M}_1)$ contains $H$. By [Res00, Theorem 4], there exists a point $\mathcal{M}_2$ in the relative interior of $\mathcal{F}$ such that $X^{\rm ss}(\mathcal{M}_1)$ contains $X^{\rm ss}(\mathcal{M}_2)$. The inclusion $X^{\rm ss}(\mathcal{M}_2) \subset X^{\rm ss}(\mathcal{M}_1)$ induces a surjective morphism $\eta : X^{\rm ss}(\mathcal{M}_2)/\!/G \longrightarrow X^{\rm ss}(\mathcal{M}_1)/\!/G$. Let $\xi' \in (X^{\rm ss}(\mathcal{M}_2)/\!/G)_{\rm pr}$ such that $\xi = \eta(\xi') \in (X^{\rm ss}(\mathcal{M}_1)/\!/G)_{\rm pr}$. Let $x$ be a point in the closed $G$-orbit in $X^{\rm ss}(\mathcal{M}_1)$ over $\xi$. The fiber in $X^{\rm ss}(\mathcal{M}_1)$ over $\xi$ is fibered over $G.x$; hence, for any $y$ in this fiber, $G_y$ is conjugated to a subgroup of $G_x$. Since this fiber contains the fiber in $X^{\rm ss}(\mathcal{M}_2)$ over $\xi'$, $H$ is conjugated to a subgroup of $G_x$. The second assertion is proved.

From now on, $\mathcal{L}$ is a point in the relative interior of $\mathcal{F}$. Let $Y$ be the subvariety of $X^H$ of Assertion (iii). By Theorem 1, $Y$ satisfies Assertion (iv). Moreover, $G.Y^+$ contains $\pi_{\mathcal{L}}^{-1}((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr})$; and, Assertion (v) is proved.
$\square$

### 4.2 Local structure of the $G$-ample cone around a face

**Notation.** Let $E$ be a prime Cartier divisor on a variety $X$ endowed with a line bundle $\mathcal{L}$ and $\sigma$ be a rational section for $\mathcal{L}$. We will denote by $\nu_E(\sigma)\in\mathbb{Z}$ the order of zero of $\sigma$ along $E$.

Let $\mathcal{P}$ be a polyhedron in a rational vector space $V$ and $\mathcal{F}$ be a face of $\mathcal{P}$. Let $x$ in the relative interior of $\mathcal{F}$. The cone of $V$ generated by the vectors $y-x$ for $y\in\mathcal{P}$ does not depends on the choice of $x$ in the relative interior of $\mathcal{F}$ and will be called *the cone of* $\mathcal{P}$ *viewed from* $\mathcal{F}$. This cone carries the geometry of $\mathcal{P}$ in a neighborhood of $x$.

Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$. Let $\mathcal{L}$ in the relative interior of $\mathcal{F}$. Let $x$ be a semistable point for $\mathcal{L}$ whose the orbit is closed in $X^{\rm ss}(\mathcal{L})$. Let $V$ be an affine $G_x$-variety satisfying Proposition 1. Consider the cone $\mathcal{AC}^{G_x}(V)$ as in Section 2.3. Notice that $V$ is not necessarily irreducible, and so $\mathcal{C}^{G_x}(V)$ is not necessarily convex.

The action of $G_x$ on the fiber over $x$ defines a morphism $\mu^\bullet(x,G_x)\,:\,\Lambda\longrightarrow X(G_x)$; and so, a linear map from $\Lambda_\mathbb{Q}$ on $X(G_x)_\mathbb{Q}$ also denoted by $\mu^\bullet(x,G_x)$.

**Theorem 4** *With above notation, the cone of $\mathcal{AC}^G(X)$ viewed from $\mathcal{F}$ equals $(\mu^\bullet(x,G_x))^{-1}(\mathcal{AC}^{G_x}(V))$. In particular, if $\mu^\bullet(x,G_x)$ is surjective then $\mathcal{AC}^{G_x}(V)$ is convex.*

**Proof.** Let $\mathcal{L}_0$ and $\mathcal{L}$ be two different ample $G$-linearized line bundles in $\Lambda$. We assume that $\mathcal{L}_0$ is the only point in the segment $[\mathcal{L};\mathcal{L}_0]$ which belongs to $\mathcal{AC}^G_\Lambda(X)$. For convenience, we set $U=X^{\rm ss}(\mathcal{L}_0)$. By assumption, there is no $G$-invariant rational section of $\mathcal{L}$ which is regular on $X$; we claim that there is no such section which is regular on $U$.

Let us prove the claim. Let us fix a non zero regular $G$-invariant section $\sigma_0$ of $\mathcal{L}_0^{\otimes m}$ for a positive integer $m$. Let $\sigma$ be a $G$-invariant rational section of $\mathcal{L}$ which is regular on $U$. For any positive integer $k$, $\sigma\otimes\sigma_0^{\otimes k}$ is a rational $G$-invariant section of $\mathcal{L}\otimes\mathcal{L}_0^{\otimes mk}$ which is regular on $U$. Let $E$ be an irreducible component of codimension one of $X-U$. By definition of $U$, $\sigma_0$ is zero along $E$; and, $\nu_E(\sigma_0)>0$. Then, $\nu_E(\sigma\otimes\sigma_0^{\otimes k})=\nu_E(\sigma)+k.\nu_E(\sigma_0)$ is positive for $k$ big enough. We deduce that $\sigma\otimes\sigma_0^{\otimes k}$ is regular on $X$ for $k$ big enough. Since by assumption $\mathcal{L}\otimes\mathcal{L}_0^{\otimes mk}$ does not belong to $\mathcal{AC}^G_\Lambda(X)$, this implies that $\sigma\otimes\sigma_0^{\otimes k}$ and finally $\sigma$ are zero. The claim is proved.

We now fix a point $\mathcal{L}_0$ in the relative interior of $\mathcal{F}$ as in the statement. By an elementary argument of convexity, there exists an open neighborhood

$\Omega$ of $\mathcal{L}_0$ in $\Lambda_{\mathbb{Q}}$ such that

(i) for any $\mathcal{L}\in\Omega$, if $\mathcal{L}$ does not belong to $\mathcal{AC}^G_\Lambda(X)$ then $\mathcal{L}_0$ is the only point in $[\mathcal{L},\mathcal{L}_0]\cap\mathcal{AC}^G_\Lambda(X)$.

By [Res00, Proposition 2.3], we may also assume that for all $\mathcal{L}\in\Omega$, $X^{\rm ss}(\mathcal{L})$ is contained in $X^{\rm ss}(\mathcal{L}_0)$. It remains to prove that for any $\mathcal{L}\in\Omega$, $\mathcal{L}\in\mathcal{AC}^G_\Lambda(X)$ if and only if $\mu^{\mathcal{L}}(x,G_x)\in\mathcal{AC}^{G_x}(V)$.

Let $\mathcal{L}\in\Omega$ which does not belongs to $\mathcal{AC}^G_\Lambda(X)$. Set $\xi=\pi_{\mathcal{L}_0}(x)$. By the beginning of the proof, for any positive $n$, $\mathrm{H}^0(U,\mathcal{L}^{\otimes n})^G=\{0\}$. Since $\pi_{\mathcal{L}_0}^{-1}(\xi)$ is closed in $U$, this implies that $\mathrm{H}^0(\pi_{\mathcal{L}_0}^{-1}(\xi),\mathcal{L}^{\otimes n})^G=\{0\}$ for all positive $n$. So, for all positive $n$, $\mathrm{H}^0(V,\mathcal{L}^{\otimes n}_{|V})^{G_x}=\{0\}$; and, so $\mu^{\mathcal{L}}(x,G_x)$ does not belong to $\mathcal{AC}^{G_x}(V)$.

Let now $\mathcal{L}\in\Omega\cap\mathcal{AC}^G_\Lambda(X)$. Since the map $\phi\,:\,X^{\rm ss}(\mathcal{L})/\!/G\longrightarrow X^{\rm ss}(\mathcal{L}_0)/\!/G$ induced by the inclusion $X^{\rm ss}(\mathcal{L})\subset X^{\rm ss}(\mathcal{L}_0)$ is surjective, there exists $y\in X^{\rm ss}(\mathcal{L})$ such that $\phi\circ\pi_{\mathcal{L}}(y)=\xi$. Up to changing $y$ by $g.y$ for some $g\in G$, one may assume that $y\in V$. Let $\sigma$ be a $G$-invariant section of $\mathcal{L}$ which is non zero at $y$. Obviously, the restriction of $\sigma$ is a $G_x$-invariant section of $\mathcal{L}_{|V}$ which is non zero. It follows that $\mu^{\mathcal{L}}(x,G_x)$ belongs to $\mathcal{AC}^{G_x}(V)$.

The last assertion follows from an obvious argue of convexity. ☐

# 5 Well covering pairs

## 5.1 The functions $\mu^\bullet(x,\lambda)$

Let $\mathcal{L}\in\mathrm{Pic^G(X)}$. Let $x$ be a point in $X$ and $\lambda$ be a one parameter subgroup of $G$. Since $X$ is complete, $\lim_{t\to0}\lambda(t)x$ exists; let $z$ denote this limit. The image of $\lambda$ fixes $z$ and so the group $\mathbb{K}^*$ acts via $\lambda$ on the fiber $\mathcal{L}_z$. This action defines a character of $\mathbb{K}^*$, that is, an element of $\mathbb{Z}$ denoted by $\mu^{\mathcal{L}}(x,\lambda)$.

The numbers $\mu^{\mathcal{L}}(x,\lambda)$ are used in [MFK94] to give a numerical criterion for stability with respect to an ample $G-$linearized line bundle $\mathcal{L}$:

$$x\in X^{\rm ss}(\mathcal{L})\iff\mu^{\mathcal{L}}(x,\lambda)\leq0\text{ for all one parameter subgroup }\lambda,$$
$$x\in X^{\rm s}(\mathcal{L})\iff\mu^{\mathcal{L}}(x,\lambda)<0\text{ for all non trivial }\lambda.$$

## 5.2 Definition

**Notation.** The set of fix points of the image of $\lambda$ will be denoted by $X^\lambda$; the centralizer of this image will be denoted by $G^\lambda$. We consider the following parabolic subgroup of $G$:

$$P(\lambda)=\left\{g\in G\ :\ \lim_{t\to 0}\lambda(t).g.\lambda(t)^{-1}\text{ exists in }G\right\}.$$

Let $C$ be an irreducible component of $X^\lambda$. Since $G^\lambda$ is connected, $C$ is a $G^\lambda$-stable closed subvariety of $X$. We set:

$$C^+:=\{x\in X\ :\ \lim_{t\to 0}\lambda(t)x\in C\}.$$

Then, $C^+$ is a locally closed subvariety of $X$ stable by $P(\lambda)$. Moreover, the map $p_\lambda\ :\ C^+\longrightarrow C$, $x\longmapsto\lim_{t\to 0}\lambda(t)x$ is a morphism satisfying:

$$\forall(l,u)\in G^\lambda\times U(\lambda)\qquad p_\lambda(lu.x)=lp_\lambda(x).$$

Consider over $G\times C^+$ the action of $G\times P(\lambda)$ given by the formula (with obvious notation):

$$(g,p).(g',y)=(gg'p^{-1},py).$$

Since the quotient map $G\longrightarrow G/P(\lambda)$ is a Zariski-locally trivial principal $P(\lambda)$-bundle; one can easily construct a quotient $G\times_{P(\lambda)}C^+$ of $G\times C^+$ by the action of $\{e\}\times P(\lambda)$. The action of $G\times\{e\}$ induces an action of $G$ on $G\times_{P(\lambda)}C^+$.

**Definition.** Consider the following $G$-equivariant map

$$\begin{array}{cccc}\eta\ : & G\times_{P(\lambda)}C^+ & \longrightarrow & X\\ & [g:x] & \longmapsto & g.x.\end{array}$$

The pair $(C,\lambda)$ is said to be *covering* (resp. *dominant*) if $\eta$ is birational (resp. dominant). It is said to be *well covering* if $\eta$ induces an isomorphism from $G\times_{P(\lambda)}\Omega$ onto a $P(\lambda)$-stable open subset of $X$ for an open subset $\Omega$ of $C^+$ intersecting $C$.

### 5.3 Face associated to $(C,\lambda)$

Let us denote by $\mu^\bullet(C,\lambda)$, the common value of the $\mu^\bullet(x,\lambda)$, for $x\in C^+$.

**Lemma 2** *Let $(C,\lambda)$ be a dominant pair. The set of $\mathcal{L}\in\mathcal{AC}^G_\Lambda(X)$ such that $\mu^{\mathcal{L}}(C,\lambda)=0$ is either empty or a face $\mathcal{F}$ of $\mathcal{AC}^G_\Lambda(X)$. Moreover, $\mathcal{F}$ is the set of $\mathcal{L}\in\mathcal{AC}^G_\Lambda(X)$ such that $X^{\rm ss}(\mathcal{L})$ intersects $C$.*

*From now on, $\mathcal{F}$ which only depends on $C$ is denoted by $\mathcal{F}(C)$.*

**Proof.** The first assertion is [Res07, Lemma 7]. If $\mathcal{L} \in \mathcal{F}$, then there exists $x \in C^+$ semistable for $\mathcal{L}$. By [Res07, Lemma 4], $\lim_{t\to 0} \lambda(t)x$ is semistable and belongs to $C$. Conversely, assume that $X^{\rm ss}(\mathcal{L}) \cap C$ contains $z$. Since $z$ is fixed by $\lambda$, $\mu^{\mathcal{L}}(z, -\lambda) = -\mu^{\mathcal{L}}(z, \lambda)$. But $z$ is semistable, so $\mu^{\mathcal{L}}(z, -\lambda) = 0$. $\square$

# 6 The case $X = Y \times G/B$

In this section, we assume that $X = Y \times G/B$, with a normal projective $G$-variety $Y$. Moreover, we assume that $\Lambda$ is abundant (see [Res07]).

## 6.1 General closed isotropy and well covering pairs

Let $S$ be a torus contained in $G$. Let $C$ be an irreducible component of $X^S$.
**Definition.** The pair $(C, S)$ is said to be *admissible* if there exists $x \in C$ such that $G_x^\circ = S$. The pair is said to be *well covering* if there exists a one parameter subgroup $\lambda$ of $S$, such that $C$ is an irreducible component of $X^\lambda$ and $(C, \lambda)$ is well covering.

A rephrasing of [Res07, Corollary 3] is

**Proposition 2** *We assume that $X = Y \times G/B$ with a normal projective $G$-variety $Y$ and $\Lambda$ is abundant (see [Res07, Section 7.4]). Let $\mathcal{F}$ be a face of $\mathcal{AC}_\Lambda^G(X)$ of codimension $r$. Then, there exists an admissible well covering pair $(C, S)$ with $S$ of dimension $r$ such that $\mathcal{F} = \mathcal{F}(C)$.*

We are now interested in the generic closed isotropy of faces of $\mathcal{AC}_\Lambda^G(X)$:

**Proposition 3** *Let $\mathcal{F}$ be a face of codimension codimension $r$. Let $(C, S)$ be an admissible well covering pair with a $r$-dimensional torus $S$ such that $\mathcal{F}(C) = \mathcal{F}$.*

*There exists a generic closed isotropy $H$ of $\mathcal{F}$ such that $H^\circ = S$.*

**Proof.** By Lemma 2, $\mathcal{F}$ is an union of GIT-classes. By [Res00], there are only finitely many such classes and they are convex; so, they exists a GIT-class $F$ which spans $\mathcal{F}$. Let $\mathcal{L} \in F$. Let $\xi \in (X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}$ and $\mathcal{O}_\xi$ be the corresponding closed $G$-orbit in $X^{\rm ss}(\mathcal{L})$.

By [Res07, Theorem 5], $\mathcal{O}_\xi$ intersects $C$. Let us fix $x \in O_\xi \cap C$. Now, consider the morphism $\mu^\bullet(x, G_x) : \Lambda_\mathbb{Q} \longrightarrow X(G_x) \otimes \mathbb{Q}$ induced by restriction and the isomorphism $X(G_x) \simeq {\rm Pic}^G(\mathcal{O}_\xi)$. By Theorem 4, ${\rm Ker}\mu^\bullet(x, G_x)$ is contained in $\mathrm{Span}(\mathcal{F})$. On the other hand, the GIT-class of $\mathcal{L}$ is contained in $\mathrm{Ker}\mu^\bullet(x, G_x)$. Finally, $\mathrm{Ker}\mu^\bullet(x, G_x) = \mathrm{Span}(\mathcal{F})$. Since $\Lambda$ is abundant, this implies that the rank of $X(G_x)$ equals $r$.

Since $G.x$ is affine, $G_x$ is reductive. Since $X = Y \times G/B$, $G_x$ is contained in a Borel subgroup of $G$. Finally, $G_x$ is diagonalisable. But the rank of $X(G_x)$ equals $r$; and, $G_x^\circ$ is a $r$-dimensional torus.

Since $x \in C$, $S$ is contained in $G_x$; it follows that $G_x^\circ = S$. □

## 6.2 Unicity

**Notation.** Let $S$ be a torus. We will denote $Y(S)$ the group of one parameter subgroups of $S$. There is a natural perfect paring $Y(S) \times X(S) \longrightarrow \mathbb{Z}$ denoted by $\langle \cdot, \cdot \rangle$.

The following proposition is a first statement of unicity.

**Lemma 3** *We assume that $Y$ (and so $X$) is smooth. Let $\mathcal{F}$ be a face of codimension $r$. Let $(C_1, S_1)$ and $(C_2, S_2)$ be two well covering pairs with two $r$-dimensional tori $S_1$ and $S_2$ such that $\mathcal{F}(C_1) = \mathcal{F}(C_2) = \mathcal{F}$.*

*Then, the there exists $g \in G$ such that $g.C_2 = C_1$ and $g.S_2.g^{-1} = S_1$.*

**Proof.** Starting the proof as Proposition 3, we obtain that $\mathcal{O}_\xi$ intersects $C_1$ and $C_2$. Up to conjugacy, we may assume that $x$ belongs to $\mathcal{O}_\xi \cap C_1 \cap C_2$. So, we obtain that $G_x^\circ = S_1 = S_2$. Then, $C_1$ equals $C_2$ since they are the irreducible component of $X^{S_1} = X^{S_2}$ containing $x$. □

Let us fix a face $\mathcal{F}$ of codimension $r$. The set of linear forms $\varphi \in \mathrm{Hom}(\Lambda_{\mathbb{Q}}, \mathbb{Q})$ such that $\varphi$ is non negative on $\mathcal{AC}^G_\Lambda(X)$ and zero on $\mathcal{F}$ is denote by $\mathcal{F}^\vee$.

Let $(C, S)$ be an admissible pair where $S$ has dimension $r$ and set $\mathcal{F} = \mathcal{F}(C)$. Let $\mathcal{C}$ denote the set of $\lambda \in Y(S) \otimes \mathbb{Q}$ such that for some positive $n$, the pair $(C_{n\lambda}, n\lambda)$ is dominant; where, $C_{n\lambda}$ denote the irreducible component of $X^{n\lambda}$ containing $C$.

**Lemma 4** *We assume that $Y$ is smooth.*

*Then, $\lambda \in \mathcal{C}$ if and only if $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$.*

**Proof.** Let $\lambda \in \mathcal{C}$ and $n$ be a positive integer such that $n\lambda \in Y(S)$. Since $(C_{n\lambda}, n\lambda)$ is dominant, [Res07, Lemma 7] imply that $\mu^\bullet(C, \lambda)$ is non negative on $\mathcal{AC}^G_\Lambda(X)$. Moreover, for any $\mathcal{L} \in \mathcal{F}$, $X^{\mathrm{ss}}(\mathcal{L})$ intersects $C$. This implies that $\mu^{\mathcal{L}}(C, \lambda) = 0$. Finally, $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$.

Conversely, let $\lambda$ be a rational one parameter subgroup and $n$ be a positive integer such that $n\lambda \in Y(S)$ and $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$. Set $C^+ = \{x \in X : \lim_{t\to 0} n\lambda(t)x \in C_{n\lambda}\}$ and $\eta : G\times_{P(n\lambda)} C^+ \longrightarrow X$. Let us fix a generic point $x \in C$. Then, $G_x$ is the generic closed isotropy of $\mathcal{F}$, its neutral component is $S$ and the $G_x$-module $T_xX/T_xG.x$ is the type of $\mathcal{F}$. Theorem 4 implies that $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$ if and only if $\langle n\lambda, \cdot\rangle$ is non negative on all weights of $S$ in $T_xX/T_xG.x$. We deduce that $T\eta_x$ is surjective. Since $Y$ is smooth, this implies that $\eta$ is dominant. $\square$

We can now state our main result of unicity:

**Proposition 4** *We assume that $Y$ is smooth. Let $p_{G/B} : X \longrightarrow G/B$ denote the projection. Let us fix a Borel subgroup $B$ of $G$ and a maximal torus $T$ of $B$.*

*Let $\mathcal{F}$ be a face of codimension $r$. Then there exists a unique well covering pair $(C, S)$ where $S$ is a $r$-dimensional subtorus of $T$, $p_{G/B}(C)$ contains $B/B$ and $\mathcal{F}(C) = \mathcal{F}$.*

*Let $\lambda_1$ and $\lambda_2$ be two one parameter subgroups of $S$ such that $(C, \lambda_i)$ is dominant and $\mathcal{F}$ equals the set of $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ such that $\mu^{\mathcal{L}}(C, \lambda_i) = 0$ for $i = 1, 2$. Then, $P(\lambda_1) = P(\lambda_2)$, $C$ is an irreducible component of $X^{\lambda_1}$ and $X^{\lambda_2}$ and $C^+(\lambda_1) = C^+(\lambda_2)$.*

**Proof.** Let $(C_1, S_1)$ and $(C_2, S_2)$ be two admissible well covering pairs with two $r$-dimensional tori $S_1$ and $S_2$ such that $\mathcal{F}(C_1) = \mathcal{F}(C_2) = \mathcal{F}$. We also assume that $p_{G/B}(C_1)$ and $p_{G/B}(C_2)$ contain $B/B$. By Lemma 3, there exists $g \in G$ such that $gS_2g^{-1} = S_1$ and $gC_2 = C_1$. Note that $g^{-1}Tg$ and $T$ contain $S_2$ and are maximal tori of $G^{S_2}$: there exists $h \in G^{S_2}$ such that $hTh^{-1} = g^{-1}Tg$. Set $\tilde{w} = gh$. One easily checks that $\tilde{w}$ normalize $T$, $\tilde{w}S_2\tilde{w}^{-1} = S_1$ and $\tilde{w}C_2 = C_1$. Now, $G^{S_1}B/B = p_{G/B}(C_1) = \tilde{w}p_{G/B}(C_2) \ni \tilde{w}B/B$. We deduce that $\tilde{w} \in G^{S_1}$. So, $S_2 = \tilde{w}^{-1}S_1\tilde{w} = S_1$ and $C_2 = \tilde{w}^{-1}C_1 = C_1$. The first assertion is proved.

Let now $C, S, \lambda_1$ and $\lambda_2$ be as in the statement. Let us first prove that $P(\lambda_1) = P(\lambda_2)$. By Lemma 4, the set of the one parameter subgroups $\lambda$ of $S$ as in the proposition is convex. So, if it is possible to have $P(\lambda_1) \neq P(\lambda_2)$; it is possible to have $P(\lambda_1) \subset P(\lambda_2)$ and $P(\lambda_1) \neq P(\lambda_2)$. In other words, we may assume that $P(\lambda_1) \subset P(\lambda_2)$. Let $C_1$ (resp. $C_2$) denote the irreducible component of $X^{\lambda_1}$ (resp. $X^{\lambda_2}$) containing $C$. By Lemma 3, we have $C_1 = C_2$. In particular, $G^{\lambda_1}B/B = p_{G/B}(C_1) = p_{G/B}(C_2) = G^{\lambda_2}B/B$;

so, $G^{\lambda_1} = G^{\lambda_2}$. It follows that $P(\lambda_1) = P(\lambda_2)$. Let $P$ denote this parabolic subgroup of $G$.

Let $x \in C$ be general. Since $\lambda_1$ fixes $x$, it acts on the tangent space $T_xX$. Consider the subspaces $(T_xX)_{>0}$ and $(T_xX)_{<0}$ of the $\xi \in T_xX$ such that $\lim_{t\to 0} \lambda_1(t)\xi = 0$ and $\lim_{t\to 0} \lambda_1(t^{-1})\xi = 0$ respectively. We have: $T_xX = (T_xX)_{<0} \oplus (T_xX)_0 \oplus (T_xX)_{>0}$. The first part identify with $T_eG/P$ and the second one with $T_xC$ as $S$-modules. But, the third part is the unique $S$-stable supplementary of the sum of the two first one. In particular, the same construction with $\lambda_2$ in place of $\lambda_1$ gives the same decomposition $T_xX = (T_xX)_{<0} \oplus (T_xX)_0 \oplus (T_xX)_{>0}$. It follows that $C^+(\lambda_1) = C^+(\lambda_2)$. $\square$

### 6.3 Inclusion of faces

**Proposition 5** *We assume that $Y$ is smooth. Let us fix a maximal torus $T$ of $G$ and a Borel subgroup $B$ containing $T$. Let $(C_1, S_1)$ and $(C_2, S_2)$ be two admissible well covering pairs with two subtori $S_1$ and $S_2$ of $T$ of dimension $r_1$ and $r_2$ such that $B/B \in p_{G/B}(C_i)$ for $i = 1, 2$. We assume that $\mathcal{F}(C_1)$ and $\mathcal{F}(C_2)$ have respectively codimension $r_1$ and $r_2$.*

*Then, the following are equivalent:*

*(i)* $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$;

*(ii)* $C_1 \subset C_2$ *and* $S_2 \subset S_1$.

**Proof.** The second assertion implies the first one by Lemma 2. Conversely, let us assume that $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$.

By Proposition 3, there exists $\mathcal{L} \in \mathcal{F}(C_1)$ and $x \in C_1$ such that $G_x^\circ = S_1$ and $G.x$ is closed in $X^{\rm ss}(\mathcal{L})$. Since $C_2$ intersects $G.x$, there exists $g \in G$ such that $g.x \in C_2$. Since $S_2$ fixes $g.x$, $S_2 \subset gS_1g^{-1}$. In particular, $S_2$ is contained in $T$ and $gTg^{-1}$; so, $T$ and $gTg^{-1}$ are maximal tori in $G^{S_2}$. There exists $l \in G^{S_2}$ such that $lTl^{-1} = gTg^{-1}$. The element $n = l^{-1}g$ normalizes $T$. Since $C_2$ is stable by $G^{S_2}$ (which is connected), $x$ belongs to $n^{-1}C_2$. Applying $p_{G/B}$ we obtain that $p_{G/B}(x)$ belongs to $n^{-1}G^{S_2}B/B \cap G^{S_1}B/B$.

Since $n^{-1}S_2n = g^{-1}S_2g \subset S_1$, we have $G^{n^{-1}S_2n} \subset G^{S_1}$. In particular, $n^{-1}G^{S_2}B/B \subset G^{S_1}n^{-1}B/B$. It follows that $G^{S_1}n^{-1}B/B = G^{S_1}B/B$.

Since $n$ normalizes $T$, this implies that $n \in G^{S_1}$. So, $S_2 \subset nS_1n^{-1} = S_1$.

Since $n \in G^{S_1}$, $nx \in C_1$. It follows that $C_1$ is the irreducible component of $X^{S_1}$ containing $nx$. On the other hand, $C_2$ is the irreducible component of $X^{S_2}$ containing $nx$. It follows that $C_1$ is contained in $C_2$. $\square$

## 7 GIT-cone and moment polytope

Let us now explain the relation mentioned in the introduction between the moment polytopes of $Y$ and some total $G$-cones of $X = Y \times G/B$.

Let us fix a maximal torus $T$ of $G$ and a Borel subgroup $B$ containing $T$. Let $\mathcal{L}$ be an ample $G$-linearized line bundle on $Y$. We consider the set $P_G(Y, \mathcal{L})$ of the points $p \in X(T)_{\mathbb{Q}}$ such that for some positive integer $n$, $np$ is a dominant character of $T$ and the dual $V_{np}^*$ of $V_{np}$ is a submodule of $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n})$. In fact, $P_G(Y, \mathcal{L})$ is a polytope, called *moment polytope*. Notice that "the dual" is not usual in the definition; but it will be practical in our context.

Consider the two projections:

$$\begin{tikzcd}
 & X \arrow[dl, "p_Y"'] \arrow[dr, "p_{G/B}"] & \\
Y & & G/B.
\end{tikzcd}$$

In Section 7, $\Lambda$ will always denote the subgroup of $\mathrm{Pic^G(X)}$ generated by $p_{G/B}^*(\mathrm{Pic^G(G/B)})$ and $p_Y^*(\mathcal{L})$. Note that $\mathrm{Pic^G(G/B)}$ identifies with $X(T)$; we will denote $\mathcal{L}_\nu$ the element of $\mathrm{Pic^G(G/B)}$ associated to $\nu \in X(T)$

**Proposition 6** *With above notation, we have:*

*(i)* $\mathcal{TC}_\Lambda^G(X) = \mathcal{SAC}_\Lambda^G(X)$;

*(ii)* $\mathcal{SAC}_\Lambda^G(X)$ *is a cone over* $P_G(Y, \mathcal{L})$; *more precisely, for all positive rational number* $m$ *and* $\nu \in X(T)$, *we have:*

$$mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu) \in \mathcal{SAC}_\Lambda^G(X) \iff \frac{\nu}{m} \in P_G(Y, \mathcal{L}).$$

**Proof.** Let $n$ be a non negative integer and $\nu$ be a character of $T$. As a $G$-module, $\mathrm{H}^0(X, p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu))$ is isomorphic to $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n}) \otimes \mathrm{H}^0(G/B, \mathcal{L}_\nu)$. In particular, if $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$ has non zero global sections then $n \geq 0$ and $\nu$ is dominant; in this case, it is semiample. The first assertion follows.

Assume now that $\nu$ is dominant. Then, $\mathrm{H}^0(G/B, \mathcal{L}_\nu) = V_\nu$. Hence, $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$ has non zero $G$-invariant sections if and only if $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n}) \otimes$

$V_\nu$ contains non zero $G$-invariant vectors; that is, if and only if $V_\nu^*$ is a submodule of $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n})$. The second assertion follows. □

**Remark.** To each face $\mathcal{F}$ of $\mathcal{AC}^G_\Lambda(X)$, one can associate a face of $P_G(Y, \mathcal{L})$ (by intersecting and taking a closure) which intersects the interior of the dominant chamber. By this way, we obtain a bijection between the set of faces of $\mathcal{AC}^G_\Lambda(X)$ and the faces of $P_G(Y, \mathcal{L})$ which intersects the interior of the dominant chamber.

## 7.1 A reduction

It is possible that $\mathcal{AC}^G_\Lambda(X)$ is empty. In this case, our results cannot be applied directly.

Let $mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$ in the relative interior of $\mathcal{SAC}^G(X)$ such that $\mathrm{H}^0(X, mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu))^G$ is non zero, that is such that $V_\nu^*$ can be equivariantly embedded in $\mathrm{H}^0(Y, \mathcal{L}^{\otimes m})$. Let $P$ be the standard parabolic subgroup of $G$ associated to the face of the dominant chamber containing $\nu$. Let $L$ denote the Levi subgroup of $P$ containing $T$ and $D$ denote its derived subgroup.

The next proposition shows that $\mathcal{SAC}^G(X)$ is always equal to such a cone satisfying $\mathcal{AC}^G(X)^n eq\emptyset$. The proof which is essentially extracted from [Bri99, Section 5] is included for completeness.

**Proposition 7** *With above notation, there exists an irreducible component $C_Y$ of $Y^D$ such that a point $\mathcal{L} \in \Lambda_\mathbb{Q}$ belongs to $\mathcal{SAC}^G(X)$ if and only if $\mathcal{L}_{|C}$ belongs to $\mathcal{SAC}^{G^D}(C_Y \times G^D.B/B)$.*

*Moreover, $G^D.B/B$ is isomorphic to the variety of complete flags of $G^D$ and $\mathcal{AC}^{G^D}(C_Y \times G^D.B/B)$ is non empty.*

**Proof.** The inclusion $V_\nu^* \subset \mathrm{H}^0(Y, \mathcal{L}^{\otimes m})$ gives a $G$-equivariant rational map $\phi : Y \longrightarrow \mathbb{P}(V_\nu)$. Let $v_\nu$ be a vector of highest weight in $V_\nu$; $P$ is the stabilizer in $G$ of $[v_\nu] \in \mathbb{P}(V_\nu)$. Let $\sigma \in V_\nu^*$ be an eigenvector of the Borel subgroup $B^-$ opposite to $B$ and containing $T$. Let $Q$ be the stabilizer in $G$ of $[\sigma] \in \mathbb{P}(V_\nu^*)$ and $Q^u$ be its unipotent radical.

Let $Y_\sigma$ denote the set of $y \in Y$ such that $\sigma(y) \neq 0$. Let $W$ be a $L$-stable supplementary subspace of $\mathbb{K}.v_\nu$ in $V_\nu$. By $w \longmapsto [v_\nu + w]$, we identify $W$ with an open subspace of $\mathbb{P}(V_\nu)$. Then $\phi$ induces by restriction $\widetilde{\phi} : Y_\sigma \longrightarrow W$.

Let $S$ be a $L$-stable supplementary to $T_{[v_\nu]}G.[v_\nu]$ in $W$ (actually, $W$ canonically identify with $T_{[v_\nu]}\mathbb{P}(V_\nu)$). Set $Z = \widetilde{\phi}^{-1}(S)$. By [Bri99, Remark in Section 5], $Z$ is point wise fixed by $D$ and the action of $Q^u$ induces an isomorphism $Q^u \times Z \simeq Y_\sigma$.

Consider $X' = Y \times G/P$. Let $\Lambda'$ be the subgroup of $\mathrm{Pic^G(X')}$ generated by $p_Y^*(\mathcal{L})$ and $p_{G/P}^*(\mathrm{Pic^G(G/P)})$ (with obvious notation). It is clear that $\mathcal{TC}^G_\Lambda(X)$ identifies with $\mathcal{TC}^G_{\Lambda'}(X')$. Moreover, $\mathcal{AC}^G_{\Lambda'}(X')$ is not empty. Consider a generic closed isotropy $H$ of $\mathcal{TC}^G_{\Lambda'}(X')$ viewed as a face $\mathcal{F}$ of itself. Since $Q^u \times Z \simeq Y_\sigma \subset X^{\rm ss}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_\nu))$, up to conjugacy, one may assume that $D \subset H \subset L$. Since $Y_\sigma \times \{P/P\} \subset X^{\rm ss}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_\nu))$, $X'_\mathcal{F}$ intersects $Z \times \{P/P\}$.

Consider the irreducible component $C_Y$ of $Y^D$ which contains $Z$. By Theorem 3, for any ample $\mathcal{L} \in \Lambda'$, $X'^{\rm ss}(\mathcal{L})$ intersects $C_Y \times \{P/P\}$ if it is non empty. By continuity, this is also true if $\mathcal{L}$ is only semiample. The proposition follows easily. ☐

## 7.2 Faces of $\mathcal{SAC}^G(X)$ if $\mathcal{AC}^G(Y)$ is non empty

From now on, we assume that $Y$ is smooth. We will first adapt the notion of covering and well covering pairs for the situation.

Recall that $T \subset B$ are fixed. Let $\lambda$ be a one parameter subgroup of $T$. Set $B(\lambda) = B \cap P(\lambda)$. Let $C$ be an irreducible component of $Y^\lambda$ and $C^+$ the associated Bialinicki-Birula cell.

**Definition.** The pair $(C, \lambda)$ is said to be *$B$-covering* if the natural map $\eta\ :\ B \times_{B(\lambda)} C^+ \longrightarrow Y$ is birational. It is said to be *well $B$-covering* is $\eta$ induces an isomorphism over an open subset of $Y$ intersecting $C$.

The proof of the following lemma is obvious.

**Lemma 5** *With above notation, the pair $(C, \lambda)$ is $B$-covering (resp. well $B$-covering) if and only if $(C \times G^\lambda B/B)$ is covering (resp. well covering).*

Let us recall that the subtori of $T$ correspond bijectively to the linear subspaces of $X(T)_\mathbb{Q}$. If $V$ is a linear subspace of $X(T)_\mathbb{Q}$, the associated torus is the neutral component of the intersection of kernels of elements in $X(T) \cap V$. If $F$ is a convex part of $X(T)_\mathbb{Q}$, the direction $\mathrm{dir}(F)$ of $F$ is the linear subspace spanned by the differences of two elements of $F$.

We will denote by $\mathcal{C}^+$ the convex cone in $X(T)_\mathbb{Q}$ generated by the dominant weights. The next proposition is an improvement of [Bri99, Theorem 1]:

**Proposition 8** *We keep the above notation and assume that $Y$ is smooth and $P_G(Y,\mathcal{L})$ intersects the interior of the dominant chamber. Let $\mathcal{F}$ be a face of codimension $d$ of $P_G(Y,\mathcal{L})$ which intersect the interior of the dominant chamber. Let $S$ the subtorus of $T$ associated to* $\mathrm{dir}(\mathcal{F})$.

*There exists a unique irreducible component $C$ of $Y^S$ and a one parameter subgroup $\lambda$ of $S$ such that $G^\lambda = G^S$ and $(C,\lambda)$ is a well $B$-covering pair such that $\mathcal{F} = P_{G^S}(C,\mathcal{L}_{|C}) \cap \mathcal{C}^+$.*

**Proof.** Let $\widetilde{\mathcal{F}}$ be the face of $\mathcal{AC}^G_\Lambda(X)$ corresponding to $\mathcal{F}$ and $r$ denote its codimension. By Proposition 2, there exists an admissible well covering pair $(C_X, S')$ such that $\widetilde{\mathcal{F}} = \mathcal{F}(C_X)$ and $S'$ is a $r$-dimensional torus. Up to conjugacy, we may assume that $C_X$ intersects $Y \times B/B$, and $S'$ is contained in $T$. Let $\lambda$ be a one parameter subgroup of $S'$ such that $(C_X,\lambda)$ is well covering. Then, $C_X = C \times G^\lambda B/B$ for some irreducible component $C$ of $Y^{S'}$.

The fact $\widetilde{\mathcal{F}} = \mathcal{F}(C_X)$ readily means that $\mathcal{F} = P_{G^{S'}}(C,\mathcal{L}_{|C}) \cap \mathcal{C}^+$. Since the direction of $P_{G^{S'}}(C,\mathcal{L}_{|C})$ is contained in $X(T)^{S'}$, this implies that $X(T)^S$ is contained in $X(T)^{S'}$. But, $S$ and $S'$ have the same rank, it follows that $S = S'$.

The unicity part is a direct consequence of Proposition 4. ☐

# 8 The case $X = \hat{G}/\hat{B} \times G/B$

## 8.1 Interpretations of the $G$-cones

From now on, we assume that $G$ is a connected reductive subgroup of a connected reductive group $\hat{G}$. Let us fix maximal tori $T$ (resp. $\hat{T}$) and Borel subgroups $B$ (resp. $\hat{B}$) of $G$ (resp. $\hat{G}$) such that $T \subset B \subset \hat{B} \supset \hat{T} \supset T$.

Let $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ denote the Lie algebras of $G$ and $\hat{G}$ respectively.

We denote by $\mathcal{LR}(\hat{G},G)$ (resp. $\mathcal{LR}^\circ(\hat{G},G)$) the cone of the pairs $(\hat{\nu},\nu) \in X(\hat{T})_\mathbb{Q} \times X(T)_\mathbb{Q}$ such that for a positive integer $n$, $n\hat{\nu}$ and $n\nu$ are dominant (resp. strictly dominant) weights such that $V_{n\hat{\nu}} \otimes V_{n\nu}$ contains non zero $G$-invariant vectors.

In this section, $X$ denote the variety $\hat{G}/\hat{B} \times G/B$ endowed with the diagonal action of $G$. We will apply the results of Section 4 to $X$ with $\Lambda = \mathrm{Pic^G(X)}$. The cones $\mathcal{TC}^G(X)$, $\mathcal{SAC}^G(X)$ and $\mathcal{AC}^G(X)$ will be denoted without the $\Lambda$ in subscribe. By [Res07, Proposition 9], $\mathcal{LR}^\circ(\hat{G},G) = \mathcal{AC}^G(X) \subset \mathcal{SAC}^G(X) = \mathcal{TC}^G(X) = \mathcal{LR}(\hat{G},G)$. Moreover, if no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$, by [Res07, Assertion $(i)$ of Theorem 9] $\mathcal{LR}^\circ(\hat{G},G)$ has non empty interior.

## 8.2 The case $X = \hat{G}/\hat{B} \times G/B$

**8.2.1** — Consider the $G$-module $\hat{\mathfrak{g}}/\mathfrak{g}$. Let $\chi_1, \cdots, \chi_n$ be the set of the non trivial weights of $T$ on $\hat{\mathfrak{g}}/\mathfrak{g}$. For $I \subset \{1, \cdots, n\}$, we will denote by $T_I$ the neutral component of the intersection of the kernels of the $\chi_i$'s with $i \in I$. A subtorus of the form $T_I$ is said to be *admissible.*

Let $\lambda$ be a one parameter subgroup of $T$. Consider the parabolic subgroups $P$ and $\hat{P}$ of $G$ and $\hat{G}$ associated to $\lambda$. Let $W_P$ be the Weyl group of $P$. The cohomology group $\mathrm{H}^*(G/P, \mathbb{Z})$ is freely generated by the Schubert classes $[\overline{BwP/P}]$ parametrized by the elements $w \in W/W_P$. Since $\hat{P} \cap G = P$, we have a canonical $G$-equivariant immersion $\iota : G/P(\lambda) \longrightarrow \hat{G}/\hat{P}(\lambda)$; and the corresponding morphism in cohomology $\iota^*$.

Let $\rho$ (resp. $\rho^\lambda$) denote the half sum of the positive roots of $G$ (resp. $G^\lambda$). Let $\Phi^+$ and $\Phi(P^u)$ denote the set of roots of the groups $B$ and $P^u$ for the torus $T$. In the same way, we define $\hat{\Phi}^+$ and $\Phi(\hat{P}^u)$. For $\hat{w} \in \hat{W}$, we set:

$$\theta^P := \sum_{\alpha \in \Phi^+ \cap \Phi(P^u)} \alpha \in X(T) \quad \text{and} \quad \theta^{\hat{P}}_{\hat{w}} := \sum_{\alpha \in \hat{w}\hat{\Phi}^+ \cap \Phi(\hat{P}^u)} \alpha \in X(\hat{T}).$$

**8.2.2** — Let $S$ be an admissible subtorus of $T$. All irreducible component $C$ of $X^S$ such that $p_{G/B}(C)$ contains $B/B$ equals $C(\hat{w}) := (\hat{G}^S.\hat{w}^{-1}\hat{B}/\hat{B} \times G^S B/B)$ for a unique element $\hat{w} \in \hat{W}/\hat{W}_{\hat{G}^S}$. Let us fix $\hat{w} \in \hat{W}/\hat{W}_{\hat{G}^S}$. The pair $(S, \hat{w})$ is said to be *admissible* if there exists a parabolic subgroup $\hat{P}$ of $\hat{G}$ such that

(i) there exists $\lambda \in Y(S)$ such that $\hat{P} = \hat{P}(\lambda)$;

(ii) $\hat{G}^S$ is a Levi subgroup of $\hat{P}$;

(iii) $G^S$ is a Levi subgroup of $\hat{P} \cap G =: P$;

(iv) $\iota^*([\overline{\hat{B}\hat{w}\hat{P}/\hat{P}}]).[\overline{BP/P}] = [\mathrm{pt}] \in \mathrm{H}^*(G/P, \mathbb{Z})$;

(v) $(\theta^{\hat{P}}_{\hat{w}})_{|S} = (\theta^P - 2(\rho - \rho^\lambda))_{|S}$.

**Lemma 6** *Let $S$ be an admissible subtorus of $T$ and $\hat{w} \in \hat{W}/\hat{W}_{\hat{G}^S}$. The pair $(S, \hat{w})$ is admissible if and only if there exists a one parameter subgroup $\lambda$ of $S$ such that $C(\hat{w})$ is an irreducible component of $X^\lambda$ and $(C(\hat{w}), \lambda)$ is a well covering pair.*

**Proof.** The proof is very analogous to [Res07, Proposition 10]: we leave details to the reader. We prove (using mainly Kleiman's Theorem) that

$\iota^*([\overline{\hat{B}\hat{w}\hat{P}/\hat{P}}]).[\overline{BP/P}] = [\mathrm{pt}] \in \mathrm{H}^*(G/P,\mathbb{Z})$ if and only if $\eta$ is birational. Now, the condition $(\theta^{\hat{P}}_{\hat{w}})_{|S} = (\theta^P - 2(\rho - \rho^\lambda))_{|S}$ means that $S$ acts trivially on the restriction over $C$ of the determinant bundle of $\eta$. ☐

**8.2.3** — To simplify, in the following statement we assume that $\mathcal{AC}^G(X)$ has a non empty interior in $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X})_{\mathbb{Q}}$. In fact, this assumption is equivalent to say that no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$.

**Theorem 5** *We assume that no ideal of* $\mathfrak{g}$ *is an ideal of* $\hat{\mathfrak{g}}$.

*The map which associates to a pair* $(S,\hat{w})$ *the set* $\mathcal{F}(S,\hat{w}) = \{(\hat{\nu},\nu) \in C^G(X)\ :\ \hat{w}\hat{\nu}_{|S} = -\nu_{|S}\}$ *is a bijection from the set of admissible pairs onto the set of faces of* $\mathcal{AC}^G_\Lambda(X)$*. Moreover, the codimension of* $\mathcal{F}(S,\hat{w})$ *equals the dimension of* $S$*.*

*The following are equivalent:*

*(i)* $\mathcal{F}(S,\hat{w}) \subset \mathcal{F}(S',\hat{w}')$*;*

*(ii)* $S' \subset S$ *and* $\hat{w}W_{G^{S'}} = \hat{w}'W_{G^{S'}}$*.*

**Proof.** Let $(S,\hat{w})$ be an admissible pair. Set $\overline{\mathcal{F}}(S,\hat{w}) = \{(\hat{\nu},\nu) \in \mathcal{LR}(G,\hat{G})\ :\ \hat{w}\hat{\nu}_{|S} = -\nu_{|S}\}$. By [Res07, Theorem 9], $\overline{\mathcal{F}}(S,\hat{w})$ is a face of $\mathcal{LR}(G,\hat{G})$ of codimension $\dim(S)$. In particular, $\widetilde{\mathcal{F}}(S,\hat{w})$ spans the vector subspace of the $(\hat{\nu},\nu) \in X(\hat{T}) \times X(T)$ such that $\hat{w}\hat{\nu}_{|S} = -\nu_{|S}\}$. Since $\mathcal{AC}^G_\Lambda(X)$ is the interior of $\mathcal{LR}(G,\hat{G})$, to prove that the map in the theorem is well defined, it is enough to prove that $\overline{\mathcal{F}}(S,\hat{w})$ intersects $\mathcal{AC}^G_\Lambda(X)$. If not, $\overline{\mathcal{F}}(S,\hat{w})$ would be contained in the boundary of the dominant chamber. Its projection on $X(\hat{T})_{\mathbb{Q}}$ or $X(T)_{\mathbb{Q}}$ would be contained in an hyperplane; which is a contradiction.

The surjectivity is a rephrasing of [Res07, Theorem 9 Assertion *(ii)*]. The injectivity is a direct application of Proposition 4.

The last assertion follows from Proposition 5. ☐

## 9 Application to the tensor product cone

**9.1** — In this section, $G$ is assumed to be semisimple. As above, $T \subset B$ are fixed maximal torus and Borel subgroup of $G$. We also fix an integer $s \geq 2$ and set $\hat{G} = G^s$, $\hat{T} = T^s$ and $\hat{B} = B^s$. We embed $G$ diagonally in $\hat{G}$. Now, $X = \hat{G}/_h B \times G/B = (G/B)^{s+1}$. Then $\mathcal{AC}^G(X) \cap X(T)^{s+1}$ identifies with the $(s+1)$-uple $(\nu_1,\cdots,\nu_{s+1}) \in X(T)^{s+1}$ such that the for $n$ big enough

$n\nu_i$'s are strictly dominant weights and $V_{n\nu_1}\otimes\cdots\otimes V_{n\nu_{s+1}}$ contains a non zero $G$-invariant vector.

A parabolic subgroup $P$ of $G$ is said to be *standard* if it contains $B$. We will denote by $Z(P)$ the neutral component of the center of the Levi subgroup of $P$ containing $T$.

**9.2** — In [BK06], Belkale and Kumar defined a new product denoted $\odot_0$ on the cohomology groups $H^*(G/P,\mathbb{Z})$ for any parabolic subgroup $P$ of $G$. We consider the set $\Theta$ of the $(P,\Lambda_{w_0},\cdots,\Lambda_{w_s})$ where $P$ is a standard parabolic subgroup of $G$ and the $\Lambda_{w_i}$'s are $s+1$ Schubert varieties of $G/P$ such that

$$[\Lambda_{w_0}]\odot_0\cdots\odot_0[\Lambda_{w_s}]=[\mathrm{pt}].$$

**9.3** — In [Res07], Theorem 9 applied to $\hat{G}=G^s$ gives Corollary 5. The same translation of Theorem 5 to this case gives the following:

**Theorem 6** *The map which associates to a $(P,\Lambda_{w_0},\cdots,\Lambda_{w_s})\in\Theta$ the set $\mathcal{F}(P,\Lambda_{w_0},\cdots,\Lambda_{w_s})$ of the $(\nu_0,\cdots,\nu_s)\in\mathcal{AC}^G(X)$ such that the restriction of $\sum_i w_i^{-1}\nu_i$ to $Z(P)$ is trivial is a bijection from $\Theta$ onto the set of faces of $\mathcal{AC}^G(X)$. Moreover, the codimension of $\mathcal{F}(P,\Lambda_{w_0},\cdots,\Lambda_{w_s})$ equals the dimension of $Z(P)$.*

*The following are equivalent:*

*(i) $\mathcal{F}(P,\Lambda_{w_0},\cdots,\Lambda_{w_s})\subset\mathcal{F}(P',\Lambda_{w'_0},\cdots,\Lambda_{w'_s})$);*

*(ii) $P\subset P'$ and $\pi(\Lambda_{w_i})=\Lambda_{w'_i}$ for all $i=0,\cdots,s$ (here, $\pi:G/P\longrightarrow G/P'$ is the natural $G$-equivariant map).*

## References

[BK06] Prakash Belkale and Shrawan Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), no. 1, 185–228.

[Bri99] Michel Brion, *On the general faces of the moment polytope*, Internat. Math. Res. Notices (1999), no. 4, 185–201.

[DH98] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56, With an appendix by Nicolas Ressayre.

[LR79] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496.

[Lun73] Domingo Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33.

[Lun75] D. Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3d ed., Springer Verlag, New York, 1994.

[PV91] V. L. Popov and È. B. Vinberg, *Algebraic Geometry IV*, Encyclopedia of Mathematical Sciences, vol. 55, ch. Invariant Theory, pp. 123–284, Springer-Verlag, 1991.

[Res00] N. Ressayre, *The GIT-equivalence for $G$-line bundles*, Geom. Dedicata **81** (2000), no. 1-3, 295–324.

[Res07] Nicolas Ressayre, *Geometric invariant theory and generalized eigenvalue problem*, Preprint (2007), no. arXiv:0704.2127, 1–45.

[Sja98] Reyer Sjamaar, *Convexity properties of the moment mapping re-examined*, Adv. Math. **138** (1998), no. 1, 46–91.

- ◇ -

N. R.
Université Montpellier II
Département de Mathématiques
Case courrier 051-Place Eugène Bataillon
34095 Montpellier Cedex 5
France
e-mail: `ressayre@math.univ-montp2.fr`