Statistics for low-lying zeros of Hecke L-functions in the level aspect

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1. Introduction

We would like to provide evidence for the fact that zeros of L-functions seem to behave statistically as eigenvalues of random matrices of large rank throughout the instance of Hecke L-functions. First, we remind you of Iwaniec-Luo-Sarnak's results on one-level densities for low-lying zeros of Hecke L-functions (see [5]) and Katz-Sarnak's results on one-level densities for eigenvalues of orthogonal random matrices (see [6]). Then, we explain that Hughes and Miller (see [1]) found a new example of a very strange phenomenon discovered by Hughes and Rudnick (see [2]) called mock-Gaussian behavior. These works were carried on by the author and Royer in the context of low-lying zeros of symmetric power L-functions in the level aspect (see [7]).

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Notation. *We write* $\mathcal{P}$ *for the set of prime numbers; the main parameter in this paper is a prime number* q*, whose name is the* level*, which goes to infinity among* $\mathcal{P}$*. For any* $\nu > 0$*,* $\mathcal{S}_\nu(\mathbb{R})$ *stands for the space of even Schwartz functions* Φ *whose Fourier transform*

$$\widehat{\Phi}(\xi) := \int_{\mathbb{R}} \Phi(x) e(-x\xi)\,\mathrm{d}x$$

is compactly supported in $[-\nu, +\nu]$*.*

2. A quick walk in the world of L-functions

2.1. **Hecke L-functions and their zeros.** Let f be a primitive cusp form of level q, even integer weight $\kappa \geq 2$ and trivial character ϵ_q say $f \in H_\kappa^*(q)$ (see [3] for the automorphic background). If $(\lambda_f(n))_{n\geq 1}$ are its (suitably normalised) Hecke eigenvalues then we define

$$L(f,s) := \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_{p\in\mathcal{P}} \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\epsilon_q(p)}{p^{2s}}\right)^{-1},$$

which is an absolutely convergent and non-vanishing Dirichlet series and Euler product on $\Re s > 1$, and also $L_\infty(f,s) := \Gamma_{\mathbb{R}}(s + (\kappa-1)/2)\,\Gamma_{\mathbb{R}}(s+(\kappa+1)/2)$ where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\,\Gamma(s/2)$ as usual. The function $\Lambda(f,s) := q^{s/2} L_\infty(f,s) L(f,s)$ is a *completed L-function* in the sense that it satisfies the following *nice* analytic properties, proved by E. Hecke:

- the function $\Lambda(f,s)$ can be extended to a holomorphic function of order 1 on $\mathbb{C}$;

- the function $\Lambda(f,s)$ satisfies a functional equation of the shape
$$\Lambda(f,s)=i^{\kappa}\epsilon_f(q)\Lambda(f,1-s)$$
where $\epsilon_f(q)=-\sqrt{q}\lambda_f(q)=\pm 1$.

Let us recall some preliminary facts on zeros of Hecke L-functions, which can be found in section 5.3 of [4]. If $\epsilon_f(q)=-1$ then the functional equation of $L(\operatorname{Sym}^r f,s)$ evaluated at the critical point $s=1/2$ provides a trivial zero. The *Generalised Riemann Hypothesis* is the main conjecture about the horizontal distribution of the zeros of $\Lambda(\operatorname{Sym}^r f,s)$ in the critical strip.

Hypothesis GRH. *For any prime number q and any f in $H_\kappa^*(q)$, all the zeros of $\Lambda(f,s)$ lie on the critical line $\{s\in\mathbb{C}\colon \Re s=1/2\}$.*

Under hypothesis GRH, it can be shown that the spacing between two consecutive zeros with imaginary part in $[0,1]$ is roughly of size $(2\pi)/\log(q)$. Thus, we normalise the zeros by defining
$$\widehat{\rho}:=\frac{\log(q)}{2i\pi}\left(\Re\rho-\frac{1}{2}+i\Im\rho\right)$$
for any zero ρ of $\Lambda(f,s)$. We aim at studying the local distribution of the zeros of $\Lambda(f,s)$ in a neighborhood of the real axis of size $1/\log q$.

2.2. **One-level density.** Fix $\Phi\in\mathcal{S}_\nu(\mathbb{R})$. Let us define the *harmonic* probability measure on $H_\kappa^*(q)$. If A is any subset of this space then its *harmonic probability measure* is defined by
$$\mu_q^h(A):=\sum_{f\in A}\omega_f(q)$$
where the *harmonic weight* associated to any f in $H_\kappa^*(q)$ is given by
$$\omega_q(f):=\frac{\Gamma(\kappa-1)}{(4\pi)^{\kappa-1}\langle f,f\rangle_q}$$
and $\langle f,f\rangle_q$ stands for the Petersson scalar product. The random variable on $\left(H_\kappa^*(q),\mu_q^h\right)$ defined by
$$\forall f\in H_\kappa^*(q),\quad D_{1,q}[\Phi](f):=\sum_{\rho,\Lambda(f,\rho)=0}\Phi\left(\widehat{\rho}\right)$$
is the *one-level density* (relatively to Φ). Its *harmonic expectation* is
$$\mathbb{E}_q^{\mathrm{h}}\left(D_{1,q}[\Phi]\right):=\sum_{f\in H_\kappa^*(q)}\omega_q(f)D_{1,q}[\Phi](f)$$
and its *m-th moments* are
$$\mathbb{M}_{q,m}^{\mathrm{h}}\left(D_{1,q}[\Phi]\right):=\mathbb{E}_q^{\mathrm{h}}\left(\left(D_{1,q}[\Phi]-\mathbb{E}_q^{\mathrm{h}}\left(D_{1,q}[\Phi]\right)\right)^m\right)$$
for any integer $m\geq 1$. We may legitimately wonder if the previous sequences of complex numbers converge as q goes to infinity among the primes. If yes, the following general notations will be used for their limits $\mathbb{E}_\infty^{\mathrm{h}}\left(D_1[\Phi]\right)$ and $\mathbb{M}_{\infty,m}^{\mathrm{h}}\left(D_1[\Phi]\right)$

for any integer $m \geq 1$. Let $\varepsilon = \pm 1$. The *signed harmonic expectation* of the one-level density is

$$\mathbb{E}_q^{\mathrm{h},\epsilon}\left(D_{1,q}[\Phi]\right) := 2 \sum_{\substack{f \in H_\kappa^\epsilon(q) \\ \epsilon_f(q)=\epsilon}} \omega_q(f) D_{1,q}[\Phi](f)$$

and its *signed m-th moments* are

$$\mathbb{M}_{q,m}^{\mathrm{h},\epsilon}\left(D_{1,q}[\Phi]\right) := \mathbb{E}_q^{\mathrm{h},\epsilon}\left(\left(D_{1,q}[\Phi] - \mathbb{E}_q^{\mathrm{h},\epsilon}\left(D_{1,q}[\Phi]\right)\right)^m\right)$$

for any integer $m \geq 1$. The possible limits of these sequences will be denoted $\mathbb{E}_\infty^{\mathrm{h},\epsilon}\left(D_1[\Phi]\right)$ and $\mathbb{M}_{\infty,m}^{\mathrm{h},\epsilon}\left(D_1[\Phi]\right)$ for any integer $m \geq 1$.

3. A very quick walk in the world of random matrices

3.1. On classical compact groups.

Let $N \geq 1$ be an integer. We define

$$\begin{aligned} U_N &:= \left\{A \in M_N(\mathbb{C}), \quad AA^* = 1_N\right\}, \\ SO_N &:= \left\{A \in U_N \cap M_N(\mathbb{R}), \quad \det(A) = +1\right\} \end{aligned}$$

where 1_N is the identity matrix of size N. These compact groups are endowed with normalised Haar measures d_{U_N} and d_{SO_N}. We consider the following sequences of probability spaces

$$\begin{aligned} O &:= \left((SO_N, \mathrm{d}_{SO_N})\right)_{N\geq 1}, \\ SO^+ &:= \left((SO_{2N}, \mathrm{d}_{SO_{2N}})\right)_{N\geq 1}, \\ SO^- &:= \left((SO_{2N+1}, \mathrm{d}_{SO_{2N+1}})\right)_{N\geq 1}. \end{aligned}$$

Note that the eigenvalues of any $A \in U_N$ can be writen as

$$\exp\left(i\theta_1(A)\right), \ldots, \exp\left(i\theta_N(A)\right)$$

where $0 \leq \theta_1(A)) \leq \ldots \leq \theta_N(A)) \leq 2\pi$. We define the normalised eigenangles by

$$\forall i \in \{1, \ldots, N\}, \quad \widehat{\theta_j}(A) := \frac{N}{2\pi}\theta_i(A).$$

since the mean spacing between eigenangles is roughly $(2\pi)/N$.

3.2. One-level density.

Fix $\Phi \in \mathcal{S}_\nu(\mathbb{R})$. If $K_N \subset U_N$ is one of the above compact groups, then the random variable on (K_N, d_{K_N}) defined by

$$\forall A \in K_N, \quad D_{1,K_N}[\Phi](A) := \sum_{j=1}^N \Phi\left(\widehat{\theta_j}(A)\right)$$

is the *one-level density* (relatively to Φ). Its *expectation* is

$$\mathbb{E}_{\mathrm{N}}(D_{1,K_N}[\Phi]) := \int_{K_N} D_{1,K_N}[\Phi](A)\mathrm{d}_{K_N}(A)$$

and its *m-th moments* are

$$\mathbb{M}_{\mathrm{N,m}}(D_{1,K_N}[\Phi]) := E_N\left(\left(D_{1,K_N}[\Phi] - E_N(D_{1,K_N}[\Phi])\right)^m\right)$$

for any integer $m \geq 1$. The limits of the sequences of complex numbers

$$\left(\mathbb{E}_{\mathrm{N}}\left(D_{1,K_N}[\Phi]\right)\right)_{N\geq 1}, \quad \left(\mathbb{M}_{\mathrm{N},\mathrm{m}}\left(D_{1,K_N}[\Phi]\right)\right)_{N\geq 1}$$

as N goes to infinity will be denoted

$$\mathbb{E}_{\infty}\left(D_{1,K}[\Phi]\right), \quad \mathbb{M}_{\infty,m}\left(D_{1,K}[\Phi]\right)$$

for any integer $m \geq 1$.

4. Iwaniec-Katz-Luo-Sarnak's results on one-level densities

Katz and Sarnak (see [6]) proved the following result.

Theorem 1. *If $\nu > 0$ is any real number and Φ belongs to $\mathcal{S}_\nu(\mathbb{R})$ then*

$$\begin{aligned}
\mathbb{E}_{\infty}\left(D_{1,O}[\Phi]\right) &= \delta_0(x) + \frac{1}{2}, \\
\mathbb{E}_{\infty}\left(D_{1,SO^+}[\Phi]\right) &= \delta_0(x) + \frac{1}{2}\eta(x), \\
\mathbb{E}_{\infty}\left(D_{1,SO^-}[\Phi]\right) &= \delta_0(x) - \frac{1}{2}\eta(x) + 1,
\end{aligned}$$

where

$$\eta(x) := \begin{cases} 1 & \text{if } |x| < 1, \\ \frac{1}{2} & \text{if } x = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2. *It should be mentioned that if Φ belongs to $\mathcal{S}_\nu(\mathbb{R})$ with $\nu < 1$ then the three densities match:*

$$\mathbb{E}_{\infty}\left(D_{1,O}[\Phi]\right) = \mathbb{E}_{\infty}\left(D_{1,SO^+}[\Phi]\right) = \mathbb{E}_{\infty}\left(D_{1,SO^-}[\Phi]\right).$$

A result similar in the world of L-functions was proved by Iwaniec and Luo and Sarnak (see [5]).

Theorem 3. *If $\nu < 2$ and Φ is in $\mathcal{S}_\nu(\mathbb{R})$ then*

$$\begin{aligned}
\mathbb{E}^{\mathrm{h}}_{\infty}\left(D_1[\Phi]\right) &= \mathbb{E}_{\infty}\left(D_{1,O}[\Phi]\right), \\
\mathbb{E}^{\mathrm{h},+1}_{\infty}\left(D_1[\Phi]\right) &= \mathbb{E}_{\infty}\left(D_{1,SO^+}[\Phi]\right), \\
\mathbb{E}^{\mathrm{h},-1}_{\infty}\left(D_1[\Phi]\right) &= \mathbb{E}_{\infty}\left(D_{1,SO^-}[\Phi]\right).
\end{aligned}$$

Remark 4. *The crucial fact is that the authors succeeded in breaking the natural barrier $\nu = 1$.*

Remark 5. *This result, which is believed to be true without any restriction on the size of the support ν, suggests that zeros of Hecke L-functions behave like eigenvalues of orthogonal random matrices of large rank. In addition, a trivial vanishing at the critical point seems to have some effect on the behaviour of low-lying zeros.*

5. Hughes-Miller's results on mock-Gaussian behaviour

For any $\Phi \in \mathcal{S}_\nu(\mathbb{R})$, one defines

$$\sigma_\Phi^2 := 2\int_{-1}^{+1} |u|\widehat{\Phi}^2(u)\,\mathrm{d}u$$

and

$$R_m(\Phi) := (-1)^{m-1}2^{m-1}\left(\int_{\mathbb{R}} \Phi(x)^m \frac{\sin(2\pi x)}{2\pi x}\,\mathrm{d}x - \frac{1}{2}\Phi(0)^m\right)$$

for any integer $m \geq 1$. Hughes and Miller proved the following striking result (see [2]).

Theorem 6. *Let $\varepsilon = \pm 1$ and $\Phi \in \mathcal{S}_\nu(\mathbb{R})$. We assume hypothesis* GRH *and the Generalized Riemann hypothesis for all Dirichlet L-functions. If $\nu < \frac{1}{m-1}$ then*

$$\mathbb{M}^{\mathrm{h}}_{\infty,m}\left(D_1[\Phi]\right) = \mathbb{M}_{\infty,m}\left(D_{1,0}[\Phi]\right) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 2\int_{\mathbb{R}} |u|\widehat{\Phi}^2(u)\,\mathrm{d}u \times \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!} & \text{otherwise.}\end{cases}$$

and

$$\mathbb{M}^{\mathrm{h},\epsilon}_{\infty,m}\left(D_1[\Phi]\right) = \mathbb{M}_{\infty,m}\left(D_{1,SO^\varepsilon}[\Phi]\right) = \begin{cases} \varepsilon \times R_m(\Phi) & \text{if } m \text{ is odd,} \\ \varepsilon \times R_m(\Phi) + 2\int_{\mathbb{R}} |u|\widehat{\Phi}^2(u)\,\mathrm{d}u \times \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!} & \text{otherwise.}\end{cases}$$

Remark 7. *It may be checked that if $\nu < \frac{1}{m}$ then $R_m(\Phi) = 0$ while if $\nu < \frac{1}{m-1}$ then $R_m(\Phi)$ is not identically zero. As a consequence, the moments of the signed one-level densities of low-lying zeros of Hecke L-functions and the moments of the one-level densities attached to SO^- and SO^+ are Gaussian if $\nu < \frac{1}{m}$ but cease to be Gaussian as soon as the support exceeds $\frac{1}{m}$. Such a phenomenon was observed for the first time by Hughes and Rudnick (see [2]) in the particular case of Dirichlet L-functions. In addition, the defect of being Gaussian is exactly balanced according to the "sign", which implies that the moments of the one-level density of low-lying zeros of Hecke L-functions and the moments of the one-level density attached to O are Gaussian if $\nu < \frac{1}{m}$.*

Remark 8. *Let us explain the different assumptions in the previous theorem. Firstly, hypothesis* GRH *may be easily removed. Secondly, the Generalized Riemann hypothesis for all Dirichlet L-functions is crucial for the following reason. The Gaussian term comes from the diagonal term in Petersson's trace formula whereas the non-Gaussian term $R_m(\Phi)$ comes from an analysis of sums of Kloosterman sums on the prime numbers. Evaluating such sums comes down to evaluating sums of characters over the prime numbers.*

References

[1] C. P. Hughes and Steven J. Miller, *Low-lying zeros of L-functions with orthogonal symmetry*, Duke Math. J. **136** (2007), no. 1, 115–172. MR MR2271297

[2] C. P. Hughes and Z. Rudnick, *Linear statistics of low-lying zeros of L-functions*, Q. J. Math. **54** (2003), no. 3, 309–333. MR MR2013141 (2005a:11131)

[3] Henryk Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997. MR MR1474964 (98e:11051)

[4] Henryk Iwaniec and Emmanuel Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR MR2061214 (2005h:11005)

[5] Henryk Iwaniec, Wenzhi Luo, and Peter Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. (2000), no. 91, 55–131 (2001). MR MR1828743 (2002h:11081)

[6] Nicholas M. Katz and Peter Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications, vol. 45, American Mathematical Society, Providence, RI, 1999. MR MR1659828 (2000b:11070)

[7] G. Ricotta and E. Royer, *Statistics for low-lying zeros of symmetric power l-functions in the level aspect*, preprint available at `http://arxiv.org/abs/math/0703760` (2007).