BLOW UP AND GRAZING COLLISION IN VISCOUS FLUID SOLID INTERACTION SYSTEMS

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Abstract. In this paper we investigate finite time blow up of strong solutions to the system describing the motion of a rigid ball inside a bounded cavity filled with a viscous incompressible fluid. The equations of motion for the fluid are of Navier–Stokes type and the equations for the motion of the rigid ball are obtained by applying Newton's laws. The whole system evolves under the action of gravity. First, we prove contact between the ball and the boundary of the cavity implies the blow up of the strong solution. Then we prove for some configurations such a contact has to occur in finite time.

1. Introduction

In this paper, we compute blowing-up solutions for the classical fluid solid interaction system. More precisely, we consider a bounded domain $\Omega \subset \mathbb{R}^3$ of class C^2 containing a viscous incompressible fluid and a rigid ball. The equations of motion for the fluid and the rigid body are the classical Navier–Stokes equations coupled with the Newton laws.

$$(1)\qquad \left\{\begin{array}{rcll} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &=& \operatorname{div}\mathbb{T}(\mathbf{u},p) + \mathbf{f} & \text{in } \mathcal{F}_t,\\ \operatorname{div}\mathbf{u} &=& 0 & \text{in } \mathcal{F}_t,\\ \mathbf{u} &=& \dot{\mathbf{G}} + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}), & \text{on } \partial\mathcal{B}_t,\\ \mathbf{u} &=& 0, & \text{on } \partial\Omega, \end{array}\right.$$

$$(2)\qquad \left\{\begin{array}{rcl} -\displaystyle\int_{\partial\mathcal{B}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\;\mathrm{d}\sigma + \int_{\mathcal{B}} \rho_{\mathcal{B}}\mathbf{f}\;\mathrm{d}\mathbf{x} &=& m\ddot{\mathbf{G}},\\ -\displaystyle\int_{\partial\mathcal{B}} (\mathbf{x}-\mathbf{G})\times\mathbb{T}(\mathbf{u},p)\mathbf{n}\;\mathrm{d}\sigma + \int_{\mathcal{B}} \rho_{\mathcal{B}}(\mathbf{x}-\mathbf{G})\times\mathbf{f}\;\mathrm{d}\mathbf{x} &=& J\dot{\boldsymbol{\omega}}. \end{array}\right.$$

In the above system, the set $\mathcal{B}_t$ stands for the domain of the solid $\mathcal{B}$: it is a ball with center $\mathbf{G}(t)$ and radius 1. Its complement in Ω, $\mathcal{F}_t = \Omega\setminus\overline{\mathcal{B}}_t$, is the domain occupied by the fluid.

The velocity/pressure field of the fluid is denoted by $(\mathbf{u},p)$ and satisfies the Navier–Stokes system with no slip boundary conditions (1). The fluid has a constant density $\rho_{\mathcal{F}} = 1$ and its stress tensor is given by:

$$\mathbb{T}(\mathbf{u},p) = \mu(\nabla\mathbf{u} + [\nabla\mathbf{u}]^\top) - p\mathrm{I}_3 = 2\mu D(\mathbf{u}) - p\mathrm{I}_3,$$

where μ is the viscosity of the fluid. For any matrix M, we denote by $M^\top$ the transpose of M. The solid is homogeneous with constant density $\rho_{\mathcal{B}} > 1$ so that $\mathbf{G}(t)$ is the position of

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the center of mass of $\mathcal{B}$ at time t and

$$m = \rho_{\mathcal{B}}|\mathcal{B}_t|, \quad J = \left[\int_{\mathcal{B}_t} \rho_{\mathcal{B}}|\mathbf{x} - \mathbf{G}(t)|^2 \, \mathrm{d}\mathbf{x}\right] \mathrm{I}_3 \quad \forall t \geqslant 0.$$

The vector $\boldsymbol{\omega}$ stands for the angular velocity of $\mathcal{B}$. We take into account the action of the fluid in the Newton laws. The whole system evolves under the action of gravity $\mathbf{f} = -g\mathbf{e}_3$.

The main unknown in the system (1)–(2) are $(\mathbf{u}, \mathbf{G}, \boldsymbol{\omega})$. This system is completed by the following initial conditions:

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \mathbf{G}(0) = \mathbf{G}_0, \quad \dot{\mathbf{G}}(0) = \dot{\mathbf{G}}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \tag{3}$$

1.1. **Previous results.** We call Fluid Solid Interaction System ((FSIS) for short) the set of equations (1)–(2). This system, and its many-bodies variant, is relevant on the theoretical level as in applications. It is the motivation for many recent studies. First, some authors obtain existence of weak solutions (in a sense which will be made precise later on) up to collision between two bodies [1, 4, 7, 8, 17]. Then these results "up-to-collision" are extended to a global one by San Martín–Starovoitov–Tucsnak in [14] in dimension 2 and by E. Feireisl in [5]. More precisely, they prove there exist global weak solutions to (FSIS) regardless collisions. The two-dimensional result is slightly more general than the three-dimensional one. Indeed, in [5], the author impose that if there is contact between rigid solids then these solids remain "stuck" forever.

These results show that the existence of collisions is a major issue in (FSIS). Such a question has been already tackled in two ways, to our knowledge. A first method uses the fact that, in these fluid solid interaction systems, the bodies follow characteristics of the extended velocity field:

$$\overline{\mathbf{u}} = \mathbf{1}_{\mathcal{F}_t}\mathbf{u} + \mathbf{1}_{\mathcal{B}_t}\mathbf{u}_{\mathcal{B}_t}[\dot{\mathbf{G}}, \boldsymbol{\omega}],$$

where $\mathbf{u}_{\mathcal{B}_t}[\mathbf{V}, \boldsymbol{\omega}] = \mathbf{V} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}(t))$. If this velocity field is sufficiently smooth ($\mathcal{C}^1$ uniformly in time, for example), the Cauchy–Lipschitz theorem implies two particles following the characteristics cannot meet each other in finite time. Hence, collision is impossible. We emphasize such a regularity is unexpectable here because the Newton laws impose a jump in the derivatives of $\overline{\mathbf{u}}$ on $\partial\mathcal{B}$. Even though restricting to the fluid domain, estimates on derivatives of $\overline{\mathbf{u}}$ in a solution to (FSIS) are known to depend drastically on the distance between solids (see [3]). Nevertheless, a criterion based on these ideas is derived by V.N. Starovoitov [16]. It does not enable to prevent solution to (FSIS) from collision between solids, but, it follows from this criterion that a certain class of strong solutions cannot persist through collisions in the two-dimensional as in the three-dimensional case. This argument is developed for our class of strong solutions in Section 2.

In the second method, one takes further advantage of the Newton laws. More precisely, in solutions to the above (FSIS) the least one can expect is decrease of the total energy of the system:

$$E := \int_{\Omega} \overline{\rho}|\overline{\mathbf{u}}|^2 + \int_{\mathcal{B}} [\rho_{\mathcal{B}} - 1] \, g\mathbf{e}_3,$$

where $\overline{\rho} := \mathbf{1}_{\mathcal{F}_t} + \mathbf{1}_{\mathcal{B}_t}\rho_{\mathcal{B}}$. In particular, in the toy-model of a ball falling over a horizontal ramp $\mathcal{P}$, this yields that the speed of the ball remains bounded. Then integrating the Newton law on the linear momentum with respect to time, we deduce

$$\int_0^T \int_{\partial\mathcal{B}_t} \mathbb{T}(\mathbf{u},p)\mathbf{n}\ \mathrm{d}\sigma \ \cdot \mathbf{e}_3\ \mathrm{d}t < C_0, \tag{4}$$

where $\mathbf{e}_3$ is the vertical direction and C_0 is a constant fixed by initial data. In the slow motion regime, computations due to Cooley and O'Neill [2] imply that

$$\int_{\partial\mathcal{B}_t} \mathbb{T}(\mathbf{u},p)\mathbf{n}\ \mathrm{d}\sigma \ \cdot \ \mathbf{e}_3 \sim -\frac{\kappa\dot{h}(t)}{h^\alpha(t)},$$

where $h(t) = \mathrm{dist}(\mathcal{B}_t, \mathcal{P})$. The factor κ depends on the radius of $\mathcal{B}$ and the exponent α depends on the dimension. Cooley and O'Neill computed $\alpha = 1$ in the case of a ball falling over a ramp in the three-dimensional case. While, in the case of a disk over a ramp in $\mathbb{R}^2$, there holds $\alpha = 3/2$. In both cases, (4) implies $\dot{h}/h \in L^1(0,T)$ so that h cannot go to 0 in finite time. These arguments are adapted rigorously to the full non-linear system in the two-dimensional case in [10] and in the three-dimensional case in [11]. They are also extended in [6] to more singular geometries yielding a threshold for the body-shape regularity under which collision can occur. These results are in contrast with the non viscous case in which it is proved that collision can occur with non-zero relative velocity [12]. In all these articles, only frontal collisions are taken into account. The aim of this paper is to show that in the three-dimensional setting, grazing collisions between smooth bodies can occur (see Theorem 2). Combining this result with the arguments mentioned above, we finally obtain the following result.

Theorem 1. *The global Fluid Solid Interaction System is ill-posed* i.e., *there exists initial conditions for (FSIS) for which global strong solutions to (FSIS) do not exist.*

In the three-dimensional context, it is not clear whether collision is the most important responsible for ill-posedness of strong solutions. Indeed, non-linear convective terms in the Navier–Stokes system could make strong solutions to blow up before collision. However our result does not depend on the size of initial data. In particular, blow up of strong solutions occurs for small as for large data, contrary to the pure Navier–Stokes system. In the frame of weak solutions, collision occurrence is also an important phenomenon because it is known that it causes failure of uniqueness [15]. We emphasize this result does not contradict [11] because the geometric configuration under consideration here does not enter in the frame of this former result.

1.2. **Description of the geometry and formal arguments.** The geometry of the problem is crucial to obtain Theorem 2. For simplicity, we set

$$\Omega = B((0,0,0),M) \setminus \overline{(B((2,0,0),1) \cup B((-2,0,0),1))},$$

with M sufficiently large. However, our techniques extend to more general geometries. The main assumptions underlying our results are:

G1. The cavity Ω is symmetric with respect to some line D.

G2. The cavity Ω has exactly two holes $\mathcal{B}^l$ and $\mathcal{B}^r$ symmetric with respect to D. These holes have the shape of balls with radius 1. The distance between the holes is 2.

G3. The gravity $\mathbf{f}$ is parallel do D.

G4. Near $\partial D \cap \partial\Omega$ the boundary $\partial\Omega$ is flat.

Another example of such a geometry is represented in Figure 1.

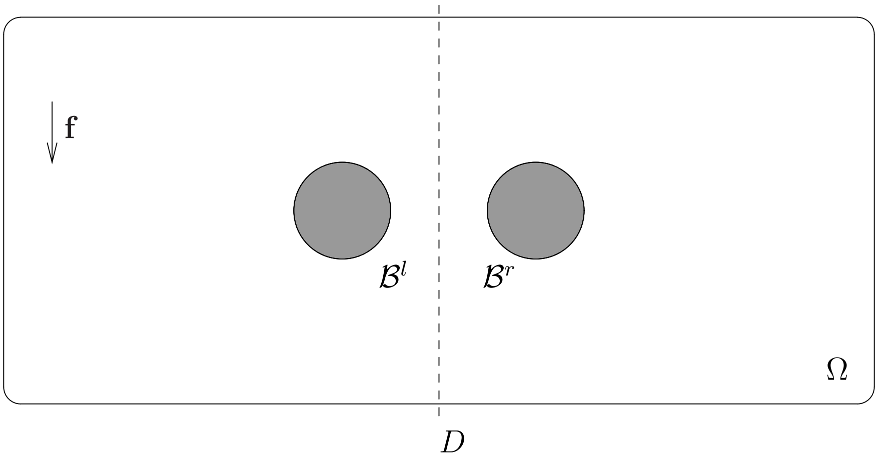

FIGURE 1. Example of Ω

In the following, we denote by $\mathbf{G}^l = (-2, 0, 0)$ and $\mathbf{G}^r = (2, 0, 0)$ the centers of the holes and the corresponding holes by $\mathcal{B}^l = B((-2, 0, 0), 1)$ and $\mathcal{B}^r = B((2, 0, 0), 1)$. We emphasize the distance between the holes is chosen so that $\mathcal{B}$ can fill exactly the gap between $\mathcal{B}^l$ and $\mathcal{B}^r$. We introduce $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the orthonormal basis corresponding to our coordinates for $\mathbb{R}^3$. In particular $\mathbf{e}_1$ is a direction of the line joining the two hole centers and the last unit vector $\mathbf{e}_3$ is a direction of the gravity. We underline D is the line which is parallel to the gravity and passes through the origin of our system of coordinates.

Our computations are motivated by the following formal arguments. If the ball $\mathcal{B}$ moves along the axis D, we have $\mathbf{G}(t) = (0, 0, d(t))$. In our coordinates $d(t)$ is the "altitude" of $\mathcal{B}$ at time t. We denote by $h(t)$ the distance between $\mathcal{B}$ and the holes $\mathcal{B}^l$ and $\mathcal{B}^r$ at time t. With these conventions, contact occurs between $\mathcal{B}$ and the holes if d or h vanishes. We do not envisage other kinds of collision between $\mathcal{B}$ and $\partial\Omega$ because they are precluded by former arguments (see [11]).

In a first approximation, when $\mathcal{B}$ comes close to the holes, the viscous force can be divided into the sum of two contributions. One is due to the vicinity of $\mathcal{B}^l$ and the other one to the vicinity of $\mathcal{B}^r$. Concerning $\mathcal{B}^l$ for example, we split the force in a frontal resistance preventing $\mathcal{B}$ from going closer to $\mathcal{B}^l$ and a friction. It stems from computations in the lubrication theory we can neglect the frictions in what follows [13] and the frontal resistance reads [2]:

$$-\frac{1}{(|\mathbf{G}^l - \mathbf{G}| - 2)} \frac{(\dot{\mathbf{G}}^l - \dot{\mathbf{G}}) \cdot (\mathbf{G}^l - \mathbf{G})}{|\mathbf{G}^l - \mathbf{G}|} \frac{(\mathbf{G}^l - \mathbf{G})}{|\mathbf{G}^l - \mathbf{G}|}.$$

We have an equivalent formula for the second contribution with $\mathbf{G}^r$. Eventually, the projection of the Newton laws on the linear momentum along $\mathbf{e}_3$ reads:

$$\ddot{d} = -\frac{2\dot{d}d^2}{(\sqrt{d^2+4}-2)(d^2+4)} - (m-|\mathcal{B}|)g, \tag{5}$$

where we take into account the Archimede law. This equation is complemented with initial conditions $d(0) = d_0$ and $\dot{d}(0) = \dot{d}_0$. Standard Cauchy–Lipschitz arguments imply the system is locally well-posed for initial conditions $d_0 \in \mathbb{R} \setminus \{0\}$. Moreover, maximal solutions to this system may blow up at finite time T_* in three ways

$$\limsup_{t\to T_*} |\dot{d}(t)| = \infty, \quad \limsup_{t\to T_*} |d(t)| = \infty, \quad \liminf_{t\to T_*} |d(t)| = 0.$$

However, multiplying (5) by $\dot{d}$, we obtain that this simplified model dissipates the total energy of the particle $\mathcal{B}$. This implies the only way solutions to (5) may blow up is the third one. Furthermore, we remark $\tilde{d}(t) = d_0$ for all $t \geqslant 0$ is a global supersolution to (5) regardless of the value of $d_0 \neq 0$. In particular, if $d_0 > 0$ and $\dot{d}_0 < 0$, then $d(t) \in [0, d_0]$ until blow up of the solution. So, under this assumption the only way the maximal solution may blow up in finite time T_* is

$$\lim_{t\to T_*} d(t) = 0.$$

In the following, we assume $d_0 > 0$ and $\dot{d}_0 < 0$.

Integrating (5) between 0 and t, we obtain

$$\dot{d}(t) - \dot{d}_0 = -\int_0^t \left[\frac{2\dot{d}d^2}{(\sqrt{d^2+4}-2)(d^2+4)} + (m-|\mathcal{B}|g\right] \mathrm{d}s = -P(t) - (m-|\mathcal{B}|)gt, \tag{6}$$

where, after a straightforward change of variable:

$$P(t) = \tilde{P}(|d(t)|^2) = \int_{|d_0|^2}^{|d(t)|^2} \frac{\sqrt{r}\mathrm{d}r}{(\sqrt{r+4}-2)(r+4)}.$$

Standard computations lead to $|\tilde{P}(z)| \leqslant C$ for all $z \in (0, d_0]$ Finally, assuming the function d is defined globally, (6) together with dissipation of total energy implies:

$$C \leqslant P(t) \leqslant K_0 - (m-|\mathcal{B}|)gt \quad \forall\, t \in (0,\infty),$$

with a constant K_0 depending only on initial data. Because the solid is heavier than the fluid, we obtain a contradiction for large times and d must vanish in finite time. We emphasize considering a three-dimensional example is critical here. Indeed, in the two-dimensional case we would get a function $\tilde{P}(\alpha)$ which diverges when α goes to 0.

1.3. **Notations.** We use the classical Lebesgue and Sobolev spaces $L^q(\mathcal{O})$, $W^{m,q}(\mathcal{O})$, $H^m(\mathcal{O})$, $H^m_0(\mathcal{O})$ for any open set $\mathcal{O} \subset \mathbb{R}^3$. We define

$$\mathcal{H} = \{\boldsymbol{\phi} \in L^2(\Omega)\ ;\ \operatorname{div} \boldsymbol{\phi} = 0,\ \boldsymbol{\phi}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \mathcal{V} = \{\boldsymbol{\phi} \in H^1_0(\Omega)\ ;\ \operatorname{div} \boldsymbol{\phi} = 0\}.$$

We recall that $\mathcal{H}$ and $\mathcal{V}$ are closed subspace of $L^2(\Omega)$ and $H^1_0(\Omega)$ respectively. Thus, they form Hilbert spaces with respect to the induced inner products. For an open subset $\mathcal{O} \subset \Omega$, we also consider the following Hilbert spaces:

$$\mathbb{H}(\mathcal{O}) = \{\boldsymbol{\phi} \in \mathcal{H}\ ;\ D(\boldsymbol{\phi}) = 0 \text{ in } \mathcal{O}\}, \quad \mathbb{V}(\mathcal{O}) = \{\boldsymbol{\phi} \in \mathcal{V}\ ;\ D(\boldsymbol{\phi}) = 0 \text{ in } \mathcal{O}\}.$$

To simplify, if $\mathbf{G} \in \Omega$ we set $\mathcal{B}_{\mathbf{G}} := (B(\mathbf{G}, 1))$ and $\mathcal{F}_{\mathbf{G}} := \Omega \setminus \overline{\mathcal{B}}_{\mathbf{G}}$. Moreover, if $\mathcal{B}_{\mathbf{G}} \subset \Omega$, we define $\mathbb{H}(\mathbf{G}) = \mathbb{H}(\mathcal{B}_{\mathbf{G}})$, $\quad \mathbb{V}(\mathbf{G}) = \mathbb{V}(\mathcal{B}_{\mathbf{G}})$. Under the same assumption, we also denote by $\rho_{\mathbf{G}}$ the function

$$\rho_{\mathbf{G}}(\mathbf{x}) = \begin{cases} \rho_{\mathcal{B}} & \text{if } \mathbf{x} \in \mathcal{B}_{\mathbf{G}}, \\ 1 & \text{if } \mathbf{x} \in \mathcal{F}_{\mathbf{G}}. \end{cases}$$

If $\mathbf{v} \in \mathbb{H}(\mathbf{G})$, from [18, p.18], there exists a unique pair $(\mathbf{V}[\mathbf{v}], \boldsymbol{\omega}[\mathbf{v}]) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\mathbf{v}|_{\mathcal{B}_{\mathbf{G}}} = \mathbf{V}[\mathbf{v}] + \boldsymbol{\omega}[\mathbf{v}] \times (\mathbf{x} - \mathbf{G}).$$

In particular, if $(\mathbf{u}, \mathbf{v}) \in \mathbb{H}(\mathbf{G})^2$,

$$\int_{\Omega} \rho_{\mathbf{G}} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} = \int_{\Omega \setminus \mathcal{B}_{\mathbf{G}}} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} + m \mathbf{V}[\mathbf{u}] \cdot \mathbf{V}[\mathbf{v}] + J \boldsymbol{\omega}[\mathbf{u}] \cdot \boldsymbol{\omega}[\mathbf{v}].$$

2. Cauchy theory and main result

As classical in Navier–Stokes like systems, there exist two concepts of solution. First, we can define the weak solutions.

Definition 2.1. *Assume* $\mathbf{G}_0 \in \Omega$ *such that* $\operatorname{dist}(\mathbf{G}_0, \partial\Omega) \geqslant 1$ *and* $\mathbf{u}_0 \in \mathbb{H}(\mathbf{G}_0)$, *a pair* $(\mathbf{u}, \mathbf{G})$ *is called weak solution to (FSIS) on* $(0, T)$ *with initial data* $(\mathbf{u_0}, \mathbf{G_0})$ *if*

$$\mathbf{G} \in W^{1,\infty}(0, T; \Omega), \quad \textit{with} \quad \mathbf{G}(0) = \mathbf{G}_0, \tag{7}$$

$$\operatorname{dist}(\mathbf{G}(\mathrm{t}), \partial\Omega) \geqslant 1, \quad \textit{for all} \quad t \in (0, T), \tag{8}$$

$$\mathbf{u} \in L^{\infty}(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}), \tag{9}$$

$$\mathbf{u} = \mathbf{V} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}) \quad \textit{in } \mathcal{B}_{\mathbf{G}}, \quad \textit{with} \quad \mathbf{V} = \dot{\mathbf{G}}; \tag{10}$$

if for all $\mathbf{v} \in \mathcal{C}([0, T]; H^1_0(\Omega)) \cap H^1(0, T; L^2(\Omega))$ *with compact support in* $(0, T) \times \Omega$ *and such that* $\mathbf{v} \in \mathbb{V}(\mathbf{G}(t))$ *for all* $t \in [0, T]$,

$$\begin{aligned} -\int_0^T \int_{\Omega} \rho_{\mathbf{G}} \mathbf{u} \cdot \partial_t \mathbf{v} \, \mathrm{d}\mathbf{y} \, \mathrm{d}t + 2\mu \int_0^T \int_{\Omega} D(\mathbf{u}) : D(\mathbf{v}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t \\ - \int_0^T \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : D(\mathbf{v}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t = \int_0^T \int_{\Omega} \rho_{\mathbf{G}} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{y} \, \mathrm{d}t; \end{aligned} \tag{11}$$

if for all $\mathbf{v} \in \mathcal{C}([0, T]; L^2(\Omega))$ *with compact support in* $[0, T) \times \Omega$ *and such that* $\mathbf{v} \in \mathbb{H}(\mathbf{G}(t))$ *for all* $t \in [0, T]$ *we have*

$$W : t \mapsto \int_{\Omega} \rho_{\mathbf{G}} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \in \mathcal{C}([0, T]) \quad \textit{with} \quad W(0) = \int_{\Omega} \rho_{\mathbf{G}_0} \mathbf{u}_0 \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}; \tag{12}$$

and if the energy estimate holds true:

$$\begin{aligned} \left[\frac{1}{2} \int_{\Omega} \rho_{\mathbf{G}} |\mathbf{u}|^2(t, \mathbf{x}) \, \mathrm{d}\mathbf{x} + g(m - |\mathcal{B}|) \mathbf{G}(t) \cdot \mathbf{e}_3 \right] + 2\mu \int_0^t \int_{\Omega} |D(\mathbf{u})|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}s \\ \leqslant \left[\frac{1}{2} \int_{\Omega} \rho_{\mathbf{G}_0} |\mathbf{u}_0|^2(\mathbf{x}) \, \mathrm{d}\mathbf{x} + g(m - |\mathcal{B}|) \mathbf{G}_0 \cdot \mathbf{e}_3 \right] \quad \textit{for a.a. } t \in (0, T). \end{aligned} \tag{13}$$

There could be slightly different definitions in other articles, here we use the same definition used in [11]. For instance, the main differences between Definition 2.1 and the definition used in [5] are the following. First, as we work with a constant-density fluid, we introduce the position of the center of mass $\mathbf{G}$ as unknown instead of the density $\overline{\rho}$ and isometry η. From our weak solution, one can build back these unknowns setting

$$\overline{\rho}(t,x) = \mathbf{1}_{\mathcal{F}_t}(x) + \rho_{\mathcal{B}}\mathbf{1}_{\mathcal{B}_t}(x) \quad \forall\,(t,x) \in (0,T)\times\Omega,$$

and choosing for η the composition of the translation associated to $\mathbf{G}(t) - \mathbf{G_0}$ with some rotation associated to $\boldsymbol{\omega}$. We emphasize that we actually do not need any information on this rotation because $\mathcal{B}$ is a ball. Concerning energy estimate, we have the above particular form because, in [5, (1.16)], we replace the source term $\mathbf{f}$ by the gravity with direction $\mathbf{e}_3$. Hence, after integration by parts, we get

$$\int_\Omega \rho(t,\mathbf{x})\mathbf{f}(t,\mathbf{x})\cdot\mathbf{u}(t,\mathbf{x})\ \mathrm{d}\mathbf{x} = -\int_{\mathcal{B}_t}(\rho_{\mathcal{B}}-1)g\mathbf{u}(t,\mathbf{x})\cdot\mathbf{e}_3\ \mathrm{d}\mathbf{x} = -g(m-|\mathcal{B}|)\dot{\mathbf{G}}(t)\cdot\mathbf{e}_3.$$

Finally, we can apply the result in reference [5] to obtain that weak solutions to (FSIS) do exist globally regardless of the initial position of $\mathcal{B}$ with $\mathrm{dist}(\mathbf{G_0},\partial\Omega) > 1$ and the value of initial data.

The second classical concept of solution can be rephrased as follows.

Definition 2.2. *Given* $\mathbf{G}_0 \in \Omega$ *such that* $\mathrm{dist}(\mathbf{G}_0,\partial\Omega) > 1$ *and* $\mathbf{u}_0 \in \mathbb{V}(\mathbf{G}_0)$*, a pair* $(\mathbf{u},\mathbf{G})$ *is a strong solution to (FSIS) on* $(0,T)$ *if it is a weak solution to (FSIS) with the additional regularity:*

$$\mathbf{u} \in \mathcal{C}(0,T;\mathcal{V}),\quad \text{and}\ \ \mathbf{u}(t,\cdot)\in H^2(\mathcal{F}_t),\ p(t,\cdot)\in H^1(\mathcal{F}_t),\quad \text{for a.a. } t\in(0,T), \tag{14}$$

$$\sup_{(0,t)}\|\nabla\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t\int_{\mathcal{F}_t}|\nabla^2\mathbf{u}(t,\mathbf{x})|^2 + |\nabla p(t,\mathbf{x})|^2\ \mathrm{d}\mathbf{x}\ \mathrm{d}t < \infty,\qquad \text{in } (0,T). \tag{15}$$

This is the equivalent notion to the one developed in [17] but this reformulation allows us to deal with collision. In this definition, we measure the regularity of the velocity field after restriction to the fluid domain only. We emphasize that as long as no collision occurs, both concepts are equivalent. In particular, the classical local and uniqueness result still holds in three dimensions [17]. However, contrary to the weak solutions, there is no result for large times. Indeed, for a fixed ball $\mathcal{B}$, (FSIS) becomes a particular case of the Navier–Stokes system. Consequently, (FSIS) contains the complexity of Navier–Stokes system. Moreover, as pointed out in [16], (FSIS) is more singular in the sense that collision is a second way for strong solutions to blow up.

2.1. **Main result.** We prove here this second way for strong solutions to blow up can occur. To this end, we obtain the following fundamental result on collisions:

Theorem 2. *Given* $(\mathbf{u},\mathbf{G})$ *a global weak solution to (FSIS), such that* $\mathbf{G}(t) = (0,0,d(t))$ *for all* $t\in(0,\infty)$*, with* $d(0) > 0$*, there exists* $T^* < \infty$ *such that* $\mathrm{dist}(\mathcal{B}_{T^*},\partial\Omega) = 0$.

Before going to the proof of Theorem 2 in further details, we explain how it implies **Theorem 1**. Let us assume at first **Theorem 1** is false. Hence, given any initial condition $(\mathbf{u}_0,\mathbf{G}_0)$ there would be a global strong solution to (FSIS). Due to arguments in [17], this strong solution is unique before collision.

Given any velocity field $\mathbf{v}$ defined on Ω, we denote by S_D the mapping

$$S_D[\mathbf{v}](\mathbf{x}) = (-v_1, -v_2, v_3)(-x_1, -x_2, x_3) \quad \forall\, \mathbf{x} \in \Omega,$$

and we assume that the initial data $(\mathbf{G_0}, \mathbf{u_0})$ satisfies

$$S_D[\mathbf{u_0}] = \mathbf{u_0}, \quad \mathbf{G}_0 = (0,0,d_0), \quad \text{with } d_0 > 0.$$

Let $(\mathbf{G}, \mathbf{u})$ be a global strong solution to (FSIS) with initial data $(\mathbf{G_0}, \mathbf{u_0})$. One can check that $(\tilde{\mathbf{G}}, \tilde{\mathbf{u}})$ as defined by

$$\tilde{\mathbf{G}}(t) = (-G_1, -G_2, G_3)(t), \quad \tilde{\mathbf{u}}(t, \cdot) = S_D[\mathbf{u}(t, \cdot)], \quad \forall\, t \geqslant 0,$$

is also a strong solution to (FSIS) with the same initial data. Hence $\tilde{\mathbf{G}} = \mathbf{G}$ and $\tilde{\mathbf{u}} = \mathbf{u}$ so that $(\mathbf{G}, \mathbf{u})$ is symmetric with respect to D before contact. In particular, it is a weak solution such that $\mathbf{G}(t) = (0,0,d(t))$ with $d(t) > 0$ before collision. Applying **Theorem 2**, there exists $T^* < \infty$ for which $\mathrm{dist}(\mathcal{B}_{T^*}, \partial\Omega) = 0$. Without further restriction, we assume T^* is the first time of collision and in particular $h(t) = \mathrm{dist}(\mathcal{B}_t, \partial\Omega) > 0$ and $\mathbf{G}(t) \in D$ for $t \in [0, T^*)$. Then we have (see Appendix B recognizing $\mathbf{V} \cdot \tilde{\mathbf{e}}_3 = \dot{h}$):

$$|\dot{h}(t)| \leqslant C|h(t)|^{\frac{3}{2}} \|\nabla^2 \mathbf{u}(t, \cdot)\|_{L^2(\mathcal{F}_t)} \tag{16}$$

for some universal constant C. Consequently, (16) implies h might not vanish at time T^* as

$$\int_0^{T^*} \|\nabla^2 \mathbf{u}\|_{L^2(\mathcal{F}_t)}\, \mathrm{d}t < \infty. \tag{17}$$

Thus $(\mathbf{G}, \mathbf{u})$ is not a strong solution defined until T^*.

The remainder of this paper is devoted to the proof of **Theorem 2**.

2.2. **Sketch of the proof of Theorem 2.** The proof of **Theorem 2** follows the same ideas as in [6, 11]. In the remainder of this section $(\mathbf{u}, \mathbf{G})$ is a given weak solution such that $\mathbf{G}(t) = (0,0,d(t))$ at any time. In particular, it has initial data $(\mathbf{u}(0,\cdot), \mathbf{G}(0))$ where $\mathbf{G}(0) = (0,0,d_0)$ with $d_0 > 0$. Following similar arguments as in [11], collisions between $\mathcal{B}$ and $\partial\Omega \setminus (\partial\mathcal{B}^l \cup \partial\mathcal{B}^r)$ are ruled out. On the contrary, we prove that simultaneous contact between $\mathcal{B}$ and $(\mathcal{B}^l, \mathcal{B}^r)$ occurs in finite time. So, we denote by $h(t)$ the common distance between $\mathcal{B}$ and the holes $\mathcal{B}^l$ and $\mathcal{B}^r$ at time t. Combining that $\mathbf{G}(t) = (0,0,d(t))$ with $d(t) > 0$ and $\mathrm{dist}(\mathcal{B}_t, \mathcal{B}^l) = \mathrm{dist}(\mathcal{B}_t, \mathcal{B}^r) = h(t)$ we obtain that, before contact

$$\mathbf{G}(t) = \mathbf{G}_{h(t)} = \left(0, 0, \sqrt{h(t)^2 + 4h(t)}\right). \tag{18}$$

We restrict ourselves to the case $d(t) > 0$, because we assume initially that the solid is "above" the holes. We prove by contradiction that $h(t)$ is bound to vanish in finite time.

We emphasize that, as collisions between $\mathcal{B}$ and $\partial\Omega \setminus (\partial\mathcal{B}^l \cup \partial\mathcal{B}^r)$ are impossible, there exists $h_{max} > 0$ such that $h(t) \in (0, h_{max}]$ before collision and

$$\mathrm{dist}(\mathcal{B}_h, \partial\Omega \setminus (\partial\mathcal{B}^l \cup \partial\mathcal{B}^r)) \geqslant \delta_0 > 0 \quad \forall\, h \in [0, h_{max}],$$

where $\mathcal{B}_h = \mathcal{B}_{\mathbf{G}_h}$ with the convention (18). In next section, we construct a suitable family of "stationary" test functions to use in the weak formulation. This family of test velocity fields reads $(\mathbf{w}[h])_{h>0}$ where, for arbitrary $h > 0$, there holds $\mathbf{w}[h] \in \mathbb{H}(\mathbf{G}_h)$. In the following, we replace $\mathbf{G}$ by h in notation, assuming $\mathbf{G} = \mathbf{G}_h$ implicitly. For example $\mathcal{B}_h = \mathcal{B}_{\mathbf{G}_h}$, as above, and $\mathcal{F}_h = \Omega \setminus \overline{\mathcal{B}}_h$. The test functions $\mathbf{w}[h]$ will be constructed so that they satisfy many properties. First we have the following result.

Proposition 1. *Given $h_{min} > 0$, there holds:*

(1) *for any $h \in [h_{min}, h_{max}]$, $\mathbf{w}[h] \in \mathcal{C}(\overline{\Omega})$ with*

$$\mathbf{w}[h] = \mathbf{e_3} \text{ on } \mathcal{B}_h, \quad \mathbf{w}[h] = 0 \text{ on } \partial\Omega,$$

(2) *assume $\mathcal{Q}_h := \{(h,x), \quad h \in (h_{min}, h_{max}), \quad x \in \mathcal{F}_h\}$, and*

$$\begin{array}{rccc} \tilde{\mathbf{w}} : & (0, h_{max}) \times \Omega & \longrightarrow & \mathbb{R}^3, \\ & (h, \mathbf{x}) & \longmapsto & \mathbf{w}[h](\mathbf{x}), \end{array}$$

then $\tilde{\mathbf{w}} \in \mathcal{C}^\infty(\overline{\mathcal{Q}}_h)$.

Assuming at first the function h does not vanish on $(0,T)$ (where T is arbitrary) there exists $h_{min} > 0$ such that $h(t) > h_{min}$ for any $t \in [0,T]$. Hence, for any $\chi \in \mathcal{D}(0,T)$ we can use the following test function in (11):

$$\begin{array}{rccc} \mathbf{w} : & (0, T) \times \Omega & \longrightarrow & \mathbb{R}^3, \\ & (t, \mathbf{x}) & \longmapsto & \chi(t)\, \mathbf{w}[h(t)](\mathbf{x}). \end{array}$$

This yields

$$\begin{aligned} (19) \quad & \int_0^T \int_\Omega \rho_h \mathbf{u} \cdot \partial_t \mathbf{w} \, \mathrm{d}\mathbf{y} \, \mathrm{d}t + \int_0^T \int_\Omega \rho_h \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}\mathbf{y} \, \mathrm{d}t \\ & = - \int_0^T \int_\Omega \mathbf{u} \otimes \mathbf{u} : D(\mathbf{w}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t + 2\mu \int_0^T \int_\Omega D(\mathbf{u}) : D(\mathbf{w}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t. \end{aligned}$$

We split this identity in $I_1^l + I_2^l = I_1^r + I_2^r$ where, after straightforward computations:

$$I_1^l = \int_0^T \dot{\chi} \int_\Omega \rho_h \mathbf{u} \cdot \mathbf{w}[h] \, \mathrm{d}\mathbf{y} \, \mathrm{d}t + \int_0^T \dot{h} \chi \int_\Omega \rho_h \mathbf{u} \cdot \partial_h \mathbf{w}_h \, \mathrm{d}\mathbf{y} \, \mathrm{d}t, \qquad I_2^l = -(m - |B|) g \int_0^T \chi(s) \, \mathrm{d}s.$$

In Section 4, we prove:

Proposition 2. *There exist a positive constant C depending only on $\rho_{\mathcal{B}}$ and h_{max} such that, for any $\mathbf{v} \in \mathbb{H}(\mathbf{G}_h)$, there hold:*

$$(20) \qquad \left| \int_\Omega \rho_h \mathbf{v} \cdot \mathbf{w}[h] \, \mathrm{d}\mathbf{y} \right| \leqslant C \|\mathbf{v}\|_{L^2(\Omega)},$$

$$(21) \qquad \left| \int_\Omega \rho_h \mathbf{v} \cdot \partial_h \mathbf{w}[h] \, \mathrm{d}\mathbf{y} \right| \leqslant \frac{C \|\nabla \mathbf{v}\|_{L^2(\Omega)}}{\sqrt{h}},$$

$$(22) \qquad \left| \int_\Omega \mathbf{v} \otimes \mathbf{v} : D(\mathbf{w}[h]) \, \mathrm{d}\mathbf{y} \right| \leqslant C \|\nabla \mathbf{v}\|^2_{L^2(\Omega)}.$$

Moreover, if $\mathbf{v} = \ell \mathbf{e_3}$ on $\mathcal{B}_h$ then

$$(23) \qquad \int_\Omega D(\mathbf{v}) : D(\mathbf{w}[h]) \, \mathrm{d}\mathbf{y} = \ell b(h) + R$$

with $|R| \leqslant C \|\nabla \mathbf{v}\|_{L^2(\Omega)}$ and with $0 \leqslant b(h) \leqslant C |\ln(h)|$.

The content of this proposition is twofold. First, inequalities (20)–(22) enable to dominate remainder terms in (19). Indeed, combining these inequalities and energy estimate (13), this yields

$$|I_1^l| \leqslant \int_0^T \left[C|\dot{\chi}| \; \|\mathbf{u}(t,\cdot)\|_{L^2(\Omega)} + C|\chi(t)| \; |\dot{h}(t)||h(t)|^{-\frac{1}{2}} \; \|\nabla \mathbf{u}(t,\cdot)\|_{L^2(\Omega)} \right] \, \mathrm{d}t,$$

where, applying [16, Theorem 3.1], there exists a universal constant for which

$$|\dot{h}(t)||h(t)|^{-1/2} \leqslant C\|\nabla \mathbf{u}(t,\cdot)\|_{L^2(\Omega)}.$$

Consequently

$$|I_1^l| \leqslant C_0 \left(\|\dot{\chi}\|_{L^1(0,T)} + \|\chi\|_{L^\infty(0,T)} \right)$$

with C_0 a constant depending only on initial data and the size of Ω. We emphasize that here (13) implies decrease of the total energy of the system. As Ω is bounded this implies a T-independent control on the solution in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. This uniform estimate would not persist if $\mathbf{f}$ were not deriving from such a potential. Similarly

$$|I_1^r| \leqslant \int_0^T C\|\chi\|_{L^\infty(0,T)} \|\nabla \mathbf{u}(t,\cdot)\|^2_{L^2(\Omega)} \, \mathrm{d}t \leqslant C_0 \|\chi\|_{L^\infty(0,T)}.$$

Second, inequality (23) computes the drag acted on a body in a fluid flow $\mathbf{v}$ with a precision $O(|\nabla \mathbf{v}|_{L^2(\Omega)})$. In the frame of our weak solution, this leads to

$$\left| I_2^r - 2\mu \int_0^T \chi(t)\dot{d}(t) b(h(t)) \, \mathrm{d}t \right| \leqslant C_0 \|\chi\|_{L^\infty(0,T)} \sqrt{T},$$

where $\dot{d}(t) = \dot{h}(h+2)/\sqrt{h^2+4h}$ (see (18)).

Eventually (19) reduces to:

$$\int_0^T \chi(t) \left[2\mu \frac{\dot{h}(t)(h(t)+2)b(h(t))}{\sqrt{h(t)^2+4h(t)}} + (m - |\mathcal{B}|)g \right] \, \mathrm{d}t \leqslant C_0(1+\sqrt{T}) \left[\|\chi\|_{L^\infty(0,T)} + \|\dot{\chi}\|_{L^1(0,T)} \right].$$

Using a family of functions χ converging in a monotone way toward the characteristic function of $(0,T)$, we obtain

$$\int_0^T 2\mu \frac{\dot{h}(t)(h(t)+2)}{\sqrt{h(t)^2+4h(t)}} b(h(t)) \, \mathrm{d}t \leqslant -(m-|\mathcal{B}|)gT + C_0(1+\sqrt{T}). \tag{24}$$

On the other hand, the above control on b implies

$$h \mapsto \int_{h_0}^h b(s) \frac{s+2}{\sqrt{s^2+4s}} \, \mathrm{d}s$$

is bounded continuous when h goes to 0. Hence, (24) leads to a contradiction as in our toy-model. This completes the proof of **Theorem 2**.

3. Constructing the test functions

In this section we construct the test functions used to prove Theorem 2. We build these test functions in the half space $\mathcal{P}^l := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \ ; \ x_1 \leqslant 0\}$, the constructions of the test functions in the half space $\mathcal{P}^r := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \ ; \ x_1 \geqslant 0\}$ are done by symmetry. In each half space, it is more convenient to work in a local orthonormal frame attached to the moving ball $\mathcal{B}$. The origin of this local frame is $\mathbf{G}$ and the associated direct orthonormal basis is $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$. We consider $\tilde{\mathbf{e}}_2 = \mathbf{e}_2$ and $\tilde{\mathbf{e}}_3$ is such that $(\mathbf{G} - \mathbf{G}^l) = (2+h)\tilde{\mathbf{e}}_3$. For any $\mathbf{x} \in \mathbb{R}^3$ we denote by $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ its coordinates in this new frame. In particular, the following holds:

$$\tilde{\mathbf{x}} = Q_\alpha(\mathbf{x} - \mathbf{G}) \quad \text{or} \quad \mathbf{x} = \mathbf{G} + Q_\alpha^{-1}\tilde{\mathbf{x}}$$

with Q_α the rotation with axis $\mathbb{R}\mathbf{e}_2$ and angle α (see Figure 3 for the definition of α). We also introduce (r,θ,z) the cylindrical coordinates:

$$\tilde{x}_1 = r\cos(\theta), \quad \tilde{x}_2 = r\sin(\theta), \quad \tilde{x}_3 = z.$$

In the following, we keep this convention for sets. So, if not precisely mentioned, for any set $\mathcal{S} \subset \mathbb{R}^3$ the following holds:

$$\tilde{\mathcal{S}} = Q_\alpha(\mathcal{S} - \mathbf{G}) \quad \text{or} \quad \mathcal{S} = \mathbf{G} + Q_\alpha^{-1}\tilde{\mathcal{S}}.$$

Actually, we shall only make one exception. Indeed, in this new frame the ball $\mathcal{B}$ is fixed and centered in $\mathbf{0}$ whereas the center $\mathbf{G}^l$ of $\mathcal{B}^l$ has moving coordinates $(0,0,-2-h)$. Consequently, we prefer to use $\tilde{\mathcal{B}}_*$ for the image of $\mathcal{B}$ (which is fixed) and $\tilde{\mathcal{B}}_h$ for the image of $\mathcal{B}^l$. Hence, $\tilde{\mathcal{B}}_*$ and $\tilde{\mathcal{B}}_h$ are the unit balls in $\mathbb{R}^3$ centered in the origin and in $(0,0,-2-h)$ respectively.

When $h = 0$, the fluid domain $\tilde{\mathcal{F}}_0$ has a cusp where $\tilde{\mathcal{B}}_*$ is in contact with $\tilde{\mathcal{B}}_0$. In order to surround this singularity we introduce a family of neighborhoods of the points realizing the distance between $\tilde{\mathcal{B}}_h$ and $\tilde{\mathcal{B}}_*$. Given $h \in (0, h_{max})$ and $l \in (0,1)$, we denote by $\tilde{\Omega}_{h,l}$ the cylindric domain between $\tilde{\mathcal{B}}_*$ and $\tilde{\mathcal{B}}_h$ with radius l:

$$\tilde{\Omega}_{h,l} := \{(r,\theta,z) \in \tilde{\mathcal{F}}_h \text{ such that } r \in [0,l), z \in (-(2+h),0)\}. \tag{25}$$

We remark that, given $h_{max} > 0$, there exists $l_{max} > 0$ such that $\tilde{\Omega}_{h,l_{max}} \subset \tilde{\mathcal{P}}^l$ for any $h \in [0, h_{max}]$. We assume $l_{max} > 1/2$. We emphasize that this is only for legibility. Indeed, one could replace $1/2$ by $l_{max}/2$ in what follows without changing the computations.

We notice that the upper boundary and the lower boundary of $\tilde{\Omega}_{h,l}$ are parametrized respectively by:

$$(r,\theta,z) \in \partial\tilde{\Omega}_{h,l} \cap \partial\tilde{\mathcal{B}}_* \Leftrightarrow \{r \in [0,\delta) \text{ and } z = \delta_*(r)\},$$

where

$$\delta_*(s) := -\sqrt{1-s^2} \quad \forall\, s \in [0,1), \tag{26}$$

and

$$(r,\theta,z) \in \partial\tilde{\Omega}_{h,l} \cap \partial\tilde{\mathcal{B}}_h \Leftrightarrow \{r \in [0,\delta) \text{ and } z = \delta_h(r)\},$$

where, for all $h > 0$,

$$\delta_h(s) := -(2+h) + \sqrt{1-s^2} \quad \forall\, s \in [0,1). \tag{27}$$

Finally, the remainder of the geometry (*i.e.* outside $\tilde{\Omega}_{h,1/2}$) is "smooth" in the sense that, there exists a width h_0 such that, for any distance $h \in [0, h_{max}]$ there exists a set with width

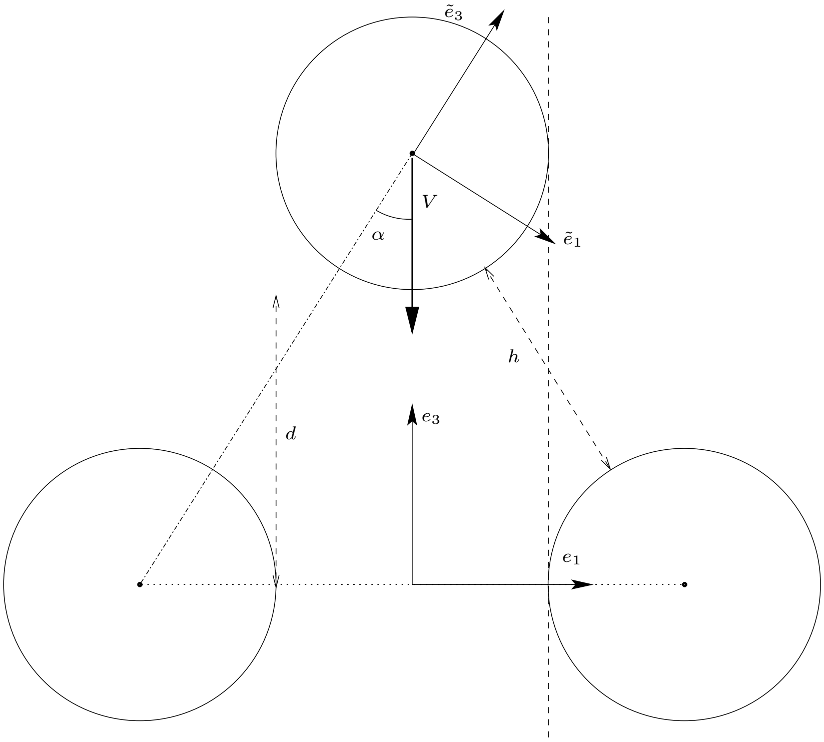

FIGURE 2. Detailed description of the geometry

h_0 which surrounds the boundaries of $\tilde{\mathcal{B}}_*$ and the hole $\tilde{\mathcal{B}}_h$. To turn this into a quantitative information, we define:

$$h_0 = \frac{1}{2} \inf_{0 \leqslant h \leqslant h_{max}} \mathrm{dist}(\partial \tilde{\mathcal{B}}_* \cap (\tilde{\Omega}_{h,\frac{1}{4}})^c, \partial \tilde{\mathcal{B}}_h \cap (\tilde{\Omega}_{h,\frac{1}{4}})^c) = \frac{\sqrt{17/16} - 1}{2}.$$

With this choice, for M large enough, for any $h \in [0, h_{max}]$ and $\tilde{\mathbf{x}} \notin \tilde{\Omega}_{h,1/4}$, if $0 < \mathrm{dist}(\tilde{\mathbf{x}}, \tilde{\mathcal{B}}_h) < h_0$ or $0 < \mathrm{dist}(\tilde{\mathbf{x}}, \tilde{\mathcal{B}}_*) < h_0$ then $\tilde{\mathbf{x}}$ is in the fluid domain $\tilde{\mathcal{F}}_h$. We have moreover that, if $\mathrm{dist}(\tilde{\mathbf{x}}, \tilde{\mathcal{B}}_h) \leqslant h_0$ then $\tilde{\mathbf{x}} \in \tilde{\mathcal{P}}^l$.

3.1. **Parallel component.** We first construct a velocity field that is rigid in $\tilde{\mathcal{B}}_*$ with rigid velocity equal to $\tilde{\mathbf{e}}_1$. At first, this velocity field is computed in the local frame, which means with coordinates computed in the local orthonormal basis $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$. Between the two spheres, our potential vector field reads, in cylindrical coordinates:

$$\tilde{\mathbf{a}}^d_{/\!/}(r, \theta, z) = \left(0, \phi_{/\!/}(r, z), \frac{1}{2} r \sin(\theta)\right) \quad \forall\, (r, \theta, z) \in \tilde{\Omega}_{h,1/2}.$$

Hence, the components of $\tilde{\mathbf{w}}^d_{/\!/}[h] = \mathrm{curl}\, \tilde{\mathbf{a}}^d_{/\!/}[h]$ read:

$$\tilde{\mathbf{w}}^d_{/\!/}(r, \theta, z) = \left(\frac{1}{2} - \partial_z \phi_{/\!/}(r, z), 0, \cos(\theta) \partial_r \phi_{/\!/}(r, z)\right) \quad \forall\, (r, \theta, z) \in \tilde{\Omega}_{h,1/2}. \tag{28}$$

In this expression, d stands for "diverging part" and $\phi_{/\!/}$ is a truncation function enabling $\tilde{\mathbf{w}}^d_{/\!/}$ to go from $(1,0,0)$ on $\partial\tilde{\mathcal{B}}_*$ to $(0,0,0)$ on $\partial\tilde{\mathcal{B}}_h$. We set, in order to fit with these boundary conditions (this shall be critical in **Lemma 3**):

$$\phi_{/\!/}(r,z) = -\frac{\chi_{/\!/}(\lambda(r,z))}{4}(\delta_*(r)-\delta_h(r)) + \frac{2+h}{4}, \tag{29}$$

with

$$\chi_{/\!/}(s) = 2s^2 - 2s + 1, \quad \forall\, s \in (0,1), \tag{30}$$

and where λ is the normalized vertical distance do $\partial\tilde{\mathcal{B}}_h$:

$$\lambda(r,z) = \frac{z-\delta_h(r)}{\delta_*(r)-\delta_h(r)}.$$

In the complement of $\tilde{\Omega}_{h,1/2}$ we set:

$$\tilde{\mathbf{a}}^s_{/\!/} = \frac{\eta_{h_0}(|\tilde{\mathbf{x}}+(0,0,2+h)|-1)}{2}(0,\ (z+2+h)/2,\ r\sin(\theta)/2) \\ + \frac{\eta_{h_0}(|\tilde{\mathbf{x}}|-1)}{2}(\tilde{\mathbf{e}}_1\times\tilde{\mathbf{x}}) \quad \forall\,\tilde{\mathbf{x}}\in\mathbb{R}^3.$$

Here and in what follows, we denote by $\eta : [0,\infty) \to [0,1]$ a smooth function such that

$$\eta(s) = \begin{cases} 1, & \text{if } s < \frac{1}{2}, \\ 0, & \text{if } s > 1, \end{cases}$$

and, we set $\eta_\alpha = \eta(\cdot/\alpha)$ for all parameter $\alpha > 0$. By definition of h_0, if $\tilde{\mathbf{x}} \notin \tilde{\Omega}_{h,1/4}$, then at most one of the functions $\eta_{h_0}(|\tilde{\mathbf{x}}+(0,0,2+h)|-1)$ and $\eta_{h_0}(|\tilde{\mathbf{x}}|-1)$ is different from 0.

Finally, we set:

$$\tilde{\mathbf{a}}_{/\!/} = \begin{cases} \eta_{1/2}(r)\tilde{\mathbf{a}}^d_{/\!/} + (1-\eta_{1/2}(r))\tilde{\mathbf{a}}^s_{/\!/}, & \text{in } \tilde{\Omega}_{h,1/2}, \\ \tilde{\mathbf{a}}^s_{/\!/}, & \text{in } \mathbb{R}^3 \setminus \left(\tilde{\Omega}_{h,1/2}\cup\tilde{\mathcal{B}}_*\cup\tilde{\mathcal{B}}_h\right), \end{cases}$$

and

$$\tilde{\mathbf{w}}_{/\!/}[h] = \begin{cases} \operatorname{curl}\tilde{\mathbf{a}}_{/\!/}, & \text{in } \mathbb{R}^3\setminus\left(\tilde{\mathcal{B}}_*\cup\tilde{\mathcal{B}}_h\right), \\ \tilde{\mathbf{e}}_1, & \text{in } \tilde{\mathcal{B}}_*, \\ 0, & \text{in } \tilde{\mathcal{B}}_h. \end{cases}$$

This family of functions $\left(\tilde{\mathbf{w}}_{/\!/}[h]\right)_{h>0}$ satisfies the following result.

Proposition 3. *For any $h > 0$, the following holds:*

(1) $\tilde{\mathbf{w}}_{/\!/}[h] \in \mathcal{C}(\mathbb{R}^3)$, *with:*

$$\tilde{\mathbf{w}}_{/\!/}[h] = \tilde{\mathbf{e}}_1 \text{ on } \tilde{\mathcal{B}}_*, \quad \tilde{\mathbf{w}}_{/\!/}[h] = 0 \text{ on } \tilde{\mathcal{B}}_h.$$

(2) *In a neighborhood of $\partial\tilde{\mathcal{P}}^l$*

$$\tilde{\mathbf{w}}_{/\!/}[h](\tilde{\mathbf{x}}) = \operatorname{curl}_{\tilde{\mathbf{x}}}\left(\frac{\eta_{h_0}(|\tilde{\mathbf{x}}|-1)}{2}\,\tilde{\mathbf{e}}_1\times\tilde{\mathbf{x}}\right). \tag{31}$$

Proof: As $h > 0$, the only difficulty to obtain (1) is to prove continuity through $\partial\tilde{\mathcal{B}}_h$ and $\partial\tilde{\mathcal{B}}_*$. In the following we drop arguments in λ and we denote by subscripts its differentiations. For example,

$$\lambda_z = \frac{1}{\delta_*(r) - \delta_h(r)} \quad \lambda_r = -\frac{\delta_h'(r)}{\delta_*(r)-\delta_h(r)} - \lambda \frac{\delta_*'(r) - \delta_h'(r)}{\delta_*(r)-\delta_h(r)}.$$

Differentiating $\phi_{/\!/}$, this yields

$$\partial_z \phi_{/\!/}(r,z) = -\frac{\chi_{/\!/}'(\lambda)}{4}\lambda_z(\delta_*(r)-\delta_h(r)) = -\frac{\chi_{/\!/}'(\lambda)}{4}, \tag{32}$$

$$\partial_r \phi_{/\!/}(r,z) = -\frac{\chi_{/\!/}'(\lambda)}{4}\lambda_r(\delta_*(r)-\delta_h(r)) - \frac{\chi_{/\!/}(\lambda)}{4}(\delta_*'(r) - \delta_h'(r)), \tag{33}$$

where:

$$\chi_{/\!/}(0) = \chi_{/\!/}(1) = 1, \quad \chi_{/\!/}'(0) = -2, \quad \chi_{/\!/}'(1) = 2.$$

As a consequence for $\lambda = 0$ ($z = \delta_h(r)$):

$$\tilde{\mathbf{a}}^d_{/\!/}(\tilde{\mathbf{x}}) = (0,\ (z + (2+h))/2,\ r\sin(\theta)/2) \quad \text{and} \quad \tilde{\mathbf{w}}^d_{/\!/}(\tilde{\mathbf{x}}) = \mathbf{0} \quad \forall\, \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_h, \tag{34}$$

and for $\lambda = 1$ ($z = \delta_*(r)$):

$$\tilde{\mathbf{a}}^d_{/\!/}(\tilde{\mathbf{x}}) = (\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{x}})/2 \quad \text{and} \quad \tilde{\mathbf{w}}^d_{/\!/}(\tilde{\mathbf{x}}) = \tilde{\mathbf{e}}_1 \quad \forall\, \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_*. \tag{35}$$

Concerning the smooth part, we recall that we chose h_0 so that outside $\tilde{\Omega}_{h,1/4}$, we have :

$$\begin{array}{ll} \eta_{h_0}(|\tilde{\mathbf{x}} - (0,0,-(2+h))| - 1) = 1, & \eta_{h_0}(|\tilde{\mathbf{x}}| - 1) = 0, \\ \eta_{h_0}'(|\tilde{\mathbf{x}} - (0,0,-(2+h))| - 1) = 0, & \eta_{h_0}'(|\tilde{\mathbf{x}}| - 1) = 0, \end{array} \quad \forall\, \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_h,$$

and

$$\begin{array}{ll} \eta_{h_0}(|\tilde{\mathbf{x}} - (0,0,-(2+h))| - 1) = 0, & \eta_{h_0}(|\tilde{\mathbf{x}}| - 1) = 1, \\ \eta_{h_0}'(|\tilde{\mathbf{x}} - (0,0,-(2+h))| - 1) = 0, & \eta_{h_0}'(|\tilde{\mathbf{x}}| - 1) = 0, \end{array} \quad \forall\, \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_*.$$

Consequently, outside $\tilde{\Omega}_{h,1/4}$:

$$\begin{cases} \tilde{\mathbf{a}}^s_{/\!/}(\tilde{\mathbf{x}}) = (0,\ (z+2+h)/2,\ r\sin(\theta)/2), & \tilde{\mathbf{w}}^s_{/\!/}(\tilde{\mathbf{x}}) = \mathbf{0}, \quad \text{on } \partial\tilde{\mathcal{B}}_h, \\ \tilde{\mathbf{a}}^s_{/\!/}(\tilde{\mathbf{x}}) = (\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{x}})/2, & \tilde{\mathbf{w}}^s_{/\!/}(\tilde{\mathbf{x}}) = \tilde{\mathbf{e}}_1, \quad \text{on } \partial\tilde{\mathcal{B}}_*. \end{cases} \tag{36}$$

The continuity of $\tilde{\mathbf{w}}_{/\!/}[h]$ through $\partial\tilde{\mathcal{B}}_*$ and $\partial\tilde{\mathcal{B}}_h$ yields by interpolation of (34)–(35) and (36).

Finally, equality (31) holds outside $\tilde{\Omega}_{h,1/2}$ and at distance larger than h_0 of $\tilde{\mathcal{B}}_h$. Due to our choice for h_0 and because we assume $1/2 < l_{max}$, this equality holds in particular in a neighborhood of $\partial\tilde{\mathcal{P}}^l$.

◇

3.2. **Normal component.** Now, we construct a velocity field that is rigid in $\tilde{\mathcal{B}}_*$ with rigid velocity equal to $\tilde{\mathbf{e}}_3$. This is the direction along which the ball $\tilde{\mathcal{B}}_*$ gets closer to the hole $\tilde{\mathcal{B}}_h$. This construction is completely similar to the one in [11]. We only change the value of λ by using the formula of the previous section. Hence, our potential vector field reads, in cylindrical coordinates:

$$\tilde{\mathbf{a}}^d_\perp(r,\theta,z) = (-\phi_\perp \sin\theta, \phi_\perp \cos\theta, 0) \quad \forall (r,\theta,z) \in \tilde{\Omega}_{h,1/2},$$

where

$$\phi_\perp(r,z) = r\chi_\perp(\lambda), \tag{37}$$

with

$$\chi_\perp(s) = \frac{s^2(3-2s)}{2} \quad (s \in (0,1)).$$

Consequently, for all $(r,\theta,z) \in \tilde{\Omega}_{h,1/2}$:

$$\tilde{\mathbf{w}}^d_\perp(r,\theta,z) = \mathrm{curl}\,\tilde{\mathbf{a}}^d_\perp = \left(-\partial_z\phi_\perp\cos\theta, -\partial_z\phi_\perp\sin\theta, \partial_r\phi_\perp + \frac{\phi_\perp}{r}\right). \tag{38}$$

In the complement of $\tilde{\Omega}_{h,1/2}$, we set:

$$\tilde{\mathbf{a}}^s_\perp = \frac{\eta_{h_0}(|\tilde{\mathbf{x}}|-1)}{2}(\tilde{\mathbf{e}}_3 \times \tilde{\mathbf{x}}) \quad \forall \tilde{\mathbf{x}} \in \mathbb{R}^3$$

and we obtain the final potential by interpolation:

$$\tilde{\mathbf{a}}_\perp = \begin{cases} \eta_{1/2}(r)\tilde{\mathbf{a}}^d_\perp + (1-\eta_{1/2}(r))\tilde{\mathbf{a}}^s_\perp, & \text{in } \tilde{\Omega}_{h,1/2}, \\ \tilde{\mathbf{a}}^s_\perp, & \text{in } \mathbb{R}^2 \setminus \left(\tilde{\Omega}_{h,1/2} \cup \tilde{\mathcal{B}}_h \cup \tilde{\mathcal{B}}_*\right). \end{cases}$$

Finally, we set:

$$\tilde{\mathbf{w}}_\perp[h] = \begin{cases} \mathrm{curl}\,\tilde{\mathbf{a}}_\perp, & \text{in } \mathbb{R}^3 \setminus \left(\tilde{\mathcal{B}}_* \cup \tilde{\mathcal{B}}_h\right), \\ \tilde{\mathbf{e}}_3, & \text{in } \tilde{\mathcal{B}}_*, \\ 0, & \text{in } \tilde{\mathcal{B}}_h. \end{cases}$$

Proposition 4. *For any $h > 0$, the following holds:*

(1) $\tilde{\mathbf{w}}_\perp[h] \in \mathcal{C}(\mathbb{R}^3)$, *with:*

$$\tilde{\mathbf{w}}_\perp[h] = \tilde{\mathbf{e}}_3 \text{ on } \tilde{\mathcal{B}}_*, \quad \tilde{\mathbf{w}}_\perp[h] = 0 \text{ on } \tilde{\mathcal{B}}_h.$$

(2) *In a neighborhood of $\partial\tilde{\mathcal{P}}^l$*

$$\tilde{\mathbf{w}}_\perp[h](\tilde{\mathbf{x}}) = \mathrm{curl}_{\tilde{\mathbf{x}}}\left(\frac{\eta_{h_0}(|\tilde{\mathbf{x}}|-1)}{2}\,\tilde{\mathbf{e}}_3 \times \tilde{\mathbf{x}}\right).$$

Proof: The proof is exactly the same as for the parallel component. We refer the reader to [11] for technical details.

$\diamond$

3.3. **Complete test function.** We recall that, by definition:

$$\mathbf{e}_3 = \cos(\alpha)\tilde{\mathbf{e}}_3 - \sin(\alpha)\tilde{\mathbf{e}}_1, \quad \tilde{\mathbf{e}}_\mathbf{3} = Q_{-\alpha}\mathbf{e}_3, \quad \tilde{\mathbf{e}}_1 = Q_{-\alpha}\mathbf{e}_1,$$

with $\alpha \in (0, \pi/2)$ given by

$$\sin(\alpha) = \frac{2}{2+h}, \quad \cos(\alpha) = \frac{\sqrt{h^2+4h}}{2+h}. \tag{39}$$

Hence, in order to obtain a velocity field with rigid velocity $\mathbf{e}_3$, we set

$$\tilde{\mathbf{w}}[h](\tilde{\mathbf{x}}) = \cos\alpha \tilde{\mathbf{w}}_\perp[h](\tilde{\mathbf{x}}) - \sin\alpha \tilde{\mathbf{w}}_{/\!/}[h](\tilde{\mathbf{x}}). \tag{40}$$

In the global frame (the one without tildas), this velocity field reads

$$\mathbf{w}[h](\mathbf{x}) = Q_{-\alpha}\tilde{\mathbf{w}}[h]\left(Q_\alpha\left(\mathbf{x} - \mathbf{G}_h\right)\right) \tag{41}$$

for all $\mathbf{x} \in \mathcal{P}^l$, or more precisely:

$$\mathbf{w}[h](\mathbf{x}) = \cos\alpha Q_{-\alpha}\tilde{\mathbf{w}}_\perp[h]\left(Q_\alpha\left(\mathbf{x} - \mathbf{G}_h\right)\right) - \sin\alpha Q_{-\alpha}\tilde{\mathbf{w}}_{/\!/}[h]\left(Q_\alpha\left(\mathbf{x} - \mathbf{G}_h\right)\right).$$

As mentioned above, we obtain our test-velocity field in the remainder of the geometry by symmetry

$$\mathbf{w}[h](\mathbf{x}) = S_D[\mathbf{w}[h]](\mathbf{x}) \quad \forall\, \mathbf{x} \in \mathcal{P}^r.$$

The family $(\mathbf{w}[h])_{h>0}$ constructed this way satisfies Proposition 1. The only difficulty to prove this, is to obtain that $\mathbf{w}$ is smooth in a neighborhood of $\partial\mathcal{P}^l = \partial\mathcal{P}^r$. But, in a neighborhood of $\partial\mathcal{P}^l$ inside $\mathcal{P}^l$, we have by substitution:

$$\begin{aligned}
\mathbf{w}[h](\mathbf{x}) &= \operatorname{curl}_\mathbf{x}\left(\eta_{h_0}(|\mathbf{x} - \mathbf{G}_h| - 1)\frac{(\mathbf{e}_3 \times (\mathbf{x} - \mathbf{G}_h))}{2}\right), \\
&= \eta'_{h_0}\left(|\mathbf{x} - \mathbf{G}_h| - 1\right)\frac{\mathbf{x} - \mathbf{G}_h}{|\mathbf{x} - \mathbf{G}_h|} \times \frac{(\mathbf{e}_3 \times (\mathbf{x} - \mathbf{G}_h))}{2} + \eta_{h_0}(|\mathbf{x} - \mathbf{G}_h| - 1)\mathbf{e}_3.
\end{aligned}$$

As $\mathbf{e}_3$ is symmetric with respect to D, the same formula holds in the other half space. Therefore, $\mathbf{w}$ is smooth in the whole fluid domain as long as $h \neq 0$. We also emphasize that $\mathbf{w}$ is symmetric with respect to D so that we only estimate the restriction of $\mathbf{w}$ to $\mathcal{P}^l$ in what follows.

4. Estimating the test functions

This section is devoted to prove Proposition 2. The method is similar for all inequalities. First, we reduce these computations in the global framework to inequalities in the local one. We then complement the study by some technical description of $\tilde{\mathbf{w}}[h]$ in $\tilde{\Omega}_{h,1/4}$.

For example computing (20), we have:

$$\int_\Omega \rho_h \mathbf{u} \cdot \mathbf{w}[h]\ \mathrm{d}\mathbf{y} = \int_{\mathcal{P}^l} \rho_h \mathbf{u} \cdot \mathbf{w}[h]\ \mathrm{d}\mathbf{y} + \int_{\mathcal{P}^r} \rho_h \mathbf{u} \cdot \mathbf{w}[h]\ \mathrm{d}\mathbf{y}.$$

We focus on the term in $\mathcal{P}^l$. The other domination is computed by symmetry. We split:

$$\int_{\mathcal{P}^l} \rho_h \mathbf{u} \cdot \mathbf{w}[h]\ \mathrm{d}\mathbf{y} = \int_{\Omega_{h,1/4}} \rho_h \mathbf{u} \cdot \mathbf{w}[h]\ \mathrm{d}\mathbf{y} + \int_{\mathcal{P}^l \setminus \Omega_{h,1/4}} \rho_h \mathbf{u} \cdot \mathbf{w}[h]\ \mathrm{d}\mathbf{y}.$$

By construction,

$$\tilde{\mathbf{a}}[h](\tilde{\mathbf{x}}) := \cos\alpha(t)\tilde{\mathbf{a}}_\perp[h](\tilde{\mathbf{x}}) - \sin\alpha(t)\tilde{\mathbf{a}}_{/\!/}[h](\tilde{\mathbf{x}}).$$

is continuous in h, smooth in the spatial variable and with compact support in

$$\{(h,\mathbf{x}) \in [0,1] \times \mathbb{R}^3 \ ; \ \mathbf{x} \notin \tilde{\Omega}_{h,1/4}\}.$$

Thus, there exists a constant $C = C(\beta)$ independent of h such that

$$\|\tilde{\mathbf{a}}\|_{H^\beta(\tilde{\mathcal{F}}_h \setminus \tilde{\Omega}_{h,1/4})} \leqslant C \quad \forall\, h < h_{max}.$$

As a consequence, we only focus on

$$\left| \int_{\Omega_{h,1/4}} \rho_h \mathbf{u} \cdot \mathbf{w}[h] \, \mathrm{d}\mathbf{y} \right| \leqslant C \|\mathbf{u}\|_{L^2(\Omega)} \ \|\mathbf{w}[h]\|_{L^2(\Omega_{h,1/4})}.$$

Using that Q_α is a unit transformation, the proof of (20), as other dominations in Proposition 2, is reduced to estimate $\tilde{\mathbf{w}}[h]$ in $\tilde{\Omega}_{h,1/4}$.

First, to end up the proof of (20) and in preparation for (23), we obtain the following result.

Proposition 5. *The function* $\tilde{\mathbf{w}}[h]$ *defined in* (40) *satisfies*

$$\|\tilde{\mathbf{w}}[h]\|_{L^2(\tilde{\Omega}_{h,1/4})} \leqslant C, \tag{42}$$

$$\|\nabla \tilde{\mathbf{w}}[h]\|_{L^2(\tilde{\Omega}_{h,1/4})} \leqslant C\sqrt{\ln(1/h)}, \tag{43}$$

for any $h > 0$, *with a constant* C *independent of* h.

Proof: To prove (42), we apply Lemmata 3 and 4. This yields that, for any $(r,\theta,z) \in \tilde{\Omega}_{h,1/4}$, and $h \in (0,1)$,

$$|\tilde{\mathbf{w}}^d_\perp| \leqslant |\partial_z \phi_\perp| + |\partial_r \phi_\perp| + \frac{|\phi_\perp|}{r} \leqslant C\left(1 + \frac{r}{\delta_*(r) - \delta_h(r)}\right) \tag{44}$$

$$|\tilde{\mathbf{w}}^d_{/\!/}| \leqslant \frac{1}{2} + \left|\partial_z \phi_{/\!/}\right| + \left|\partial_r \phi_{/\!/}\right| \leqslant C. \tag{45}$$

Both above estimates combined with Lemma 1 imply (42).

To prove (43), we first notice that for any $\mathbf{v}$,

$$|\nabla \mathbf{v}| \leqslant C\left(|\partial_r \mathbf{v}| + \frac{|\partial_\theta \mathbf{v}|}{r} + |\partial_z \mathbf{v}|\right). \tag{46}$$

From (38) and Lemma 3, we deduce

$$|\partial_r \tilde{\mathbf{w}}_\perp| \leqslant C\left(|\partial_{rz}\phi_\perp| + |\partial_{rr}\phi_\perp| + \left|\frac{\partial_r \phi_\perp}{r} - \frac{\phi_\perp}{r^2}\right|\right) \leqslant \frac{C}{\delta_* - \delta_h}, \tag{47}$$

$$\frac{|\partial_\theta \tilde{\mathbf{w}}_\perp|}{r} \leqslant \frac{|\partial_z \phi_\perp|}{r} \leqslant \frac{C}{\delta_* - \delta_h}, \tag{48}$$

$$|\partial_z \tilde{\mathbf{w}}_\perp| \leqslant C\left(|\partial_{zz}\phi_\perp| + |\partial_{rz}\phi_\perp| + \left|\frac{\partial_z \phi_\perp}{r}\right|\right) \leqslant C\left(\frac{r}{(\delta_* - \delta_h)^2} + \frac{1}{\delta_* - \delta_h}\right). \tag{49}$$

Gathering (46)–(49) yields

$$|\nabla \tilde{\mathbf{w}}_\perp| \leqslant C\left(\frac{r}{(\delta_* - \delta_h)^2} + \frac{1}{\delta_* - \delta_h}\right). \tag{50}$$

From (39) and Lemma 1, we obtain

$$\|\cos(\alpha)\nabla\tilde{\mathbf{w}}_{\perp}\|_{L^2(\tilde{\Omega}_{h,1/4})} \leqslant C\sqrt{h}\left(\sqrt{\ln 1/h} + \frac{1}{\sqrt{h}}\right) \leqslant C.$$

From (28) and Lemma 4, we get

$$\left|\partial_r \tilde{\mathbf{w}}_{/\!/}\right| \leqslant C\left(\left|\partial_{rz}\phi_{/\!/}\right| + \left|\partial_{rr}\phi_{/\!/}\right|\right) \leqslant C\left(\frac{r}{\delta_* - \delta_h} + 1\right), \tag{51}$$

$$\frac{\left|\partial_\theta \tilde{\mathbf{w}}_{/\!/}\right|}{r} \leqslant \frac{\left|\partial_r \phi_{/\!/}\right|}{r} \leqslant C, \tag{52}$$

$$\left|\partial_z \tilde{\mathbf{w}}_{/\!/}\right| \leqslant C\left(\left|\partial_{zz}\phi_{/\!/}\right| + \left|\partial_{rz}\phi_{/\!/}\right|\right) \leqslant C\left(\frac{1}{\delta_* - \delta_h} + \frac{r}{\delta_* - \delta_h}\right). \tag{53}$$

Gathering (46) and (51)–(53), we deduce

$$\left|\nabla \tilde{\mathbf{w}}_{/\!/}\right| \leqslant \frac{C}{\delta_* - \delta_h}. \tag{54}$$

From (39) and Lemma 1, we conclude

$$\|\sin(\alpha)\nabla\tilde{\mathbf{w}}_{/\!/}\|_{L^2(\tilde{\Omega}_{h,1/4})} \leqslant C\sqrt{\ln 1/h}.$$

◇

Concerning (21) and (22), we split as previously and this yields

$$\int_{\mathcal{P}^l} \rho_h \mathbf{u}\cdot\partial_h \mathbf{w}[h]\ \mathrm{d}\mathbf{y} = I_1 + I_2, \quad \int_{\mathcal{P}^l} \mathbf{u}\otimes\mathbf{u}\cdot D(\mathbf{w})[h]\ \mathrm{d}\mathbf{y} = J_1 + J_2,$$

where:

$$\begin{aligned} |I_1| &= \left|\int_{\mathcal{P}^l\setminus\Omega_{h,1/4}} \rho_h \mathbf{u}\cdot\partial_h\mathbf{w}[h]\ \mathrm{d}\mathbf{y}\right| \leqslant \|\mathbf{u}\|_{L^2(\Omega)}\|\partial_h\mathbf{w}[h]\|_{L^2(\mathcal{P}^l\setminus\Omega_{h,1/4})}, \\ |J_1| &\leqslant \|\mathbf{u}\|^2_{L^2(\Omega)}\|\mathbf{w}[h]\|_{H^3(\mathcal{P}^l\setminus\Omega_{h,1/4})}, \end{aligned}$$

and, with the same technique as in[11, Lemme 3.1]:

$$|I_2| \leqslant M_2\|\nabla\mathbf{u}\|_{L^2(\Omega)} \quad |J_2| \leqslant M_2^\infty\|\nabla\mathbf{u}\|_{L^2(\Omega)},$$

where:

$$M_2 = \left[\int_0^{\frac14}\left((\delta_h(r)-\delta_*(r))^2\left[\int_{\delta_*(r)}^{\delta_h(r)} \sup_{\theta\in(0,2\pi)}|\partial_h\mathbf{w}(r,\theta,z)|^2\ \mathrm{d}z\right]\right) r\ \mathrm{d}r\right]^{\frac12},$$

$$M_2^\infty = \sup_{r\in(0,\frac14)}(\delta_h(r)-\delta_*(r))^{3/2}\left[\int_{\delta_*(r)}^{\delta_h(r)} \sup_{\theta\in(0,2\pi)}|\nabla\mathbf{w}(r,\theta,z)|^2\ \mathrm{d}z\right]^{\frac12}.$$

Consequently, (21) and (22) are consequences of the following result.

Proposition 6. *There exists a positive constant C such that*

$$\int_0^{\frac14}\left((\delta_h(r)-\delta_*(r))^2\left[\int_{\delta_*(r)}^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}|\partial_h\mathbf{w}(r,\theta,z)|^2\,\mathrm{d}z\right]\right)r\,\mathrm{d}r\leqslant\frac{C}{h},\tag{55}$$

$$\|\partial_h\mathbf{w}[h]\|_{L^2(\mathcal{P}^l\setminus\Omega_{h,1/4})}\leqslant\frac{C}{\sqrt{h}},\tag{56}$$

$$\sup_{r\in(0,\frac14)}\left((\delta_h(r)-\delta_*(r))^{3/2}\left[\int_{\delta_*(r)}^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}|\nabla\mathbf{w}(r,\theta,z)|^2\,\mathrm{d}z\right]^{\frac12}\right)\leqslant C,\tag{57}$$

for any $h>0$.

Proof: To prove (55), we remark that in $\mathcal{P}^l$:

$$\begin{aligned}\partial_h\mathbf{w} &= \partial_h\left[Q_{-\alpha}\tilde{\mathbf{w}}[h](Q_\alpha(\mathbf{x}-\mathbf{G}_h))\right]\\ &= M_h^\top\tilde{\mathbf{w}}[h]+Q_{-\alpha}\left(M_h(\mathbf{x}-\mathbf{G}_h)-Q_\alpha\partial_h\mathbf{G}_h\right)\cdot\nabla\tilde{\mathbf{w}}[h]+Q_{-\alpha}\partial_h\tilde{\mathbf{w}}[h],\end{aligned}\tag{58}$$

with

$$\partial_h\tilde{\mathbf{w}}[h]=\partial_h(\cos\alpha)\ \tilde{\mathbf{w}}_\perp[h]+\cos\alpha\ \partial_h\tilde{\mathbf{w}}_\perp[h]-\partial_h(\sin\alpha)\ \tilde{\mathbf{w}}_{/\!/}[h]-\sin\alpha\ \partial_h\tilde{\mathbf{w}}_{/\!/}[h].$$

Due to (39) and (18), there exists a universal constant C for which:

$$|\partial_h\cos\alpha|\leqslant\frac{C}{\sqrt{h}},\quad |\partial_h\sin\alpha|\leqslant C,\quad |\partial_h\mathbf{G}_h|\leqslant\frac{C}{\sqrt{h}}.$$

Moreover, outside $\tilde{\Omega}_{h,1/4}$, $\tilde{\mathbf{w}}_{/\!/}$ and $\tilde{\mathbf{w}}_\perp$ are smooth functions of all its arguments. Consequently, the only singular terms in $\partial_h\mathbf{w}$, outside $\Omega_{h,1/4}$, are $\partial_h\cos\alpha$ and $\partial_h\mathbf{G}_h$ so that the above control leads to (56).

Finally, inside $\tilde{\Omega}_{h,1/4}$, we already estimated $\tilde{\mathbf{w}}[h]$ and $\nabla\tilde{\mathbf{w}}[h]$. Combining these dominations with:

$$|M_h(\mathbf{x}-\mathbf{G}_h)-Q_\alpha\partial_h\mathbf{G}_h|\leqslant\frac{C}{\sqrt{h}}\quad\forall\,\mathbf{x}\in\Omega_{h,1/4},$$

and the above control on $\partial_h\cos\alpha$, this yields

$$|M_h^\top\tilde{\mathbf{w}}[h]+Q_{-\alpha}\left(M_h(\mathbf{x}-\mathbf{G}_h)-Q_\alpha\partial_h\mathbf{G}_h\right)\cdot\nabla\tilde{\mathbf{w}}[h]|\leqslant C\left(1+\frac{1}{\sqrt{h}(\delta_*-\delta_h)}+\frac{r}{(\delta_*-\delta_h)^2}\right).$$

In $\partial_h\tilde{\mathbf{w}}[h]$ the same right-hand side dominates:

$$|\partial_h(\cos\alpha)\ \tilde{\mathbf{w}}_\perp[h]-\partial_h(\sin\alpha)\ \tilde{\mathbf{w}}_{/\!/}[h]|.$$

Finally, from Lemmata 3 and 4, we compute that

$$|\cos\alpha\ \partial_h\tilde{\mathbf{w}}_\perp[h]-\sin\alpha\ \partial_h\tilde{\mathbf{w}}_{/\!/}[h]|\leqslant C\left(\sqrt{h}\left[\frac{1}{\delta_*-\delta_h}+\frac{r}{(\delta_*-\delta_h)^2}\right]+\frac{r}{\delta_*-\delta_h}+\frac{1}{\delta_*-\delta_h}\right).$$

The above dominations reduce to:

$$|\partial_h\mathbf{w}|\leqslant C\left(1+\frac{1}{\sqrt{h}(\delta_*-\delta_h)}+\frac{r}{(\delta_*-\delta_h)^2}\right)\qquad\text{in }\tilde{\Omega}_{h,1/4}.$$

From Lemma 1, we finally obtain (55).

To prove, (57), we use (50) and (54):

$$(\delta_h(r)-\delta_*(r))^3\int_{\delta_*(r)}^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}|\nabla \mathbf{w}(r,\theta,z)|^2\ \mathrm{d}z \leqslant C\left((\delta_*-\delta_h)^2+r^2\right).$$

$\diamond$

In order to prove (23), we first construct a suitable pressure field:

Proposition 7. *Given $h>0$, there exists a smooth pressure-field $\tilde{q}[h]$ such that*

$$-\Delta\tilde{\mathbf{w}}[h]+\nabla\tilde{q}[h]=\tilde{\mathbf{f}}^{\mathbf{1}}+\tilde{\mathbf{f}}^{\mathbf{2}},\qquad \text{in } \tilde{\mathcal{P}}^l \tag{59}$$

with

$$\int_0^{\frac14}\left((\delta_h(r)-\delta_*(r))^2\left[\int_{\delta_*(r)}^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}|\tilde{\mathbf{f}}^{\mathbf{1}}(r,\theta,z)|^2\ \mathrm{d}z\right]\right)r\ \mathrm{d}r \quad \text{and} \quad \|\tilde{\mathbf{f}}^{\mathbf{2}}\|_{L^{6/5}(\tilde{\Omega}_{h,1/4})} \tag{60}$$

uniformly bounded for $h\in(0,1)$.

Proof: With arguments similar to those in [11, Lemma 3.8], we first construct a pressure field $q_\perp[h]$ such that

$$-\Delta\tilde{\mathbf{w}}_\perp+\nabla\tilde{q}_\perp=\tilde{\mathbf{f}}_\perp\quad \text{in } \tilde{\mathcal{P}}^l$$

with:

$$\int_0^{\frac14}\left((\delta_h(r)-\delta_*(r))^2\left[\int_{\delta_*(r)}^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}|\tilde{\mathbf{f}}_\perp(r,\theta,z)|^2\ \mathrm{d}z\right]\right)r\ \mathrm{d}r$$

uniformly bounded.

Then by definition of $\tilde{\mathbf{w}}_{/\!/}$ we have

$$-\Delta\tilde{\mathbf{w}}_{/\!/}=\begin{pmatrix}\Delta(\partial_z\phi_{/\!/})\\ 0\\ -\Delta(\cos(\theta)\partial_r\phi_{/\!/})\end{pmatrix}.$$

First

$$\Delta(\partial_z\phi_{/\!/})=\partial_{rrz}\phi_{/\!/}+\frac{1}{r}\partial_{rz}\phi_{/\!/}+\partial_{zzz}\phi_{/\!/}.$$

Using that $\partial_{zzz}\phi_{/\!/}=0$ and Lemma 4, we deduce

$$\left|\Delta(\partial_z\phi_{/\!/})\right|\leqslant\frac{C}{\delta_*-\delta_h}.$$

Second

$$\Delta(\cos(\theta)\partial_r\phi_{/\!/})=\cos(\theta)\left(\partial_{rrr}\phi_{/\!/}+\frac{1}{r}\partial_{rr}\phi_{/\!/}-\frac{1}{r^2}\partial_r\phi_{/\!/}+\partial_{zzr}\phi_{/\!/}\right).$$

Using again Lemma 4, we obtain that

$$\left|\cos(\theta)\left(\partial_{rrr}\phi_{/\!/}+\partial_{zzr}\phi_{/\!/}\right)\right|\leqslant C\frac{r}{(\delta_*-\delta_h)^2}$$

and

$$\left|\cos(\theta)\left(\frac{1}{r}\partial_{rr}\phi_{/\!/}-\frac{1}{r^2}\partial_r\phi_{/\!/}\right)\right|\leqslant\frac{C}{r}.$$

From Lemma 1,

$$\tilde{\mathbf{f}}^{\mathbf{1}}_{/\!/} = \begin{pmatrix} \Delta(\partial_z \phi_{/\!/}) \\ 0 \\ -\cos(\theta)\left(\partial_{rrr}\phi_{/\!/} + \partial_{zzr}\phi_{/\!/}\right) \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{f}}^{\mathbf{2}}_{/\!/} = \begin{pmatrix} 0 \\ 0 \\ -\cos(\theta)\left(\frac{1}{r}\partial_{rr}\phi_{/\!/} - \frac{1}{r^2}\partial_r \phi_{/\!/}\right) \end{pmatrix}$$

satisfy

$$\int_0^{\frac{1}{4}} \left((\delta_h(r) - \delta_*(r))^2 \left[\int_{\delta_*(r)}^{\delta_h(r)} \sup_{\theta \in (0,2\pi)} |\tilde{\mathbf{f}}^{\mathbf{1}}_{/\!/}(r,\theta,z)|^2 \, \mathrm{d}z \right] \right) r \, \mathrm{d}r \leqslant C,$$

and

$$\|\tilde{\mathbf{f}}^{\mathbf{2}}_{/\!/}\|_{L^{6/5}(\tilde{\Omega}_{h,1/4})} \leqslant C.$$

Finally, we set $\tilde{q}[h] = \cos\alpha \tilde{q}_{\perp}[h]$ so that $\tilde{\mathbf{f}}^{\mathbf{1}} = \sin\alpha \tilde{\mathbf{f}}^{\mathbf{1}}_{/\!/} + \cos\alpha \tilde{\mathbf{f}}_{\perp}$ and $\tilde{\mathbf{f}}^{\mathbf{2}} = \sin\alpha \tilde{\mathbf{f}}^{\mathbf{2}}_{/\!/}$. This ends up the proof.

◇

To complete the proof of (23), let $\mathbf{v} \in \mathbb{H}(\mathbf{G}_h)$ with $\mathbf{v} = \ell \mathbf{e}_3$ on $\mathcal{B}_h$, and consider

$$I = \int_\Omega D(\mathbf{v}) : D(\mathbf{w}[h]) \, \mathrm{d}\mathbf{y}.$$

We split this integral as previously $I = I^l + I^r$ with obvious notation. Then we introduce $\tilde{\mathbf{w}}[h]$, and $\tilde{\mathbf{v}}$ in the same fashion. Because Q_α is a unit transformation, we have:

$$I^l = \int_{\tilde{\mathcal{F}}_h \cap \tilde{\mathcal{P}}^l} D(\tilde{\mathbf{v}}) : D(\tilde{\mathbf{w}}[h]) \, \mathrm{d}\tilde{\mathbf{y}}.$$

Integrating by parts, this yields

$$I^l = \int_{\partial(\tilde{\mathcal{F}}_h \cap \tilde{\mathcal{P}}^l)} (2D(\tilde{\mathbf{w}}[h])\mathbf{n} - \tilde{q}[h]\mathbf{n}) \cdot \tilde{\mathbf{v}} \, \mathrm{d}\tilde{\sigma} - \int_{\tilde{\mathcal{F}}_h \cap \tilde{\mathcal{P}}^l} (\Delta \tilde{\mathbf{w}}[h] - \tilde{q}[h]) \cdot \tilde{\mathbf{v}} \, \mathrm{d}\tilde{\mathbf{y}}.$$

For symmetry reasons, after compensation with I^r the relevant boundary integral is:

$$\int_{\partial \tilde{\mathcal{B}}_* \cap \tilde{\mathcal{P}}^l} (2D(\tilde{\mathbf{w}}[h])\mathbf{n} - \nabla \tilde{q}[h]\mathbf{n}) \cdot \tilde{\mathbf{v}} \, \mathrm{d}\tilde{\sigma},$$

we notice that it is fixed by h and proportional to ℓ. Consequently, we can rewrite as $\ell b(h)/2$ with some function b to be made precise. Moreover, applying the previous proposition, and similar technique to [11, Lemma 3.9], we obtain that

$$\int_{\tilde{\mathcal{F}}_h \cap \tilde{\mathcal{P}}^l} (\Delta \tilde{\mathbf{w}}[h] - \nabla \tilde{q}[h]) \cdot \tilde{\mathbf{v}} \, \mathrm{d}\tilde{\mathbf{y}} \leqslant C \|\nabla \tilde{\mathbf{v}}\|_{L^2(\tilde{\mathcal{P}}^l \cap \tilde{\Omega})} \leqslant C \|\nabla \mathbf{v}\|_{L^2(\Omega)}.$$

Computing similarly $\mathcal{P}^r$, we finally obtain $I = \ell b(h) + R$ with $|R| \leqslant C\|\nabla \mathbf{v}\|_{L^2(\Omega)}$ where C is an absolute constant.

Taking in particular $\mathbf{v} = \mathbf{w}[h]$ we might compute our integral in the same way. This yields

$$b(h) = \int_\Omega |D(\mathbf{w})|^2 + R \quad \text{ with } |R| \leqslant C\|\nabla \mathbf{w}\|_{L^2(\Omega)}.$$

From the control on this $L^2(\Omega)$ norm obtained in Proposition 5, we finally obtain that

$$b(h) \leqslant C\|\nabla \mathbf{w}\|^2_{L^2(\Omega)} + C\|\nabla \mathbf{w}\|_{L^2(\Omega)} \leqslant C|\ln h| + C\sqrt{|\ln h|} \leqslant C|\ln h|.$$

Appendix A. Detailed description of potentials $\phi_{/\!/}$ and $\phi_\perp$

This appendix is very similar to the one given in [11], however there are some differences since we estimate not only the size of $\phi_\perp$ and its derivatives, but also the size of $\phi_{/\!/}$ and its derivatives. However, since the proofs are completely similar, we only state the results used in this paper.

We emphasize that $\phi_{/\!/}$ and $\phi_\perp$ depend on h, even if the dependency is not explicitly mentioned. In order to compare such functions in what follows, we introduce the following conventions. Given families $(f_h : \tilde{\Omega}_{h,1/4} \to \mathbb{R})_{h\in(0,1)}$ and $(g_h : \tilde{\Omega}_{h,1/4} \to \mathbb{R})_{h\in(0,1)}$, we denote by $f_h \prec g_h$ if there exists an absolute constant C such that

$$|f_h(\mathbf{x})| \leqslant C g_h(\mathbf{x}) \quad \forall\, \mathbf{x} \in \tilde{\Omega}_{h,1/4} \text{ and } h < 1.$$

Given non negative functions $f : (0,1) \to \mathbb{R}^+$ and $g : (0,1) \to \mathbb{R}^+$, we also denote by

$$f(s) \sim g(s) \quad \forall s \in (0,1),$$

if there exist two positive constants c and C such that

$$cf(s) \leqslant g(s) \leqslant Cf(s) \quad \forall s \in (0,1).$$

First, we compute typical $L^1(0,1/4)$-sizes of functions $r \mapsto r^\alpha/(\delta_*(r) - \delta_h(r))^\beta$.

Lemma 1. *Given $(\alpha,\beta) \in (\mathbb{R}_+)^2$, we have the following estimations for all $h \in (0,1)$:*

$$\int_0^{\frac14} \frac{r^\alpha}{(\delta_*(r) - \delta_h(r))^\beta}\,\mathrm{d}r \sim \begin{cases} 1 & \text{if } \alpha > 2\beta - 1, \\ \ln(1/h) & \text{if } \alpha = 2\beta - 1, \\ h^{\frac{(\alpha+1)-2\beta}{2}} & \text{if } \alpha < 2\beta - 1, \end{cases}$$

We now compare $\lambda(r,z) = \frac{z-\delta_h(r)}{\delta_*(r)-\delta_h(r)}$ to functions $(r,\theta,z) \mapsto r^\alpha/(\delta_*(r) - \delta_h(r))^\beta$ in $\tilde{\Omega}_{h,1/4}$.

Lemma 2. *We have the following sizes*

$$\begin{array}{llll} \lambda \prec 1, & \lambda_r \prec r/(\delta_* - \delta_h), & \lambda_z \prec 1/(\delta_* - \delta_h), & \lambda_h \prec 1/(\delta_* - \delta_h), \\ \lambda_{rh} \prec r/(\delta_* - \delta_h)^2, & \lambda_{zh} \prec 1/(\delta_* - \delta_h)^2, & \lambda_{rr} \prec 1/(\delta_* - \delta_h), & \lambda_{rz} \prec r/(\delta_* - \delta_h)^2, \\ \lambda_{rrz} \prec 1/(\delta_* - \delta_h)^2, & \lambda_{rrr} \prec r/(\delta_* - \delta_h)^2. & & \end{array}$$

Then we obtain the following lemmata.

Lemma 3. *We have the following sizes:*

$$\begin{array}{lll} \phi_\perp \prec r, & \partial_r \phi_\perp \prec 1, & \partial_z \phi_\perp \prec r/(\delta_* - \delta_h), \\ \partial_r(\phi_\perp/r) \prec r/(\delta_* - \delta_h), & \partial_h \phi_\perp \prec r/(\delta_* - \delta_h), & \partial_{rh} \phi_\perp \prec 1/(\delta_* - \delta_h), \\ \partial_{zh} \phi_\perp \prec r/(\delta_* - \delta_h)^2, & \partial_{rz}(\phi_\perp/r) \prec r/(\delta_* - \delta_h)^2, & \partial_{rr} \phi_\perp \prec r/(\delta_* - \delta_h), \\ \partial_{rz} \phi_\perp \prec 1/(\delta_* - \delta_h), & \partial_{zz} \phi_\perp \prec r/(\delta_* - \delta_h)^2, & \partial_{rrr} \phi_\perp \prec 1/(\delta_* - \delta_h), \\ \partial_{rzz} \phi_\perp \prec 1/(\delta_* - \delta_h)^2, & \partial_{rrz} \phi_\perp \prec r/(\delta_* - \delta_h)^2, & \partial_{zzz} \phi_\perp \prec r/(\delta_* - \delta_h)^3. \end{array}$$

Lemma 4. *We have the following sizes:*

$$\begin{array}{lll} \phi_{/\!/} \prec (\delta_* - \delta_h), & \partial_r \phi_{/\!/} \prec r, & \partial_z \phi_{/\!/} \prec 1, \\ \partial_h \phi_{/\!/} \prec 1, & \partial_{rh} \phi_{/\!/} \prec r/(\delta_* - \delta_h), & \partial_{zh} \phi_{/\!/} \prec 1/(\delta_* - \delta_h), \\ \partial_{rr} \phi_{/\!/} \prec 1, & \partial_{rz} \phi_{/\!/} \prec r/(\delta_* - \delta_h), & \partial_{zz} \phi_{/\!/} \prec 1/(\delta_* - \delta_h), \\ \partial_{rrr} \phi_{/\!/} \prec r/(\delta_* - \delta_h), & \partial_{rzz} \phi_{/\!/} \prec r/(\delta_* - \delta_h)^2, & \partial_{rrz} \phi_{/\!/} \prec 1/(\delta_* - \delta_h). \end{array}$$

Appendix B. Computation of Inequality (16)

The following computations are inspired by [9]. For simplicity we consider symmetric geometries and we apply notation introduced in Section 3.

Proposition 8. *There exists a universal constant C for which, given $\mathbf{G} \in D$ and $\mathbf{u} \in \mathcal{H} \cap H^2(\mathcal{F}_{\mathbf{G}})$ such that*

$$\left\{ \begin{array}{rcll} \mathbf{u}(\mathbf{x}) & = & 0, & \text{on } \partial\Omega, \\ \mathbf{u}(\mathbf{x}) & = & \mathbf{V} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}), & \text{on } \partial\mathcal{B}_{\mathbf{G}}, \end{array} \right.$$

whenever $h := dist(\mathcal{B}_{\mathbf{G}}, \partial\Omega) < 1$, there holds:

$$|\mathbf{V}_{\mathbf{u}} \cdot \tilde{\mathbf{e}}_3| \leqslant C|h|^{\frac{3}{2}} \left[\int_{\mathcal{F}_{\mathbf{G}}} |\nabla^2 \mathbf{u}(\mathbf{y})|^2 \, \mathrm{d}y \right]^{\frac{1}{2}}.$$

Proof: For simplicity, we assume $\mathbf{u}$ is smooth in the fluid domain. The result is then obtained by a density argument. We also introduce the orthonormal basis $(\tilde{\mathbf{e}}_r, \tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_z)$ associated to cylindrical coordinates.

Integrating $\mathrm{div}(\mathbf{u}) = 0$ in $\Omega_{h,l}$, this yields

$$\int_{\partial\mathcal{B}_{\mathbf{G}} \cap \partial\Omega_{h,l}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\sigma = -l\Phi(l)$$

where:

$$\Phi(l) = \int_{\delta_h(l)}^{\delta_*(l)} \varphi(z,l) \, \mathrm{d}z, \quad \text{with} \quad \varphi(z,l) = \int_{-\pi}^{\pi} \mathbf{u}(l,\theta,z) \cdot \tilde{\mathbf{e}}_r \, \mathrm{d}\theta.$$

We remark that no slip boundary conditions together with symmetry arguments imply

$$\varphi(\delta_*(l), l) = 0, \quad \varphi(\delta_h(l), l) = 0, \qquad \forall \, l \in (0,1).$$

As a straightforward consequence, we obtain

$$|\varphi(z,l)| \leqslant (\delta_*(l) - \delta_h(l))^{\frac{3}{2}} \left[\int_{\delta_h(l)}^{\delta_*(l)} |\partial_{zz}\varphi(\alpha,l)|^2 \, \mathrm{d}\alpha \right]^{\frac{1}{2}},$$

and

$$\left| \int_{\partial\mathcal{B}_{\mathbf{G}}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\sigma \right| \leqslant Cl \, (\delta_*(l) - \delta_h(l))^{\frac{5}{2}} \left[\int_{\delta_h(l)}^{\delta_*(l)} \int_{-\pi}^{\pi} |\partial_{zz}\mathbf{u}(r,\theta,z)|^2 \, \mathrm{d}r \, \mathrm{d}\theta \right]^{\frac{1}{2}}. \tag{61}$$

Moreover, one might compute

$$\int_{\partial\mathcal{B}_{\mathbf{G}} \cap \partial\Omega_{h,l}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\sigma = 2\pi l^2 \mathbf{V} \cdot \tilde{\mathbf{e}}_3,$$

so that, (61) reads:

$$2\pi l^2 |\mathbf{V}\cdot\tilde{\mathbf{e}}_3| \leqslant Cl\left(\delta_*(l)-\delta_h(l)\right)^{\frac{5}{2}}\left[\int_{\delta_h(l)}^{\delta_*(l)}\int_{-\pi}^{\pi}|\partial_{zz}\mathbf{u}(r,\theta,z)|^2\,\mathrm{d}z\,\mathrm{d}\theta\right]^{\frac{1}{2}}.$$

Finally, we integrate the above inequality over $l\in[0,r]$. This yields

$$|\mathbf{V}\cdot\tilde{\mathbf{e}}_3| \leqslant \frac{C\left(\delta_*(r)-\delta_h(r)\right)^{\frac{5}{2}}}{r^2}|\nabla^2\mathbf{u}|_{L^2(\mathcal{F}_{\mathbf{G}})}.$$

We optimize this last inequality taking $r=\sqrt{h}$ and obtain the expected inequality when $h<1$.

◇

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