J.-M. Delort
Université Paris 13, Institut Galilée
CNRS, UMR 7539, Laboratoire Analyse Géométrie et Applications
99, Avenue J.-B. Clément
F-93430 Villetaneuse

A QUASI-LINEAR BIRKHOFF NORMAL FORMS METHOD. APPLICATION TO THE QUASI-LINEAR KLEIN-GORDON EQUATION ON $\mathbb{S}^1$

J.-M. Delort, Université Paris 13, Institut Galilée, CNRS, UMR 7539, Laboratoire Analyse Géométrie et Applications, 99, Avenue J.-B. Clément, F-93430 Villetaneuse

2000 *Mathematics Subject Classification.* — 35L70, 35B45, 37K05, 35S50.

Key words and phrases. — Birkhoff normal forms, Quasi-linear Hamiltonian equations, Almost global existence, Klein-Gordon equation.

This work was partially supported by the ANR project *Equa-disp*.

A QUASI-LINEAR BIRKHOFF NORMAL FORMS METHOD. APPLICATION TO THE QUASI-LINEAR KLEIN-GORDON EQUATION ON $\mathbb{S}^1$

J.-M. Delort
Université Paris 13, Institut Galilée
CNRS, UMR 7539, Laboratoire Analyse Géométrie et Applications
99, Avenue J.-B. Clément
F-93430 Villetaneuse

Abstract. — Consider a nonlinear Klein-Gordon equation on the unit circle, with smooth data of size $\epsilon \to 0$. A solution u which, for any $\kappa \in \mathbb{N}$, may be extended as a smooth solution on a time-interval $]-c_\kappa \epsilon^{-\kappa}, c_\kappa \epsilon^{-\kappa}[$ for some $c_\kappa > 0$ and for $0 < \epsilon < \epsilon_\kappa$, is called an almost global solution. It is known that when the nonlinearity is a polynomial depending only on u, and vanishing at order at least 2 at the origin, any smooth small Cauchy data generate, as soon as the mass parameter in the equation stays outside a subset of zero measure of $\mathbb{R}_+^*$, an almost global solution, whose Sobolev norms of higher order stay uniformly bounded. The goal of this paper is to extend this result to general Hamiltonian *quasi-linear* nonlinearities. These are the only *Hamiltonian* non linearities that depend not only on u, but also on its space derivative. To prove the main theorem, we develop a Birkhoff normal form method for quasi-linear equations.

Résumé. — Considérons une équation de Klein-Gordon non-linéaire sur le cercle unité, à données régulières de taille $\epsilon \to 0$. Appelons solution presque globale toute solution u, qui se prolonge pour tout $\kappa \in \mathbb{N}$ sur un intervalle de temps $]-c_\kappa \epsilon^{-\kappa}, c_\kappa \epsilon^{-\kappa}[$, pour un certain $c_\kappa > 0$ et $0 < \epsilon < \epsilon_\kappa$. Il est connu que de telles solutions existent, et restent uniformément bornées dans des espaces de Sobolev d'ordre élevé, lorsque la non-linéarité de l'équation est un polynôme en u nul à l'ordre 2 à l'origine, et lorsque le paramètre de masse de l'équation reste en dehors d'un sous-ensemble de mesure nulle de $\mathbb{R}_+^*$. Le but de cet article est d'étendre ce résultat à des non-linéarités *quasi-linéaires* Hamiltoniennes générales. Il s'agit en effet des seules non-linéarités *Hamiltoniennes* qui puissent dépendre non seulement de u, mais aussi de sa dérivée en espace. Nous devons, pour obtenir le théorème principal, développer une méthode de formes normales de Birkhoff pour des équations quasi-linéaires.

CONTENTS

CHAPTER 0

INTRODUCTION

The main objective of this paper is the construction of a Birkhoff normal forms method, applying to quasi-linear Hamiltonian equations. We use this method to obtain almost global solutions for quasi-linear Hamiltonian Klein-Gordon equations, with small Cauchy data, on the circle $\mathbb{S}^1$.

Let us first present the general framework we are interested in. Let Δ be the Laplace-Beltrami operator on $\mathbb{R}^d$ or on a compact manifold, and consider the evolution equation

$$\begin{aligned}(\partial_t^2 - \Delta + m^2)v &= F(v, \partial_t v, \partial_x v, \partial_t\partial_x v, \partial_x^2 v)\\ v|_{t=0} &= \epsilon v_0\\ \partial_t v|_{t=0} &= \epsilon v_1,\end{aligned} \tag{0.0.1}$$

where v_0, v_1 are smooth functions, $\epsilon > 0$ is small, F is a polynomial non-linearity with affine dependence in $(\partial_t\partial_x v, \partial_x^2 v)$, so that the equation is quasi-linear. We are interested in finding a solution defined on the largest possible time-interval when $\epsilon \to 0+$. If F vanishes at order $\alpha + 1$ at the origin, local existence theory implies that the solution exists at least over an interval $]-c\epsilon^{-\alpha}, c\epsilon^{-\alpha}[$, if $v_0 \in H^{s+1}$, $v_1 \in H^s$ with s large enough, and that $\|v(t,\cdot)\|_{H^{s+1}} + \|\partial_t v(t,\cdot)\|_{H^s}$ stays bounded on such an interval. The problem we are interested in is the construction of almost global solutions, i.e. solutions defined on $]-c_\kappa\epsilon^{-\kappa}, c_\kappa\epsilon^{-\kappa}[$ for any κ.

This problem is well understood when one can make use of dispersion, e.g. when one studies (0.0.1) on $\mathbb{R}^d$, with v_0, v_1 smooth and quickly decaying at infinity (for instance $v_0, v_1 \in C_0^\infty(\mathbb{R}^d)$). When dimension d is larger or equal to three, Klainerman [**15**] and Shatah [**19**] proved independently global existence for small enough $\epsilon > 0$. Their methods were quite different: the main ingredient of Klainerman's proof was the use of vector fields commuting to the linear part of the equation. On the other hand, Shatah introduced in the subject normal form methods, which are classical tools in ordinary differential equations. Both approaches have been combined by Ozawa, Tsutaya and Tsutsumi [**18**] to prove global existence for the same equation in two space dimensions. We also refer to [**9**] and references therein for the case of dimension 1.

A second line of investigation concerns equation (0.0.1) on a compact manifold. In this case, no dispersion is available. Nevertheless, two trails may be used to obtain solutions, defined on time-intervals larger than the one given by local existence theory, and whose higher order Sobolev norms are uniformly bounded. The first one is to consider special Cauchy data giving rise to periodic or quasi-periodic (hence global) solutions. A lot of work has been devoted to these questions in dimension one, i.e. for $x \in \mathbb{S}^1$, when the non-linearity in (0.0.1) depends only on v. We refer to the work of Kuksin [**16, 17**], Craig and Wayne [**8**], Wayne [**20**], and for a state of the art around 2000, to the book of Craig [**7**] and references therein. More recent results may be found in the book of Bourgain [**6**].

The second approach concerns the construction of almost global H^s-small solutions for the Cauchy problem (0.0.1) on $\mathbb{S}^1$, when the non-linearity depends only on v. In this case, small H^1 Cauchy data give rise to global solutions, and the question is to keep uniform control of the H^s-norm of the solution, over time-intervals of length $\epsilon^{-\kappa}$, for any κ and large enough s. This has been initiated by Bourgain [**5**], who stated a result of almost global existence and uniform control for $(\partial_t^2 - \partial_x^2 + m^2)v = F(v)$ on $\mathbb{S}^1$, when m stays outside a subset of zero measure, and the Cauchy data are small and smooth enough. A complete proof has been given by Bambusi [**1**], Bambusi-Grébert [**3**] (see also Grébert [**14**]). It relies on the use of a Birkhoff normal form method, exploiting the fact that when the non-linearity depends only on v, the equation may be written as a Hamiltonian system.

Let us mention that some of the results we described so far admit extensions to higher dimensions. Actually, constructions of periodic or quasi-periodic solutions for equations of type $(i\partial_t - \Delta)v = F(v)$ or $(\partial_t^2 - \Delta + m^2)v = F(v)$ have been performed by Eliasson-Kuksin [**13**] and Bourgain [**6**] on higher dimensional tori. Almost global solutions for the Cauchy problem on spheres and Zoll manifolds have been obtained by Bambusi, Delort, Grébert and Szeftel [**2**] for almost all values of m.

We are interested here in the Cauchy problem when the non-linearity is a function not only of v, but also of derivatives of v. Some results have been proved by Delort and Szeftel [**11, 12**] for semi-linear non-linearities of the form $F(v, \partial_t v, \partial_x v)$ on $\mathbb{S}^d$ or on Zoll manifolds. For instance, it has been proved that if F is homogeneous of even order $\alpha + 1$, then the solution exists over an interval of length $\epsilon^{-2\alpha}$, when the mass m stays outside a subset of zero measure. Similar statements have been obtained in one space dimension for quasi-linear equations in [**10**]. Nevertheless, no result of almost global existence was known up to now, for non-linearities depending on the derivatives. This is related to the fact that, in contrast with the case of non-linearities $F(v)$, the normal form method used to pass from a time-length $\epsilon^{-\alpha}$ (corresponding to local existence theory) to $\epsilon^{-2\alpha}$ cannot be easily iterated. Actually, for non-linearities depending only on v, the iteration may be performed using a Birkhoff normal forms approach permitted by the Hamiltonian structure. To try to obtain almost global existence for equations involving derivatives in their right hand side, it is thus natural to limit oneself to systems of the form of (0.0.1) for which the non-linearity is Hamiltonian. This obliges one to consider quasi-linear equations, as the only semi-linear non-linearities enjoying the Hamiltonian structure of theorem 1.1.1 below are those depending only on v.

The main result of this paper asserts that the quasi-linear Klein-Gordon equation on $\mathbb{S}^1$, with Hamiltonian non-linearity, admits almost global solutions for small enough, smooth enough Cauchy data, when the mass is outside a subset of zero measure (see section 1.1 for a more precise statement). The main novelty in this paper, compared with the semi-linear setting, is the introduction of a Birkhoff method adapted to quasi-linear equations. We shall describe below the idea of the method on a model case, which can be used as a road-map for the more technical approach that will be followed in the bulk of the paper. Roughly speaking, the idea is to combine the usual Birkhoff normal forms method with the strategy used to obtain quasi-linear energy inequalities (namely (para)diagonalization of the nonlinear principal symbol of the operator). The latter was used in [**10**] in the non-Hamiltonian framework. Here, as we need to preserve the Hamiltonian structure of our problem, such a diagonalization will have to be performed respecting the underlying symplectic form.

Let us describe the organization of the paper and the idea of the proof on a model problem. Chapter one is devoted to the statement of the main theorem and to the introduction of the symplectic framework. In this presentation, let us consider the symplectic form on the Sobolev space $H^s(\mathbb{S}^1;\mathbb{C})$ $(s \geq 0)$

$$\omega_0(c, c') = 2\mathrm{Im} \int_{\mathbb{S}^1} c(x)\overline{c'(x)}dx.$$

If F, G are two C^1 functions defined on an open subset of $H^s(\mathbb{S}^1;\mathbb{C})$, whose gradients belong to L^2, we define the Poisson bracket

$$\{F, G\} = i(\partial_u F \nabla_{\bar{u}} G - \partial_u G \nabla_{\bar{u}} F).$$

For a given C^1 Hamiltonian G on $H^s(\mathbb{S}^1;\mathbb{C})$, the associated evolution equation defined by its symplectic gradient is

$$\dot{u} = i\nabla_{\bar{u}} G(u, \bar{u}). \tag{0.0.2}$$

Let us study as a model the case when

$$G(u, \bar{u}) = \int_{\mathbb{S}^1} (\Lambda_m u)\bar{u}dx + \mathrm{Re} \int_{\mathbb{S}^1} (a(u, \bar{u})\Lambda_m u)\bar{u}dx + \mathrm{Re} \int_{\mathbb{S}^1} (b(u, \bar{u})\Lambda_m u)udx \tag{0.0.3}$$

where a, b are polynomials in $(u, \bar{u})$ and $\Lambda_m = \sqrt{-\partial_x^2 + m^2}$. The associated evolution equation is

$$\begin{aligned}\frac{\partial u}{\partial t} &= i\Lambda_m u + \frac{i}{2}[a\Lambda_m + \Lambda_m \bar{a}]u + \frac{i}{2}[\bar{b}\Lambda_m + \Lambda_m \bar{b}]\bar{u} \\ &+ \frac{i}{2}\Big(\frac{\partial a}{\partial \bar{u}}\Big)(\Lambda_m u)\bar{u} + \frac{i}{2}\Big(\frac{\partial \bar{a}}{\partial \bar{u}}\Big)(\Lambda_m \bar{u})u \\ &+ \frac{i}{2}\Big(\frac{\partial b}{\partial \bar{u}}\Big)(\Lambda_m u)u + \frac{i}{2}\Big(\frac{\partial \bar{b}}{\partial \bar{u}}\Big)(\Lambda_m \bar{u})\bar{u}.\end{aligned} \tag{0.0.4}$$

This equation is, if $a(0) = b(0) = 0$ and if u is small enough, a small perturbation of the linear hyperbolic equation $\frac{\partial u}{\partial t} = i\Lambda_m u$. Moreover, since the non-linearity involves first order derivatives, this is a quasi-linear equation.

To prove that (0.0.4), with a Cauchy data $u|_{t=0} = \epsilon u_0$ with $u_0 \in H^s(\mathbb{S}^1; \mathbb{C})$, has a solution defined on an interval $] - c\epsilon^{-\kappa}, c\epsilon^{-\kappa}[$ for any given $\kappa \in \mathbb{N}$, it is enough to prove an a priori bound $\Theta_s^0(u(t, \cdot)) \leq C\epsilon^2$ when $|t| \leq c\epsilon^{-\kappa}$, where

$$\Theta_s^0(u) = \frac{1}{2}\langle \Lambda_m^s u, \Lambda_m^s u\rangle \tag{0.0.5}$$

is equivalent to the square of the Sobolev norm of u. Let us recall how such a uniform control may be obtained in the case of semi-linear equations (i.e. when the last two terms in (0.0.3) are replaced by Re $\int_{\mathbb{S}^1} a(u, \bar{u})u\bar{u}dx$ + Re $\int_{\mathbb{S}^1} b(u, \bar{u})uudx$). One introduces an auxiliary C^1-function F and solves the Hamiltonian equation

$$\dot{\Phi}(t, u) = X_F(\Phi(t, u)),\ \Phi(0, u) = u, \tag{0.0.6}$$

where X_F is the Hamiltonian vector field associated to F. Then $\chi_F(u) = \Phi(1, u)$ is a canonical transformation, defined on a neighborhood of zero in $H^s(\mathbb{S}^1, \mathbb{C})$, with $\chi(0) = 0$, and one wants to choose F so that $\Theta_s(u) = \Theta_s^0 \circ \chi_F(u)$ satisfies, for a given arbitrary κ,

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t, \cdot)) &= O(\|u(t, \cdot)\|_{H^s}^{\kappa+2}) \\ |\Theta_s(u) - \Theta_s^0(u)| &= O(\|u(t, \cdot)\|_{H^s}^3).\end{aligned} \tag{0.0.7}$$

These two equalities imply that, for small enough Cauchy data, $\|u(t, \cdot)\|_{H^s}$ stays bounded by $C\epsilon$ over an interval of time of length $c\epsilon^{-\kappa}$. One wants to apply a Birkhoff method. Since by (0.0.2) $\dot{u} = X_G(u(t, \cdot))$, one has

$$\frac{d}{dt}\Theta_s^0 \circ \chi_F(u(t, \cdot)) = \{\Theta_s^0 \circ \chi_F, G\}(u(t, \cdot)) = \{\Theta_s^0, G \circ \chi_F^{-1}\}(\chi_F(u(t, \cdot))), \tag{0.0.8}$$

and one would like to choose F so that $\{\Theta_s^0, G \circ \chi_F^{-1}\}(u)$ vanishes at order $\kappa + 2$ when $u \to 0$. If F satisfies convenient smoothness assumptions, one may deduce from Taylor expansion that

$$G \circ \chi_F^{-1}(u) = \sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} \cdot G(u) + \frac{1}{(\kappa-1)!}\int_0^1 (1-\tau)^{\kappa-1}(\mathrm{Ad}^\kappa F \cdot G)(\Phi(-\tau, u))d\tau, \tag{0.0.9}$$

where $\mathrm{Ad}F\cdot G=\{F,G\}$. When considering *semi-linear* equations, one looks for $F=\sum_{\ell=1}^{\kappa-1}F_\ell(u,\bar u)$, with F_ℓ homogeneous of degree $\ell+2$, such that

$$\Big\{\Theta_s^0,\sum_{k=0}^{\kappa-1}\frac{\mathrm{Ad}^kF}{k!}\cdot G(u)\Big\}=O(\|u(t,\cdot)\|_{H^s}^{\kappa+2}),\ u\to 0. \tag{0.0.10}$$

Decomposing the second argument of the above Poisson bracket in terms of increasing degree of homogeneity, one gets

$$G_0+\sum_{\ell\geq 1}(\{F_\ell,G_0\}+H_\ell),$$

where $G_0(u)=\int_{\mathbb{S}^1}(\Lambda_m u)\bar u dx$ and where H_ℓ is homogeneous of degree $\ell+2$, and depends on the homogeneous component G_k of degree k of G, for $k=1,\dots,\ell$ and on $F_1,\dots,F_{\ell-1}$. In that way, (0.0.10) can be reduced to

$$\{\Theta_s^0,\{F_\ell,G_0\}+H_\ell\}=0,\ \ell=1,\dots,\kappa-1. \tag{0.0.11}$$

This homological equation can easily be solved in the semi-linear case, as soon as the parameter m in $\Lambda_m=\sqrt{-\partial_x^2+m^2}$ is taken outside a subset of zero measure, to avoid resonances.

Let us examine now the quasi-linear case, i.e. the case when G is given by (0.0.3). Equation (0.0.11) for $\ell=1$ may be written

$$\{\Theta_s^0,\{F_1,G_0\}+G_1\}=0, \tag{0.0.12}$$

where

$$G_1(u,\bar u)=\mathrm{Re}\Big[\int_{\mathbb{S}^1}(a_1(u,\bar u)\Lambda_m u)\bar u dx+\int_{\mathbb{S}^1}(b_1(u,\bar u)\Lambda_m u)u dx\Big], \tag{0.0.13}$$

with a_1,b_1 homogeneous of degree 1 in $u,\bar u$. Let us look for F_1 given by

$$\mathrm{Re}\int_{\mathbb{S}^1}(\tilde A_1(u,\bar u)u)\bar u dx+\mathrm{Re}\int_{\mathbb{S}^1}(\tilde B_1(u,\bar u)u)u dx, \tag{0.0.14}$$

where $\tilde A_1,\tilde B_1$ are operators depending on $u,\bar u$ to be determined. We have

$$\begin{aligned}\Big\{\int_{\mathbb{S}^1}(\tilde A_1(u,\bar u)u)\bar u dx,G_0\Big\}&=i\int_{\mathbb{S}^1}\big([\tilde A_1(u,\bar u)\Lambda_m-\Lambda_m\tilde A_1(u,\bar u)]u\big)\bar u dx\\&+i\int_{\mathbb{S}^1}\big([\partial_u\tilde A_1(u,\bar u)\cdot\Lambda_m u-\partial_{\bar u}\tilde A_1(u,\bar u)\cdot\Lambda_m\bar u]u\big)\bar u dx\end{aligned} \tag{0.0.15}$$

and

$$\begin{aligned}\Big\{\int_{\mathbb{S}^1}(\tilde B_1(u,\bar u)u)u dx,G_0\Big\}&=i\int_{\mathbb{S}^1}\big([\tilde B_1(u,\bar u)\Lambda_m+\Lambda_m\tilde B_1(u,\bar u)]u\big)u dx\\&+i\int_{\mathbb{S}^1}([\partial_u\tilde B_1(u,\bar u)\cdot\Lambda_m u-\partial_{\bar u}\tilde B_1(u,\bar u)\cdot\Lambda_m\bar u]u)u dx.\end{aligned} \tag{0.0.16}$$

Let us try to solve (0.0.12) finding F_1 such that $\{F_1, G_0\} + G_1 = 0$. It would be enough to determine $\tilde{A}_1, \tilde{B}_1$ such that, according to (0.0.13), (0.0.15), (0.0.16),

(0.0.17)
$$i[\tilde{A}_1, \Lambda_m] + i\partial_u \tilde{A}_1(u,\bar{u}) \cdot (\Lambda_m u) - i\partial_{\bar{u}} \tilde{A}_1(u,\bar{u}) \cdot (\Lambda_m \bar{u}) = -a_1(u,\bar{u})\Lambda_m$$
$$i[\tilde{B}_1\Lambda_m + \Lambda_m\tilde{B}_1] + i\partial_u \tilde{B}_1(u,\bar{u}) \cdot (\Lambda_m u) - i\partial_{\bar{u}} \tilde{B}_1(u,\bar{u}) \cdot (\Lambda_m \bar{u}) = -b_1(u,\bar{u})\Lambda_m.$$

Note that if $\tilde{A}_1$ (resp. $\tilde{B}_1$) is an operator of order α (resp. β), then $\partial_u \tilde{A}_1(u,\bar{u})\cdot(\Lambda_m u)$, $\partial_{\bar{u}} \tilde{A}_1(u,\bar{u}) \cdot (\Lambda_m \bar{u})$ (resp. $\partial_u \tilde{B}_1(u,\bar{u}) \cdot (\Lambda_m u)$, $\partial_{\bar{u}} \tilde{B}_1(u,\bar{u}) \cdot (\Lambda_m \bar{u})$) is also of order α (resp. β), since the loss of one derivative coming from Λ_m affects the smoothness of the coefficients, and not the order of the operator. On the other hand $[\tilde{A}_1, \Lambda_m]$ (resp. $[\tilde{B}_1\Lambda_m + \Lambda_m\tilde{B}_1]$) is of order α (resp. $\beta + 1$). Since the right hand sides on (0.0.17) are operators of order 1, we may expect, if we can solve (0.0.17), to find $\tilde{A}_1$ of order 1 and $\tilde{B}_1$ of order zero. This would give F_1 by expression (0.0.14). Let us switch to (0.0.11) for $\ell = 2$. Then H_2 will contain, because of (0.0.10), a contribution of form $\{F_1, G_1\}$. Denote to simplify notations

$$A_1 = \frac{1}{2}(a_1(u,\bar{u})\Lambda_m + \Lambda_m\overline{a_1(u,\bar{u})}), B_1(u,\bar{u}) = b_1(u,\bar{u})\Lambda_m.$$

Let us compute the Poisson brackets (0.0.15), (0.0.16) with G_0 replaced by G_1:

(0.0.18)
$$\begin{aligned}\Big\{\textstyle\int_{\mathbb{S}^1}(\tilde{A}_1(u,\bar{u})u)\bar{u}dx, \int_{\mathbb{S}^1}(A_1(u,\bar{u})u)\bar{u}dx + \frac{1}{2}\int_{\mathbb{S}^1}(B_1(u,\bar{u})u)udx + \frac{1}{2}\int_{\mathbb{S}^1}(\overline{B_1(u,\bar{u})}\bar{u})\bar{u}dx\Big\}&\\ = i\int_{\mathbb{S}^1}([\tilde{A}_1, A_1](u,\bar{u})u)\bar{u}dx + \frac{i}{2}\int_{\mathbb{S}^1}(\tilde{A}_1(\overline{B}_1 + {}^t\overline{B}_1)(u,\bar{u})\bar{u})\bar{u}dx&\\ -\frac{i}{2}\int_{\mathbb{S}^1}((B_1 + {}^tB_1)\tilde{A}_1(u,\bar{u})u)udx + \text{ other terms}&\end{aligned}$$

and

(0.0.19)
$$\begin{aligned}\Big\{\textstyle\int_{\mathbb{S}^1}(\tilde{B}_1(u,\bar{u})u)udx, \int_{\mathbb{S}^1}(A_1(u,\bar{u})u)\bar{u}dx + \frac{1}{2}\int_{\mathbb{S}^1}(B_1(u,\bar{u})u)udx + \frac{1}{2}\int_{\mathbb{S}^1}(\overline{B_1(u,\bar{u})}\bar{u})\bar{u}dx\Big\}&\\ = i\int_{\mathbb{S}^1}([(\tilde{B}_1A_1 + {}^tA_1\tilde{B}_1)(u,\bar{u})u)udx + \frac{i}{2}\int_{\mathbb{S}^1}((\overline{B}_1 + {}^t\overline{B}_1)({}^t\tilde{B}_1 + \tilde{B}_1)(u,\bar{u})u)\bar{u}dx&\\ + \text{ other terms.}&\end{aligned}$$

Note that since $\tilde{A}_1$ and B_1 are of order 1, the right hand side of (0.0.18) has a structure similar to G_1, except that the expressions which are bilinear in u or in $\bar{u}$ are now of order 2. In other words, if we solve (0.0.12) for a quasi-linear Hamiltonian, we get in (0.0.11) with $\ell = 2$ a contribution to H_2 which loses two derivatives, instead of just one. Obviously, if we repeat the process, we shall loose one new derivative at each step, which apparently ruins the method. Observe nevertheless that we can avoid such losses if, in a first attempt, we choose F_ℓ in order to eliminate in (0.0.10) only those terms homogeneous of degree $1, 2, \dots, \kappa - 1$ coming from *the second* contribution on the right hand side of (0.0.13). In other words, we look for F_1 given by (0.0.14) with

$\tilde{A}_1 = 0$, and want to solve only the second equation in (0.0.17). As already noticed, we shall find an operator $\tilde{B}_1$ of order *zero*. If we look at the contribution induced by this $\tilde{B}_1$ at the following step, we have to consider (0.0.19), whose right hand side may be written essentially

$$\int_{\mathbb{S}^1} (\tilde{A}_2(u,\bar{u})u)\bar{u}dx + \int_{\mathbb{S}^1} (\tilde{B}_2(u,\bar{u})u)udx + \text{ other terms}$$

where $\tilde{A}_2 = (\overline{B}_1 + {}^t\overline{B}_1)({}^t\tilde{B}_1 + \tilde{B}_1)$ and $\tilde{B}_2 = \tilde{B}_1 A_1 + {}^t A_1 \tilde{B}_1$ are of *order 1*. We obtain again an expression of type (0.0.14), without any loss of derivatives, and a gain on the degree of homogeneity. Of course, we have completed only part of our objective, since the b_1 contribution to (0.0.13) has been removed, but not the a_1 one. In other words, the best we may expect is to choose F in such a way that in (0.0.10)

$$\sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} \cdot G(u) = \sum_{k=0}^{\kappa-1} G'_k(u) + R_\kappa(u), \tag{0.0.20}$$

with $G'_0 = G_0$ and $G'_k(u) = \mathrm{Re} \int_{\mathbb{S}^1} (A'_k(u,\bar{u})u)\bar{u}dx$, with A'_k operator of order 1, homogeneous of degree k in $(u,\bar{u})$. The remainder R_κ will be of type

$$\mathrm{Re} \int_{\mathbb{S}^1} (A'_\kappa(u,\bar{u})u)\bar{u}dx + \mathrm{Re} \int_{\mathbb{S}^1} (B'_\kappa(u,u)u)udx, \tag{0.0.21}$$

with A'_κ, B'_κ of order 1, homogeneous of degree κ. The reduction to such a form, for the true problem we study, will be performed in section 5.2 of the paper.

The next step is to eliminate in (0.0.21) the B'_κ contribution. We cannot repeat the preceding method, as it would induce another remainder of the same type, with an higher degree of homogeneity. Instead, we shall use a diagonalization process. When one wants to obtain an energy inequality for an equation of type (0.0.4), the b-contributions of the right hand side already cause trouble. Actually, if one multiplies (0.0.4) by $\Lambda_m^{2s}\bar{u}$, integrates on $\mathbb{S}^1$ and takes the real part, the contributions coming from the a-term is controlled by some power of $\|u\|_{H^s}$, since it may be written in terms of the commutator $[a + \bar{a}, \Lambda_m]$. On the other hand, the contribution coming from b cannot be expressed in such a way, and loses one derivative. The way to avoid such a difficulty is well-known: one writes the system in $(u,\bar{u})$ corresponding to equation (0.0.4), diagonalizes the principal symbol of the right hand side, and performs the energy method on the diagonalized system. We adapt here a similar strategy to the Hamiltonian framework: We look for a change of variable close to zero in H^s, $(v,\bar{v}) \to (u = \psi(v), \bar{u} = \overline{\psi(v)})$, to transform (0.0.21) into

$$\mathrm{Re} \int_{\mathbb{S}^1} (A''_\kappa(v,\bar{v})v)\bar{v}dx, \tag{0.0.22}$$

where A''_κ is an operator of order 1. This is done looking for $\psi(v) = (\mathrm{Id} + R(v,\bar{v}))v$, where R is some operator, determined by a symbol diagonalizing the principal symbol of the Hamiltonian equation associated to (0.0.20). Since we need to preserve the Hamiltonian structure, i.e. to construct ψ as an (almost) canonical transformation, this diagonalization has to be performed in an (almost) symplectic way. The argument is given in section 5.3, using the results obtained in chapter 4 concerning symplectic reductions. To exploit this, we shall consider instead of $\Theta_s(u) = \Theta_s^0 \circ \chi_F(u)$ in (0.0.7), (0.0.8) a quantity $\Theta_s(u) = \Theta_s^1 \circ \psi^{-1} \circ \chi_F(u)$, for some Θ_s^1 that will be chosen later on. Then (0.0.10) has to be replaced by

$$\Big\{\Theta_s^1 \circ \psi^{-1}, \textstyle\sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} \cdot G\Big\} = O(\|u\|_{H^s}^{\kappa+2}). \tag{0.0.23}$$

Because of (0.0.20), this is equivalent to

$$\Big\{\Theta_s^1 \circ \psi^{-1}, \textstyle\sum_{k=0}^{\kappa-1} G_k'(u) + R_\kappa(u)\Big\} = O(\|u\|_{H^s}^{\kappa+2}),$$

and since ψ is canonical, this is also equivalent to

$$\Big\{\Theta_s^1, \textstyle\sum_{k=0}^{\kappa-1} G_k'(\psi(v)) + R_\kappa(\psi(v))\Big\} = O(\|v\|_{H^s}^{\kappa+2}). \tag{0.0.24}$$

The remainder $R_\kappa(\psi(v))$ is given by (0.0.22). Since Θ_s^1 will be constructed under the form $\int (\Omega(v,\bar{v})v)\bar{v}dx$, where Ω is a self-adjoint operator of order $2s$, $\{\Theta_s^1, R_\kappa(\psi(v))\}$ will be seen to be controlled by the right hand side of (0.0.24) (again, the structure of Θ_s^1 and of (0.0.22) allows one to express the Poisson bracket from a commutator $[\Omega, A_\kappa'']$ of order $2s$, vanishing at order κ at $v = 0$). Similar statements hold for $G_k'(\psi(v)) - G_k'(v)$, so that (0.0.24) is equivalent to

$$\Big\{\Theta_s^1, \textstyle\sum_{k=0}^{\kappa-1} G_k'(v)\Big\} = O(\|v\|_{H^s}^{\kappa+2}). \tag{0.0.25}$$

We are reduced to finding $\Theta_s^1(v)$, equivalent to $\|v\|_{H^s}^2$ for small v's, such that (0.0.25) holds when all G_k' are of type Re $\int_{\mathbb{S}^1}(A_k'(v,\bar{v})v)\bar{v}dx$. If we look for $\Theta_s^1 = \Theta_0^s \circ \chi_H$, for some auxiliary function H, we get formally by (0.0.9), (0.0.10), that (0.0.25) is equivalent to

$$\Big\{\Theta_s^0, \textstyle\sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k H}{k!} \cdot G'\Big\} = O(\|v\|_{H^s}^{\kappa+2}). \tag{0.0.26}$$

with $G' = \sum_{k=0}^{\kappa-1} G_k'(v)$. As in (0.0.11), (0.0.12), this equality may be reduced to a family of homological equations, the first one being

$$\{\Theta_s^0, \{H_1, G_0'\} + G_1'\} = 0. \tag{0.0.27}$$

The gain in comparison to (0.0.12), (0.0.13), is that G_1' is given by Re $\int_{\mathbb{S}^1}(A_1'(v,\bar{v})v)\bar{v}dx$, i.e. *does not* contain any component in $\int_{\mathbb{S}^1}(B_1'(v,\bar{v})v)vdx$. If one looks for some H_1 of type Re $\int_{\mathbb{S}^1}(\tilde{A}_1'(v,\bar{v})v)\bar{v}dx$, with $\tilde{A}_1'$ of order 1, all Poisson brackets involved in (0.0.27) may be expressed from commutators, so that the overall order never increases. In particular, the second homological equation may be written

$$\{\Theta_s^0, \{H_2, G_0'\} + \tilde{G}_2'\} = 0,$$

where $\tilde{G}_2'$ is given in terms of G_2' and of the Poisson brackets of H_1, G_1, and so is still of the form Re $\int_{\mathbb{S}^1}(A_2'(v,\bar{v})v)\bar{v}dx$ with A_2' of order 1. In other words, the reduction performed in the first two steps of the proof made disappear the terms of higher order in (0.0.18). In that way, one determines recursively $H_1, H_2, \ldots$, all of these functions being expressed from quantities Re $\int_{\mathbb{S}^1}(A'_j(v,\bar{v})v)\bar{v}dx$ with A'_j of order 1. There is nevertheless a technical difficulty in the implementation of this strategy: it turns out that one cannot define the canonical transformation χ_H from some Hamiltonian H, as the value at time 1 of the solution of (0.0.6) (with F replaced by H). Actually, since H is given in terms of quantities $\int_{\mathbb{S}^1}(\tilde{A}'(v,\bar{v})v)\bar{v}dx$, with $\tilde{A}'$ an operator of order 1, $X_H(v)$ is given by an operator of order 1 acting on v, so that $\dot{\Phi}(t,v) = X_H(\Phi(t,v))$ is no longer an ordinary differential equation. We get around this difficulty in section 5.3, defining a substitute to χ_H in terms of expressions involving finitely many Poisson brackets, which allows us to proceed as described above, without constructing the flow of X_H.

Let us conclude this introduction with some more technical details. As explained above, our quasi-linear Birkhoff normal forms method uses Hamiltonians given by expressions of form $\int_{\mathbb{S}^1}(A(u,\bar{u})u)\bar{u}dx$, $\int_{\mathbb{S}^1}(B(u,\bar{u})u)udx$, where A, B are operators depending on $u, \bar{u}$. Chapters 2 and 3 are devoted to the construction of the classes of operators that we need. These are para-differential operators on $\mathbb{S}^1$, whose symbols depend multilinearly on $u, \bar{u}$. Such classes have been already introduced in [**10**] (see also [**11**]). We have to modify here their definition for the following technical reason. When one uses a Birkhoff normal form method in the semi-linear case, one does not need to know much about the structure of the remainder given by the integral in (0.0.9). On the other hand, for quasi-linear problems, we need to be able to write for the remainder a quite explicit expression, of the form of (0.0.21). It is not clear how to do so from the integral expression in (0.0.9), as it involves the flow Φ of X_F. To overcome this difficulty, we use instead of (0.0.9) a full Taylor expansion of $G \circ \chi_F^{-1}$. The remainder is then $\sum_{\kappa}^{+\infty} \frac{\mathrm{Ad}^k F}{k!} \cdot G(u)$, and we need estimates to make converge the series. Since F, G are expressed in terms of para-differential operators, we have to introduce classes of symbols $a_k(u,\bar{u};x,n)$, which vanish at order k at $u = 0$, and are controlled by $C^k k!\|u\|_{H^s}^k$. Each a_k is itself an infinite sum of the type $\sum_{j\geq k} a_k^j(u,\bar{u};x,n)$, where a_k^j is homogeneous of degree j in $(u,\bar{u})$ and satisfies bounds of the form $B^j k!$ (For technical reasons, the actual (j,k)-dependence of our bounds will be more involved than that). The construction of these classes of symbols, the study of their symbolic calculus and of the Poisson brackets of functions defined in terms of the associated operators, occupies chapters 2 and 3 of this paper.

Finally, let us mention that an index of notations is provided at the end of the paper.

CHAPTER 1

ALMOST GLOBAL EXISTENCE

1.1. Statement of the main theorem

Let $H(x, X, Y)$ be a polynomial in (X, Y) with real coefficients which are smooth functions of $x \in \mathbb{S}^1$. Assume that $(X, Y) \to H(x, X, Y)$ vanishes at least at order 3 at zero. Let $m \in]0, +\infty[$. For s a large enough real number, (v_0, v_1) an element of the unit ball of $H^{s+\frac{1}{2}}(\mathbb{S}^1; \mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{S}^1; \mathbb{R})$, $\epsilon \in]0, 1[$, consider the solution $(t, x) \to v(t, x)$ defined on $[-T, T] \times \mathbb{S}^1$ for some $T > 0$ of the equation

$$\begin{aligned}(\partial_t^2 - \partial_x^2 + m^2)v &= \frac{\partial}{\partial x}\Big[\frac{\partial H}{\partial Y}(x, v, \partial_x v)\Big] - \frac{\partial H}{\partial X}(x, v, \partial_x v)\\ v|_{t=0} &= \epsilon v_0\\ \partial_t v|_{t=0} &= \epsilon v_1.\end{aligned} \tag{1.1.1}$$

The right hand side of the first equation in (1.1.1) is a quasi-linear non-linearity. Its special form will allow us to write (1.1.1) as a Hamiltonian equation in section 1.2 below. Note that the only *semi-linear* non-linearities of the form of the right hand side of (1.1.1) are those depending only on v. Our main result is:

Theorem 1.1.1. — *There is a subset $\mathcal{N} \subset]0, +\infty[$ of zero measure and, for any H as in (1.1.1), for any $m \in]0, +\infty[-\mathcal{N}$, for any $\kappa \in \mathbb{N}$, there is $s_0 \in \mathbb{N}$ such that for any integer $s \geq s_0$, there are $\epsilon_0 \in]0, 1[$, $c > 0$, satisfying the following:*

For any $\epsilon \in]0, \epsilon_0[$, for any pair (v_0, v_1) in the unit ball of $H^{s+\frac{1}{2}}(\mathbb{S}^1; \mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{S}^1; \mathbb{R})$, equation (1.1.1) has a unique solution v, defined on $] - T_\epsilon, T_\epsilon[\times\mathbb{S}^1$ with $T_\epsilon \geq c\epsilon^{-\kappa}$, and belonging to the space

$$C_b^0(] - T_\epsilon, T_\epsilon[, H^{s+\frac{1}{2}}(\mathbb{S}^1; \mathbb{R})) \times C_b^1(] - T_\epsilon, T_\epsilon[, H^{s-\frac{1}{2}}(\mathbb{S}^1; \mathbb{R}))$$

(where $C_b^j(] - T_\epsilon, T_\epsilon[, E)$ denotes the space of C^j functions on the interval $] - T_\epsilon, T_\epsilon[$ with values in the space E, whose derivatives up to order j are bounded in E uniformly on $] - T_\epsilon, T_\epsilon[$).

Remarks. — As pointed out in the introduction, when $\frac{\partial H}{\partial Y} \equiv 0$ theorem 1.1.1 is well-known. It is stated in Bourgain [**5**] and a complete proof has been given by Bambusi [**1**], Bambusi-Grébert [**3**] (see also the lectures of Grébert [**14**]).

— Results involving a semi-linear non-linearity depending also on first order derivatives (i.e. equation (1.1.1) in which the right hand side is replaced by $f(v, \partial_t v, \partial_x v)$) have been obtained by Delort and Szeftel [**11, 12**], included for equations on $\mathbb{S}^d, (d \geq 1)$ instead of $\mathbb{S}^1$. One obtains then a lower bound for the existence time in terms of some non-negative power of ϵ – better (when convenient assumptions are satisfied) than the one given by local existence theory – depending on the order of vanishing of the non-linearity at zero. In particular, one *does not* get almost global solutions for such non-linearities. For some examples of polynomial non-linearities depending on v and its first order derivatives, the lower bound of the existence time given by local existence theory (namely $T_\epsilon \geq c\epsilon^{-a}$ when v vanishes at order $a+1$ at zero) is even optimal.

— In the same way, for more general quasi-linear equations than (1.1.1), it is shown in [**10**] that the existence time is bounded from below by $c\epsilon^{-2a}$ when the non-linearity vanishes at some even order $a+1$ at zero.

— The proofs of the almost global existence results of Bourgain, Bambusi, Bambusi-Grébert refered to above rely in an essential way on the fact that the equation under consideration may be written as a Hamiltonian system. This is also the key to extend these lower bounds on the time of existence of solutions to the case of equations on $\mathbb{S}^d$, as in Bambusi, Delort, Grébert and Szeftel [**2**]. In our problem (1.1.1), we shall use the special form of the non-linearity to write the equation as a Hamiltonian system.

1.2. Hamiltonian formulation

We shall describe here the Hamiltonian formulation of our problem. Let us introduce some notation. Set

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{1.2.1}$$

and if Z, Z' are two L^2-functions on $\mathbb{S}^1$ with values in $\mathbb{R}^2$, define

$$\omega_0(Z, Z') = \langle {}^t JZ, Z' \rangle = \langle Z, JZ' \rangle \tag{1.2.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the $L^2(\mathbb{S}^1; \mathbb{R}^2)$ scalar product. Let $s \geq 0$, U be an open subset of $H^s(\mathbb{S}^1; \mathbb{R}^2)$ and $F : U \to \mathbb{R}$ be a C^1 map. Assume that for any $u \in U$, $dF(u)$ extends as a bounded linear map on $L^2(\mathbb{S}^1; \mathbb{R}^2)$. We define then $X_F(u)$ as the unique element of $L^2(\mathbb{S}^1; \mathbb{R}^2)$ such that for any $Z \in L^2(\mathbb{S}^1; \mathbb{R}^2)$

$$\omega_0(X_F(u), Z) = dF(u) \cdot Z. \tag{1.2.3}$$

In an equivalent way

$$X_F(u) = J\nabla F(u). \tag{1.2.4}$$

If $G : U \to \mathbb{R}$ is another function of the same type, we set

$$\{F, G\} = dF(u) \cdot X_G(u) = dF(u)J\nabla G(u). \tag{1.2.5}$$

Let us rewrite equation (1.1.1) as a Hamiltonian system. Set

$$\Lambda_m = \sqrt{-\Delta + m^2} \text{ on } \mathbb{S}^1. \tag{1.2.6}$$

If v solves (1.1.1), define

$$u(t, x) = \begin{bmatrix} \Lambda_m^{-1/2}\partial_t v \\ \Lambda_m^{1/2} v \end{bmatrix} = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}. \tag{1.2.7}$$

For $u \in H^s(\mathbb{S}^1; \mathbb{R}^2)$ with $s > 1$ set

$$G(u) = \frac{1}{2}\langle \Lambda_m u, u\rangle + \int_{\mathbb{S}^1} H(x, \Lambda_m^{-1/2} u^2, \partial_x \Lambda_m^{-1/2} u^2)dx. \tag{1.2.8}$$

By (1.2.7), (1.2.8), equation (1.1.1) is equivalent to

$$\begin{aligned} \partial_t u &= X_G(u) \\ u|_{t=0} &= \epsilon u_0 \end{aligned} \tag{1.2.9}$$

where $u_0(t, x) = \begin{bmatrix} \Lambda_m^{-1/2} v_1 \\ \Lambda_m^{1/2} v_0 \end{bmatrix}$ is in $H^s(\mathbb{S}^1; \mathbb{R}^2)$. To prove theorem 1.1.1, it is enough to get a priori uniform bounds for the Sobolev norm $\|u(t, \cdot)\|_{H^s}$ when s is large enough. We shall do that designing a Birkhoff normal forms method adapted to quasi-linear Hamiltonian equations.

Let us end this section writing equation (1.2.9) in complex coordinates. We identify $H^s(\mathbb{S}^1; \mathbb{R}^2)$ to $H^s(\mathbb{S}^1; \mathbb{C})$ through the map

$$u = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \to w = \frac{\sqrt{2}}{2}[u^1 + iu^2]. \tag{1.2.10}$$

More precisely, we identify $H^s(\mathbb{S}^1; \mathbb{R}^2)$ to the submanifold $\{w_1 = \overline{w}_2\}$ inside the product $H^s(\mathbb{S}^1; \mathbb{C}) \times H^s(\mathbb{S}^1; \mathbb{C})$ through

$$u = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \to \begin{bmatrix} w = \frac{\sqrt{2}}{2}[u^1 + iu^2] \\ \overline{w} = \frac{\sqrt{2}}{2}[u^1 - iu^2] \end{bmatrix}. \tag{1.2.11}$$

If we set for a real C^1 function F defined on an open subset U of $H^s(\mathbb{S}^1; \mathbb{R}^2)$

$$\begin{aligned} d_w F &= \frac{\sqrt{2}}{2}(d_{u^1}F - id_{u^2}F),\ d_{\overline{w}}F = \frac{\sqrt{2}}{2}(d_{u^1}F + id_{u^2}F) \\ \nabla_w F &= \frac{\sqrt{2}}{2}(\nabla_{u^1}F - i\nabla_{u^2}F),\ \nabla_{\overline{w}}F = \frac{\sqrt{2}}{2}(\nabla_{u^1}F + i\nabla_{u^2}F) \end{aligned} \tag{1.2.12}$$

we see that the identification (1.2.11) sends $\nabla_u F$ to $\begin{bmatrix}\nabla_{\overline{w}}F\\ \nabla_w F\end{bmatrix}$ and $X_F(u)$ to $i\begin{bmatrix}\nabla_{\overline{w}}F\\ -\nabla_w F\end{bmatrix}$. If $Z = \begin{bmatrix}c\\ \bar{c}\end{bmatrix}$ and $Z' = \begin{bmatrix}c'\\ \bar{c'}\end{bmatrix}$ are two vector fields tangent to $\{w_2 = \overline{w}_1\}$ in $H^s(\mathbb{S}^1;\mathbb{C}) \times H^s(\mathbb{S}^1;\mathbb{C})$, the symplectic form coming from ω_0 through (1.2.11), computed at (Z, Z'), is given by

$$\omega_0(Z, Z') = 2\mathrm{Im}\int_{\mathbb{S}^1} c(s)\overline{c'(x)}dx. \tag{1.2.13}$$

Moreover, if F and G are two C^1 functions on U, whose differentials extend to bounded linear maps on $L^2(\mathbb{S}^1;\mathbb{R}^2)$, we have

$$\begin{aligned}\{F, G\} &= [d_w F \ d_{\overline{w}}F]\begin{bmatrix}i\nabla_{\overline{w}}G\\ -i\nabla_w G\end{bmatrix}\\ &= i(d_w F\cdot\nabla_{\overline{w}}G - d_{\overline{w}}F\cdot\nabla_w G).\end{aligned} \tag{1.2.14}$$

Finally, if G is a C^1 function on U, the Hamiltonian equation $\dot{u} = X_G(u)$ may be written in complex coordinates

$$\dot{w} = i\nabla_{\overline{w}}G(w, \overline{w}). \tag{1.2.15}$$

CHAPTER 2

SYMBOLIC CALCULUS

We shall introduce in this chapter classes of symbols of para-differential operators in the sense of Bony [**4**]. These symbols will be formal power series of multilinear functions on $C^\infty(\mathbb{S}^1;\mathbb{R}^2)$, the general term of theses series obeying analytic estimates that will ensure convergence on a neighborhood of zero in a convenient Sobolev space.

2.1. Multilinear para-differential symbols and operators

Let us introduce some notations. If $a:\mathbb{Z}\to\mathbb{C}$ is a function, we define the finite difference operator

$$\partial_n a(n) = a(n) - a(n-1), n\in\mathbb{Z}. \tag{2.1.1}$$

Its adjoint, for the scalar product $\sum_{n=-\infty}^{+\infty} a(n)\overline{b(n)}$, is

$$\partial_n^* a(n) = -(\partial_n a)(n+1) = -\tau_{-1}\circ\partial_n a(n) \tag{2.1.2}$$

where for $j,n\in\mathbb{Z}$ we set $\tau_j b(n) = b(n-j)$. We have

$$\begin{aligned}\partial_n^*[a(-n)] &= (\partial_n a)(-n)\\ \partial_n[ab] &= (\partial_n a)b + (\tau_1 a)(\partial_n b).\end{aligned} \tag{2.1.3}$$

Let us remark that the second formula above may be written

$$\partial_n[ab] = (\partial_n a)b + a(\partial_n b) - (\partial_n a)(\partial_n b).$$

We generalize this expression to higher order derivatives in order to obtain a Leibniz formula.

Lemma 2.1.1. — *For any integer $\beta\in\mathbb{N}$, there are real constants $\widetilde{C}^{\beta}_{\beta_1,\beta_2,\beta_3}$, indexed by integers β_1,β_2,β_3 satisfying $\beta_1+\beta_2=\beta, 0\leq\beta_3\leq\beta$, such that for any functions*

a, b from $\mathbb{Z}$ to $\mathbb{C}$, any $\beta \in \mathbb{N}^*$

$$\partial_n^\beta[ab] = (\partial_n^\beta a)b + a(\partial_n^\beta b) + \sum_{\substack{\beta_1>0,\beta_2>0\\ \beta_1+\beta_2=\beta\\ 0\le\beta_3\le\beta}} \widetilde{C}^\beta_{\beta_1,\beta_2,\beta_3}\left[(1-\tau_1)^{\beta_3}\partial_n^{\beta_1}a\right]\left[\partial_n^{\beta_2}b\right]. \tag{2.1.4}$$

Proof. — For $\beta \in \mathbb{N}, \beta_1 \le \beta_2$ denote by $C_\beta^{\beta_1,\beta_2}$ the value at $X = -1$ of

$$(-1)^{\beta_1+\beta_2}\frac{(X^{\beta_1}\partial_X^{\beta_1})(X^{\beta_2}\partial_X^{\beta_2})}{\beta_1!\beta_2!}[(1+X)^\beta]. \tag{2.1.5}$$

When $\beta_1 > \beta_2$, set $C_\beta^{\beta_1,\beta_2} = C_\beta^{\beta_2,\beta_1}$. Let us show that

$$\partial_n^\beta[ab] = \sum_{\beta_1\ge 0}\sum_{\beta_2\ge 0} C_\beta^{\beta_1,\beta_2}(\partial_n^{\beta_1}a)(\partial_n^{\beta_2}b).$$

Since by definition $C_\beta^{\beta_1,\beta_2} = 0$ when $\beta_1+\beta_2 < \beta$ and when $\beta_1 > \beta$ or $\beta_2 > \beta$, the sum in the above expression is actually for $\beta \le \beta_1+\beta_2, \beta_1 \le \beta, \beta_2 \le \beta$. By (2.1.1), $\mathrm{Id} - \partial_n = \tau_1$, so that

$$\begin{aligned}\partial_n^\beta[ab] &= [\mathrm{Id}-\tau_1]^\beta(ab)\\ &= \sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}(-1)^{\beta'}\tau_1^{\beta'}[ab] = \sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}(-1)^{\beta'}(\tau_1^{\beta'}a)(\tau_1^{\beta'}b)\end{aligned}$$

whence

$$\begin{aligned}\partial_n^\beta[ab] &= \sum_{\beta'=0}^{\beta}\sum_{\beta_1=0}^{\beta'}\sum_{\beta_2=0}^{\beta'}(-1)^{\beta'+\beta_1+\beta_2}\binom{\beta}{\beta'}\binom{\beta'}{\beta_1}\binom{\beta'}{\beta_2}(\partial_n^{\beta_1}a)(\partial_n^{\beta_2}b)\\ &= \sum_{\beta_1}\sum_{\beta_2}C_\beta^{\beta_1,\beta_2}(\partial_n^{\beta_1}a)(\partial_n^{\beta_2}b)\end{aligned} \tag{2.1.6}$$

with

$$C_\beta^{\beta_1,\beta_2} = (-1)^{\beta_1+\beta_2}\sum_{\max(\beta_1,\beta_2)\le\beta'\le\beta}(-1)^{\beta'}\binom{\beta}{\beta'}\binom{\beta'}{\beta_1}\binom{\beta'}{\beta_2}.$$

Since $X^{\beta_2}\partial_X^{\beta_2}[(1+X)^\beta] = \sum_{\beta_2\le\beta'\le\beta}\binom{\beta}{\beta'}\frac{\beta'!}{(\beta'-\beta_2)!}X^{\beta'}$, this coefficient is the value at $X=-1$ of (2.1.5). In (2.1.6), we have $0\le\beta_1\le\beta$, $0\le\beta_2\le\beta, \beta_1+\beta_2\ge\beta$. When $\beta_1+\beta_2>\beta$, we write

$$\partial_n^{\beta_1}a\partial_n^{\beta_2}b = \big([(\mathrm{Id}-\tau_1)^{\beta_1+\beta_2-\beta}\partial_n^{\beta-\beta_2}]a\big)(\partial_n^{\beta_2}b)$$

which shows that the corresponding contributions to (2.1.6) may be written as one of the terms on (2.1.4), up to a change of notations. When $\beta_1+\beta_2=\beta$, we get the first two terms of the right hand side of (2.1.4) when $\beta_1=0$ or $\beta_2=0$ and contributions to the sum in that formula. This concludes the proof. □

For $n \in \mathbb{Z}$, we denote by $\theta_n(x)$ the function on $\mathbb{S}^1$ defined by

$$\theta_n(x) = \frac{e^{inx}}{\sqrt{2\pi}} \tag{2.1.7}$$

and for $\alpha \in \mathbb{Z}$ and $x \neq 0 \mod 2\pi$ we put

$$\theta_{n,\alpha}(x) = \frac{\theta_n(x)}{(1-e^{-ix})^\alpha}. \tag{2.1.8}$$

When $\alpha \in \mathbb{Z}, \beta \in \mathbb{N}$ we have

$$\partial_n^\beta \theta_{n,\alpha} = \theta_{n,\alpha-\beta}. \tag{2.1.9}$$

If $u \in L^2(\mathbb{S}^1;\mathbb{R})$ (resp. $u \in L^2(\mathbb{S}^1;\mathbb{R}^2)$) we set $\hat{u}(n) = \int_{\mathbb{S}^1} e^{-inx}u(x)\,dx$ and

$$\Pi_n u = \int_{\mathbb{S}^1} u(y)\theta_{-n}(y)dy\theta_n(x) = \hat{u}(n)\frac{e^{inx}}{2\pi}$$

the orthonormal projection on the subspace of $L^2(\mathbb{S}^1;\mathbb{C})$ (resp. $L^2(\mathbb{S}^1;\mathbb{C}^2)$) spanned by θ_n (resp. $\theta_n \begin{bmatrix}1\\0\end{bmatrix}$ and $\theta_n \begin{bmatrix}0\\1\end{bmatrix}$).

Let us introduce some notations and definitions. Let $(x,n) \to a(x,n), (x,n) \to b(x,n)$ be two C^∞ functions on $\mathbb{S}^1 \times \mathbb{Z}$. By formula (2.1.4) and the usual Leibniz formula for ∂_x-derivatives, there are real constants $\widetilde{C}^{\alpha,\beta}_{\alpha',\beta',\gamma}$ indexed by $\alpha, \beta \in \mathbb{N}, \alpha', \beta', \gamma \in \mathbb{N}$ with $0 \leq \alpha' \leq \alpha$, $0 \leq \beta' \leq \beta$, $0 \leq \gamma \leq \beta$, $0 < \alpha' + \beta' < \alpha + \beta$ such that for any a, b as above, any $\alpha, \beta \in \mathbb{N}$

$$\begin{aligned}\partial_x^\alpha \partial_n^\beta [ab(x,n)] &= (\partial_x^\alpha\partial_n^\beta a)b + a(\partial_x^\alpha\partial_n^\beta b)\\ &+ \sum_{\substack{0\leq\alpha'\leq\alpha\\ 0\leq\beta'\leq\beta\\ 0\leq\gamma\leq\beta\\ 0<\alpha'+\beta'<\alpha+\beta}} \widetilde{C}^{\alpha,\beta}_{\alpha',\beta',\gamma}[(\mathrm{Id}-\tau_1)^\gamma(\partial_x^{\alpha'}\partial_n^{\beta'}a)](\partial_x^{\alpha-\alpha'}\partial_n^{\beta-\beta'}b).\end{aligned} \tag{2.1.10}$$

We shall fix some $C_0^\infty(\mathbb{R})$ functions $\chi, \tilde{\chi}, \chi_1$ with $0 \leq \chi, \tilde{\chi}, \chi_1 \leq 1$, with small enough supports, identically equal to one close to zero. We denote by $C_\cdot(\chi_1)$ a sequence of positive constants such that for any $n \in \mathbb{Z}$, any $\lambda \in \mathbb{R}$, any $\gamma \in \mathbb{N}$

$$\Big|\partial_n^\gamma \chi_1\Big(\frac{\lambda}{\langle n\rangle}\Big)\Big| \leq C_\gamma(\chi_1)\langle n\rangle^{-\gamma}.$$

Moreover, we define from χ the kernel

$$K_n(z) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} e^{ikz}\chi\Big(\frac{k}{\langle n\rangle}\Big) \tag{2.1.11}$$

with $z \in \mathbb{S}^1$ identified with $[-\pi,\pi]$. We denote by $C_{\cdot,M}(\chi)$ a sequence of positive constants such that for any $\gamma, M \in \mathbb{N}$ for any $n \in \mathbb{Z}, z \in [-\pi,\pi]$

$$|\partial_n^\gamma K_n(z)| \leq C_{\gamma,M}(\chi)\langle n\rangle^{1-\gamma}(1+\langle n\rangle|z|)^{-M}. \tag{2.1.12}$$

Definition 2.1.2. — Let $\Upsilon, M \in \mathbb{R}_+$ be given. We say that a sequence $(D_p)_{p\in\mathbb{N}}$ of positive constants is a "(Υ, M)-conveniently increasing sequence" if $(D_p)_{p\in\mathbb{N}}$ is an increasing sequence of real numbers with $D_0 \geq 1$, satisfying the following three inequalities:

For any $p \in \mathbb{N}$, for any $\alpha, \beta \in \mathbb{N}$ with $\alpha + \beta = p$,

$$2^p \sum_{\substack{0\leq\alpha'\leq\alpha\\ 0\leq\beta'\leq\beta\\ 0\leq\gamma\leq\beta\\ 0<\alpha'+\beta'<p}} \left|\widetilde{C}^{\alpha,\beta}_{\alpha',\beta',\gamma}\right|(2\langle p\rangle)^{\Upsilon+pM} D_{\alpha'+\beta'} D_{p-\alpha'-\beta'} \leq D_p, \tag{2.1.13}$$

$$\sum_{\substack{0<\beta'<\beta\\ 0\leq\gamma\leq\beta}} \left|\widetilde{C}^{\alpha,\beta}_{\alpha,\beta',\gamma}\right|(4\langle p\rangle)^{\Upsilon+pM} C_{\beta-\beta'}(\chi_1) D_{\alpha+\beta'} \leq D_p, \tag{2.1.14}$$

$$\sum_{\substack{0<\beta'\leq\beta\\ 0\leq\gamma\leq\beta}} \left|\widetilde{C}^{\alpha,\beta}_{0,\beta',\gamma}\right|(4\langle p\rangle)^{p}[C_{\beta',2}(\chi) + C_{\beta',2}(\tilde{\chi})] D_{p-\beta'} \leq D_p. \tag{2.1.15}$$

Note that since the left hand side of the above three equations depends only on $D_0, \dots, D_{p-1}$, we may always construct a conveniently increasing sequence whose terms dominate those of a given sequence.

We shall use several times that if j', j'', k', k'' are in $\mathbb{N}^*$,

$$\begin{aligned} \frac{(k'+j'-1)!}{(j'+1)!}\frac{(k''+j''-1)!}{(j''+1)!} &\leq \frac{(k'+k''+j'+j''-2)!}{(j'+1)(j'+j'')!(j''+1)} \\ &\leq \frac{1}{(j'+1)(j''+1)}\frac{(k'+k''+j'+j''-1)!}{(j'+j''+1)!}. \end{aligned} \tag{2.1.16}$$

We set for $j \in \mathbb{N}$, $c_1(j) = \frac{1}{8(\pi+1)}\frac{1}{1+j^2}$, so that for any $j \in \mathbb{Z}, c_1 * c_1(j) \leq c_1(j)$. For K_0 a constant that will be chosen later on large enough, we put $c(j) = K_0^{-1}c_1(j)$. Then for any $j \in \mathbb{Z}$

$$c * c(j) \leq K_0^{-1}c(j). \tag{2.1.17}$$

Definition 2.1.3. — Let $d \in \mathbb{R}, \nu \in \mathbb{R}_+, \zeta \in \mathbb{R}_+, \sigma \in \mathbb{R}$ with $\sigma \geq \nu + \zeta + 2, j, k \in \mathbb{N}^*, j \geq k, N_0 \in \mathbb{N}, B \in \mathbb{R}^*_+, D. = (D_p)_{p\in\mathbb{N}}$ a $(\nu+|d|+\sigma, N_0+1)$-conveniently increasing sequence. We denote by $\Sigma^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.)$ the set of all maps

$$\begin{aligned} (u_1, \dots, u_j) &\to ((x,n) \to a(u_1, \dots, u_j; x, n)) \\ C^\infty(\mathbb{S}^1;\mathbb{R}^2)^j &\to C^\infty(\mathbb{S}^1 \times \mathbb{Z}; \mathbb{C}) \end{aligned} \tag{2.1.18}$$

which are j-linear and symmetric in $(u_1, \dots, u_j)$, such that there is a constant $C > 0$ so that for any $u_1, \dots, u_j \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$, any $n_1 \dots, n_j \in \mathbb{Z}$,

$$a(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; x, n) \equiv 0 \text{ if } \max|n_\ell| > \frac{1}{4}|n|, \tag{2.1.19}$$

and for any $p \in \mathbb{N}$, any $\sigma' \in [\nu+\zeta+2,\sigma]$, any $(x,n) \in \mathbb{S}^1 \times \mathbb{Z}$

$$\begin{aligned}&\sup_{\alpha+\beta=p} |\partial_x^\alpha \partial_n^\beta a(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)|\\ &\leq C\frac{(k+j-1)!}{(j+1)!}c(j)D_pB^j\langle n\rangle^{d-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}\prod_{\ell=1}^{j}\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2},\end{aligned} \tag{2.1.20}$$

and for any $\ell = 1,\dots,j$

$$\begin{aligned}\sup_{\alpha+\beta=p} |\partial_x^\alpha \partial_n^\beta a(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)| &\leq C\frac{(k+j-1)!}{(j+1)!}c(j)D_pB^j\langle n\rangle^{d-\beta+\alpha+\nu+N_0\beta+\sigma'}\\ &\times\Big(\prod_{\substack{1\leq\ell'\leq j\\ \ell'\neq\ell}}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|\Big)\langle n_\ell\rangle^{-\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}.\end{aligned} \tag{2.1.21}$$

The best constant $C > 0$ in (2.1.20), (2.1.21) will be denoted by

$$\mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.;a). \tag{2.1.22}$$

Remarks. — We extend systematically our multilinear maps of form (2.1.18) to $\mathbb{C}$-multilinear maps on $C^\infty(\mathbb{S}^1;\mathbb{C}^2)^j$ to be able to compute them at complex arguments.

— By definition for $\alpha \geq 0$, $\sigma \geq \nu+\zeta+\alpha+2$,

$$\Sigma^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.) \subset \Sigma^{d,\nu+\alpha}_{(k,j),N_0}(\sigma,\zeta,B,D.). \tag{2.1.23}$$

— When $N_0 = 0$, the above inequalities define a class of para-differential symbols: by (2.1.20), if $u_1,\dots,u_j$ belong to some Sobolev space H^s, then the symbol $a(u_1,\dots,u_j;x,n)$ obeys estimates of pseudo-differential symbols as long as the number of x-derivatives is smaller than $s-\frac{1}{2}-\nu$. For higher order derivatives, one loses a power of $\langle n\rangle$. Moreover (2.1.21) shows that if one of the u_ℓ is in a Sobolev space of negative index H^{-s}, one gets estimates of symbols of order essentially $d+s$, with a loss of one extra power for each ∂_x-derivative.

— The precise form of the factors $\frac{(k+j-1)!}{(j+1)!}$ in the above definitions is not essential. The important fact is that these quantities are bounded by $k!$ (times a power $k+j$ of some fixed constant). For $u \in H^s$, with s large enough and $\|u\|_{H^s}$ small enough, this will allow us to make converge the sum in $j \geq k$ of such quantities, and to obtain bounds in $C^k k!\|u\|_{H^s}^k$ i.e. bounds verified by the derivatives at zero of an analytic function defined on a neighborhood of zero.

We shall define below other classes of symbols given by infinite series whose general terms will be given from elements of $\Sigma^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.)$. We shall need precise dependence of the constants in (2.1.20), (2.1.21) in j,k to obtain convergence of these series. But we shall also use polynomial symbols, defined as finite sums, for which explicit dependence of the constants is useless. Because of that we introduce another notation.

Definition 2.1.4. — Let $d \in \mathbb{R}, \nu, \zeta \in \mathbb{R}_+, N_0 \in \mathbb{N}, j \in \mathbb{N}$. We denote by $\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$ the space of j-linear maps $(u_1, \dots, u_j) \to ((x,n) \to a(u_1, \dots, u_j; x, n))$ defined on $C^\infty(\mathbb{S}^1; \mathbb{R}^2)^j$ with values in $C^\infty(\mathbb{S}^1 \times \mathbb{Z}; \mathbb{C})$ satisfying the following conditions:

• For any $n_1, \dots, n_j, n$ with $\max |n_\ell| > \frac{1}{4}|n|$, for any $u_1, \dots, u_j$ in $C^\infty(\mathbb{S}^1; \mathbb{R}^2)$,

$$a(\Pi_{n_1} u_1, \dots, \Pi_{n_j} u_j; x, n) \equiv 0. \tag{2.1.24}$$

• For any $\alpha, \beta \in \mathbb{N}$, any $\sigma \geq \nu + \zeta + 2$, there is a constant $C > 0$ such that for any $n_1, \dots, n_j, n \in \mathbb{Z}$, any $x \in \mathbb{S}^1$, any $u_1, \dots, u_j$ in $C^\infty(\mathbb{S}^1; \mathbb{R})$,

(2.1.25)
$$|\partial_x^\alpha \partial_n^\beta a(\Pi_{n_1} u_1, \dots, \Pi_{n_j} u_j; x, n)| \leq C \langle n \rangle^{d-\beta+(\alpha+\nu+N_0\beta-\sigma)_+} \prod_{\ell=1}^{j} \langle n_\ell \rangle^\sigma \|\Pi_{n_\ell} u_\ell\|_{L^2},$$

and for any $\ell = 1, \dots, j$

(2.1.26)
$$\begin{aligned} |\partial_x^\alpha \partial_n^\beta a(\Pi_{n_1} u_1, \dots, \Pi_{n_j} u_j; x, n)| \leq & C \langle n \rangle^{d-\beta+\alpha+\nu+N_0\beta+\sigma} \\ & \times \Big(\prod_{\substack{1 \leq \ell' \leq j \\ \ell' \neq \ell}} \langle n_{\ell'} \rangle^\sigma \|\Pi_{n_{\ell'}} u_{\ell'}\| \Big) \langle n_\ell \rangle^{-\sigma} \|\Pi_{n_\ell} u_\ell\|_{L^2}. \end{aligned}$$

Let us now define from the preceding classes symbols depending only on one argument u.

Definition 2.1.5. — Let $d \in \mathbb{R}, \nu, \zeta \in \mathbb{R}_+, N_0 \in \mathbb{N}, \sigma \in \mathbb{R}, \sigma \geq \nu + \zeta + 2, k \in \mathbb{N}^*, B > 0, D.$ a $(\nu + |d| + \sigma, N_0 + 1)$-conveniently increasing sequence. We denote by $S^{d,\nu}_{(k),N_0}(\sigma, \zeta, B, D.)$ the set of formal series depending on $u \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)$, $(x,n) \in \mathbb{S}^1 \times \mathbb{R}$,

$$a(u; x, n) \stackrel{\text{def}}{=} \sum_{j \geq k} a_j(\underbrace{u, \dots, u}_{j}; x, n) \tag{2.1.27}$$

where $a_j \in \Sigma^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.)$ are such that

$$\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma, \zeta, B, D.; a) \stackrel{\text{def}}{=} \sup_{j \geq k} \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.; a_j) < +\infty. \tag{2.1.28}$$

Note that if $s_0 > \nu + \zeta + \frac{5}{2}$ and if u stays in $B_{s_0}(R)$, the ball of center 0 and radius R in $H^{s_0}(\mathbb{S}^1, \mathbb{R}^2)$, each a_j extends as a bounded multilinear map on $H^{s_0}(\mathbb{S}^1, \mathbb{R}^2)$ and by (2.1.20), one has estimates

$$|\partial_x^\alpha \partial_n^\beta a_j(u, \dots, u; x, n)| \leq C_{\alpha,\beta} \frac{(k+j-1)!}{(j+1)!} c(j) B^j \langle n \rangle^{d-\beta+(\alpha+N_0\beta-2)_+} R^j,$$

so that if $2BR < 1$ the sum in $j \geq k$ of the preceding quantities converges, and is bounded by $C(4RB)^k(k-1)!$.

We introduce a similar definition for polynomial symbols.

Definition 2.1.6. — Let $d \in \mathbb{R}, \nu, \zeta \in \mathbb{R}_+, N_0 \in \mathbb{N}, k \in \mathbb{N}^*$. We denote by $\widetilde{S}^{d,\nu}_{(k),N_0}(\zeta)$ the space of finite sums

$$a(u;x,n) = \sum_{\substack{j\geq k\\ \text{finite}}} a_j(\underbrace{u,\dots,u}_{j};x,n) \tag{2.1.29}$$

where $a_j \in \widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$.

Quantization of symbols

Definition 2.1.7. — Let $\chi \in C_0^\infty(]-1,1[)$, χ even. Let $a_j \in \Sigma^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.)$ (resp. $a = \sum_{j\geq k} a_j \in S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)$). We define

$$\begin{aligned} a_{j,\chi}(u_1,\dots,u_j;x,n) &= \chi\Big(\frac{D}{\langle n\rangle}\Big)a_j(u_1,\dots,u_j;x,n)\\ a_\chi(u;x,n) &= \sum_{j\geq k} a_{j,\chi}(\underbrace{u,\dots,u}_{j};x,n). \end{aligned} \tag{2.1.30}$$

Let us remark that $a_{j,\chi}$ (resp. a_χ) still belongs to $\Sigma^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.)$ (resp. $S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)$) and that

$$\begin{aligned} \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.;a_{j,\chi}) &\leq C_0\mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.;a_j)\\ \mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;a_\chi) &\leq C_0\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;a) \end{aligned} \tag{2.1.31}$$

for a constant C_0 depending only on χ. Actually, if $K_n(z)$ is the kernel defined by (2.1.11), and if we set $U' = (u_1,\dots,u_j)$, $n' = (n_1,\dots,n_j)$, $\Pi_{n'}U' = (\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)$, we have

$$\begin{aligned} a_{j,\chi}(\Pi_{n'}U';x,n) &= \chi\Big(\frac{D}{\langle n\rangle}\Big)\big[a_j(\Pi_{n'}U';x,n)\big]\\ &= K_n * a_j(\Pi_{n'}U';\cdot,n) \end{aligned}$$

where the convolution is made with respect to the x-variable on $\mathbb{S}^1$. By (2.1.10), we may write

$$\begin{aligned} \partial_x^\alpha\partial_n^\beta a_{j,\chi}(\Pi_{n'}U';x,n) &= \partial_n^\beta K_n * \partial_x^\alpha a_j(\Pi_{n'}U';x,n) + K_n * \partial_x^\alpha\partial_n^\beta a_j(\Pi_{n'}U';x,n)\\ &\quad + \sum_{\substack{0<\beta'<\beta\\ 0\leq\gamma\leq\beta}} \widetilde{C}^{\alpha,\beta}_{0,\beta',\gamma}[(\mathrm{Id}-\tau_1)^\gamma\partial_n^{\beta'}K_n] * (\partial_x^\alpha\partial_n^{\beta-\beta'}a_j(\Pi_{n'}U';x,n)). \end{aligned} \tag{2.1.32}$$

We may write

$$|(\mathrm{Id}-\tau_1)^\gamma \partial_n^{\beta'} K_n| \leq \sum_{0\leq\gamma'\leq\gamma} \binom{\gamma}{\gamma'} |\partial_n^{\beta'} \tau_1^{\gamma'} K_n|.$$

Using (2.1.12) with $M=2$, we bound for $\gamma \leq p, \beta' \leq p$

$$|(\mathrm{Id}-\tau_1)^\gamma \partial_n^{\beta'} K_n| \leq C_{\beta',2}(\chi) \sum_{0\leq\gamma'\leq\gamma} \binom{\gamma}{\gamma'} \langle n-\gamma'\rangle^{1-\beta'}(1+\langle n-\gamma'\rangle|z|)^{-2} \tag{2.1.33}$$

Note that

$$\frac{1}{2\langle\gamma'\rangle}\langle n\rangle \leq \langle n-\gamma'\rangle \leq 2\langle\gamma'\rangle\langle n\rangle \tag{2.1.34}$$

so that the $L^1(dz)$ norm of (2.1.33) is smaller than

$$2C_{\beta',2}(\chi)(4\langle p\rangle)^p \langle n\rangle^{-\beta'}. \tag{2.1.35}$$

If we plug this in (2.1.32), use (2.1.20) or (2.1.21) to estimate $|\partial_x^\alpha \partial_n^{\beta-\beta'} a_j|$ and remind that we assume that $\widetilde{C}^{\alpha,\beta}_{0,\beta',\gamma}$ satisfies (2.1.15), we obtain for $\partial_x^\alpha\partial_n^\beta a_{j,\chi}$ estimates of type (2.1.20), (2.1.21) with the constant C replaced by C_0C, for some uniform $C_0 \geq 6$.

Let us quantize our symbols.

Definition 2.1.8. — Let $\chi \in C_0^\infty(]-\frac{1}{4},\frac{1}{4}[), 0\leq\chi\leq 1$, χ even, $\chi\equiv 1$ close to zero. If $a_j \in \Sigma^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.)$ we define for $u_1,\dots,u_{j+1}\in C^\infty(\mathbb{S}^1,\mathbb{R}^2)$

$$\mathrm{Op}[a(u_1,\dots,u_j;\cdot)]u_{j+1}(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{inx} a(u_1,\dots,u_j;x,n)\hat{u}_{j+1}(n). \tag{2.1.36}$$

If $a=\sum_{j\geq k} a_j$ belongs to $S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)$ we define $\mathrm{Op}[a(u;\cdot)]$ as the formal series of operators

$$\sum_{j\geq k} \mathrm{Op}[a_j(\underbrace{u,\dots,u}_{j};\cdot)]. \tag{2.1.37}$$

Finally, we define $\mathrm{Op}_\chi[a_j(u_1,\dots,u_j;\cdot)]$ (resp. $\mathrm{Op}_\chi[a(u;\cdot)]$) replacing in (2.1.36) (resp. (2.1.37)) a_j by $a_{j,\chi}$ (resp. a by a_χ).

Let us study the L^2-action of the above operators.

Proposition 2.1.9. — *Let $d\in\mathbb{R}, \nu,\zeta\in\mathbb{R}_+, \sigma\in\mathbb{R}, \sigma\geq\nu+\zeta+2, N_0\in\mathbb{N}, j,k\in\mathbb{N}^*, j\geq k, B>0, D.$ a $(\nu+|d|+\sigma, N_0+1)$-conveniently increasing sequence. There is a universal constant C_0 such that for any $a\in\Sigma^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.)$, any $n_0,\dots,n_{j+1}\in$*

$\mathbb{Z}$, *any* $u_1, \ldots, u_j \in C^\infty(\mathbb{S}^1; \mathbb{R}^2)$, *any* $N \in \mathbb{N}$, *any* $\sigma' \in [\nu + \zeta + 2, \sigma]$,

$$\begin{aligned}
&\|\Pi_{n_0} \mathrm{Op}_\chi[a(\Pi_{n_1} u_1, \ldots, \Pi_{n_j} u_j; \cdot)]\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \\
&\qquad\leq C_0 D_N \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.; a) \frac{(k+j-1)!}{(j+1)!} c(j) B^j \\
&\qquad\quad\times \frac{\langle n_{j+1}\rangle^{d+(\nu+N-\sigma')_+}}{\langle n_0 - n_{j+1}\rangle^N} \prod_{\ell=1}^{j} \langle n_\ell\rangle^{\sigma'} \|\Pi_{n_\ell} u_\ell\|_{L^2} \\
&\qquad\quad\times \mathbf{1}_{\{|n_0 - n_{j+1}| < \frac{1}{4}\langle n_{j+1}\rangle, \max(|n_1|,\ldots,|n_j|) \leq \frac{1}{4}|n_{j+1}|\}}
\end{aligned} \tag{2.1.38}$$

and for any $\ell = 1, \ldots, j$,

$$\begin{aligned}
&\|\Pi_{n_0} \mathrm{Op}_\chi[a(\Pi_{n_1} u_1, \ldots, \Pi_{n_j} u_j; \cdot)]\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \\
&\leq C_0 D_N \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.; a) \frac{(k+j-1)!}{(j+1)!} c(j) B^j \\
&\quad\times \frac{\langle n_{j+1}\rangle^{d+\nu+N+\sigma'}}{\langle n_0 - n_{j+1}\rangle^N} \Big(\prod_{\substack{1 \leq \ell' \leq j \\ \ell' \neq \ell}} \langle n_{\ell'}\rangle^{\sigma'} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \Big) \langle n_\ell\rangle^{-\sigma'} \|\Pi_{n_\ell} u_\ell\|_{L^2} \\
&\quad\times \mathbf{1}_{\{|n_0 - n_{j+1}| < \frac{1}{4}\langle n_{j+1}\rangle, \max(|n_1|,\ldots,|n_j|) \leq \frac{1}{4}|n_{j+1}|\}}.
\end{aligned} \tag{2.1.39}$$

Proof. — We denote $U' = (u_1, \ldots, u_j)$, $n' = (n_1, \ldots, n_j)$, and set $\Pi_{n'} U' = (\Pi_{n_1} u_1, \ldots, \Pi_{n_j} u_j)$. By definition 2.1.8, the Fourier transform of $\mathrm{Op}_\chi[a(\Pi_{n'} U'; \cdot)]u_{j+1}$ evaluated at n_0 may be written

$$\frac{1}{2\pi} \sum_{n_{j+1}} \widehat{a_\chi}(\Pi_{n'} U'; n_0 - n_{j+1}, n_{j+1}) \hat{u}(n_{j+1}).$$

By (2.1.30), $\widehat{a_\chi}(\Pi_{n'} U'; k, n_{j+1})$ is supported for $|k| \leq \frac{1}{4}\langle n_{j+1}\rangle$ and by (2.1.19) it is supported in $\max(|n_1|, \ldots, |n_j|) \leq \frac{1}{4}|n_{j+1}|$. Moreover integrations by parts and estimates (2.1.20) show that

$$\begin{aligned}
|\widehat{a_\chi}(\Pi_{n'} U'; k, n)| &\leq C_0 \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.; a) \frac{(k+j-1)!}{(j+1)!} c(j) B^j D_N \\
&\quad\times \langle n\rangle^{d+(N+\nu-\sigma')_+} \langle k\rangle^{-N} \prod_{\ell=1}^{j} \langle n_\ell\rangle^{\sigma'} \|\Pi_{n_\ell} u_\ell\|_{L^2}
\end{aligned}$$

for some universal constant C_0. This gives inequality (2.1.38). Estimate (2.1.39) follows in the same way from (2.1.21). □

We shall use some remainder operators that we now define.

Definition 2.1.10. — Let $\nu, \zeta \in \mathbb{R}_+, d \in \mathbb{R}_+, \sigma \in \mathbb{R}_+, \sigma \geq \nu + 2 + \max(\zeta, \frac{d}{3}), B > 0, j, k \in \mathbb{N}^*, j \geq k$. One denotes by $\Lambda^{d,\nu}_{(k,j)}(\sigma, \zeta, B)$ the set of j-linear maps M from $C^\infty(\mathbb{S}^1; \mathbb{R}^2)^j$ to $\mathcal{L}(L^2(\mathbb{S}^1; \mathbb{R}^2))$, the space of bounded linear operators on $L^2(\mathbb{S}^1; \mathbb{R}^2)$, such that there is a constant $C > 0$ and for any $u_1, \dots, u_j \in C^\infty(\mathbb{S}^1; \mathbb{R}^2)$, any $n_0, \dots, n_{j+1} \in \mathbb{Z}$, any $\ell = 0, \dots, j+1$, any $\sigma' \in [\nu + 2 + \max(\zeta, \frac{d}{3}), \sigma]$,

$$\begin{aligned} &\|\Pi_{n_0} M(\Pi_{n_1} u_1, \dots, \Pi_{n_j} u_j) \Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \\ &\leq C \frac{(k+j-1)!}{(j+1)!} c(j) B^j \langle n_\ell \rangle^{-3\sigma' + \nu + d} \prod_{\ell'=0}^{j+1} \langle n_{\ell'} \rangle^{\sigma'} \prod_{\ell'=1}^{j} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2}. \end{aligned} \tag{2.1.40}$$

The best constant $C > 0$ in the above estimate will be denoted by $\mathfrak{N}^{d,\nu}_{(k,j)}(\sigma, \zeta, B; M)$.

We also define operators depending on a sole argument.

Definition 2.1.11. — Let $\nu, \zeta \in \mathbb{R}_+, d \in \mathbb{R}_+, \sigma \in \mathbb{R}_+, \sigma \geq \nu + 2 + \max(\zeta, \frac{d}{3}), B > 0, k \in \mathbb{N}^*$. One denotes by $\mathcal{L}^{d,\nu}_{(k)}(\sigma, \zeta, B)$ the space of formal series of elements of $\mathcal{L}(L^2(\mathbb{S}^1; \mathbb{R}^2))$ depending on $u \in C^\infty(\mathbb{S}^1; \mathbb{R}^2)$

$$M(u) = \sum_{j \geq k} M_j(\underbrace{u, \dots, u}_{j}) \tag{2.1.41}$$

where $M_j \in \Lambda^{d,\nu}_{(k,j)}(\sigma, \zeta, B)$, such that

$$\mathfrak{N}^{d,\nu}_{(k)}(\sigma, \zeta, B; M) \stackrel{\text{def}}{=} \sup_{j \geq k} \mathfrak{N}^{d,\nu}_{(k,j)}(\sigma, \zeta, B; M_j) < +\infty. \tag{2.1.42}$$

Let us give an example of an operator belonging to the preceding classes. Consider an element $a_j \in \Sigma^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.)$ for some $d \geq 0$, some $\zeta \in \mathbb{R}_+$. Let χ be as in definition 2.1.8 and take $\chi_1 \in C_0^\infty(]-1, 1[)$, $\chi_1 \equiv 1$ close to zero. Define

$$a_{j,1}(u_1, \dots, u_j; x, n) = (1 - \chi_1)\Big(\frac{D}{\langle n \rangle}\Big)[a_j(u_1, \dots, u_j; x, n)].$$

Then it follows from (2.1.20) that $a_{j,1}$ satisfies estimates of the same form, with (d, ν) replaced by $(d - \gamma, \nu + \gamma)$ for any $\gamma \geq 0$, any $\sigma' \in [\nu + \zeta + 2, \sigma]$. We thus get for the operator

$$M(u_1, \dots, u_j) = \mathrm{Op}_\chi[a_{j,1}(u_1, \dots, u_j; \cdot)]$$

bounds of type (2.1.38) with $N = 0$

$$\begin{aligned} &\|\Pi_{n_0} M(\Pi_{n_1} u_1, \dots, \Pi_{n_j} u_j) \Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \\ &\leq C \frac{(k+j-1)!}{(j+1)!} c(j) B^j \langle n_{j+1} \rangle^{d - \gamma + (\nu + \gamma - \sigma')_+} \prod_{\ell=1}^{j} \langle n_\ell \rangle^{\sigma'} \|\Pi_{n_\ell} u_\ell\|_{L^2} \end{aligned} \tag{2.1.43}$$

for any $\sigma' \in [\nu + \zeta + 2, \sigma]$. Take $\gamma = \sigma' - \nu$ and assume $\sigma' \geq \nu + 2 + \max(\zeta, \frac{d}{3})$. We get a bound of type

$$C \frac{(k+j-1)!}{(j+1)!} c(j) B^j \langle n_{j+1} \rangle^{-2\sigma' + \nu + d} \langle n_0 \rangle^{-\sigma'} \prod_{\ell'=0}^{j+1} \langle n_{\ell'} \rangle^{\sigma'} \prod_{\ell'=1}^{j} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2}.$$

Since by (2.1.38), $\langle n_0\rangle \sim \langle n_{j+1}\rangle$, this gives an estimate of form (2.1.40) for $\ell = 0$ and $\ell = j+1$. To obtain the same estimate when $\ell \in \{1,\dots,j\}$ we remind that because of the cut-off in (2.1.38), we may assume $\langle n_{j+1}\rangle \geq c\langle n_\ell\rangle$, $\ell = 1,\dots,j$ which shows that in any case we obtain estimates of an element of $\Lambda^{d,\nu}_{(k,j)}(\sigma,\zeta,B)$ since $-3\sigma'+\nu+d \leq 0$.

We also define the polynomial counterpart of the preceding remainder classes.

Definition 2.1.12. — Let $\nu,\zeta \in \mathbb{R}_+, d \in \mathbb{R}_+, j,k \in \mathbb{N}^*$. We define $\tilde{\Lambda}^{d,\nu}_{(j)}(\zeta)$ to be the space of j-linear maps from $C^\infty(\mathbb{S}^1;\mathbb{R}^2)^j$ to $\mathcal{L}(L^2(\mathbb{S}^1;\mathbb{R}^2))$ satisfying for any $\sigma' \geq \nu + 2 + \max(\zeta, \frac{d}{3})$ estimates of form (2.1.40) with an arbitrary constant instead of $\frac{(k+j-1)!}{(j+1)!}c(j)B^j$. We denote by $\widetilde{\mathcal{L}}^{d,\nu}_{(k)}(\zeta)$ the space of finite sums $M(u) = \sum_{j\geq k} M_j(u,\dots,u)$ where $M_j \in \tilde{\Lambda}^{d,\nu}_{(j)}(\zeta)$.

We have defined operators as formal series in (2.1.37), (2.1.41). Let us show that for u in a small enough ball of a convenient Sobolev space, these series do converge.

Proposition 2.1.13. — *Let $d \in \mathbb{R}, \nu,\zeta \in \mathbb{R}_+, \sigma \in \mathbb{R}_+, \sigma \geq \nu+\zeta+2, B > 0, N_0 \in \mathbb{N}, D.$ a $(|d|+\nu+\sigma, N_0+1)$-conveniently increasing sequence, $k \in \mathbb{N}^*$.*

(i) Let $\delta > 0$ be a small positive number. There are constants $r > 0, \tilde{C} > 0$, depending only on B,ν,ζ,δ, such that if $u \in H^{\nu+\zeta+\frac{5}{2}+\delta}(\mathbb{S}^1;\mathbb{R}^2)$ and $\|u\|_{H^{\nu+\zeta+\frac{5}{2}+\delta}} < r$, $\mathrm{Op}_\chi[a(u;\cdot)]$ defines a bounded linear map from $H^s(\mathbb{S}^1;\mathbb{R}^2)$ to $H^{s-d}(\mathbb{S}^1;\mathbb{R}^2)$ for any $s \in \mathbb{R}$, and one has the estimate

(2.1.44)
$$\|\mathrm{Op}_\chi[a(u;\cdot)]\|_{\mathcal{L}(H^s,H^{s-d})} \leq C(s)(\tilde{C}B)^k(k-1)!\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;a)\|u\|^k_{H^{\nu+\zeta+\frac{5}{2}+\delta}}$$

for some constant $C(s)$. The same estimate holds for $\|\mathrm{Op}_\chi[\partial_u a(u;\cdot)\cdot V]\|_{\mathcal{L}(H^s,H^{s-d})}$ if $V \in H^{\nu+\zeta+\frac{5}{2}+\delta}$, with $\|u\|^k_{H^{\nu+\zeta+\frac{5}{2}+\delta}}$ replaced by $\|u\|^{k-1}_{H^{\nu+\zeta+\frac{5}{2}+\delta}}\|V\|_{H^{\nu+\zeta+\frac{5}{2}+\delta}}$.

(ii) Let $\sigma' \in [\nu+\zeta+2, \sigma-\frac{1}{2}[$ and $\delta > 0$ such that $\sigma'+\frac{1}{2}+\delta < \sigma$. There are $\tilde{C} > 0, r > 0$ depending only on σ',δ,B such that for any $u \in H^{\sigma'+\frac{1}{2}+\delta}$ with $\|u\|_{H^{\sigma'+\frac{1}{2}+\delta}} < r$, any $V \in H^{-\sigma'+\frac{1}{2}+\delta}$, the operator $\mathrm{Op}_\chi[\partial_u a(u;\cdot)\cdot V]$ defines for any $s \in \mathbb{R}$ a bounded linear map from $H^s(\mathbb{S}^1;\mathbb{R}^2)$ to $H^{s-(d+\nu+\sigma'+2)}(\mathbb{S}^1;\mathbb{R}^2)$ with an estimate

(2.1.45)
$$\|\mathrm{Op}_\chi[\partial_u a(u;\cdot)\cdot V]\|_{\mathcal{L}(H^s,H^{s-(d+\nu+\sigma'+2)})} \leq C(s)(\tilde{C}B)^k(k-1)!\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;a) \times\|u\|^{k-1}_{H^{\sigma'+\frac{1}{2}+\delta}}\|V\|_{H^{-\sigma'+\frac{1}{2}+\delta}}.$$

Moreover, for any $\delta > 0$, there are $\tilde{C},\rho_0 > 0$ depending on δ,ν,B, such that for any $u \in H^{\nu+\zeta+\frac{5}{2}+\delta}$ with $\|u\|_{H^{\nu+\zeta+\frac{5}{2}+\delta}} < \rho_0$, any $s > \nu+\zeta+\frac{3}{2}$, any $V \in H^{-s}(\mathbb{S}^1;\mathbb{R}^2)$, $\mathrm{Op}_\chi[\partial_u a(u;\cdot)\cdot V]$ defines a bounded linear map from H^s to $H^{-d-\nu-\frac{5}{2}-\delta}$ with an *estimate*

(2.1.46)
$$\|\mathrm{Op}_\chi[\partial_u a(u;\cdot)\cdot V]\|_{\mathcal{L}(H^s,H^{-d-\nu-\frac{5}{2}-\delta})} \leq C(s)(\tilde{C}B)^k(k-1)!\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;a) \times\|u\|^{k-1}_{H^{\nu+\zeta+\frac{5}{2}+\delta}}\|V\|_{H^{-s}}.$$

(iii) Assume $d\geq 0$, $\sigma>\nu+\frac{5}{2}+\max(\zeta,\frac{d}{3})$. Let $\sigma'\in[\nu+2+\max(\zeta,\frac{d}{3}),\sigma-\frac{1}{2}[$ and $\delta>0$ such that $\sigma'+\frac{1}{2}+\delta<\sigma$. There are $\tilde{C}>0,r>0$ depending only on σ',δ,B, such that for any $u\in H^{\sigma'+\frac{1}{2}+\delta}$ with $\|u\|_{H^{\sigma'+\frac{1}{2}+\delta}}<r$, any $M\in\mathcal{L}^{d,\nu}_{(k)}(\sigma,\zeta,B)$, the operator $M(u)$ defines a bounded linear map from $H^{\sigma'+\frac{1}{2}+\delta}$ to $H^{2\sigma'-\nu-\frac{1}{2}-\delta-d}$ with the estimate

(2.1.47)
$$\|M(u)\|_{\mathcal{L}(H^{\sigma'+\frac{1}{2}+\delta},H^{2\sigma'-\nu-\frac{1}{2}-\delta-d})}\leq C(\sigma')(\tilde{C}B)^k(k-1)!\mathfrak{N}^{\nu}_{(k)}(\sigma,\zeta,B;M)\|u\|^k_{H^{\sigma'+\frac{1}{2}+\delta}}.$$

In addition, for any $V\in H^{\sigma'+\frac{1}{2}+\delta}$, $\partial_u M(u)\cdot V$ is a bounded linear map from $H^{\sigma'+\frac{1}{2}+\delta}$ to $H^{2\sigma'-\nu-\frac{1}{2}-\delta-d}$ and its operator norm is smaller than the right hand side of (2.1.47) with $\|u\|^k_{H^{\sigma'+\frac{1}{2}+\delta}}$ replaced by $\|u\|^{k-1}_{H^{\sigma'+\frac{1}{2}+\delta}}\|V\|_{H^{\sigma'+\frac{1}{2}+\delta}}$.

Moreover, for any $s\in]\nu+d+\frac{3}{2},\sigma[$ satisfying $s>\nu+\frac{5}{2}+\max(\zeta,\frac{d}{3})$, there are $\tilde{C},\rho_0>0$ depending on s,ν,B such that for any $u\in H^s$ satisfying $\|u\|_{H^s}<\rho_0$, the linear maps $M(u)$ and $V\to(\partial_u M(u)\cdot V)u$ belong to $\mathcal{L}(H^{-s},H^{-(2+\nu+d)})$ and satisfy

$$\begin{aligned}&\|M(u)\cdot V\|_{H^{-2-\nu-d}}+\|((\partial_u M(u))\cdot V)u\|_{H^{-2-\nu-d}}\\ &\leq C(\tilde{C}B)^k(k-1)!\mathfrak{N}^{\nu}_{(k)}(\sigma,\zeta,B;M)\|u\|^k_{H^s}\|V\|_{H^{-s}}.\end{aligned}\tag{2.1.48}$$

Proof. — (i) We write $a=\sum_{j\geq k}a_j$ with $a_j\in\Sigma^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.)$. We apply (2.1.38) with $\sigma'=\nu+\zeta+2,N=2$ and estimate $\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}$ by $\langle n_\ell\rangle^{-\frac{1}{2}-\delta}c_{n_\ell}\|u_\ell\|_{H^{\nu+\zeta+\frac{5}{2}+\delta}}$ for a sequence $(c_{n_\ell})_{n_\ell}$ in the unit ball of ℓ^2. Summing (2.1.38) in $n_1,\dots,n_j$ we obtain

$$\begin{aligned}&\|\Pi_{n_0}\mathrm{Op}_\chi[a_j(u,\dots,u;\cdot)]\Pi_{n_{j+1}}w\|_{L^2}\\ &\leq C_0D_2\mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma,\zeta,B,D.;a_j)2^{k+j-1}(k-1)!B^j\big(C_0'\|u\|_{H^{\nu+\zeta+\frac{5}{2}+\delta}}\big)^j\\ &\qquad\times\|\Pi_{n_{j+1}}w\|_{L^2}\langle n_{j+1}\rangle^d\langle n_0-n_{j+1}\rangle^{-2}\mathbf{1}_{|n_0-n_{j+1}|<\frac{1}{4}\langle n_{j+1}\rangle}\end{aligned}$$

for some uniform constant C_0'. We deduce from this and (2.1.28) that

$$\begin{aligned}\|\mathrm{Op}_\chi[a(u;\cdot)]\|_{\mathcal{L}(H^s,H^{s-d})}&\leq C(s)2^k(k-1)!\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;a)\\ &\qquad\times\sum_{j\geq k}(2BC_0'\|u\|_{H^{\nu+\zeta+\frac{5}{2}+\delta}})^j\end{aligned}$$

which gives the first conclusion of (i). The second one is obtained in the same way.

(ii) We decompose again $a=\sum_{j\geq k}a_j$, and write $\partial_u a_j(u,\cdot)\cdot V$ as a sum of j terms

$$a_j(u,\dots,u,V,u,\dots,u;x,n).\tag{2.1.49}$$

We apply estimate (2.1.39) with $N = 2$, the special index ℓ corresponding to the place where is located V. We bound $\|\Pi_{n_{\ell'}} u\|_{L^2} \langle n_{\ell'} \rangle^{\sigma'} \leq c_{n_{\ell'}} \langle n_{\ell'} \rangle^{-\frac{1}{2}-\delta} \|u\|_{H^{\sigma'+\frac{1}{2}+\delta}}$, $\|\Pi_{n_\ell} V\|_{L^2} \langle n_\ell \rangle^{-\sigma'} \leq c_{n_\ell} \langle n_\ell \rangle^{-\frac{1}{2}-\delta} \|V\|_{H^{-\sigma'+\frac{1}{2}+\delta}}$, for sequences $(c_{n_\ell})_{n_\ell}$ in the unit ball of ℓ^2. Summing (2.1.39) in $n_1, \dots, n_j$ and taking into account the fact that we have j terms of form (2.1.49), we get

$$\begin{aligned}
&\|\Pi_{n_0} \mathrm{Op}_\chi[\partial_u a_j(u, \dots, u; \cdot) \cdot V]\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \\
&\quad \leq C_0 D_2 \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.; a_j) 2^{k+j-1}(k-1)! B^j \|u\|^{j-1}_{H^{\sigma'+\frac{1}{2}+\delta}} (C_0')^j \\
&\quad \times \|V\|_{H^{-\sigma'+\frac{1}{2}+\delta}} \langle n_{j+1} \rangle^{d+\nu+\sigma'+2} \langle n_0 - n_{j+1} \rangle^{-2} \mathbf{1}_{|n_0 - n_{j+1}| < \frac{1}{4}\langle n_{j+1} \rangle}
\end{aligned}$$

for some uniform constant C_0'. Summing in $j \geq k$ when $\|u\|_{H^{\sigma'+\frac{1}{2}+\delta}}$ is small enough, we get estimate (2.1.45).

To obtain (2.1.46), we apply again (2.1.39) with $\sigma' = \nu + \zeta + 2, N = 2$, the special index being located on the V term. We bound for $\ell' \neq \ell$ $\langle n_{\ell'} \rangle^{\sigma'} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \leq c_{n_{\ell'}} \langle n_{\ell'} \rangle^{-\frac{1}{2}-\delta} \|u\|_{H^{\sigma'+\frac{1}{2}+\delta}}$ and

$$\langle n_\ell \rangle^{-\sigma'} \|\Pi_{n_\ell} V\|_{L^2} \leq c_{n_\ell} \langle n_\ell \rangle^{-\sigma'+s+\frac{1}{2}+\delta} \|V\|_{H^{-s}} \langle n_\ell \rangle^{-\frac{1}{2}-\delta}$$

with ℓ^2 sequences $(c_{n_\ell})_{n_\ell}, (c_{n_{\ell'}})_{n_{\ell'}}$. Using that $\langle n_\ell \rangle \leq \langle n_{j+1} \rangle$, we get summing (2.1.39) in $n_1, \dots, n_j$

$$\begin{aligned}
&\|\Pi_{n_0} \mathrm{Op}_\chi[\partial_u a_j(u, \dots, u; \cdot) \cdot V]\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \\
&\quad \leq C_0 D_2 \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \sigma, B, D.; a_j) 2^{k+j-1}(k-1)! B^j \|u\|^{j-1}_{H^{\sigma'+\frac{1}{2}+\delta}} (C_0')^j \\
&\quad \times \|V\|_{H^{-s}} \langle n_{j+1} \rangle^{d+\nu+s+\frac{5}{2}+\delta} \langle n_0 - n_{j+1} \rangle^{-2} \mathbf{1}_{|n_0 - n_{j+1}| < \frac{1}{2}\langle n_{j+1} \rangle}.
\end{aligned}$$

We sum next in $j \geq k$ for $\|u\|_{H^{\nu+\zeta+\frac{5}{2}+\delta}}$ small enough. We obtain the bound of (2.1.46) for the $\mathcal{L}(H^s, H^{-d-\nu-\frac{5}{2}-\delta})$-norm of $\mathrm{Op}_\chi[\partial_u a(u; \cdot) \cdot V]$.

(iii) We decompose $M = \sum_{j \geq k} M_j$ with $M_j \in \Lambda^{d,\nu}_{(k,j)}(\sigma, \zeta, B)$. We apply estimate (2.1.40) with $\ell = 0$, bounding $\|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \langle n_{\ell'} \rangle^{\sigma'}$ by $\langle n_{\ell'} \rangle^{-\frac{1}{2}-\delta} \|u_{\ell'}\|_{H^{\sigma'+\frac{1}{2}+\delta}} c_{n_{\ell'}}$ for a sequence $(c_{n_{\ell'}})_{n_{\ell'}}$ in the unit ball of ℓ^2. Summing in $n_1, \dots, n_j$ we get

(2.1.50)
$$\begin{aligned}
\|\Pi_{n_0} M_j(u, \dots, u)\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} &\leq (k-1)! 2^{k+j-1} \langle n_0 \rangle^{-2\sigma'+\nu+d} \langle n_{j+1} \rangle^{\sigma'} \\
&\quad \times \mathfrak{N}^{\nu}_{(k,j)}(\sigma, \zeta, B; M_j)(C_0' \|u\|_{H^{\sigma'+\frac{1}{2}+\delta}})^j B^j
\end{aligned}$$

for some constant C_0'. If we make act the resulting operator on some w in $H^{\sigma'+\frac{1}{2}+\delta}$ and sum in n_{j+1} and in $j \geq k$, we get that

$$\|M(u)\|_{\mathcal{L}(H^{\sigma'+\frac{1}{2}+\delta}, H^{2\sigma'-\nu-\frac{1}{2}-\delta-d})} \leq C(\sigma') \mathfrak{N}^{d,\nu}_{(k)}(\sigma, \zeta, B; M)(\tilde{C}B)^k (k-1)! \|u\|^k_{H^{\sigma'+\frac{1}{2}+\delta}}$$

if $\|u\|_{H^{\sigma'+\frac{1}{2}+\delta}} < r$ small enough.

To estimate $\partial_u M(u) \cdot V$, we have to study expressions of form (2.1.50), with one of the arguments u replaced by V. The rest of the computation is identical.

We still have to prove (2.1.48). We write again $M = \sum_{j\geq k} M_j$ and use estimate (2.1.40), taking for n_ℓ the index for which $|n_\ell| \geq |n_{\ell'}|$, $\ell' = 0,\dots,j+1$. We obtain if we take in (2.1.40) $\sigma' = s - \frac{1}{2} - \delta$ for some $\delta > 0$ small enough

(2.1.51)
$$\begin{aligned}\|\Pi_{n_0} M_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}V\|_{L^2} &\leq C(k-1)!2^{k+j-1}\mathfrak{N}^{d,\nu}_{(k)}(\sigma,\zeta,B;M)B^j\\ &\times\Big(\prod_{\ell'=1}^{j}\langle n_{\ell'}\rangle^{-\frac{1}{2}-\delta}c^{\ell'}_{n_{\ell'}}\Big)\langle n_{j+1}\rangle^s c^{j+1}_{n_{j+1}}\prod_1^j\|u_{\ell'}\|_{H^s}\|V\|_{H^{-s}}\\ &\times\langle n_0\rangle^{s-\frac{1}{2}-\delta}\langle n_{j+1}\rangle^{s-\frac{1}{2}-\delta}\langle n_\ell\rangle^{-3s+\frac{3}{2}+3\delta+\nu+d}\end{aligned}$$

where $(c^{\ell'}_{n_{\ell'}})_{n_{\ell'}}$ $\ell' = 1,\dots,j+1$ are ℓ^2 sequences. We obtain a bound in terms of a constant times $2^k(k-1)!(2B)^j\prod_1^j\|u_{\ell'}\|_{H^s}\|V\|_{H^{-s}}\mathfrak{N}^{d,\nu}_{(k)}(\sigma,\zeta,B;M)$ times

$$\Big(\prod_{\ell'=1}^{j+1}\langle n_{\ell'}\rangle^{-\frac{1}{2}-\delta}c^{\ell'}_{n_{\ell'}}\Big)[\langle n_{j+1}\rangle^{2s}\langle n_0\rangle^{s-\frac{1}{2}-\delta}\langle n_\ell\rangle^{-3s+\frac{3}{2}+3\delta+\nu+d}].$$

Because of the choice of n_ℓ, and since $s > d+\nu+\frac{3}{2}$, the factor between brackets is bounded by $\langle n_0\rangle^{1+\nu+2\delta+d} \leq \langle n_0\rangle^{\frac{3}{2}+3\delta+\nu+d}\tilde{c}_{n_0}$ with an ℓ^2-sequence $(\tilde{c}_{n_0})_{n_0}$. Summing in $n_0,\dots,n_{j+1}$ we obtain

$$\|M_j(u)\cdot V\|_{H^{-\nu-2-d}} \leq C\mathfrak{N}^{d,\nu}_{(k)}(\sigma,\zeta,B;M)(2B\|u\|_{H^s})^j\|V\|_{H^{-s}}(k-1)!2^k.$$

Summing in $j \geq k$ when $\|u\|_{H^s}$ is small enough, we get the wanted upper bound. To estimate in the same way $(\partial_u M(u)\cdot V)u$, we remark that we have to estimate j expressions of form (2.1.51), except that the argument V replaces now one of the u_j, so that in the right hand side of (2.1.51) we have to exchange the roles of $\langle n_{j+1}\rangle$ and of one of the $\langle n_{\ell'}\rangle$. The rest of the proof is identical. □

2.2. Substitution in symbols

In this section, we shall study the effect of substituting a multi-linear map to one or several arguments inside a multi-linear symbol.

Let us fix some notations. Let $B > 0, \nu, \zeta \in \mathbb{R}_+, \sigma \geq \nu+\zeta+2, d \in \mathbb{R}, N_0 \in \mathbb{N}, D.$ a $(|d|+\nu+\sigma, N_0+1)$-conveniently increasing sequence. Let $b \in S^{d,\nu}_{(\kappa),N_0}(\sigma,\zeta,B,D.)$ for some $\kappa \in \mathbb{N}^*$. According to definition 2.1.5, we decompose

$$b(u;x,n) = \sum_{j\geq\kappa} b_j(\underbrace{u,\dots,u}_{j};x,n)$$

with $b_j \in \Sigma^{d,\nu}_{(\kappa,j),N_0}(\sigma,\zeta,B,D.)$. For $u_1,\dots,u_{j+1} \in C^\infty(\mathbb{S}^1,\mathbb{R}^2)$ we set

(2.2.1) $$V_j(u_1,\dots,u_{j+1}) = \mathrm{Op}_\chi[b_j(u_1,\dots,u_j;\cdot)]u_{j+1}$$

or

(2.2.2) $$V_j(u_1,\dots,u_{j+1}) = {}^t\mathrm{Op}_\chi[b_j(u_1,\dots,u_j;\cdot)]u_{j+1}$$

where $\chi \in C_0^\infty(]-\frac{1}{4},\frac{1}{4}[)$, χ even, $\chi \equiv 1$ close to zero.

Let us apply inequalities (2.1.38) and (2.1.39) with $N = 2$. There is a sequence $(Q_n)_n$ in the unit ball of ℓ^1 and for any $s \in \mathbb{R}$ a constant $K_2 \geq 1$, depending only on s and D_2, such that for any $\sigma' \in [\nu+\zeta+2,\sigma]$ one has estimates

$$\begin{aligned}
&\langle n_0\rangle^{s-d}\|\Pi_{n_0}V_j(\Pi_{n_1}u_1,\cdots,\Pi_{n_{j+1}}u_{j+1})\|_{L^2}\\
&\leq K_2\mathfrak{N}^{d,\nu}_{(\kappa,j),N_0}(\sigma,\zeta,B,D.;b_j)\frac{(\kappa+j-1)!}{(j+1)!}c(j)B^jQ_{n_0-n_{j+1}}\\
&\times\Big(\prod_{\ell=1}^{j}\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}\Big)\langle n_{j+1}\rangle^{s}\|\Pi_{n_{j+1}}u_{j+1}\|_{L^2}\\
&\times\mathbf{1}_{\{|n_0-n_{j+1}|<\frac{1}{4}\langle n_{j+1}\rangle,\max(|n_1|,\dots,|n_j|)\leq\frac{1}{4}|n_{j+1}|\}}
\end{aligned}\tag{2.2.3}$$

and for any $\ell = 1,\dots,j$

$$\begin{aligned}
&\langle n_0\rangle^{s-d}\|\Pi_{n_0}V_j(\Pi_{n_1}u_1,\cdots,\Pi_{n_{j+1}}u_{j+1})\|_{L^2}\\
&\leq K_2\mathfrak{N}^{d,\nu}_{(\kappa,j),N_0}(\sigma,\zeta,B,D.;b_j)\frac{(\kappa+j-1)!}{(j+1)!}c(j)B^jQ_{n_0-n_{j+1}}\\
&\times\Big(\prod_{\substack{1\leq\ell'\leq j\\ \ell'\neq\ell}}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\Big)(\langle n_\ell\rangle^{-\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2})\\
&\times\langle n_{j+1}\rangle^{s+\sigma'+\nu+2}\|\Pi_{n_{j+1}}u_{j+1}\|_{L^2}.
\end{aligned}\tag{2.2.4}$$

Set now when $d=0,\zeta=0,N_0=0,\kappa=k_0\geq 1$

$$V(u) = u + \sum_{j\geq k_0}V_j(\underbrace{u,\dots,u}_{j+1})\tag{2.2.5}$$

as a formal series of homogeneous terms. Note that by (2.2.3) with $\sigma' = \nu+2$, we have if $u \in H^{\nu+\frac{5}{2}+\delta}\cap H^s$ for some $\delta>0$ that $\|V_j(u)\|_{H^s} \leq C\|u\|^j_{H^{\nu+\frac{5}{2}+\delta}}(2B)^j\|u\|_{H^s}$, so that (2.2.5) is actually converging in H^s if $\|u\|_{H^{\nu+\frac{5}{2}+\delta}}$ is small enough relatively to $1/B$.

Proposition 2.2.1. — *Let $d\in\mathbb{R},\nu,\zeta\in\mathbb{R}_+,k\in\mathbb{N}^*,a\in S^{d,\nu}_{(k),0}(\sigma,\zeta,B,D.)$. Define*

$$c(u;x,n) = a(V(u);x,n).\tag{2.2.6}$$

Assume that the constant K_0 in (2.1.17) is large enough with respect to σ, D_2 and $\mathfrak{N}^{0,\nu}_{(1),0}(\sigma,0,B,D.;b)$.

Then $c\in S^{d,\nu}_{(k+k_0-1),0}(\sigma,\zeta,B,D.)$. Moreover

$$\mathfrak{N}^{d,\nu}_{(k+k_0-1),0}(\sigma,\zeta,B,D.;c)\leq C\mathfrak{N}^{d,\nu}_{(k),0}(\sigma,\zeta,B,D.;a)\mathfrak{N}^{0,\nu}_{(k_0),0}(\sigma,0,B,D.;b)\tag{2.2.7}$$

with a constant C depending only on $\mathfrak{N}^{0,\nu}_{(1),0}(\sigma,0,B,D.;b)$.

Proof. — We decompose $a(u;x,n) = \sum_{i\geq k} a_i(u,\dots,u;x,n)$ so that c is by definition the formal series $\sum_{j\geq k} c_j(u,\dots,u;x,n)$ where

$$c_j(u_1,\dots,u_j;x,n) = \sum_{i=k}^{j} \sum_{j_1+\cdots+j_i=j-i} a_i(V_{j_1}(U^{j_1}),\dots,V_{j_i}(U^{j_i});x,n)_S \tag{2.2.8}$$

where we used the following notations:

If $j=0$, $V_0(u)=u$. If $j_\ell>0$, we have set

$$U^{j_\ell} = (u_{j_1+\cdots+j_{\ell-1}+\ell},\dots,u_{j_1+\cdots+j_\ell+\ell}),\ \ell=1,\dots,i \tag{2.2.9}$$

and S in (2.2.8) denotes symmetrization in $(u_1,\dots,u_j)$. To further simplify notations set

$$\begin{aligned} &n^{j_\ell} = (n_1^{j_\ell},\dots,n_{j_\ell+1}^{j_\ell}) \in \mathbb{Z}^{j_\ell+1}\\ &\text{with } n_q^{j_\ell} = n_{j_1+\cdots+j_{\ell-1}+\ell+q-1}, 1\leq q\leq j_\ell+1 \end{aligned} \tag{2.2.10}$$

and

$$\Pi_{n^{j_\ell}} U^{j_\ell} = (\Pi_{n_q^{j_\ell}} u_{n_q^{j_\ell}})_{1\leq q\leq j_\ell+1}. \tag{2.2.11}$$

We shall estimate $c_j(u_1,\dots,u_j;x,n) - a_j(u_1,\dots,u_j;x,n)$, which is given by (2.2.8) where the $(j_1,\dots,j_i)$ sum is taken only for $j_1+\cdots+j_i>0$. Then, for $\alpha+\beta=p$,

$$\partial_x^\alpha \partial_n^\beta [(c_j-a_j)(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)] \tag{2.2.12}$$

will be given by the sum

(2.2.13)

$$\sum_{i=k}^{j} \sum_{0<j_1+\cdots+j_i=j-i} \sum_{n_0^{j_1}} \cdots \sum_{n_0^{j_i}} \partial_x^\alpha \partial_n^\beta a_i(\Pi_{n_0^{j_1}} V_{j_1}(\Pi_{n^{j_1}} U^{j_1}),\dots,\Pi_{n_0^{j_i}} V_{j_i}(\Pi_{n^{j_i}} U^{j_i});x,n)$$

where we no longer write symmetrization. We apply (2.1.20) to a_i and (2.2.3) with $s=\sigma'$ to V_{j_ℓ} to bound the modulus of the general term of (2.2.13) by the product of

$$\mathfrak{N}_{(k),0}^{d,\nu}(\sigma,\zeta,B,D_\cdot;a)\mathfrak{N}_{(1),0}^{0,\nu}(\sigma,0,B,D_\cdot;b)^{\tilde{\imath}-1}\mathfrak{N}_{(k_0),0}^{0,\nu}(\sigma,0,B,D_\cdot;b)D_p$$

(where $\tilde{\imath}$ is the number of $j_\ell\neq 0$, so that $1\leq\tilde{\imath}\leq i$) and of

$$\begin{aligned} &\frac{(k_0+j_1-1)!}{j_1!}\frac{(k+i-1)!}{(i+1)!}c(i)\prod_{\ell=1}^{i}\frac{1}{(j_\ell+1)}c(j_\ell)B^jK_2^{\tilde{\imath}}\langle n\rangle^{d-\beta+(\alpha+\nu-\sigma')_+}\\ &\qquad\qquad\times\prod_{\ell=1}^{i}Q_{n_0^{j_\ell}-n_{j_{\ell+1}}^{j_\ell}}\Big(\prod_{\ell=1}^{j}\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}\Big). \end{aligned} \tag{2.2.14}$$

(We have considered V_{j_1} as defined in terms of a symbol of valuation k_0 and $V_{j_2},\dots,V_{j_i}$ as defined by symbols of valuation 1 or 0, assuming that $j_1>0$). We sum in

$n_0^{j_1}, \dots, n_0^{j_i}$. We use also that by (2.1.16)

$$\frac{1}{(i+1)\prod_1^i (j_\ell+1)} \leq \frac{1}{j+1}$$
$$\frac{(k_0+j_1-1)!}{j_1!}\frac{(k+i-1)!}{i!} \leq \frac{(k+(k_0-1)+i+j_1-1)!}{(j_1+i)!} \leq \frac{(k+k_0-1+j-1)!}{j!}$$

to bound (2.2.13) by $\mathfrak{N}^{d,\nu}_{(k),0}(\sigma,\zeta,B,D.;a)$ times

(2.2.15)
$$D_p B^j \frac{(k+(k_0-1)+j-1)!}{(j+1)!}\sum_{i=k}^{j} \max[1,\mathfrak{N}^{0,\nu}_{(1),0}(\sigma,0,B,D.;b)]^{i-1}\mathfrak{N}^{0,\nu}_{(k_0),0}(\sigma,0,B,D.;b)$$
$$\times\Big(K_2^i \sum_{j_1+\cdots+j_i=j-i} c(i)\prod_{\ell=1}^{i} c(j_\ell)\langle n\rangle^{d-\beta+(\alpha+\nu-\sigma')_+}\prod_{\ell=1}^{j}\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}\Big).$$

with a new value of K_2. By (2.1.17), the inner sum in (2.2.15) is bounded by $\frac{c(i)c(j-i)}{K_0^{i-1}}$. If we assume that K_0 is large enough so that

$$K_2\max[1,\mathfrak{N}^{0,\nu}_{(1),0}(\sigma,0,B,D.;b)] < K_0$$

we obtain the bounds (2.1.20) for a symbol in $\Sigma^{d,\nu}_{(k+k_0-1,j),N_0}(\sigma,\zeta,B,D.)$.

Let us get bounds of type (2.1.21) for (2.2.13), when for instance the special index ℓ corresponds to one of the arguments of U^{j_1}. We apply to a_i estimate (2.1.21) with $\ell=1$. This obliges us to bound $\langle n_0^{j_1}\rangle^{-\sigma'}\|\Pi_{n_0^{j_1}}V_{j_1}(\Pi_{n^{j_1}}U^{j_1})\|_{L^2}$. We control this expression using (2.2.3) (resp. (2.2.4)) with $s=-\sigma'$ if we want to make appear the power $\langle n_\ell^{j_1}\rangle^{-\sigma'}$ with $\ell=j_1+1$ (resp. $1\leq\ell\leq j_1$). We obtain a bound of type (2.2.14), except that the power of $\langle n\rangle$ is now $\langle n\rangle^{d-\beta+\alpha+\nu+\sigma'}$ and that one of the $\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}$ is replaced by $\langle n_\ell\rangle^{-\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}$. We conclude as before.

We still have to check that the support property (2.1.19) holds. Remark that in (2.2.13) we have $|n_0^{j_\ell}|\leq\frac14|n|$ by (2.1.19) for a, and $|n_q^{j_\ell}|\leq\frac14|n^{j_\ell}_{j_{\ell+1}}|, q=1,\dots,j_\ell$, $|n^{j_\ell}_{j_{\ell+1}}|\leq 2|n_0^{j_\ell}|$ because of the cut-off in (2.2.3). This implies that (2.2.12) is supported for $|n_\ell|\leq\frac14|n|$, $\ell=1,\dots,j$ as wanted. □

Our next goal is to study quantities of form $\partial_u a(u;x,n)\cdot V(u)$ where a belongs to some $S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.)$ and V is defined by a formula of type (2.2.5).

Proposition 2.2.2. — *Let* $d',d''\in\mathbb{R}, d''\geq 0, d=d'+d'', \iota=\min(1,d''), \nu,\zeta\in\mathbb{R}_+, \sigma\geq\iota+\nu+\zeta+2, k',k''\in\mathbb{N}^*, N_0\in\mathbb{N}, B>0, D.$ *a* $(\nu+|d'|+|d''|+\sigma,N_0+1)$-*conveniently increasing sequence. Define*

(2.2.16)
$$V(u)=\sum_{j''\geq k''}V_{j''}(\underbrace{u,\dots,u}_{j''+1})$$

(as a formal series), where $V_{j''}$ *is defined by (2.2.1) from the components of a symbol* $b = \sum_{j''\geq k''} b_{j''}$ *satisfying* $b \in S^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.)$*. Let also* a *be an element of* $S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.)$*. Then*

$$c(u;x,n) = \partial_u a(u;x,n)\cdot V(u) \tag{2.2.17}$$

defines an element of $S^{d-\iota,\nu+\iota}_{(k'+k''),N_0}(\sigma,\zeta,B,D.)$*.*

Proof. — We decompose $a(u;x,n) = \sum_{j'\geq k'} a_{j'}(u,\dots,u;x,n)$. Since

$$\partial_u a_{j'}(u,\dots,u;x,n)\cdot V(u) = j' a_{j'}(V(u),u,\dots,u;x,n),$$

we may write with $k = k'+k''$

(2.2.18)

$$c(u,:x,n) = \sum_{j\geq k} c_j(u,\dots,u;x,n)$$

$$c_j(u_1,\dots,u_j;x,n) = \sum_{\substack{j'+j''=j\\ j'\geq k',j''\geq k''}} j' a_{j'}(V_{j''}(u_1,\dots,u_{j''+1}),u_{j''+2},\dots,u_j;x,n)_S$$

where S stands again for symmetrization. Write

$$\partial_x^\alpha\partial_n^\beta c_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)$$

as

(2.2.19)

$$\sum_{n_0=-\infty}^{+\infty}\sum_{j'+j''=j} j'\partial_x^\alpha\partial_n^\beta a_{j'}(\Pi_{n_0}V_{j''}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j''+1}}u_{j''+1}),$$
$$\Pi_{n_{j''+2}}u_{j''+2},\dots,\Pi_{n_j}u_j;x,n)_S.$$

We estimate the general term of the above sum. We apply (2.1.20) to $a_{j'}$ with σ' replaced by $\sigma'-\iota\geq\nu+\zeta+2$, and (2.2.3) to $V_{j''}$ with $s=\sigma'$. We get for (2.2.19) a bound given by the product of

$$\mathfrak{N}^{d',\nu}_{(k',j'),N_0}(\sigma-\iota,\zeta,B,D.;a_{j'})\mathfrak{N}^{d'',\nu}_{(k'',j''),N_0}(\sigma,\zeta,B,D.;b_{j''}) \tag{2.2.20}$$

times

$$\sum_{n_0}\sum_{j'+j''=j} K_2 j'\frac{(k''+j''-1)!}{(j''+1)!}\frac{(k'+j'-1)!}{(j'+1)!}B^j c(j')c(j'')Q_{n_0-n_{j''+1}}$$
$$\times\langle n\rangle^{d-\iota-\beta+(\alpha+\nu+\iota+N_0\beta-\sigma')}\prod_{\ell=1}^j\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2} \tag{2.2.21}$$

using that $\langle n_0\rangle^{d''-\sigma'}\leq\langle n\rangle^{d''-\iota}\langle n_0\rangle^{\iota-\sigma'}$ because of (2.1.19). Using (2.1.16) and (2.1.17) with $K_0\geq K_2$, we obtain an estimate of type (2.1.20) for (2.1.26).

We also need to prove bounds of form (2.1.21). Consider first the case when the special index ℓ in (2.1.21) is between $j''+2$ and j, for instance $\ell=j$. We apply

(2.1.21) to $a_{j'}$ and (2.2.3) to $V_{j''}$, taking $s = \sigma' + d''$. We get a bound given by $\mathfrak{N}^{d',\nu}_{(k',j'),N_0}(\sigma,\zeta,B,D.;a_{j'})\mathfrak{N}^{d'',\nu}_{(k''j''),N_0}(\sigma,\zeta,B,D.;b_{j''})$ times

(2.2.22)
$$\sum_{n_0}\sum_{j'+j''=j} K_2 j' \frac{(k''+j''-1)!}{(j''+1)!}\frac{(k'+j'-1)!}{(j'+1)!}B^j c(j')c(j'')Q_{n_0-n_{j''+1}} \times \langle n\rangle^{d'-\beta+\alpha+\nu+N_0\beta+\sigma'}\langle n_{j''+1}\rangle^{d''}\prod_{\ell'=1}^{j-1}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\langle n_j\rangle^{-\sigma'}\|\Pi_{n_j}u_j\|_{L^2}.$$

Moreover, by the cut-off in (2.2.3) $\langle n_{j''+1}\rangle \le 2\langle n_0\rangle$ and by (2.1.19) for $a_{j'}$, $|n_0| \le \frac{1}{4}|n|$. Since $d'' \ge 0$, we bound $\langle n_{j''+1}\rangle^{d''}$ by $(2\langle n\rangle)^{d''}$. Using then as in (2.2.21) inequalities (2.1.16) and (2.1.17), we get a bound of type (2.1.21) for a symbol belonging to $\Sigma^{d-\iota,\nu+\iota}_{(k'+k'',j),N_0}(\sigma,\zeta,B,D.)$.

Consider now the case when the special index ℓ of (2.1.21) is between 1 and $j''+1$. If $\ell = j''+1$, we apply (2.1.21) to $a_{j''}$ taking the negative power $-\sigma'$ on $\langle n_0\rangle$, and (2.2.3) with $s = -\sigma' + d''$ to $V_{j''}$. Since $\langle n_0\rangle \sim \langle n_{j''+1}\rangle$, we get a bound of form (2.2.22) with $\langle n_{j''+1}\rangle^{\sigma'+d''}$ (resp. $\langle n_j\rangle^{-\sigma'}$) replaced by $\langle n_{j''+1}\rangle^{-\sigma'+d''}$ (resp. $\langle n_j\rangle^{\sigma'}$) and conclude as above. If the special index ℓ is between 1 and j'', we apply (2.1.21) to $a_{j'}$ (taking the negative power $-\sigma'$ on $\langle n_0\rangle$) and (2.2.4) with $s = -\sigma' + d''$. We obtain the upper bound

$$\sum_{n_0}\sum_{j'+j''=j} K_2 j' \frac{(k''+j''-1)!}{(j''+1)!}\frac{(k'+j'-1)!}{(j'+1)!}B^j c(j')c(j'')Q_{n_0-n_{j''+1}} \times\langle n\rangle^{d'-\beta+\alpha+\nu+N_0\beta+\sigma'}\prod_{\substack{1\le\ell'\le j\\ \ell'\neq j''+1\\ \ell'\neq\ell}}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\langle n_\ell\rangle^{-\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2} \times\langle n_{j''+1}\rangle^{\nu+d''+2}\|\Pi_{n_{j''+1}}u_{j''+1}\|_{L^2}.$$

We write using the support condition (2.1.19) and (2.2.3) $\langle n_{j''+1}\rangle^{\nu+d''+2} \le \langle n\rangle^{d''-\iota}\langle n_{j''+1}\rangle^{\nu+2+\iota}$. Since $\nu+2+\iota \le \sigma'$, we deduce again from that the wanted estimate of form (2.1.21). Since the support condition (2.1.19) is seen to be satisfied as at the end of the proof of proposition 2.2.1, this concludes the proof of proposition 2.2.2. □

We shall need a version of proposition 2.2.2 when $V_{j''}$ in (2.2.16) is replaced by a multi-linear map defined in a slightly different way. If V_j is defined by (2.2.1), let $W_j(u_1,\dots,u_{j+1})$ be the multi-linear map given by

(2.2.23)
$$\langle W_j(u_1,\dots,u_{j+1}),u_0\rangle = \langle V_j(u_0,u_2,\dots,u_{j+1}),u_1\rangle$$

for any $u_0,\dots,u_{j+1}$ in $C^\infty(\mathbb{S}^1,\mathbb{R}^2)$. Let us prove:

Lemma 2.2.3. — *For any $\sigma' \in [\nu+\zeta+2, \sigma]$ there is a constant K_2, depending only on σ', such that for any $u_1, \dots, u_{j+1}$ in $C^\infty(\mathbb{S}^1, \mathbb{R}^2)$, any $n_0, \dots, n_{j+1} \in \mathbb{Z}$, $\|\Pi_{n_0} W_j(\Pi_{n_1} u_1, \dots, \Pi_{n_{j+1}} u_{j+1})\|_{L^2}$ is bounded from above by the product of*

$$K_2 \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.; b_j) \frac{(k+j-1)!}{(j+1)!} c(j) B^j \tag{2.2.24}$$

times

$$\langle n_0 \rangle^{-\sigma'} \langle n_{j+1} \rangle^{d+\nu-\sigma'} \prod_{\ell'=1}^{j+1} \langle n_{\ell'} \rangle^{\sigma'} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \tag{2.2.25}$$

resp. times, for any $\ell = 1, \dots, j$

$$\langle n_0 \rangle^{\sigma'} \langle n_{j+1} \rangle^{d+\nu-\sigma'} \prod_{\substack{1 \le \ell' \le j+1 \\ \ell' \neq \ell}} \langle n_{\ell'} \rangle^{\sigma'} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \langle n_\ell \rangle^{-\sigma'} \|\Pi_{n_\ell} u_\ell\|_{L^2}. \tag{2.2.26}$$

Moreover, on the support of $\Pi_{n_0} W_j(\Pi_{n_1} u_1, \dots, \Pi_{n_{j+1}} u_{j+1})$

$$\max(|n_0|, |n_2|, \dots, |n_j|) < \frac{1}{4}|n_{j+1}|, |n_1 - n_{j+1}| \le \frac{1}{4}\langle n_{j+1} \rangle. \tag{2.2.27}$$

Finally, if $\tilde{\chi} \in C_0^\infty(]-\frac{1}{4}, \frac{1}{4}[)$, and if $C_{\gamma,2}(\tilde{\chi})$ is defined by (2.1.12), we may bound for any $\gamma \in \mathbb{N}, \beta' \in \mathbb{N}, \gamma \le p, \beta' \le p$

$$\|(\mathrm{Id} - \tau_1)^\gamma \partial_n^{\beta'} \Pi_{n_0} W_j(\Pi_{n_1} u_1, \dots, \Pi_{n_{j+1}} \tilde{\chi}\Big(\frac{D}{\langle n \rangle}\Big) u_{j+1})\|_{L^2} \tag{2.2.28}$$

by the product of (2.2.24) and (2.2.25) (resp. (2.2.26)) with

$$2 \mathbf{1}_{|n_{j+1}| \le \langle n \rangle / 4} C_{\beta',2}(\tilde{\chi}) \langle n \rangle^{-\beta'} (4\langle p \rangle)^p. \tag{2.2.29}$$

Proof. — Inequalities (2.2.27) follow from (2.2.23) and (2.1.19). Let us prove (2.2.25). We compute for $\|u_0\|_{L^2} \le 1$

$$|\langle \mathrm{Op}_\chi[b_j(\Pi_{n_0} u_0, \Pi_{n_2} u_2, \dots, \Pi_{n_j} u_j; \cdot)] \Pi_{n_{j+1}} u_{j+1}, \Pi_{n_1} u_1 \rangle|. \tag{2.2.30}$$

We apply (2.1.39) with $N = 0$, taking for the special index the one corresponding to the first argument of b_j, and we get the bound

$$\begin{aligned} &C_0 D_0 \mathfrak{N}^{d,\nu}_{(k,j),N_0}(\sigma, \zeta, B, D.; b_j) \frac{(k+j-1)!}{(j+1)!} c(j) B^j \\ &\times \langle n_{j+1} \rangle^{d+\nu+\sigma'} \langle n_0 \rangle^{-\sigma'} \|\Pi_{n_0} u_0\|_{L^2} \prod_{\ell'=2}^{j} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \langle n_{\ell'} \rangle^{\sigma'} \|\Pi_{n_{j+1}} u_{j+1}\|_{L^2} \|\Pi_{n_1} u_1\|_{L^2}. \end{aligned} \tag{2.2.31}$$

Since $|n_1 - n_{j+1}| \le \frac{1}{4}\langle n_{j+1} \rangle$, we obtain (2.2.25) with a constant K_2 depending only on σ'.

To obtain (2.2.26), we use (2.1.39) with $N = 0$ and the special index corresponding to one of the arguments $u_2, \dots, u_j$ of b_j in (2.2.30), for instance $\ell = 2$. We get a bound given by the first line of (2.2.31) times

$$\begin{aligned}(2.2.32)\quad &\langle n_{j+1}\rangle^{d+\nu+\sigma'}\langle n_0\rangle^{\sigma'}\|\Pi_{n_0}u_0\|_{L^2}\langle n_2\rangle^{-\sigma'}\|\Pi_{n_2}u_2\|_{L^2}\\ &\times\prod_{\ell'=3}^{j}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{j+1}}u_{j+1}\|_{L^2}\|\Pi_{n_1}u_1\|_{L^2}.\end{aligned}$$

and we conclude as above. Note that (2.2.26) for $\ell = 1$ follows from (2.1.38) and the fact that $\langle n_1\rangle \sim \langle n_{j+1}\rangle$.

To estimate (2.2.28), we insert inside (2.2.30) the cut-off $\tilde\chi\big(\frac{D}{\langle n\rangle}\big)$ against u_{j+1} and write $\chi\big(\frac{D}{\langle n\rangle}\big)\Pi_{n_{j+1}}u_{j+1} = \tilde K_n * \Pi_{n_{j+1}}u_{j+1}$ where $\tilde K_n$ is defined by (2.1.11) with χ replaced by $\tilde\chi$. We then make ∂_n-derivatives act and use (2.1.33), (2.1.35) to make appear the gain (2.2.29) in estimates (2.2.31), (2.2.32). □

Proposition 2.2.4. — *Let $d', d'', \nu, \zeta, \sigma, k', k'', N_0, B, D., \iota$ be as in the statement of proposition 2.2.2. Assume $\sigma \geq \nu + 3 + \max(\zeta, \frac{d'_+ + d''}{3})$. Let $a \in S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.)$, $b \in S^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.)$ and define from $W_{j''}$ given by (2.2.23)*

$$(2.2.33)\qquad W(u) = \sum_{j''\geq k''} j'' W_{j''}(\underbrace{u,\dots,u}_{j''+1}).$$

There is a symbol $c \in S^{d'+d''-\iota,\nu+\iota}_{(k'+k''),N_0}(\sigma,\zeta,B,D.)$ and a multi-linear map $M(u) \in \mathcal{L}^{d'_+ + d'',\nu+1}_{(k'+k'')}(\sigma,\zeta,B)$ such that

$$(2.2.34)\qquad \mathrm{Op}_\chi[\partial_u a(u;\cdot)\cdot W(u)] = \mathrm{Op}_\chi[c(u;\cdot)] + M(u).$$

Proof. — Consider the symbol $c(u;x,n) = \sum_{j\geq k} c_j(u,\dots,u;x,n)$ where

$$\begin{aligned}(2.2.35)\quad &c_j(u_1,\dots,u_j;x,n)\\ &= \sum_{j'+j''=j} j'j'' a_{j'}[W_{j''}(u_1,\dots,\tilde\chi(D/\langle n\rangle)u_{j''+1}), u_{j''+2},\dots,u_j;x,n]_S,\end{aligned}$$

$\tilde\chi$ being a function in $C_0^\infty(]-\frac14,\frac14[)$, with small enough support, $\tilde\chi \equiv 1$ close to zero. By (2.1.19) applied to $a_{j'}$ and (2.2.27), c_j will satisfy (2.1.19) if the support of $\tilde\chi$ is small enough. Let us prove that $\partial_x^\alpha\partial_n^\beta c_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)$ obeys estimates (2.1.20) and (2.1.21). From now on, we no longer write the symmetrization operator. We make $\partial_x^\alpha\partial_n^\beta$ act on $c_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)$ for $\alpha+\beta = p$. For $0 \leq \beta' \leq \beta, 0 \leq \gamma \leq \beta$ set

$$(2.2.36)\quad \widetilde{W}^{\beta',\gamma}_{j''}(n_0,\dots,n_{j''+1},n) = (\mathrm{Id}-\tau_1)^\gamma\partial_n^{\beta'}\Pi_{n_0}W_{j''}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j''+1}}\tilde\chi(D/\langle n\rangle)u_{j''+1}).$$

We use (2.1.10) to write $\partial_x^\alpha\partial_n^\beta c_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)$ as the sum for $j'+j'' = j$ and for $n_0 \in \mathbb{Z}$ of

$$(2.2.37)\qquad j'j''(\partial_x^\alpha\partial_n^\beta a_{j'})[\widetilde{W}^{0,0}_{j''}(n_0,\dots,n_{j''+1},n),\Pi_{n_{j''+2}}u_{j''+2},\dots,\Pi_{n_j}u_j;x,n]$$

and of

$$\sum_{\substack{0<\beta'\leq\beta\\0\leq\gamma\leq\beta}} \widetilde{C}^{\alpha,\beta}_{0,\beta',\gamma} j'j''(\partial_x^\alpha \partial_n^{\beta-\beta'} a_{j'})[\widetilde{W}^{\beta',\gamma}_{j''}(n_0,\dots,n_{j''+1},n), \Pi_{n_{j''+2}} u_{j''+2},\dots,\Pi_{n_j}u_j;x,n]. \tag{2.2.38}$$

We estimate the general term of (2.2.38) applying (2.1.20) to $a_{j'}$ and bounding

$$\langle n_0\rangle^{\sigma'} \|\widetilde{W}^{\beta',\gamma}_{j''}(n_0,\dots,n_{j''+1},n)\|_{L^2} \tag{2.2.39}$$

using the product of (2.2.24), (2.2.25), (2.2.29) in lemma 2.2.3. We obtain a bound given by the product of

$$K_2\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;b)\prod_{\ell'=1}^{j}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2} \tag{2.2.40}$$

and of the sum for $0<\beta'\leq\beta, 0\leq\gamma\leq\beta$ of

$$2D_{\alpha+\beta-\beta'}\widetilde{C}^{\alpha,\beta}_{0,\beta',\gamma}j'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')B^j(4\langle p\rangle)^pC_{\beta',2}(\tilde{\chi}) \tag{2.2.41}$$

multiplied by

$$\langle n\rangle^{d'-\beta+(\alpha+\nu+N_0(\beta-\beta')-\sigma')_+}\langle n_{j''+1}\rangle^{d''+\nu-\sigma'}\prod_{\ell'=1}^{j}\langle n_{\ell'}\rangle^{\sigma'}. \tag{2.2.42}$$

Since by the cut-off in (2.2.29), $|n_{j''+1}|\leq\langle n\rangle$, we bound $\langle n_{j''+1}\rangle^{d''+\nu-\sigma'}\leq\langle n\rangle^{d''-\iota}\langle n_{j''+1}\rangle^{\nu+\iota-\sigma'}\leq\langle n\rangle^{d''-\iota}\langle n_{j''+1}\rangle^{-2}$. As by (2.2.27), $|n_{j''+1}|\geq c|n_0|$, the last factor will make converge the n_0-series. Consequently, the sum for $n_0\in\mathbb{Z}, j'+j''=j$ of (2.2.38) will be controlled by the product of (2.2.40), of

$$\langle n\rangle^{d'+d''-\iota-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}\prod_{\ell'=1}^{j}\langle n_{\ell'}\rangle^{\sigma'}$$

and of the sum for $j'+j''=j$, $0<\beta'\leq\beta, 0\leq\gamma\leq\beta$ of (2.2.41). Using (2.1.15) and (2.1.16), (2.1.17) with a large enough K_0 (independent of any parameter), we get for (2.2.42) an estimate of form (2.1.20).

We still have to bound the contribution (2.2.37). We proceed as above, estimating the $\widetilde{W}^{0,0}_{j''}$ term by the product of (2.2.24) and (2.2.25). We get a bound in terms of the product of (2.2.40) multiplied by

$$D_pj'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')B^j$$

and by (2.2.42) with $\beta'=0$. We end the computation as above.

Let us prove that (2.1.21) is valid for c_j. Take first the special index ℓ in this estimate be equal to some index between 1 and $j''+1$, for instance $\ell=1$. We apply

(2.1.21) to $a_{j'}$, making appear the $-\sigma'$ exponent on the index corresponding to the first argument of $c_{j'}$. We obtain an upper bound in terms of

$$\langle n_0\rangle^{-\sigma'}\|\widetilde{W}_{j''}^{\beta',\gamma}(n_0,\dots,n_{j''+1},n)\|_{L^2}$$

that we bound using the product of (2.2.24), (2.2.26) (with $\ell=1$) and (2.2.29). We obtain for (2.2.38) an estimate in terms of the product of (2.2.40) by the sum for $0<\beta'\leq\beta, 0\leq\gamma\leq\beta$ of (2.2.41) multiplied by

$$\langle n\rangle^{d'-\beta+\alpha+\nu+N_0(\beta-\beta')+\sigma'}\langle n_{j''+1}\rangle^{d''+\nu+\sigma'}\prod_{\ell'=2}^{j}\langle n_{\ell'}\rangle^{\sigma'}\langle n_1\rangle^{-\sigma'}.$$

Bounding as above the last factor before the product by $\langle n\rangle^{d''-\iota}\langle n_{j''+1}\rangle^{-2}$, we obtain a control of the sum in $n_0, j'+j''=j$ of (2.2.38) by the product of (2.2.40), of

$$\langle n\rangle^{d'+d''-\iota-\beta+\alpha+N_0\beta+\sigma'}\prod_{\ell'=2}^{j}\langle n_{\ell'}\rangle^{\sigma'}\langle n_1\rangle^{-\sigma'} \tag{2.2.43}$$

and of the sum for $j'+j''=j$, $0<\beta'\leq\beta, 0\leq\gamma\leq\beta$ of (2.2.41). We again deduce from that the looked for estimate of type (2.1.21). The contribution coming from (2.2.37) is treated similarly.

We still have to obtain an estimate of form (2.1.21) when the special index ℓ is between $j''+2$ and j, say $\ell=j$. We apply (2.1.21) to $a_{j'}$, with $\ell=j$ corresponding to the last argument, and obtain a bound in terms of (2.2.39), that we control from (2.2.24), (2.2.25), (2.2.29). We get then similar bounds as in the case $\ell=1$, except that in (2.2.43) $\langle n_j\rangle^{\sigma'}\langle n_1\rangle^{-\sigma'}$ has to be replaced by $\langle n_j\rangle^{-\sigma'}\langle n_1\rangle^{\sigma'}$. This concludes the proof of the fact that c belongs to $S^{d-\iota,\nu+\iota}_{(k'+k''),N_0}(\sigma,\zeta,B,D.)$.

Define now

(2.2.44)

$$\tilde c(u;x,n)=\sum_{j\geq k}\tilde c_j(u,\dots,u;x,n)$$

$$\tilde c_j(u_1,\dots,u_j;x,n)=$$
$$\sum_{j'+j''=j}j'j''a_{j'}\big[W_{j''}\big(u_1,\dots,u_{j''},(1-\tilde\chi)\Big(\frac{D}{\langle n\rangle}\Big)u_{j''+1}\big),u_{j''+2},\dots,u_j;x,n\big]_S$$

and set

$$\begin{aligned}M_j(u_1,\dots,u_j)&=\mathrm{Op}_\chi[\tilde c_j(u_1,\dots,u_j;\cdot)]\\ M(u)&=\sum_{j\geq k}M_j(\underbrace{u,\dots,u}_{j}).\end{aligned} \tag{2.2.45}$$

Let $\sigma'\geq\nu+3+\max(\zeta,\frac{d'_++d''}{3})$. Using (2.1.38) for $\mathrm{Op}_\chi[a_{j'}(u;\cdot)]$, we bound

$$\|\Pi_{n_0}M_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \tag{2.2.46}$$

by the sum for $j'+j''=j$ and $n_0'\in\mathbb{Z}$ of

$$C_0D_N\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)j'j''\frac{(k'+j'-1)!}{(j'+1)!}c(j')B^{j'}\frac{\langle n_{j+1}\rangle^{d'+(\nu+N-\sigma')_+}}{\langle n_0-n_{j+1}\rangle^N} \tag{2.2.47}$$

multiplied by

$$\begin{aligned}&\langle n_0'\rangle^{\sigma'}\|\Pi_{n_0'}W_{j''}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j''}}u_{j''},(1-\tilde\chi)\Big(\frac{D}{\langle n_{j+1}\rangle}\Big)\Pi_{n_{j''+1}}u_{j''+1})\|_{L^2}\\&\qquad\qquad\times\prod_{\ell'=j''+2}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}.\end{aligned} \tag{2.2.48}$$

The cut-off in (2.1.38) shows moreover that we may assume

$$|n_0'|,|n_{j''+2}|,\dots,|n_j|\le\frac14\langle n_{j+1}\rangle \text{ and } \langle n_0\rangle\sim\langle n_{j+1}\rangle. \tag{2.2.49}$$

The support conditions (2.2.27) on $W_{j''}$ imply moreover that

$$|n_0'|,|n_2|,\dots,|n_{j''}|\le C\langle n_{j''+1}\rangle \text{ and } \langle n_{j''+1}\rangle\sim\langle n_1\rangle. \tag{2.2.50}$$

Finally, the cut-off $1-\tilde\chi$ in (2.2.48) implies that $|n_{j''+1}|\ge c\langle n_{j+1}\rangle$ for some $c>0$. Altogether, these inequalities show that $\langle n_{j''+1}\rangle\ge c\langle n_\ell\rangle$ for any $\ell=0,\dots,j$. Consequently, to prove that $M_j(u_1,\dots,u_j)$ is in $\Lambda^{d'_++d'',\nu+1}_{(k,j)}(\sigma,\zeta,B)$ we have to obtain (2.1.40) with $\ell=j''+1$, ν replaced by $\nu+1$.

We estimate (2.2.48) using (2.2.26) with $\ell=1$. We obtain a bound given by (2.2.24) multiplied by

$$\langle n_0'\rangle^{2\sigma'}\langle n_{j''+1}\rangle^{d''+\nu-\sigma'}\langle n_1\rangle^{-2\sigma'}\prod_{\ell'=1}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}.$$

By (2.2.50), $\langle n_{j''+1}\rangle\sim\langle n_1\rangle$. Going back to the estimate of (2.2.46) by the product of (2.2.47) – where we take $N=0$ – and of (2.2.48), we see that $\|\Pi_{n_0}M_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)}$ is bounded by the sum for $j'+j''=j$ and $n_0'\in\mathbb{Z}$ of the product of

$$\begin{aligned}&K_2C_0D_0\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;b)\\&\times j'j''\frac{(k'+j'-1)!}{(j'+1)!}\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')B^j\end{aligned} \tag{2.2.51}$$

and of

$$\langle n_{j+1}\rangle^{d'}\langle n_0'\rangle^{2\sigma'}\langle n_{j''+1}\rangle^{d''+\nu-3\sigma'}\langle n_0\rangle^{-\sigma'}\langle n_{j+1}\rangle^{-\sigma'}\prod_{\ell'=0}^{j+1}\langle n_{\ell'}\rangle^{\sigma'}\prod_{\ell'=1}^{j}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}. \tag{2.2.52}$$

Using that by (2.2.49) $\langle n_0'\rangle\le\langle n_{j+1}\rangle\sim\langle n_0\rangle$, the sum in n_0' of $\langle n_0'\rangle^{2\sigma'}\langle n_0\rangle^{-\sigma'}\langle n_{j+1}\rangle^{-\sigma'}$ is smaller than $C\langle n_{j+1}\rangle\le C\langle n_{j''+1}\rangle$. If we sum (2.2.51) for $j'+j''=j$ using (2.1.16), (2.1.17) we obtain an estimate of form (2.1.40) with d replaced by $d'_+ + d''$, ν replaced by $\nu + 1$. $\square$

2.3. Composition and transpose of operators

In this section, we shall study $\mathrm{Op}_\chi[a(u;\cdot)] \circ \mathrm{Op}_\chi[b(u;\cdot)]$ and ${}^t\mathrm{Op}_\chi[a(u;\cdot)]$ where $a \in S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.)$ and $b \in S^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.)$.

Theorem 2.3.1. — *Let* $d', d'' \in \mathbb{R}, N_0 \in \mathbb{N}, \nu, \zeta \in \mathbb{R}_+, k', k'' \in \mathbb{N}^*, \sigma \in \mathbb{R}$ *with* $\sigma \geq N_0 + \nu + \zeta + 2, B > 0$. *Let* $D.$ *be a* $(\nu + |d'| + |d''| + \sigma, N_0 + 1)$*-conveniently increasing sequence. Assume that the constant* K_0 *in (2.1.17) satisfies* $K_0 \geq 100(2D_0 + 1)$.

(i) For any $a \in S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.), b \in S^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.)$, *the product* $ab \in S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)$ *with* $d = d' + d'', k = k' + k''$. *Moreover*

$$\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;ab) \leq \frac{1}{100}\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;b). \tag{2.3.1}$$

(ii) Assume $\sigma \geq N_0 + \nu + 5 + \max(\zeta, \frac{d_+}{3})$. *There is a* $(\nu + N_0 + 3 + |d'| + |d''| + \sigma, N_0 + 1)$*-conveniently increasing sequence* $\tilde{D}.$, *a symbol* $e \in S^{d-1,\nu+N_0+3}_{(k),N_0}(\sigma,\zeta,B,\tilde{D}.)$ *and an operator* $M \in \mathcal{L}^{d_+,\nu+N_0+3}_{(k)}(\sigma,\zeta,B)$ *such that*

$$\mathrm{Op}_\chi[a(u;\cdot)] \circ \mathrm{Op}_\chi[b(u;\cdot)] = \mathrm{Op}_\chi[ab(u;\cdot)] + \mathrm{Op}_\chi[e(u;\cdot)] + M(u). \tag{2.3.2}$$

Proof of (i). — Decompose $a(u;x,n) = \sum_{j' \geq k'} a_{j'}(u,\dots,u;x,n)$, $b(u;x,n) = \sum_{j'' \geq k''} b_{j''}(u,\dots,u;x,n)$ according to definition 2.1.5. Then

$$ab = \sum_{j \geq k} c_j(u,\dots,u;x,n)$$

with

$$c_j(u_1,\dots,u_j;x,n) = \sum_{j'+j''=j} [a_{j'}(u_1,\dots,u_{j'};x,n)b_{j''}(u_{j'+1},\dots,u_j;x,n)]_S$$

where S stands for symmetrization in $(u_1,\dots,u_j)$. Let $\alpha, \beta \in \mathbb{N}$ with $\alpha + \beta = p$, and compute $\partial_x^\alpha \partial_n^\beta (a_{j'} b_{j''})$ using (2.1.10). Let us prove upper bounds of type (2.1.20). Let $\sigma' \in [\nu + \zeta + 2, \sigma]$. When we estimate $(\partial_x^\alpha \partial_n^\beta a_{j'}) b_{j''}$ or $a_{j'}(\partial_x^\alpha \partial_n^\beta b_{j''})$ from (2.1.20) for $a_{j'}, b_{j''}$, we get a bound given by the product of

$$\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;b) \tag{2.3.3}$$

and of

$$\begin{aligned}\frac{(k'+j'-1)!}{(j'+1)!}\frac{(k''+j''-1)!}{(j''+1)!}&B^j c(j')c(j'')D_p D_0 \langle n\rangle^{d-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}\\ &\times \prod_{\ell=1}^{j} \langle n_\ell\rangle^{\sigma'} \|\Pi_{n_\ell} u_\ell\|_{L^2}.\end{aligned} \tag{2.3.4}$$

If we sum for $j'+j''=j$ and use (2.1.16) and (2.1.17), we obtain a bound given by the product of (2.3.3) and of

$$\frac{D_0}{K_0}\frac{(k+j-1)!}{(j+1)!}B^j c(j)D_p\langle n\rangle^{d-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}\prod_{\ell=1}^{j}\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}. \tag{2.3.5}$$

Consider now a contribution to $\partial_x^\alpha\partial_n^\beta(a_{j'}b_{j''})$ corresponding to terms in the sum in (2.1.10) i.e.

$$|\widetilde{C}^{\alpha,\beta}_{\alpha',\beta',\gamma}||(\mathrm{Id}-\tau_1)^\gamma\partial_x^{\alpha'}\partial_n^{\beta'}a_{j'}||\partial_x^{\alpha-\alpha'}\partial_n^{\beta-\beta'}b_{j''}|. \tag{2.3.6}$$

By (2.1.20) for $a_{j'}$ and (2.1.34)

$$\begin{aligned}|(\mathrm{Id}-\tau_1)^\gamma\partial_x^{\alpha'}\partial_n^{\beta'}a_{j'}| &\le \sum_{\gamma'=0}^{\gamma}\binom{\gamma}{\gamma'}\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,b,D.;a)\frac{(k'+j'-1)!}{(j'+1)!}D_{\alpha'+\beta'}\\ &\qquad\times c(j')B^{j'}\langle n-\gamma'\rangle^{d'-\beta'+(\alpha'+\nu+N_0\beta'-\sigma')_+}\\ &\le 2^\gamma\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,b,D.;a)(2\langle\gamma\rangle)^{|d'|+\beta'+(\alpha'+\nu+N_0\beta'-\sigma')_+}\frac{(k'+j'-1)!}{(j'+1)!}D_{\alpha'+\beta'}\\ &\qquad\times c(j')B^{j'}\langle n\rangle^{d'-\beta'+(\alpha'+\nu+N_0\beta'-\sigma')_+}\end{aligned} \tag{2.3.7}$$

Using also (2.1.20) to estimate the $b_{j''}$ contribution, we bound (2.3.6) by the product of (2.3.3) and of

(2.3.8)
$$\begin{aligned}&2^p|\widetilde{C}^{\alpha,\beta}_{\alpha',\beta',\gamma}|(2\langle\gamma\rangle)^{|d'|+\beta'+(\alpha'+\nu+N_0\beta'-\sigma')_+}D_{\alpha'+\beta'}D_{p-(\alpha'+\beta')}\\ &\qquad\times\frac{(k'+j'-1)!}{(j'+1)!}\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')B^j\langle n\rangle^{d-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}\\ &\qquad\times\prod_{\ell=1}^{j}\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}\end{aligned}$$

where we have used

$$(\alpha'+\nu+N_0\beta'-\sigma')_+ + (\alpha''+\nu+N_0\beta''-\sigma')_+ \le (\alpha+\nu+N_0\beta-\sigma')_+$$

since $\sigma'\ge\nu$. Remark that the first line in (2.3.8) is smaller than

$$2^p|\widetilde{C}^{\alpha,\beta}_{\alpha',\beta',\gamma}|(2\langle p\rangle)^{|d'|+\nu+p(N_0+1)}D_{\alpha'+\beta'}D_{p-(\alpha'+\beta')}$$

and so the sum in α',β',γ of these quantities will be bounded, according to (2.1.13) and the assumptions by D_p. Summing also (2.3.8) for $j'+j''=j$, we get a bound of form (2.3.5) with $\frac{D_0}{K_0}$ replaced by $\frac{1}{K_0}$. If we assume $\frac{2D_0+1}{K_0}\le\frac{1}{100}$, we obtain for $\partial_x^\alpha\partial_n^\beta c_j$ the estimate (2.1.20), with the bound (2.3.1) for $\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;ab)$. We must next get bound (2.1.21). The proof proceeds in the same way as above, except that one uses an estimate of form (2.1.20) (resp. (2.1.21)) for $\partial_x^{\alpha'}\partial_n^{\beta'}a_{j'}$ and (2.1.21) (resp. (2.1.20)) for $\partial_x^{\alpha-\alpha'}\partial_n^{\beta-\beta'}b_{j''}$. This concludes the proof of assertion (i) of the theorem. □

Remark. — When we estimate the sum for $j'+j''=j$ in (2.3.4), (2.3.8), we may use the first inequality in (2.1.16). In that way, we get a bound for c_j in terms of $\frac{(k-1+j-1)!}{(j+1)!}$ i.e. we have, instead of (2.3.1)

$$\mathfrak{N}^{d,\nu}_{(k-1),N_0}(\sigma,\zeta,B,D.;ab)\leq \mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;b). \tag{2.3.9}$$

Before proving (ii) of the theorem, let us establish some intermediate results.

Proposition 2.3.2. — *Let $\tilde d',\tilde d''\in\mathbb{R}$, $\sigma,\nu,\zeta,B,D.$ be as in the statement of theorem 2.3.1, set $\tilde d=\tilde d'+\tilde d''$. Let $\nu'\geq\nu$ be given, assume $\sigma\geq\nu'+\zeta+2$ and let*

$$\tilde a(u;x,\ell,n)=\sum_{j'\geq k'}\tilde a_{j'}(\underbrace{u,\dots,u}_{j'};x,\ell,n),\tilde b(u;x,y,n)=\sum_{j''\geq k''}\tilde b_{j''}(\underbrace{u,\dots,u}_{j''};x,y,n) \tag{2.3.10}$$

be formal series defined in terms of multi-linear maps satisfying the following conditions: $\partial_x^\alpha\partial_n^{\beta_1}\partial_\ell^{\beta_2}\tilde a_{j'}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j'}}u_{j'};x,\ell,n)$ with $\beta_1+\beta_2=\beta$, $\alpha+\beta=p$ (resp. $\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_n^{\beta}\tilde b_{j''}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j''}}u_{j''};x,y,n)$ with $\alpha_1+\alpha_2=\alpha$, $\alpha+\beta=p$) satisfies (2.1.20) and (2.1.21) with d,j,k,ν replaced by $\tilde d',j',k',\nu'$ (resp. $\tilde d'',j'',k'',\nu'$). Assume moreover that $\tilde a_{j'}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j'}}u_{j'};x,\ell,n)\equiv 0$ (resp. $\tilde b_{j''}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j''}}u_{j''};x,y,n)\equiv 0$) if $\max_{i=1,\dots,j'}(|n_i|)>\frac12|n|$ or if $|\ell|>\frac12\langle n\rangle$ (resp. if $\max_{i=1,\dots,j''}(|n_i|)>\frac14|n|$). Assume also that the x-Fourier transform of these functions is supported in the interval of $\mathbb{Z}$ of center 0, and radius $\frac12\langle n\rangle$. Define

$$\tilde c(u;x,n)=\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\int_{\mathbb{S}^1}e^{-i\ell y}\tilde a(u;x,\ell,n)\tilde b(u;x,y,n)dy. \tag{2.3.11}$$

Then $\tilde c(u;x,n)=\sum_{j\geq k=k'+k''}\tilde c_j(u,\dots,u;x,n)$, where each $\tilde c_j$ satisfies estimates (2.1.20), (2.1.21) of an element of $\Sigma^{\tilde d,\nu'+2}_{(k,j),N_0}(\sigma,\zeta,B,\tilde D.)$ for a new increasing sequence $\tilde D.$, depending on $D.,\tilde d',\tilde d'',\nu,\sigma,N_0$. Moreover the support condition (2.1.19) is verified with $\frac14|n|$ replaced by $\frac12|n|$.

Proof. — We define

$$\tilde c_j(u_1,\dots,u_j;x,n)=\sum_{\substack{j'+j''=j\\ j'\geq k',j''\geq k''}}\sum_{\ell=-\infty}^{+\infty}\frac{1}{2\pi}\int_{\mathbb{S}^1}e^{-i\ell y}[\tilde a_{j'}(u_1,\dots,u_{j'};x,\ell,n)\times\tilde b_{j''}(u_{j'+1},\dots,u_j;x,y,n)]_S dy \tag{2.3.12}$$

where S denotes symmetrization in $(u_1,\dots,u_j)$. Let $p\in\mathbb{N}$ and for $(\alpha,\beta)\in\mathbb{N}\times\mathbb{N}$ with $\alpha+\beta=p$, $0\leq\alpha'\leq\alpha, 0\leq\beta'\leq\beta, 0\leq\gamma\leq\beta$, set

$$\Gamma^{\alpha,\beta,\ell}_{\alpha',\beta',\gamma}(\tilde a_{j'},\tilde b_{j''})=\int_{\mathbb{S}^1}e^{-i\ell y}[(\mathrm{Id}-\tau_1)^\gamma\partial_x^{\alpha'}\partial_n^{\beta'}\tilde a_{j'}(u_1,\dots,u_{j'};x,\ell,n)]\times\partial_x^{\alpha-\alpha'}\partial_n^{\beta-\beta'}\tilde b_{j''}(u_{j'+1},\dots,u_j;x,y,n)dy \tag{2.3.13}$$

when $0 < \alpha' + \beta' < p$,

(2.3.14)
$$\Gamma^{\alpha,\beta,\ell}_{0,0}(\tilde{a}_{j'}, \tilde{b}_{j''}) = \int_{\mathbb{S}^1} e^{-i\ell y} \tilde{a}_{j'}(u_1, \dots, u_{j'}; x, \ell, n) \partial_x^\alpha \partial_n^\beta \tilde{b}_{j''}(u_{j'+1}, \dots, u_j; x, y, n) dy$$

and denote by $\Gamma^{\alpha,\beta,\ell}_{\alpha,\beta}(\tilde{a}_{j'}, \tilde{b}_{j''})$ the quantity of the same form obtained when all derivatives fall on $\tilde{a}_{j'}$. By (2.1.10)

(2.3.15)
$$\partial_x^\alpha \partial_n^\beta \tilde{c}_j(u_1, \dots, u_j; x, n) = \frac{1}{2\pi} \sum_{j'+j''=j} \sum_{\ell=-\infty}^{+\infty} \Big[\Gamma^{\alpha,\beta,\ell}_{0,0}(\tilde{a}_{j'}, \tilde{b}_{j''}) + \Gamma^{\alpha,\beta,\ell}_{\alpha,\beta}(\tilde{a}_{j'}, \tilde{b}_{j''}) + \sum_{\substack{0\le\alpha'\le\alpha, 0\le\beta'\le\beta \\ 0\le\gamma\le\beta, 0<\alpha'+\beta'<p}} \widetilde{C}^{\alpha,\beta}_{\alpha',\beta',\gamma} \Gamma^{\alpha,\beta,\ell}_{\alpha',\beta',\gamma}(\tilde{a}_{j'}, \tilde{b}_{j''})\Big].$$

Let us estimate (2.3.15).

We make in (2.3.13), (2.3.14) two integrations by parts using the vector field $L = \frac{1-\ell D_y}{1+\ell^2}$. In that way, we gain a $\langle \ell \rangle^{-2}$ factor in the integral and lose on $\tilde{b}_{j''}$ up to two ∂_y-derivatives. We use that $(\mathrm{Id} - \tau_1)^\gamma \partial_x^{\alpha'} \partial_n^{\beta'} \tilde{a}_{j'}$ (resp. $\partial_x^{\alpha''} \partial_n^{\beta''} \partial_y^\delta \tilde{b}_{j''}$ $(\delta = 0, 1, 2)$) obeys estimates of type (2.3.7) (resp. (2.1.20)) to bound (2.3.13) by the product of

$$\mathfrak{N}^{\tilde{d}',\nu}_{(k'),N_0}(\sigma, \zeta, B, D.; \tilde{a}) \mathfrak{N}^{\tilde{d}'',\nu}_{(k''),N_0}(\sigma, \zeta, B, D.; \tilde{b}) \tag{2.3.16}$$

and of

$$\begin{aligned} &C(p)\langle \ell \rangle^{-2} c(j') \frac{(k'+j'-1)!}{(j'+1)!} B^{j'} D_{\alpha'+\beta'} \langle n \rangle^{\tilde{d}'-\beta'+(\alpha'+\nu'+N_0\beta'-\sigma')_+} \\ &\times c(j'') \frac{(k''+j''-1)!}{(j''+1)!} B^{j''} D_{\alpha''+\beta''+2} \langle n \rangle^{\tilde{d}''-\beta''+(\alpha''+2+\nu'+N_0\beta''-\sigma')_+} \\ &\times \prod_{\ell'=1}^{j} \langle n_{\ell'} \rangle^{\sigma'} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \end{aligned} \tag{2.3.17}$$

for some constant $C(p)$ depending on $\tilde{d}', \tilde{d}'', \nu, \sigma, N_0$ and for any σ' in the interval $[\nu' + \zeta + 2, \sigma]$. We remark that

$$(\alpha' + \nu' + N_0\beta' - \sigma')_+ + (\alpha'' + 2 + \nu' + N_0\beta'' - \sigma')_+ \le (\alpha + \nu' + 2 + N_0\beta - \sigma')_+ \tag{2.3.18}$$

since $\sigma' \ge \nu'$. Summing (2.3.17) for $j' + j'' = j, \ell \in \mathbb{Z}$, using (2.1.16), (2.1.17) we obtain a bound given by the product of (2.3.16) and of

$$\begin{aligned} &\frac{c(j)}{j} \frac{(k+j-1)!}{(j+1)!} B^j \tilde{D}_p \langle n \rangle^{\tilde{d}-\beta+(\alpha+N_0\beta+\nu'+2-\sigma')_+} \\ &\times \prod_{\ell'=1}^{j} \langle n_{\ell'} \rangle^{\sigma'} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \end{aligned} \tag{2.3.19}$$

for a new constant $\tilde{D}_p$ depending on p but not on j. This gives an estimate of type (2.1.20) for $\tilde{c}_j$. To get an estimate of form (2.1.21), we argue in the same way, bounding either $a_{j'}$ or $b_{j''}$ using (2.1.20) and the other one using (2.1.21). The only difference is that we have to replace (2.3.18) by either

$$(\alpha'+\nu'+N_0\beta'-\sigma')_+ + \alpha''+\nu'+N_0\beta''+2+\sigma' \leq \alpha+\nu'+2+N_0\beta+\sigma'$$

or

$$\alpha'+\nu'+N_0\beta'+\sigma'+(\alpha''+2+\nu'+N_0\beta''-\sigma')_+ \leq \alpha+\nu'+2+N_0\beta+\sigma'$$

which again holds true because $\sigma' \geq \nu'$. This concludes the proof of the proposition. □

End of proof of theorem 2.3.1. — (ii) We have by definition

$$\mathrm{Op}_\chi[a(u;\cdot)]\circ\mathrm{Op}_\chi[b(u;\cdot)] = \mathrm{Op}[c(u;\cdot)]$$

where

$$c(u;x,n) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\int e^{-i\ell y}a_\chi(u;x,n-\ell)b_\chi(u;x-y,n)dy. \tag{2.3.20}$$

Since the Fourier transform of $x \to b_\chi(u;x,n)$ is supported inside $\{\ell; \frac{\ell}{\langle n\rangle} \in \mathrm{Supp}\,\chi\}$, we may insert inside the sum in (2.3.20) a factor $\tilde\chi(\ell/\langle n\rangle)$ for some cut-off function $\tilde\chi \in C_0^\infty(]-1/2,1/2[)$, $\tilde\chi \equiv 1$ close to $\mathrm{Supp}\,\chi$. We may then write

$$\begin{aligned} &c(u;x,n)-(a_\chi b_\chi)(u;x,n) = \\ &\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\int e^{-i\ell y}\tilde\chi\Big(\frac{\ell}{\langle n\rangle}\Big)a_\chi(u;x,n-\ell)[b_\chi(u;x-y,n)-b_\chi(u;x,n)]dy. \end{aligned} \tag{2.3.21}$$

Define

$$\begin{aligned} \tilde b(u;x,y,n) &= \frac{b_\chi(u;x-y,n)-b_\chi(u;x,n)}{1-e^{-iy}} \\ \tilde a(u;x,\ell,n) &= \partial_\ell\big[\tilde\chi\Big(\frac{\ell}{\langle n\rangle}\Big)a_\chi(u;x,n-\ell)\big] \end{aligned} \tag{2.3.22}$$

It follows from the definition of symbols that $\tilde a$ (resp $\tilde b$) satisfies the assumptions of proposition 2.3.2 with $\tilde d' = d'-1$, $\nu'=\nu+N_0$ (resp. $\tilde d''=d''$, $\nu'=\nu+1$) and with $D.$ replaced by a new sequence. Thus we may write

$$c(u;x,n) = (a_\chi b_\chi)(u;x,n)+\tilde c(u;x,n) \tag{2.3.23}$$

for a symbol $\tilde c$ satisfying the conclusion of proposition 2.3.2 i.e. $\tilde c = \sum \tilde c_j$ with $\tilde c_j$ obeying estimates (2.1.20), (2.1.21) of an element of $\Sigma^{d'+d''-1,\nu+N_0+3}_{(k,j),N_0}(\sigma,\zeta,B,\tilde D.)$ for some increasing sequence $\tilde D.$, and verifying (2.1.19) with $\frac14|n|$ replaced by $\frac12|n|$. It remains to show that

$$\mathrm{Op}[c(u;\cdot)] = \mathrm{Op}_\chi[ab(u;\cdot)]+\mathrm{Op}_\chi[e(u;\cdot)]+M(u) \tag{2.3.24}$$

with the notations of the statement of the theorem. Note first that, by the example following definition 2.1.11, $\mathrm{Op}[\tilde c(u;\cdot)]-\mathrm{Op}_\chi[\tilde c(u;\cdot)]$ may be written as $M(u)$ for some

$M \in \mathcal{L}^{d_+,\nu+N_0+3}_{(k)}(\sigma,\zeta,B)$ (the fact that the support condition verified by $\tilde{c}_j$ is (2.1.19) with $\frac{1}{4}|n|$ replaced by $\frac{1}{2}|n|$ does not affect the result). Moreover, modulo another contribution $M(u)$ of the same type, we may write $\mathrm{Op}_\chi[\tilde{c}(u;\cdot)] = \mathrm{Op}_\chi[e(u;\cdot)]$ for some $e \in S^{d-1,\nu+N_0+3}_{(k),N_0}(\sigma,\zeta,B,\tilde{D}.)$: actually, we define $e = \sum_{j\geq k} e_j$ with

$$e_j(u_1,\dots,u_j;x,n) = \sum_{n_1}\cdots\sum_{n_j}\theta\Big(\frac{\max(|n_1|,\dots,|n_j|)}{\langle n\rangle}\Big)\tilde{c}(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;x,n)$$

where $\theta \in C_0^\infty(]-\frac{1}{4},\frac{1}{4}[)$, $\theta \equiv 1$ close to zero, $0 \leq \theta \leq 1$. Then, at the difference of $\tilde{c}$, e_j satisfies the support condition (2.1.19). Moreover, if we apply (2.1.39) to $a = \tilde{c}_j - e_j$, choosing as a special index ℓ one for which $|n_\ell| \geq c\langle n\rangle$, we deduce from (2.1.39) a bound of type (2.1.40), so that $\mathrm{Op}_\chi[\tilde{c}(u;\cdot)] - \mathrm{Op}_\chi[e(u;\cdot)]$ is of form $M(u)$.

To show that (2.3.24) holds true, it remains to prove, because of (2.3.23), that

$$\mathrm{Op}[a_\chi b_\chi(u;\cdot)] - \mathrm{Op}_\chi[ab(u;\cdot)] \tag{2.3.25}$$

may be written as another contribution of type $M(u)$. Since

$$a_\chi b_\chi - (ab)_\chi = [a_\chi b_\chi - (a_\chi b_\chi)_\chi] + [(a_\chi - a)b_\chi]_\chi + [a(b_\chi - b)]_\chi$$

and since we may again apply to the first term in the right hand side and to $a_\chi - a$, $b_\chi - b$ the example following definition 2.1.11, we conclude again that (2.3.25) contributes to $M(u)$ in (2.3.2). This ends the proof of the theorem. □

Let us study transpose of operators.

Proposition 2.3.3. — *Let $d \in \mathbb{R}, \nu,\zeta \in \mathbb{R}_+, k \in \mathbb{N}^*, N_0 \in \mathbb{N}, \sigma \geq \nu + N_0 + 5 + \max(\zeta, \frac{d_+}{3}), B > 0, D.$ a $(|d|+\sigma+\nu, N_0+1)$-conveniently increasing sequence. Let $a \in S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)$ and denote*

$$a^\vee(u;x,n) = a(u;x,-n). \tag{2.3.26}$$

There is a $(|d|+\sigma+\nu+N_0+3, N_0+1)$-conveniently increasing sequence $\tilde{D}.$, depending only on $D., d, \nu, \sigma, N_0$, a symbol e in $S^{d-1,\nu+N_0+3}_{(k),N_0}(\sigma,\zeta,B,\tilde{D}.)$ and $M \in \mathcal{L}^{d_+,\nu+N_0+3}_{(k)}(\sigma,\zeta,B)$ such that

$$^t\mathrm{Op}_\chi[a(u;\cdot)] = \mathrm{Op}_\chi[a^\vee(u;\cdot)] + \mathrm{Op}_\chi[e(u;\cdot)] + M(u). \tag{2.3.27}$$

Proof. — We may write $^t\mathrm{Op}_\chi[a(u;\cdot)] = \mathrm{Op}[c(u;\cdot)]$ where

$$c(u;x,n) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\int_{\mathbb{S}^1} e^{-i\ell y}a_\chi(u;x-y,-n+\ell)dy. \tag{2.3.28}$$

We have

(2.3.29)

$$c(u;x,n) - a^\vee_\chi(u;x,n) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\int_{\mathbb{S}^1} e^{-i\ell y}[a_\chi(u;x-y,-n+\ell) - a_\chi(u,x,-n+\ell)]dy.$$

Define $\tilde{a}(u;x,y,n) = \frac{a_\chi(u;x-y,n)-a_\chi(u;x,n)}{1-e^{-iy}}$. Then (2.3.29) may be written

$$\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\int_{\mathbb{S}^1} e^{-i\ell y}\partial_\ell[\tilde{a}(u;x,y,-n+\ell)]dy. \tag{2.3.30}$$

Since in (2.3.29), $\chi \in C_0^\infty(]-\frac{1}{4},\frac{1}{4}[)$, in the ℓ-sum, $|\ell|$ stays smaller than $\frac{\langle n\rangle}{2}$, so we may insert inside the integral (2.3.30) a cut-off $\tilde{\chi}\Big(\frac{\ell}{\langle n\rangle}\Big)$ for some $\tilde{\chi} \in C_0^\infty(]-\frac{1}{2},\frac{1}{2}[)$. We perform next two integrations by parts using $L(\ell, D_y) = \langle \ell\rangle^{-2}(1-\ell\cdot D_y)$. In that way, we gain a $\langle\ell\rangle^{-2}$ factor, loosing up to two ∂_y derivatives on $\tilde{a}$. Making $\partial_x^\alpha\partial_n^\beta$ act on (2.3.29), (2.3.30) for $\alpha+\beta = p$, we estimate using (2.1.20) the component homogeneous of order j evaluated at $(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)$ by the sum in ℓ of

$$\langle\ell\rangle^{-2}Cc(j)\frac{(k+j-1)!}{(j+1)!}D_{p+4}B^j\langle n\rangle^{d-1-\beta+(\alpha+3+\nu+N_0+N_0\beta-\sigma')_+} \times \prod_{\ell'=1}^{j}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\langle n_{\ell'}\rangle^{\sigma'}$$

where the replacement of ν by $\nu+N_0+3$ comes from the losses due to one ∂_ℓ and up to three ∂_y derivatives. We get in that way the estimate (2.1.20) of a symbol in $\Sigma^{d-1,\nu+N_0+3}_{(k,j),N_0}(\sigma,\zeta,B,\tilde{D}.)$ for a new sequence $\tilde{D}.$. One proves in the same way a bound of form (2.1.21). Moreover, the support condition (2.1.19) is satisfied with $\frac{1}{4}|n|$ replaced by $\frac{1}{2}|n|$. We have thus written

$$^t\mathrm{Op}_\chi[a(u;\cdot)] - \mathrm{Op}_\chi[a^\vee(u;\cdot)] = \mathrm{Op}[e^1(u;\cdot)]$$

for a symbol e^1 whose component homogeneous of order j satisfies (2.1.20), (2.1.21) and a weakened form of (2.1.19). Arguing as at the end of the proof of theorem 2.3.1, we write

$$\mathrm{Op}[e^1(u;\cdot)] = \mathrm{Op}_\chi[e(u;\cdot)] + M(u)$$

with e, M satisfying the conditions of the statement of the proposition. □

2.4. Analytic functions of zero order symbols

We shall establish a stability property for symbols of order zero under composition with an analytic function. Let $k \in \mathbb{N}^*$ be given, $\nu \in \mathbb{R}_+, \sigma \geq \nu+2, B > 0, D.$ a $(\nu+\sigma,1)$-conveniently increasing sequence. If a symbol a is in $S^{0,\nu}_{(k),0}(\sigma,0,B,D.)$, we may also consider it as an element of $S^{0,\nu}_{(1),0}(\sigma,0,2B,D.)$ since in (2.1.20), (2.1.21) we may write

$$\frac{(k+j-1)!}{(j+1)!}B^j \leq \frac{(k-1)!}{j+1}2^{k-1}(2B)^j$$

and we have

$$\mathfrak{N}^{0,\nu}_{(1),0}(\sigma,0,2B,D.;a) \leq (k-1)!2^{k-1}\mathfrak{N}^{0,\nu}_{(k),0}(\sigma,0,B,D.;a). \tag{2.4.1}$$

Proposition 2.4.1. — *Let F be an analytic function defined on a neighborhood of zero in $\mathbb{C}$, satisfying $F(0)=0$, $|F^{(\ell)}(0)|\leq R^{-\ell-1}\ell!$ for some $R>0$. Let $a\in S^{0,\nu}_{(k),0}(\sigma,0,B,D.)$ with $\mathfrak{N}^{0,\nu}_{(k),0}(\sigma,0,B,D.;a)(k-1)!2^{k-1}<R$. Assume that the constant K_0 of (2.1.17) satisfies $K_0\geq 2D_0+1$. Then $F(a)\in S^{0,\nu}_{(k),0}(\sigma,0,2B,D.)$.*

Proof. — We write

$$F(a)=\sum_{\ell=1}^{+\infty}\frac{F^{(\ell)}(0)}{\ell!}a^{\ell}. \tag{2.4.2}$$

According to (2.4.1), we may consider, in a product a^{ℓ}, one of the factors as an element of $S^{0,\nu}_{(k),0}(\sigma,0,B,D.)$ and the other ones as symbols in $S^{0,\nu}_{(1),0}(\sigma,0,2B,D.)$, so that, by (i) of theorem 2.3.1 and (2.3.9), $a^{\ell}\in S^{0,\nu}_{(k),0}(\sigma,0,2B,D.)$ with

$$\mathfrak{N}^{0,\nu}_{(k),0}(\sigma,0,2B,D.;a^{\ell})\leq[(k-1)!2^{k-1}]^{\ell-1}\mathfrak{N}^{0,\nu}_{(k),0}(\sigma,0,B,D.;a)^{\ell}. \tag{2.4.3}$$

We decompose each $a^{\ell}=\sum_{j\geq k}a_{\ell,j}(u,\dots,u;x,n)$ and write

$$F(a)=\sum_{j\geq k}c_j(u,\dots,u;x,n) \tag{2.4.4}$$

with

$$c_j(u_1,\dots,u_j;x,n)=\sum_{\ell=1}^{+\infty}\frac{F^{(\ell)}(0)}{\ell!}a_{\ell,j}(u_1,\dots,u_j;x,n). \tag{2.4.5}$$

We have to show that c_j satisfies (2.1.19), (2.1.20), (2.1.21). The support condition is clearly verified. If we apply (2.1.20) to each term in the right hand side of (2.4.5), and use (2.4.3), we get for $|\partial_x^{\alpha}\partial_n^{\beta}c_j(u_1,\dots,u_j;x,n)|$ a bound

$$\sum_{\ell=1}^{+\infty}\frac{|F^{(\ell)}(0)|}{\ell!}[(k-1)!2^{k-1}]^{\ell-1}\mathfrak{N}^{0,\nu}_{(k),0}(\sigma,0,B,D.;a)^{\ell}$$
$$\times\frac{(k+j-1)!}{(j+1)!}c(j)(2B)^jD_p\langle n\rangle^{-\beta+(\alpha+\nu-\sigma')_+}\prod_{\ell'=0}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}$$

where $p=\alpha+\beta$. The choice of R implies convergence of the series. One obtains estimates of type (2.1.21) in the same way. □

CHAPTER 3

COMPOSITION AND POISSON BRACKETS

The aim of this chapter is to study composition of operators associated to symbols with remainder maps, and to apply this to Poisson brackets of functions defined in terms of such operators.

3.1. External composition with a remainder map

Proposition 3.1.1. — *Let* $d', d'' \in \mathbb{R}_+, d = d' + d'', \nu, \zeta \in \mathbb{R}_+, \sigma \in \mathbb{R}, \sigma \geq \nu + 2 + \max(\zeta, \frac{d}{3}), B > 0, k', k'' \in \mathbb{N}^*, N_0 \in \mathbb{N}, D.$ *a* $(d+\nu+\sigma, N_0+1)$*-conveniently increasing sequence. Assume that the constant* K_0 *of (2.1.17) is large enough.*

(i) Let $M' \in \mathcal{L}^{d',\nu}_{(k')}(\sigma, \zeta, B)$, $M'' \in \mathcal{L}^{d'',\nu}_{(k'')}(\sigma, \zeta, B)$*. Then* $M'(u) \circ M''(u)$ *belongs to* $\mathcal{L}^{d,\nu}_{(k)}(\sigma, \zeta, B)$ *where* $k = k' + k''$ *and*

$$\mathfrak{N}^{d,\nu}_{(k)}(\sigma, \zeta, B; M' \circ M'') \leq \mathfrak{N}^{d',\nu}_{(k')}(\sigma, \zeta, B; M')\mathfrak{N}^{d'',\nu}_{(k'')}(\sigma, \zeta, B; M''). \tag{3.1.1}$$

(ii) Let $a \in S^{d',\nu}_{(k'),N_0}(\sigma, \zeta, B, D.)$ *and* $M'' \in \mathcal{L}^{d'',\nu}_{(k'')}(\sigma, \zeta, B)$*. Then* $\mathrm{Op}_\chi[a(u;\cdot)] \circ M''(u)$ *belongs to* $\mathcal{L}^{d,\nu}_{(k)}(\sigma, \zeta, B)$ *and*

$$\mathfrak{N}^{d,\nu}_{(k)}(\sigma, \zeta, B; \mathrm{Op}_\chi[a(u;\cdot)] \circ M'') \leq \mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma, \zeta, B, D.; a)\mathfrak{N}^{d'',\nu}_{(k'')}(\sigma, \zeta, B; M'') \tag{3.1.2}$$

if K_0 *is large enough relatively to* D_2, σ, d.

(iii) Under the same assumption as in (ii), $M''(u) \circ \mathrm{Op}_\chi[a(u;\cdot)]$ *belongs to* $\mathcal{L}^{d,\nu}_{(k)}(\sigma, \zeta, B)$ *and and* $\mathfrak{N}^{d,\nu}_{(k)}(\sigma, \zeta, B; M'' \circ \mathrm{Op}_\chi[a(u;\cdot)])$ *is bounded by the right hand side of (3.1.2).*

Moreover conclusions (i), (ii), (iii) above hold true more generally if we assume that M', M'' *(resp. a) is given instead of (2.1.41) (resp. (2.1.27)) by a series* $M'(u) = \sum_{j' \geq k'} j' M'_{j'}(u, \ldots, u)$, $M''(u) = \sum_{j'' \geq k''} j'' M''_{j''}(u, \ldots, u)$ *(resp.* $a(u; x, n) = \sum_{j' \geq k'} j' a_{j'}(u, \ldots, u; x, n)$*) with* $M'_{j'} \in \Lambda^{d',\nu}_{(k',j')}(\sigma, \zeta, B)$,

$M''_{j''} \in \Lambda^{d'',\nu}_{(k'',j'')}(\sigma,\zeta,B)$ (resp. $a_j \in \Sigma^{d',\nu}_{(k',j'),N_0}(\sigma,\zeta,B,D.)$) satisfying estimates (2.1.42) (resp. (2.1.28)).

Remark. — Let us explain, before starting the proof, why we allow, in the last part of the statement, series of form $\sum_{j'} j' M_{j'}$, $\sum_{j'} j' a_{j'}$. It turns out that we shall be using proposition 3.1.1 to estimate Poisson brackets of functions given for instance by expressions of type $\langle M'(u)u,u\rangle$. These brackets will be expressed from the (symplectic) gradient of such functions, so in particular from $\langle J\nabla M'(u)u,u\rangle$. Because of the homogeneity of each component of $M'(u)$, the gradient acting on it makes lose a factor j' on the j'-th component.

Proof. — We prove the proposition using for M', M'' the more general expressions of the end of the statement.

(i) We decompose

$$M'(u) = \sum_{j'\geq k'} j' M'_{j'}(u,\dots,u), M''(u) = \sum_{j''\geq k''} j'' M''_{j''}(u,\dots,u)$$

and define

$$M_j(u_1,\dots,u_j) = \sum_{j'+j''=j} [j' M'_{j'}(u_1,\dots,u_{j'}) \circ (j'' M''_{j''}(u_{j'+1},\dots,u_j))]_S \tag{3.1.3}$$

where S stands for symmetrization. We bound, denoting

$$\Pi_{n'}U' = (\Pi_{n_1}u_1,\dots,\Pi_{n_{j'}}u_{j'}), \Pi_{n''}U'' = (\Pi_{n_{j'+1}}u_{j'+1},\dots,\Pi_{n_j}u_j)$$

and forgetting symmetrization to simplify notations

$$\begin{aligned}&\|\Pi_{n_0}M_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)}\\ &\leq \sum_{n\in\mathbb{Z}}\sum_{j'+j''=j} j'j''\|\Pi_{n_0}M'_{j'}(\Pi_{n'}U')\Pi_n\|_{\mathcal{L}(L^2)}\|\Pi_n M''_{j''}(\Pi_{n''}U'')\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)}.\end{aligned}\tag{3.1.4}$$

We apply (2.1.40) to both factors in the above sum. We bound in this way the right hand side of (3.1.4) by the sum in n and in $j'+j''=j$ of the product of the right hand side of (3.1.1) and of

$$\begin{aligned}&j'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}B^j c(j')c(j'')\\ &\times\left[\langle n\rangle^{2\sigma'}\langle\max(|n_0|,\dots,|n_{j'}|,|n|)\rangle^{-3\sigma'+d'+\nu}\langle\max(|n|,|n_{j'+1}|,\dots,|n_j|)\rangle^{-3\sigma'+d''+\nu}\right]\\ &\times\prod_{\ell'=0}^{j+1}\langle n_{\ell'}\rangle^{\sigma'}\prod_{\ell'=1}^{j}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}.\end{aligned}$$

Since $\frac{d'+d''}{3}+\nu+2\leq\sigma'$, the n sum of the factor between brackets is bounded by

$$C_0\langle\max(|n_0|,\dots,|n_j|)\rangle^{-3\sigma'+d+\nu}.$$

Using then (2.1.16), (2.1.17) when summing for $j'+j''=j$, we conclude that $M_j \in \Lambda^{d,\nu}_{(k,j)}(\sigma,\zeta,B)$, and (3.1.1) holds if $K_0^{-1}C_0\leq 1$.

(ii) We decompose as above $M''(u) = \sum_{j''\geq k''} j'' M''_{j''}(u,\dots,u)$ and, according to (2.1.27), $a(u,\cdot) = \sum_{j'\geq k'} j' a_{j'}(u,\dots,u;\cdot)$. Set

$$M_j(u_1,\dots,u_j) = \sum_{j'+j''=j} j'j''[\mathrm{Op}_\chi[a_{j'}(u_1,\dots,u_{j'};\cdot)]\circ M''_{j''}(u_{j'+1},\dots,u_j)]_S.$$

We need to bound, instead of (3.1.4),

(3.1.5)
$$\sum_{n\in\mathbb{Z}}\sum_{j'+j''=j} j'\|\Pi_{n_0}\mathrm{Op}_\chi[a_{j'}(\Pi_{n'}U';\cdot)]\Pi_n\|_{\mathcal{L}(L^2)} j''\|\Pi_n M''_{j''}(\Pi_{n''}U'')\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)}.$$

Let ℓ be such that $|n_\ell|\geq|n_{\ell'}|$ for any $0\leq\ell'\leq j+1$. To prove for (3.1.5) an estimate of type (2.1.40) when $\ell=0$ or $j'+1\leq\ell\leq j+1$ we apply to the first (resp. second) factor above inequality (2.1.38) with $N=2$ (resp. inequality (2.1.40)). We get a bound given by the right hand side of (3.1.2) times

$$\begin{aligned}&C_0D_2 j'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}B^j c(j')c(j'')\langle n_0-n\rangle^{-2}\\ &\qquad\times\langle n\rangle^{d'}\langle n_\ell\rangle^{-3\sigma+\nu+d''}\prod_{\ell'=0}^{j+1}\langle n_{\ell'}\rangle^\sigma\prod_{\ell'=1}^{j}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\end{aligned}\tag{3.1.6}$$

(where we have applied (2.1.40) to $M''_{j''}$ with the special index taken to be n_ℓ when $\ell=j'+1,\dots,j+1$, and taken to be n when $\ell=0$, using that in this case $\langle n_0\rangle\sim\langle n\rangle$), C_0 being a constant depending on σ,ν,d. Since $\langle n\rangle^{d'}\leq C\langle n_0\rangle^{d'}\leq C\langle n_\ell\rangle^{d'}$, we obtain summing in n and in $j'+j''=j$, and using (2.1.16), (2.1.17) an estimate of form (2.1.40), if K_0 is large enough relatively to D_2,σ,d,ν. To conclude the proof, we just need to note that estimate (2.1.40) with $\ell=0$ implies the same estimate for any ℓ between 1 and j', since the support condition (2.1.19) satisfied by $a_{j'}$ implies that $|n_\ell|\leq 2|n_0|$, $\ell=1,\dots,j'$.

(iii) The proof is similar. □

3.2. Substitution

We study in this section the effect of substituting to one argument of a symbol a quantity of form $M(u)u$, where M is a remainder operator.

Proposition 3.2.1. — *Let $d',d''\in\mathbb{R}_+, d=d'+d'', \iota=\min(1,d''), \nu,\zeta\in\mathbb{R}_+, \sigma\geq\nu+\max(\zeta,\frac{d}{3})+3, B>0, N_0\in\mathbb{N}, D.$ a $(\sigma+d'+\nu,N_0+1)$-conveniently increasing sequence, $k',k''\in\mathbb{N}^*$.*

For every $a\in S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.)$, for every $M(u)=\sum_{j''\geq k''}j''M_{j''}(u,\dots,u)$ with $M_{j''}\in\Lambda^{d'',\nu}_{(k'',j'')}(\sigma,\zeta,B)$ and

$$\tilde{\mathfrak{N}}^{d'',\nu}_{(k'')}(\sigma,\zeta,B;M)\stackrel{\mathrm{def}}{=}\sup_{j''\geq k''}\mathfrak{N}^{d'',\nu}_{(k'',j'')}(\sigma,\zeta,B;M_{j''})<+\infty,$$

there are a symbol $\tilde a \in S^{d-\iota,\nu+\iota}_{(k),N_0}(\sigma,\tilde\zeta,B,D.)$, *with* $\tilde\zeta = \max(\zeta,\frac{d}{3})$, *and an operator* $\widetilde M \in \mathcal{L}^{d,\nu+1}_{(k)}(\sigma,\zeta,B)$, *with* $k = k'+k''$, *such that*

$$\mathrm{Op}_\chi[\partial_u a(u;\cdot)\cdot[M(u)u]] = \mathrm{Op}_\chi[\tilde a(u;\cdot)] + \widetilde M(u). \tag{3.2.1}$$

Moreover, if the constant K_0 *in (2.1.17) is large enough relatively to* σ,

$$\begin{aligned}\mathfrak{N}^{d-\iota,\nu+\iota}_{(k),N_0}(\sigma,\tilde\zeta,B,D.;\tilde a) &\leq \mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\tilde{\mathfrak{N}}^{d'',\nu}_{(k'')}(\sigma,\zeta,B;M)\\ \mathfrak{N}^{d,\nu+1}_{(k)}(\sigma,\zeta,B;\widetilde M) &\leq \mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\tilde{\mathfrak{N}}^{d'',\nu}_{(k'')}(\sigma,\zeta,B;M).\end{aligned} \tag{3.2.2}$$

Proof. — We decompose

$$a = \sum_{j'\geq k'} a_{j'}(u,\dots,u;x,n),\, M(u) = \sum_{j''\geq k''} j'' M_{j''}(u,\dots,u)$$

with $a_{j'} \in \Sigma^{d'\nu}_{(k',j'),N_0}(\sigma,\zeta,B,D.)$, $M_{j''}\in \Lambda^{d'',\nu}_{(k'',j'')}(\sigma,\zeta,B)$. We write

$$M_{j''}(u_1,\dots,u_{j''}) = M^1_{j''}(u_1,\dots,u_{j''},n) + M^2_{j''}(u_1,\dots,u_{j''},n) \tag{3.2.3}$$

where

$$\begin{aligned}M^1_{j''}(u_1,\dots,u_{j''},n) = \sum_{n_0}\cdots\sum_{n_{j''+1}}\chi_1\Big(\frac{\max(|n_0|,\dots,|n_{j''+1}|)}{\langle n\rangle}\Big)\\ \times\Pi_{n_0}M_{j''}(\Pi_{n_1}u_1,\dots,\Pi_{n_{j''}}u_{j''})\Pi_{n_{j''+1}}\end{aligned} \tag{3.2.4}$$

with $\chi_1\in C_0^\infty(\mathbb{R})$, $\chi_1\equiv 1$ close to zero, $\mathrm{Supp}\,\chi_1$ small enough, $0\leq\chi_1\leq 1$. Set $M^\ell(u,n) = \sum_{j''\geq k''} M^\ell_{j''}(u,\dots,u,n)$ and decompose

$$(\partial_u a)(u;x,n)\cdot[M(u)u] = (\partial_u a)(u;x,n)\cdot[M^1(u,n)u] + (\partial_u a)(u;x,n)\cdot[M^2(u,n)u]. \tag{3.2.5}$$

We study first $\widetilde M(u) = \sum_{j\geq k}\widetilde M_j(u,\dots,u)$ where

$$\widetilde M_j(u_1,\dots,u_j) = \sum_{j'+j''=j} j'j''\mathrm{Op}_\chi[a_{j'}(u_1,\dots,u_{j'-1},M^2_{j''}(u_{j'},\dots,u_{j-1},\cdot)u_j;\cdot)]_S \tag{3.2.6}$$

with S denoting symmetrization. Denote $U' = (u_1,\dots,u_{j'-1})$, $U'' = (u_{j'},\dots,u_{j-1})$, $n' = (n_1,\dots,n_{j'-1})$, $n'' = (n_{j'},\dots,n_{j-1})$ and use the natural notation $\Pi_{n'}U',\Pi_{n''}U''$. Applying (2.1.38) with $N=0$, we bound $\|\Pi_{n_0}\widetilde M_j(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)}$ by the product of $\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)$ and of

$$\begin{aligned}C_0D_0\sum_{n=-\infty}^{+\infty}\sum_{j'+j''=j} j'\frac{(k'+j'-1)!}{(j'+1)!}c(j')B^{j'}\langle n_{j+1}\rangle^{d'}\\ \times\prod_{\ell=1}^{j'-1}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2} j''\langle n\rangle^{\sigma'}\|\Pi_n M^2_{j''}(\Pi_{n''}U'',n_{j+1})\Pi_{n_j}u_j\|_{L^2}.\end{aligned} \tag{3.2.7}$$

for any $\sigma' \in [\nu + 2 + \tilde{\zeta}, \sigma]$. By (2.1.19) we have on the sum

$$\max(|n_1|, \dots, |n_{j'-1}|, |n|) < \frac{1}{4}|n_{j+1}|, \langle n_0 \rangle \sim \langle n_{j+1} \rangle \tag{3.2.8}$$

and by (3.2.3), (3.2.4)

$$\max(|n_{j'}|, \dots, |n_j|, |n|) \geq c\langle n_{j+1} \rangle. \tag{3.2.9}$$

Let ℓ be such that $|n_\ell|$ is the largest among $|n_0|, \dots, |n_{j+1}|$. Inequality (3.2.8) shows that we may assume that $j' \leq \ell \leq j+1$. If we estimate the last factor in (3.2.7) using (2.1.40), we bound the second line of (3.2.7) by

$$j'' \frac{(k'' + j'' - 1)!}{(j'' + 1)!} B^{j''} c(j'') \langle n_\ell \rangle^{-3\sigma' + \nu + d''} \langle n \rangle^{2\sigma'} \langle n_0 \rangle^{-\sigma'} \langle n_{j+1} \rangle^{-\sigma'} \times \prod_{\ell'=0}^{j+1} \langle n_{\ell'} \rangle^{\sigma'} \prod_{\ell'=1}^{j} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2}.$$

Plugging in (3.2.7), using (3.2.8), (3.2.9) and (2.1.16), (2.1.17) when summing for $j' + j'' = j$, we see that we obtain for $\|\Pi_{n_0} \widetilde{M}_j(\Pi_{n_1} u_1, \dots, \Pi_{n_j} u_j)\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)}$ bounds of form (2.1.40) with ν replaced by $\nu + 1$. If the constant K_0 of (2.1.17) is large enough in function of d, σ, we get the second estimate (3.2.2).

We are left with studying the contribution of the first term in the right hand side of (3.2.5) to (3.2.1). Let us show that

$$\tilde{a}_j(u_1, \dots, u_j; x, n) = \sum_{j'+j''=j} j'j''[a_{j'}(u_1, \dots, u_{j'-1}, M^1_{j''}(u_{j'}, \dots, u_{j-1}, n)u_j); x, n]_S$$

belongs to $\Sigma^{d'+d''-\iota, \nu+\iota}_{(k,j),N_0}(\sigma, \tilde{\zeta}, B, D.)$. Forgetting again symmetrization in the notations, we have by (2.1.10), for $\alpha + \beta = p$

(3.2.10)

$$\begin{aligned} \partial_x^\alpha \partial_n^\beta \tilde{a}_j(u_1, \dots, u_j; x, n) = & \\ & \sum_{j'+j''=j} j'j''(\partial_x^\alpha \partial_n^\beta a_{j'})[u_1, \dots, u_{j'-1}, M^1_{j''}(u_{j'}, \dots, u_{j-1}, n)u_j; x, n] \\ + \sum_{j'+j''=j} \sum_{\substack{0 \leq \beta' < \beta \\ 0 \leq \gamma \leq \beta}} & \widetilde{C}^{\alpha,\beta}_{\alpha,\beta',\gamma} j'((\mathrm{Id} - \tau_1)^\gamma \partial_x^\alpha \partial_n^{\beta'} a_{j'})[u_1, \dots, u_{j'-1}, \\ & j'' \partial_n^{\beta-\beta'} M^1_{j''}(u_{j'}, \dots, u_{j-1}, n)u_j); x, n]. \end{aligned}$$

We replace u_ℓ by $\Pi_{n_\ell} u_\ell$ in (3.2.10), $\ell = 1, \dots, j$. We note that if $\mathrm{Supp}\,\chi_1$ is small enough, the support property (2.1.19) will be verified by $\tilde{a}_j$. We write in (3.2.10) $M^1_{j''} = \sum_{n_0} \Pi_{n_0} M^1_{j''}$ and note that by (3.2.4)

$$\begin{aligned} &\|\partial_n^{\beta-\beta'} \Pi_{n_0} M^1_{j''}(\Pi_{n''} U'', n)\Pi_{n_j}\|_{\mathcal{L}(L^2)} \\ &\leq C_{\beta-\beta'}(\chi_1) \langle n \rangle^{-(\beta-\beta')} \|\Pi_{n_0} M_{j''}(\Pi_{n''} U'')\Pi_{n_j}\|_{\mathcal{L}(L^2)} \end{aligned} \tag{3.2.11}$$

for some sequence $C.(\chi_1)$ depending only on χ_1, with $C_0(\chi_1) = 1$.

Let us bound the first term in the right hand side of (3.2.10). Let $\sigma' \geq \nu + 2 + \iota + \max(\zeta, \frac{d}{3})$. Using (2.1.20) to estimate $a_{j'}$ and (2.1.40) to bound the last factor in (3.2.11), we obtain an estimate by the product of

$$\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;a)\mathfrak{N}^{d'',\nu}_{(k'')}(\sigma,\zeta,B;M) \tag{3.2.12}$$

and of the sum in $n_0, j' + j'' = j$ of

(3.2.13)
$$j'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')D_pB^jC_0(\chi_1)\langle n\rangle^{d'-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}$$
$$\times[\langle\max(|n_0|,|n_{j'}|,\dots,|n_j|)\rangle^{-3\sigma'+\nu+d''}\langle n_0\rangle^{2\sigma'}]\prod_{\ell'=0}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}$$

since by assumption $\sigma' \geq \nu + 2 + \max(\zeta, \frac{d''}{3})$.

Since $-3\sigma' + \nu + d'' \leq 0$, we bound the term between brackets by

$$C\langle n_0\rangle^{-\sigma'+\iota+\nu}\langle n_0\rangle^{d''-\iota} \leq C\langle n_0\rangle^{-\sigma'+\iota+\nu}\langle n\rangle^{d''-\iota}$$

(because of the cut-off χ_1 in (3.2.4)). Since $\sigma' \geq \nu+\iota+2$, the sum in n_0 and $j'+j''=j$ of (3.2.13) will be smaller, by (2.1.16), (2.1.17) than the product of (3.2.12) and

$$\frac{1}{2}D_p\frac{(k+j-1)!}{(j+1)!}c(j)B^j\langle n\rangle^{d-\iota-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}\prod_{\ell'=0}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2} \tag{3.2.14}$$

if the constant K_0 of (2.1.17) is large enough. To obtain estimates (2.1.20) for (3.2.10), we have to bound by (3.2.14) the second term in the right hand side of (3.2.10). We write $(\mathrm{Id} - \tau_1)^\gamma = \sum_{\gamma'=0}^{\gamma}\binom{\gamma}{\gamma'}(-1)^{\gamma'}\tau_1^{\gamma'}$, estimate $a_{j'}$ using (2.1.20) and (2.1.34), and bound the right hand side of (3.2.11) using (2.1.40). We get for the second sum in (3.2.10) a bound given by the product of (3.2.12) and of the sum in n_0 and $j'+j''=j$ of

(3.2.15)
$$\sum_{\substack{0\leq\beta'<\beta\\0\leq\gamma\leq\beta}}\left|\widetilde{C}^{\alpha,\beta}_{\alpha,\beta',\gamma}\right|\binom{\gamma}{\gamma'}(2\langle\gamma\rangle)^{d'+(N_0+1)p}$$
$$\times j'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')D_{\alpha+\beta'}B^jC_{\beta-\beta'}(\chi_1)$$
$$\times\langle n\rangle^{d'-\beta+(\alpha+\nu+N_0\beta-\sigma')_+}[\langle\max(|n_0|,|n_{j'}|,\dots,|n_j|)\rangle^{-3\sigma'+\nu+d''}\langle n_0\rangle^{2\sigma'}]$$
$$\times\prod_{\ell'=0}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}.$$

By definition (2.1.14) of $(\sigma+d'+\nu, N_0+1)$-conveniently increasing sequences

$$\sum_{\substack{0\le\beta'<\beta\\0\le\gamma\le\beta}} \left|\widetilde{C}^{\alpha,\beta}_{\alpha,\beta',\gamma}\right| \binom{\gamma}{\gamma'} (2\langle\gamma\rangle)^{d'+(N_0+1)p} D_{\alpha+\beta'}C_{\beta-\beta'}(\chi_1) \le D_p. \tag{3.2.16}$$

Using again (2.1.16) (2.1.17) we obtain for the sum in n_0, $j'+j''=j$ of (3.2.15) an estimate of form (2.1.20), (3.2.14) if $\sigma'\ge\nu+\iota+\max(\zeta,\frac{d''}{3})+2$ and the constant K_0 of (2.1.17) is large enough.

Let us prove bounds of type (2.1.21). If the special index ℓ is between 1 and $j'-1$, we bound (3.2.10) computed at $(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)$ using (2.1.21) to estimate $a_{j'}$ and (3.2.11), (2.1.40) to control $M^1_{j''}$. We obtain an upper bound given by the product of (3.2.12) and of (3.2.13) or (3.2.15), where the power of $\langle n\rangle$ is now $d'-\beta+\alpha+\nu+N_0\beta+\sigma'$ and where $\langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}$ has been replaced by $\langle n_\ell\rangle^{-\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}$. We conclude then as above.

Assume next that the special index ℓ is between j' and j. We apply (2.1.21) to $a_{j'}$, but we take the special index in this estimate to be the one corresponding to the last argument of $a_{j'}$. We estimate the first term in the right hand side of (3.2.10). We use (3.2.11) and (2.1.40), in which we make appear the $-3\sigma'+\nu+d''$ exponent on $\langle n_\ell\rangle$ if $|n_\ell|\ge|n_0|$ and on $\langle n_0\rangle$ if $|n_0|\ge|n_\ell|$. We obtain an upper bound given by the product of (3.2.12) and of the sum in n_0 and $j'+j''=j$ of

$$\begin{aligned} &j'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')D_{\alpha+\beta}B^j\langle n\rangle^{d'-\beta+\alpha+\nu+N_0\beta+\sigma'}\\ &\times\prod_{\substack{1\le\ell'\le j\\ \ell'\ne\ell}}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\langle n_\ell\rangle^{-\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}\langle\max(|n_0|,|n_\ell|)\rangle^{-\sigma'+\nu+d''}. \end{aligned} \tag{3.2.17}$$

We write

$$\langle\max(|n_0|,|n_\ell|)\rangle^{-\sigma'+\nu+d''}\le\langle n\rangle^{d''-\iota}\langle\max(|n_0|,|n_\ell|)\rangle^{-\sigma'+\nu+\iota}$$

and sum next in n_0 (using $\sigma'\ge\nu+\iota+2$) and in $j'+j''=j$ (using (2.1.16), (2.1.17)) to obtain for (3.2.17) an estimate of type (3.2.14), where the power of $\langle n\rangle$ is now $d-\iota-\beta+\alpha+N_0\beta+\sigma'+\nu$.

To estimate the last sum in (3.2.10), we proceed in the same way except that we have to use (3.2.7) to bound the powers of $\langle n-\gamma\rangle$ coming from $(\mathrm{Id}-\tau_1)^\gamma$. We obtain an estimate

(3.2.18)
$$\sum_{\substack{0\leq\beta'<\beta\\0\leq\gamma\leq\beta}}\Big|\widetilde{C}^{\alpha,\beta}_{\alpha,\beta',\gamma}\Big|\binom{\gamma}{\gamma'}(2\langle\gamma\rangle)^{d'+\nu+\sigma'+(N_0+1)p}$$
$$\times j'\frac{(k'+j'-1)!}{(j'+1)!}j''\frac{(k''+j''-1)!}{(j''+1)!}c(j')c(j'')D_{\alpha+\beta'}B^jC_{\beta-\beta'}(\chi_1)\langle n\rangle^{d'-\beta+\alpha+\nu+N_0\beta+\sigma'}$$
$$\times\prod_{\substack{1\leq\ell'\leq j\\\ell'\neq\ell}}^{j}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}\langle n_\ell\rangle^{-\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2}\langle\max(|n_0|,|n_\ell|)\rangle^{-\sigma'+\nu+d''}$$

We conclude as after (3.2.17) above, using (2.1.14) to obtain a bound of type (3.2.14) with a power of $\langle n\rangle$ given by $d-\iota-\beta+\alpha+N_0\beta+\sigma'+\nu$.

This concludes the proof of the proposition. □

3.3. Poisson brackets of functions

This section is devoted to the study of Poisson brackets of functions defined in terms of para-differential operators or of remainder operators. Let us fix some notation. We set

(3.3.1)
$$I'=\begin{bmatrix}1&0\\0&-1\end{bmatrix},J=\begin{bmatrix}0&-1\\1&0\end{bmatrix},J'=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

so that any 2×2 matrix may be written as a scalar combination

(3.3.2)
$$\lambda I+\mu J+\alpha I'+\beta J'.$$

We denote by $S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$ the space of 2×2 matrices whose entries belong to $S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)$. If A is a matrix valued symbol, we decompose it in terms of scalar symbols according to (3.3.2) and define $\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;A)$ as the supremum of the four corresponding quantities for the four coefficient in (3.3.2). If $s\in\mathbb{R}$, $\rho>0$, we denote by $B_s(\rho)$ the ball of center 0 and radius ρ in $H^s(\mathbb{S}^1;\mathbb{R}^2)$.

Proposition 3.3.1. — *Let $\nu\in\mathbb{R}_+$, $N_0\in\mathbb{R}_+$. There is $\tilde\nu\geq\nu$ and for any $\zeta\in\mathbb{R}_+$, any $d',d''\in\mathbb{N}$ with $d=d'+d''\geq 1$ any $\sigma\geq\tilde\nu+2+\max(\zeta,\frac{d}{3})$, any $(\sigma+\nu+d,N_0+1)$-conveniently increasing sequence $D.$, there is a $(\sigma+\tilde\nu+d,N_0+1)$-conveniently increasing sequence $\tilde D.$ and for any $B>0,k',k''\in\mathbb{N}^*$, for any $A'\in S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$, $A''\in S^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$ with $\overline{A}'^\vee=A',\overline{A}''^\vee=A''$, one may find $A_1\in S^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$, $A_0\in S^{d-1,\tilde\nu}_{(k),N_0}(\sigma,\zeta,B,\tilde D.)\otimes\mathcal{M}_2(\mathbb{R})$ and a map*

$M \in \mathcal{L}^{d,\tilde{\nu}}_{(k)}(\sigma,\zeta,B)$, with $k=k'+k''$, such that

(3.3.3)
$$\{\langle \mathrm{Op}_\chi[A'(u;\cdot)]u,u\rangle, \langle \mathrm{Op}_\chi[A''(u;\cdot)]u,u\rangle\} = \langle \mathrm{Op}_\chi[A_1(u;\cdot)]u,u\rangle + \langle \mathrm{Op}_\chi[A_0(u;\cdot)]u,u\rangle + \langle M(u)u,u\rangle$$

and $\overline{A}_1{}^\vee = A_1, \overline{A}_0{}^\vee = A_0$. Moreover

$$\mathfrak{N}^{d,\nu}_{(k),N_0}(\sigma,\zeta,B,D.;A_1) \le \mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;A')\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;A'') \tag{3.3.4}$$

and for a uniform constant C_0,

(3.3.5)
$$\mathfrak{N}^{d-1,\tilde{\nu}}_{(k),N_0}(\sigma,\zeta,B,\tilde{D}.;A_0)+\mathfrak{N}^{d,\tilde{\nu}}_{(k)}(\sigma,\zeta,B;M) \le C_0\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;A')\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;A'').$$

Remark. — The assumptions $\overline{A}'^\vee = A', \overline{A}''^\vee = A''$ just mean that the operators $\mathrm{Op}_\chi[A'(u;\cdot)], \mathrm{Op}_\chi[A''(u;\cdot)]$ send real valued functions to real valued functions.

We shall prove first a formula similar to (3.3.3) when the matrices $A'(,\cdot)$, $A''(u,\cdot)$ are given by the product of a scalar symbol and a constant coefficient matrix.

Lemma 3.3.2. — *Let $d',d'' \in \mathbb{R}_+, d = d'+d'', \iota' = \min(d',1), \iota'' = \min(d'',1)$. Assume $\sigma \ge \nu+\zeta+3$. Let E',E'' be matrices of $\mathcal{M}_2(\mathbb{R})$, $e' \in S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.)$, $e'' \in S^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.)$. One may find symbols*

$$\tilde{e}' \in S^{d-\iota'',\nu+\iota''}_{(k),N_0}(\sigma,\zeta,B,D.), \tilde{e}'' \in S^{d-\iota',\nu+\iota'}_{(k),N_0}(\sigma,\zeta,B,D.) \tag{3.3.6}$$

and a remainder map

$$\widetilde{M}(u) \in \mathcal{L}^{d,\nu+1}_{(k)}(\sigma,\zeta,B),$$

such that

(3.3.7)
$$\begin{aligned}\{\langle \mathrm{Op}_\chi[e'(u;\cdot)]E'u,u\rangle, &\langle \mathrm{Op}_\chi[e''(u;\cdot)]E''u,u\rangle\}\\ = \langle[(\mathrm{Op}_\chi[e'(u;\cdot)]E' &+ {}^t\mathrm{Op}_\chi[e'(u;\cdot)]^tE')J(\mathrm{Op}_\chi[e''(u;\cdot)]E'' + {}^t\mathrm{Op}_\chi[e''(u;\cdot)]^tE'')]u,u\rangle\\ &+\langle[\mathrm{Op}_\chi[\tilde{e}'(u;\cdot)]E' + \mathrm{Op}_\chi[\tilde{e}''(u;\cdot)]E'']u,u\rangle + \langle M(u)u,u\rangle.\end{aligned}$$

Moreover $\mathfrak{N}^{d-\iota'',\nu+\iota''}_{(k),N_0}(\sigma,\zeta,B,D.;\tilde{e}')$ (resp. $\mathfrak{N}^{d-\iota',\nu+\iota'}_{(k),N_0}(\sigma,\zeta,B,D.;\tilde{e}'')$) may be estimated by

$$C_0[\mathfrak{N}^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.;e')\mathfrak{N}^{d'',\nu}_{(k''),N_0}(\sigma,\zeta,B,D.;e'')]$$

for some universal constant C_0.

Proof. — Denote $C_1(u) = \mathrm{Op}_\chi[e'(u;\cdot)]E', C_2(u) = \mathrm{Op}_\chi[e''(u;\cdot)]E''$ and set

$$\underline{C}_1(u) = C_1(u) + {}^tC_1(u),\ \underline{C}_2(u) = C_2(u) + {}^tC_2(u).$$

We write for $j = 1, 2$

$$\partial_u \langle C_j(u)u, u\rangle \cdot U = \partial_w \langle C_j(u)w, w\rangle|_{w=u} \cdot U + \langle (\partial_u C_j(u) \cdot U)u, u\rangle \tag{3.3.8}$$

whence by (1.2.5)

$$\begin{aligned}\{\langle C_1(u)u, u\rangle, \langle C_2(u)u, u\rangle\} &= \partial_w \langle C_1(u)w, w\rangle|_{w=u} \cdot J\nabla_u \langle C_2(u)u, u\rangle \\ &+ \langle \partial_u C_1(u) \cdot (J\nabla_u \langle C_2(u)u, u\rangle) \cdot u, u\rangle.\end{aligned} \tag{3.3.9}$$

We write the first term in the right hand side as

$$\begin{aligned}\int_{\mathbb{S}^1} \nabla_w \langle C_1(u)w, w\rangle|_{w=u} \cdot J\nabla_u \langle C_2(u)u, u\rangle dx \\ = -(\partial_u \langle C_2(u)u, u\rangle) \cdot J\nabla_w \langle C_1(u)w, w\rangle|_{w=u}\end{aligned} \tag{3.3.10}$$

since ${}^tJ = -J$. Using the notation $\underline{C}_j$ introduced above, we may write

$$\begin{aligned}\partial_u \langle C_2(u)u, u\rangle \cdot U &= \langle \underline{C}_2(u)u, U\rangle + \langle (\partial_u C_2(u) \cdot U)u, u\rangle \\ \nabla_w \langle C_1(u)w, w\rangle &= \underline{C}_1(u)w\end{aligned}$$

so that (3.3.10) may be written

$$-\langle \underline{C}_2(u)u, J\underline{C}_1(u)u\rangle - \langle \partial_u C_2(u)(J\underline{C}_1(u)u)u, u\rangle.$$

Coming back to (3.3.9), we get

$$\begin{aligned}\{\langle C_1(u)u, u\rangle, \langle C_2(u)u, u\rangle\} &= \langle \underline{C}_1(u)J\underline{C}_2(u)u, u\rangle \\ - \langle \partial_u C_2(u)(J\underline{C}_1(u)u)u, u\rangle &+ \langle \partial_u C_1(u)(J\nabla_u \langle C_2(u)u, u\rangle)u, u\rangle.\end{aligned} \tag{3.3.11}$$

The first term in the right hand side is the first term in the right hand side of (3.3.7). Let us check that the last two terms in (3.3.11) contribute to the last terms in (3.3.7). If we set $V(u) = J\underline{C}_1(u)u$ we get by (2.2.1), (2.2.2), (2.2.16) a quantity to which proposition 2.2.2 applies. Consequently, by this proposition

$$\begin{aligned}\partial_u C_2(u) \cdot V(u) = E'' \mathrm{Op}_\chi[\partial_u e''(u; \cdot) \cdot [JE' \mathrm{Op}_\chi[e'(u; \cdot)]u]]u \\ + E'' \mathrm{Op}_\chi[\partial_u e''(u; \cdot) \cdot [J^t E'^t \mathrm{Op}_\chi[e'(u; \cdot)]u]]u\end{aligned}$$

may be written as

$$\mathrm{Op}_\chi[\tilde{e}''(u; \cdot)]E'' u$$

for some $\tilde{e}'' \in S^{d-\iota', \nu+\iota'}_{(\kappa), N_0}(\sigma, \zeta, B, D.)$. This gives the wanted conclusion for the second term in the right hand side of (3.3.11). Consider now the last term in (3.3.11). We may write

$$(\partial \langle C_2(u)u, u\rangle) \cdot U = \langle \underline{C}_2(u)u, U\rangle + \langle \mathrm{Op}_\chi[\partial_u e''(u; \cdot) \cdot U]E'' u, u\rangle.$$

By (2.2.1) and (2.2.23) the last term may be written as $\int_{\mathbb{S}^1} W(u)U dx$ where $W(u)$ is given by (2.2.33). Moreover, as we have seen above, $\underline{C}_2(u)u$ is a quantity of form $V(u)$ i.e. of type (2.2.16). The last term in (3.3.11) is thus

$$\langle \partial_u C_1(u) \cdot (J(V(u) + W(u)))u, u\rangle = \langle E' \mathrm{Op}_\chi[\partial_u e'(u; \cdot) \cdot (J(V(u) + W(u)))]u, u\rangle.$$

If we apply (2.2.17) and (2.2.34), we write this as

$$\langle \mathrm{Op}_\chi[\tilde{e}'(u;\cdot)]E'u,u\rangle + \langle \widetilde{M}(u)u,u\rangle$$

where $\tilde{e}' \in S^{d-\iota'',\nu+\iota''}_{(\kappa),N_0}(\sigma,\zeta,B,D.)$, $\widetilde{M}(u) \in \mathcal{L}^{d,\nu+1}_{(k)}(\sigma,\zeta,B)$.

This concludes the proof of the lemma. □

Proof of proposition 3.3.1. — We decompose the matrices A', A'' of the statement using (3.3.2) and apply lemma 3.3.2. The last term in (3.3.7) contributes to the last term in (3.3.3). When $d''=0$ (resp. $d'=0$) the $\tilde{e}'$ (resp. $\tilde{e}''$) contribution to (3.3.7) is of the form of the A_1 term in the right hand side of (3.3.3). When $d''\geq 1$ (resp. $d'\geq 1$) we get instead contributions to the A_0 term of (3.3.3). We are left with examining the first duality bracket in the right hand side of (3.3.7). Using theorem 2.3.1, proposition 2.3.3 and proposition 3.1.1, we may write as well this expression as contributions to the three terms in the right hand side of (3.3.3). Note that the decomposition of A', A'' using (3.3.2) gives 16 terms of the form of the left hand side of (3.3.7). The first duality bracket in the right hand side of (3.3.7) gives, using the results of symbolic calculus (theorem 2.3.1 and proposition 2.3.3), for each of these terms four contributions of type

$$\langle \mathrm{Op}_\chi[f(u;\cdot)]Fu,u\rangle$$

where $F \in \{I, I', J, J'\}$ and $f = e'e''$ or $e'^\vee e''$ or $e'e''^\vee$ or $e'^\vee e''^\vee$, plus contributions to the last two terms in (3.3.3). Using estimate (2.3.1), we see that we obtain the bound (3.3.4). This concludes the proof of the proposition since the conditions $\bar{A}_1^\vee = A_1, \bar{A}_0^\vee = A_0$ may always be satisfied, using that the left hand side of (3.3.3) is real valued, which allows to replace in the right hand side $\langle \mathrm{Op}_\chi[A_j(u;\cdot)]u,u\rangle$ by

$$\frac{1}{2}\langle [\mathrm{Op}_\chi[A_j(u;\cdot)] + \overline{\mathrm{Op}_\chi[A_j(u;\cdot)]}]u,u\rangle = \Big\langle \mathrm{Op}_\chi\Big[\frac{A_j(u,\cdot)+\overline{A_j(u,\cdot)^\vee}}{2}\Big]u,u\Big\rangle.$$

□

Proposition 3.3.1 provides for the Poisson bracket of two quantities given in terms of symbols of order d', d'' an expression involving a symbol of order $d'+d''$. We cannot expect anything better if we consider arbitrary matrices A', A''. On the other hand, if we limit ourselves to matrices that are linear combinations of I and J, we may write the first term in the right hand side of (3.3.3) from a commutator of $\mathrm{Op}_\chi[A'(u;\cdot)]$ and $\mathrm{Op}_\chi[A''(u;\cdot)]$, gaining in that way one derivative. We shall develop that below, limiting ourselves to polynomial symbols in u, as this is the only case we shall have to consider in applications.

Definition 3.3.3. — Let $d\in\mathbb{R}_+, k\in\mathbb{N}^*, \nu,\zeta\in\mathbb{R}_+, N_0\in\mathbb{N}, s_0\in\mathbb{R}, s_0>\nu+\frac{5}{2}+\max(\zeta,\frac{d}{3})$ and $s_0\geq\frac{d}{2}$.

(i) One denotes by $\mathcal{H}'^{d,\nu}_{(k),N_0}(\zeta)$ the space of functions $u\to F(u)$ defined on $H^{s_0}(\mathbb{S}^1;\mathbb{R}^2)$ with values in $\mathbb{R}$, such that there are symbols $\lambda(u;\cdot)$, $\mu(u;\cdot)$ belonging to $\widetilde{S}^{d,\nu}_{(k),N_0}(\zeta)$, satisfying $\bar{\lambda}^\vee = \lambda$, $\bar{\mu}^\vee = \mu$ and an element $M(u) \in \widetilde{\mathcal{L}}^{d,\nu}_{(k)}(\zeta)$ such that for any $u \in H^{s_0}(\mathbb{S}^1;\mathbb{R}^2)$

$$F(u) = \frac{1}{2}\langle \mathrm{Op}_\chi[\lambda(u;\cdot)I + \mu(u;\cdot)J]u,u\rangle + \frac{1}{2}\langle M(u)u,u\rangle. \tag{3.3.12}$$

(ii) One denotes by $\mathcal{H}^{d,\nu}_{(k),N_0}(\zeta)$ the space of functions $u \to F(u)$ defined on $H^{s_0}(\mathbb{S}^1;\mathbb{R}^2)$ with values in $\mathbb{R}$, such that there are a symbol $A(u;\cdot) \in \widetilde{S}^{d,\nu}_{(k),N_0}(\zeta) \otimes \mathcal{M}_2(\mathbb{R})$ satisfying $\bar{A}^\vee = A$ and a map $M(u) \in \widetilde{\mathcal{L}}^{d,\nu}_{(k)}(\zeta)$ such that

$$F(u) = \frac{1}{2}\langle \mathrm{Op}_\chi[A(u;\cdot)]u,u\rangle + \frac{1}{2}\langle M(u)u,u\rangle. \tag{3.3.13}$$

Remark. — By proposition 2.1.13 (or its special case concerning polynomial symbols) the left half of each duality bracket in (3.3.12), (3.3.13) belongs to $H^{s_0-d}(\mathbb{S}^1;\mathbb{R}^2)$, so the assumptions made on s_0 show that $F(u)$ is well defined

Let us study the stability of the preceding classes under Poisson brackets.

Proposition 3.3.4. — *Let $d_1, d_2 \in \mathbb{R}_+, k_1, k_2 \in \mathbb{N}^*, \nu, \zeta \in \mathbb{R}_+, N_0 \in \mathbb{N}$. Set $\tilde{\zeta} = \max(\zeta, \frac{d_1+d_2}{3})$. There is some $\nu' > \nu$, depending only on ν, N_0 such that for any $s_0 > \nu' + \frac{5}{2} + \tilde{\zeta}$ the following holds:*

(i) Assume $d_1 \geq 1, d_2 \geq 1, s_0 \geq \frac{d_1+d_2-1}{2}$ and take $F_j \in \mathcal{H}'^{d_j}_{(k_j),N_0}(\zeta)$, $j = 1,2$. Then $\{F_1,F_2\}$ is in $\mathcal{H}'^{d_1+d_2-1,\nu'}_{(k_1+k_2),N_0}(\tilde{\zeta})$.

(ii) Assume $d_1, d_2 \in \mathbb{N}, d_1 + d_1 \geq 1, s_0 \geq \frac{d_1+d_2}{2}$ and take $F_j \in \mathcal{H}^{d_j,\nu}_{(k_j),N_0}(\zeta)$, $j = 1,2$. Then $\{F_1,F_2\}$ is in $\mathcal{H}^{d_1+d_2,\nu'}_{(k_1+k_2),N_0}(\tilde{\zeta})$.

Before starting the proof, we study Poisson brackets of quantities involving remainder operators.

Lemma 3.3.5. — *Let $d', d'' \in \mathbb{R}_+, d = d' + d'', \nu, \zeta \in \mathbb{R}_+, \sigma \geq \nu + 2 + \max(\zeta, \frac{d}{3}), D.$ a $(d + \nu + \sigma, N_0 + 1)$-conveniently increasing sequence, $k', k'' \in \mathbb{N}^*, E \in \mathcal{M}_2(\mathbb{R}), e \in S^{d',\nu}_{(k'),N_0}(\sigma,\zeta,B,D.), M'' \in \mathcal{L}^{d'',\nu}_{(k'')}(\sigma,\zeta,B)$. Denote $k = k' + k'', \iota = \min(1,d'')$.*

(i) Assume $\sigma \geq \nu + 3 + \max(\zeta, \frac{d}{3})$. There are a symbol $\tilde{e} \in S^{d-\iota,\nu+\iota}_{(k),N_0}(\sigma,\tilde{\zeta},B,D.)$, with $\tilde{\zeta} = \max(\zeta, \frac{d}{3})$, a remainder operator $\widetilde{M} \in \mathcal{L}^{d,\nu+1}_{(k)}(\sigma,\zeta,B)$ such that

$$\{\langle \mathrm{Op}_\chi[e(u;\cdot)]Eu,u\rangle, \langle M''(u)u,u\rangle\} = \langle \mathrm{Op}_\chi[\tilde{e}(u;\cdot)]Eu,u\rangle + \langle \widetilde{M}(u)u,u\rangle. \tag{3.3.14}$$

(ii) Let $M' \in \mathcal{L}^{d',\nu}_{(k')}(\sigma,\zeta,B)$. There is $\widetilde{M} \in \mathcal{L}^{d,\nu}_{(k)}(\sigma,\zeta,B)$ such that

$$\{\langle M'(u)u,u\rangle, \langle M''(u)u,u\rangle\} = \langle \widetilde{M}(u)u,u\rangle. \tag{3.3.15}$$

Finally, if e, M', M'' are polynomial i.e. belong to $\widetilde{S}^{d',\nu}_{(k'),N_0}(\zeta), \widetilde{\mathcal{L}}^{d',\nu}_{(k')}(\zeta), \widetilde{\mathcal{L}}^{d'',\nu}_{(k'')}(\zeta)$, then $\tilde{e} \in \widetilde{S}^{d-\iota,\nu+\iota}_{(k),N_0}(\tilde{\zeta}), \widetilde{M} \in \widetilde{\mathcal{L}}^{d,\nu+1}_{(k)}(\zeta)$ in (i) and $\widetilde{M} \in \widetilde{\mathcal{L}}^{d,\nu}_{(k)}(\zeta)$ in (ii).

Proof. — (i) By definitions 2.1.10, 2.1.11, we may write

$$\langle M''(u)u,u\rangle = \sum_{j''\geq k''} L_{j''}(\underbrace{u,\dots,u}_{j''+2})$$

where $L_{j''}$ is $(j''+2)$-linear and satisfies for any $\sigma' \in [\nu+2+\max(\zeta,\frac{d''}{3}),\sigma]$

$$|L_{j''}(\Pi_{n_0}u_0,\dots,\Pi_{n_{j''+1}}u_{j''+1})| \leq \mathfrak{N}^{d'',\nu}_{(k'')}(\sigma,B;M'')\frac{(k''+j''-1)!}{(j''+1)!}c(j'')B^{j''}$$
$$\times \langle n_\ell\rangle^{-3\sigma'+\nu+d''}\prod_{\ell'=0}^{j''+1}\langle n_{\ell'}\rangle^{\sigma'}\|\Pi_{n_{\ell'}}u_{\ell'}\|_{L^2}$$

for any $\ell = 0,\dots,j''+1$. This implies that we may write $J\nabla\langle M''(u)u,u\rangle$ as $\widehat{M}''(u)u$ where $\widehat{M}''(u) = \sum_{j''\geq k''} j''\widehat{M}''_{j''}(u,\dots,u)$ with $\widehat{M}''_{j''} \in \Lambda^{d'',\nu}_{(k'',j'')}(\sigma,\zeta,B)$ with

$$\sup_{j''\geq k''} \mathfrak{N}^{d'',\nu}_{(k'',j'')}(\sigma,B;\widehat{M}''_{j''}) \leq C\mathfrak{N}^{d'',\nu}_{(j'')}(\sigma,B;M'')$$

with a uniform constant C. Denote

$$C'(u) = \mathrm{Op}_\chi[e(u;\cdot)]E,\ \underline{C}'(u) = C'(u) + {}^tC'(u).$$

By (1.2.5)

$$\begin{aligned}\{\langle C'(u)u,u\rangle,\langle M''(u)u,u\rangle\} &= \partial_u\langle C'(u)u,u\rangle\cdot(\widehat{M}''(u)u)\\ &= \langle\underline{C}'(u)\cdot(\widehat{M}''(u)u),u\rangle + \langle[(\partial_u C'(u))\cdot(\widehat{M}''(u)u)]u,u\rangle.\end{aligned}\tag{3.3.16}$$

The first bracket in the right hand side may be written

$$\langle C'(u)\widehat{M}''(u)u,u\rangle + \langle u,{}^t\widehat{M}''(u)C'(u)u\rangle$$

and so, by (ii) and (iii) of proposition 3.1.1, has the structure of the last term in the right hand side of (3.3.14). The last duality bracket in (3.3.16) is $\langle\mathrm{Op}_\chi[\partial_u e(u;\cdot)\cdot(\widehat{M}''(u)u)]Eu,u\rangle$ and so, by proposition 3.2.1, has the structure of the right hand side of (3.3.14). This concludes the proof of (i).

(ii) We have written above $J\nabla\langle M''(u)u,u\rangle = \widehat{M}''(u)u$ for some $\widehat{M}''$. We may find in the same way a similar $\widehat{M}'(u)$ such that for any v

$$\partial_u(\langle M'(u)u,u\rangle)\cdot v = \langle\widehat{M}'(u)u,v\rangle.$$

Consequently, the left hand side of (3.3.15) may be written

$$\langle\widehat{M}'(u)u,\widehat{M}''(u)u\rangle = \langle{}^t\widehat{M}''(u)\widehat{M}'(u)u,u\rangle.$$

If we apply (i) of proposition 3.1.1, we get the right hand side of (3.3.15). This concludes the proof. □

Before giving the proof of proposition 3.3.4, we state and prove a lemma, giving a similar statement, for the more general case when F_1, F_2 are defined in terms of symbols that are not necessarily polynomial.

Lemma 3.3.6. — *Let* $d_1, d_2 \in \mathbb{R}_+, d_1 \geq 1, d_2 \geq 1, k_1, k_2 \in \mathbb{N}^*, N_0 \in \mathbb{N}, \nu, \zeta \in \mathbb{R}_+, \sigma \geq \nu + 2N_0 + 8 + \max(\zeta, \frac{d_1+d_2}{3})$. *Let* $D.$ *be a* $(\nu + d_1 + d_2 + \sigma, N_0 + 1)$-*conveniently increasing sequence,* $B > 0$. *Denote* $\tilde{\zeta} = \max(\zeta, \frac{d_1+d_2}{3})$. *Let* $\lambda_j, \mu_j \in S^{d_j,\nu}_{(k_j),N_0}(\sigma, \zeta, B, D.)$ *with* $\bar{\lambda}_j^\vee = \lambda_j, \bar{\mu}_j^\vee = \mu_j, j = 1, 2$ *and let* $M_2 \in \mathcal{L}^{d_2,\nu}_{(k_2)}(\sigma, \zeta, B)$. *Consider the Poisson bracket*

(3.3.17)
$$\Big\{\frac{1}{2}\langle \mathrm{Op}_\chi[\lambda_1(u;\cdot)I + \mu_1(u;\cdot)J]u, u\rangle,\\ \frac{1}{2}\langle \mathrm{Op}_\chi[\lambda_2(u;\cdot)I + \mu_2(u;\cdot)J]u, u\rangle + \frac{1}{2}\langle M_2(u)u, u\rangle\Big\}.$$

One may find $\nu' = \nu+2N_0+6$, *a new conveniently increasing sequence* $\tilde{D}.$, *and symbols* $\lambda, \mu \in S^{d_1+d_2-1,\nu'}_{(k_1+k_2),N_0}(\sigma, \tilde{\zeta}, B, \tilde{D}.)$ *satisfying* $\bar{\lambda}^\vee = \lambda, \bar{\mu}^\vee = \mu$ *and* $M \in \mathcal{L}^{d_1+d_2,\nu'}_{(k_1+k_2)}(\sigma, \zeta, B)$ *such that (3.3.17) equals*

$$\frac{1}{2}\langle \mathrm{Op}_\chi[\lambda(u;\cdot)I + \mu(u;\cdot)J]u, u\rangle + \frac{1}{2}\langle M(u)u, u\rangle. \tag{3.3.18}$$

Proof. — Let us study first the contribution coming from $\langle M_2(u)u, u\rangle$ in the second argument of the bracket (3.3.17). By (i) of lemma 3.3.5 we get a contribution to (3.3.18), with symbols $\lambda, \mu \in S^{d_1+d_2-1,\nu+1}_{(k_1+k_2),N_0}(\sigma, \tilde{\zeta}, B, D.)$ and $M \in \mathcal{L}^{d_1+d_2,\nu+1}_{(k_1+k_2)}(\sigma, \zeta, B)$. This is of the wanted form. Consider now the contribution to the bracket coming from

$$\frac{1}{4}\{\langle \mathrm{Op}_\chi[\lambda_1(u;\cdot)I + \mu_1(u;\cdot)J]u, u\rangle, \langle \mathrm{Op}_\chi[\lambda_2(u;\cdot)I + \mu_2(u;\cdot)J]u, u\rangle\}. \tag{3.3.19}$$

Apply lemma 3.3.2 with E' and E'' equal to I and J. The last two brackets in the right hand side of (3.3.7) give contributions of form (3.3.18). Let us study the contributions of the first duality bracket in the right hand side of (3.3.7). If we set

$$C_j(u) = \mathrm{Op}_\chi[\lambda_j(u;\cdot)I + \mu_j(u;\cdot)J],\ \underline{C}_j(u) = \frac{1}{2}[C_j(u) + {}^tC_j(u)]$$

this may be written

$$\langle \underline{C}_1(u)J\underline{C}_2(u)u, u\rangle = \frac{1}{2}\langle [\underline{C}_1(u)J\underline{C}_2(u) - \underline{C}_2(u)J\underline{C}_1(u)]u, u\rangle. \tag{3.3.20}$$

If we set

$$A_j(u) = \frac{1}{2}[\mathrm{Op}_\chi[\lambda_j(u;\cdot)] + {}^t\mathrm{Op}_\chi[\lambda_j(u;\cdot)]]$$
$$B_j(u) = \frac{1}{2}[\mathrm{Op}_\chi[\mu_j(u;\cdot)] - {}^t\mathrm{Op}_\chi[\mu_j(u;\cdot)]]$$

so that $\underline{C}_j(u) = A_j(u) + JB_j(u)$, (3.3.20) equals

$$\Big\langle \big(([B_2, A_1] + [A_2, B_1]) + J([A_1, A_2] - [B_1, B_2])\big)u, u\Big\rangle. \tag{3.3.21}$$

We apply proposition 2.3.3 to write

$$A_j(u) = \mathrm{Op}_\chi[\frac{1}{2}(\lambda_j + \lambda_j^\vee)(u;\cdot)] + \mathrm{Op}_\chi[e_j(u;\cdot)] + M_j^A(u)$$
(3.3.22)
$$B_j(u) = \mathrm{Op}_\chi[\frac{1}{2}(\mu_j - \mu_j^\vee)(u;\cdot)] + \mathrm{Op}_\chi[f_j(u;\cdot)] + M_j^B(u)$$

with $M_j^A, M_j^B \in \mathcal{L}_{(k_j)}^{d_j,\nu+N_0+3}(\sigma,\zeta,B)$ and $e_j, f_j \in S_{(k_j),N_0}^{d_j-1,\nu+N_0+3}(\sigma,\zeta,B,\tilde{D}.)$ (for a new sequence $\tilde{D}.$) since $\sigma \geq \nu + N_0 + 5 + \tilde{\zeta}$. By theorem 2.3.1 the contributions of the para-differential operators in (3.3.22) to (3.3.21) may be written as (3.3.18) with symbols λ, μ in $S_{(k_1+k_2),N_0}^{d_1+d_2-1,\nu+2N_0+6}(\sigma,\zeta,B,\tilde{D}.)$ (for another $\tilde{D}.$) and $M \in \mathcal{L}_{(k_1+k_2)}^{d_1+d_2,\nu+2N_0+6}(\sigma,\zeta,B)$. On the other hand, the contributions to (3.3.21) of M_j^A, M_j^B may be dealt with using proposition 3.1.1, and give expressions of form $\langle M(u)u,u\rangle$ for $M \in \mathcal{L}_{(k_1+k_2)}^{d_1+d_2,\nu+N_0+3}(\sigma,\zeta,B)$. This concludes the proof of the lemma. □

Proof of proposition 3.3.4. — (i) By definition of $\mathcal{H}'^{d_j,\nu}_{(k_j),N_0}(\zeta)$,

(3.3.23) $$F_j(u) = \frac{1}{2}\langle \mathrm{Op}_\chi[\lambda_j(u;\cdot)I + \mu_j(u;\cdot)J]u,u\rangle + \frac{1}{2}\langle M_j(u)u,u\rangle.$$

with $\lambda_j, \mu_j \in \widetilde{S}_{(k_j),N_0}^{d_j,\nu}(\zeta)$ satisfying $\bar{\lambda}_j^\vee = \lambda_j, \bar{\mu}_j^\vee = \mu_j$ and $M_j \in \widetilde{\mathcal{L}}_{(k_j)}^{d_j,\nu}(\zeta)$. We may apply lemma 3.3.6 and (ii) of lemma 3.3.5 to $\{F_1, F_2\}$ using that here the symbols and remainder operators are polynomial ones. We obtain the conclusion of the proposition.

(ii) We have to study the Poisson bracket of two functions of form

$$F_j(u) = \frac{1}{2}\langle \mathrm{Op}_\chi[A_j(u;\cdot)]u,u\rangle + \frac{1}{2}\langle M_j(u)u,u\rangle$$

with $A_j(u;\cdot) \in \widetilde{S}_{(k_j),N_0}^{d_j,\nu}(\otimes)\mathcal{M}_2(\mathbb{R})$ with $\bar{A}_j^\vee = A_j$. Lemma 3.3.5 shows that the contributions coming from a Poisson bracket involving at least one term $\langle M_j(u)u,u\rangle$ may be written as the right hand side of (3.3.13), with a symbol A belonging to $\widetilde{S}_{(k_1+k_2),N_0}^{d_1+d_2-\iota,\nu+\iota}(\tilde{\zeta}) \otimes \mathcal{M}_2(\mathbb{R}) \subset \widetilde{S}_{(k_1+k_2),N_0}^{d_1+d_2,\nu+\iota}(\tilde{\zeta}) \otimes \mathcal{M}_2(\mathbb{R})$ (where $\iota \in [0,1]$). On the other hand, the contribution coming from

$$\frac{1}{4}\{\langle \mathrm{Op}_\chi[A_1(u;\cdot)]u,u\rangle, \langle \mathrm{Op}_\chi[A_2(u;\cdot)]u,u\rangle\}$$

is of the form of the left hand side of (3.3.3), with polynomial symbols. It follows from proposition 3.3.1 (applied to polynomial symbols), that this quantity may be written under the form of an element of $\mathcal{H}_{(k_1+k_2),N_0}^{d_1+d_2,\nu'}(\tilde{\zeta})$ for some ν' depending only on ν, N_0. □

We shall make use below of the following lemma.

Lemma 3.3.7. — *Let $\nu, \zeta \geq 0, N_0 \in \mathbb{N}$. There is $s_0 > 0$ large enough, $\rho_0 > 0$ and for any $B > 0$, for any $(d,s) \in \mathbb{R}_+ \times [s_0,+\infty[$ satisfying either $d \leq 1$ or $2s \geq d \geq 2s-1$, for any $\sigma > s$, any $(\sigma + d + \nu, N_0 + 1)$-conveniently increasing sequence $D.$,* *any* $k \in \mathbb{N}^*$ *the following holds: Let* $\tilde{\zeta} = \max(\zeta, \frac{d}{3})$, *a (resp. M) be an element of* $S^{d,\nu}_{(k),N_0}(\sigma, \tilde{\zeta}, B, D.) \otimes \mathcal{M}_2(\mathbb{R})$ *(resp.* $\mathcal{L}^{d,\nu}_{(k)}(\sigma, \zeta, B)$*). Define*

$$F(u) = \langle \mathrm{Op}_\chi[a(u;\cdot)]u, u\rangle + \langle M(u)u, u\rangle. \tag{3.3.24}$$

Then for any $s \geq s_0$ *the map* $u \to DF(u)$ *(resp.* $u \to \nabla F(u)$*) is* C^1 *on* $B_s(\rho_0)$ *with values in* $\mathcal{L}(H^{-s+d}, \mathbb{R})$ *(resp.* H^{s-d}*). Moreover, there is* $C > 0$ *such that for any* $u \in B_s(\rho_0)$

$$|F(u)| \leq C\|u\|^{k+2}_{H^s}. \tag{3.3.25}$$

Proof. — Let us show that $DF(u)$ extends as a linear form on H^{-s+d}. If $V \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ we may write $DF(u) \cdot V$ in terms of

$$\langle \mathrm{Op}_\chi[a(u;\cdot)]V, u\rangle, \langle \mathrm{Op}_\chi[a(u;\cdot)]u, V\rangle \tag{3.3.26}$$

$$\langle \mathrm{Op}_\chi[\partial_u a(u;\cdot) \cdot V]u, u\rangle \tag{3.3.27}$$

$$\langle M(u)V, u\rangle, \langle M(u)u, V\rangle \tag{3.3.28}$$

$$\langle (\partial_u M(u) \cdot V,)u, u\rangle \tag{3.3.29}$$

Let us check that these expressions may be extended to V in H^{-s+d}.

By (2.1.44) with s replaced by $-s+d$, the first duality bracket in (3.3.26) is a $H^{-s} - H^s$ pairing. The second one is a $H^{s-d} - H^{-s+d}$ pairing. Note that the conditions $u \in H^{\nu+\frac{5}{2}+\tilde{\zeta}+\delta}$ and $\sigma \geq \nu + \tilde{\zeta} + 2$ of proposition 2.1.13 hold true if $s > s_0$ large enough since, because of our assumption on d, $\tilde{\zeta} \leq \max(\zeta, \frac{2s}{3})$.

Consider (3.3.27). Assume first that $0 \leq d \leq 1$. If $s > \nu + \tilde{\zeta} + \frac{5}{2}$ we may apply (2.1.46) with s replaced by $s - d$. If we assume $s > d + \nu + \tilde{\zeta} + \frac{5}{2}$, we see that this inequality implies that (3.3.27) is a $H^{-s} - H^s$ pairing. Consider now the case when $2s \geq d \geq 2s - 1$. Then $V \in H^{-s+d} \subset H^{s-1} \subset H^{\nu+\tilde{\zeta}+\frac{5}{2}+\delta}$ ($\delta > 0$ small) if $s > s_0$ large enough, depending only on ν, ζ. By (i) of proposition 2.1.13, we get that $\mathrm{Op}_\chi[\partial_u a(u;\cdot) \cdot V]$ is in $\mathcal{L}(H^s, H^{s-d}) \subset \mathcal{L}(H^s, H^{-s})$ so that (3.3.27) is a $H^{-s} - H^s$ pairing.

Let us study (3.3.28). By (2.1.47), for $s > s_0$ large enough in function of ν, ζ, $M(u)u \in H^{s-d}$ so that the second bracket in (3.3.28) is a $H^{s-d} - H^{-s+d}$ pairing. Consider now the first one. When $d \leq 1$, (2.1.48) shows that for $s > s_0$ large enough relatively to ν, $M(u)\cdot V \in H^{-s}$, so that we have a $H^{-s} - H^s$ pairing. If $2s-1 \leq d \leq 2s$, $V \in H^{-s+d}$ so that applying (2.1.47) with $\sigma' + \frac{1}{2} + \delta = -s + d$, we see that $M(u) \cdot V$ belongs to H^{-s}. Consequently (3.3.28) is a $H^{-s} - H^s$ pairing.

To treat (3.3.29), we use when $0 \leq d \leq 1$ (2.1.48) to see that for $V \in H^{-s+d}$, $(\partial_u M(u) \cdot V)u$ belongs to H^{-s} for $s \geq s_0$ large enough. When $2s - 1 \leq d \leq 2s$, $V \in H^{-s+d} \subset H^{s-1}$ so that $(\partial_u M(u) \cdot V)u$ belongs also to H^{-s} if $s \geq s_0$ large enough relatively to ν, ζ by the statement after (2.1.47).

This shows that $DF(u) \in \mathcal{L}(H^{-s+d}, \mathbb{R})$. The fact that $u \to DF(u)$ is in fact C^1 follows differentiating once more (3.3.26) to (3.3.29) in u, and making act this differential on some $W \in H^s(\mathbb{S}^1, \mathbb{R}^2)$. Since a, M are converging series, this just means replacing in the general term of their development one argument $u \in H^s(\mathbb{S}^1; \mathbb{R}^2)$ by $W \in H^s(\mathbb{S}^1; \mathbb{R}^2)$ which does not change the boundedness properties. □

Remark. — We shall use below the following consequences of the study of (3.3.28), (3.3.29). If $F(u) = \langle M(u)u, u\rangle$ with $M \in \mathcal{L}^{1,\nu}_{(k)}(\sigma, \zeta, B)$ and if $s \geq s_0$ is large enough relatively to ν, ζ, then $u \to \nabla F(u)$ is a C^1 map from $B_s(\rho)$ to $H^s(\mathbb{S}^1; \mathbb{R}^2)$. Actually, in (3.3.28), we have $M(u)u \in H^s$ by (2.1.47) if s_0 is large enough. Moreover, we have seen in the proof that $M(u) \cdot V$ and $(\partial_u M(u) \cdot V)u$ belong to H^{-s} if $V \in H^{-s}$.

3.4. Division of symbols

The aim of this section is to construct from a symbol or an operator another symbol or operator defined by division by a convenient function. We recall first some notations and results of [**5**], [**1**], [**11**].

If $n_0, \dots, n_{j+1} \in \mathbb{Z}$, denote

$$\begin{aligned} \max\nolimits_2(|n_0|, \dots, |n_{j+1}|) &= \max\{|n_0|, \dots, |n_{j+1}|\} - \{|n_{\ell_0}|\}) \\ \mu(n_0, \dots, n_{j+1}) &= 1 + \max(\{|n_0|, \dots, |n_{j+1}|\} - \{|n_{\ell_0}|, |n_{\ell_1}|\}) \end{aligned} \tag{3.4.1}$$

where ℓ_0 is an index such that $|n_{\ell_0}| = \max(|n_0|, \dots, |n_{j+1}|)$ and ℓ_1 is an index different from ℓ_0, such that $|n_{\ell_1}| = \max_2(|n_0|, \dots, |n_{j+1}|)$. In other words, $\mu(n_0, \dots, n_{j+1})$ is essentially the third largest among $|n_0|, \dots, |n_{j+1}|$.

If $m \in]0, +\infty[$, $j \in \mathbb{N}, n_0, \dots, n_{j+1} \in \mathbb{Z}$, $0 \leq \ell \leq j+1$ we set

$$F^\ell_m(n_0, \dots, n_{j+1}) = \sum_{\ell'=0}^{\ell} \sqrt{m^2 + n^2_{\ell'}} - \sum_{\ell'=\ell+1}^{j+1} \sqrt{m^2 + n^2_{\ell'}}. \tag{3.4.2}$$

It follows from [**5**], [**1**] Theorem 6.5, [**12**] Proposition 2.2.1 that the following proposition holds true:

Proposition 3.4.1. — *There is a subset $\mathcal{N} \subset]0, +\infty[$ of zero measure, and for every $m \in]0, +\infty[- \mathcal{N}$, there are $N_1 \in \mathbb{N}$, $c > 0$ such that the inequality*

$$|F^\ell_m(n_0, \dots, n_{j+1})| \geq c\mu(n_0, \dots, n_{j+1})^{-N_1} \tag{3.4.3}$$

holds in the following two cases:

- *When j is odd, or j is even and $\ell \neq \frac{j}{2}$, for any $(n_0, \dots, n_{j+1}) \in \mathbb{Z}^{j+2}$.*
- *When j is even and $\ell = \frac{j}{2}$ for any $(n_0, \dots, n_{j+1}) \in \mathbb{Z}^{j+2} - Z(j)$, where*

(3.4.4)

$$Z(j) = \{(n_0, \dots, n_{j+1}) \in \mathbb{Z}^{j+2}; \textit{there is a bijection } \sigma : \{0, \dots, \ell\} \to \{\ell+1, \dots, j+1\} \textit{ such that } |n_{\sigma(j)}| = |n_j| \textit{ for any } j = 0, \dots, \ell\}.$$

Note that a much better lower bound for $|F_m^\ell(n_0,\dots,n_{j+1})|$ holds when the largest two among $|n_0|,\dots,|n_{j+1}|$ are much bigger than the other ones, and correspond to square roots affected of the same sign in (3.4.2). To fix ideas, let us assume that $\ell \geq 1$ in (3.4.2). Then for any $m > 0$, there are constants $C > 0, c > 0$ such that for any $(n_0,\dots,n_{j+1}) \in \mathbb{Z}^{j+2}$ satisfying

$$|n_0| \geq C(1+|n_2|+\cdots+|n_{j+1}|), |n_1| \geq C(1+|n_2|+\cdots+|n_{j+1}|) \tag{3.4.5}$$

one has

$$|F_m^\ell(n_0,\dots,n_{j+1})| \geq c(1+|n_0|+\cdots+|n_{j+1}|). \tag{3.4.6}$$

Recall that we introduced in definitions 2.1.4 and 2.1.12 classes of multi-linear symbols $\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$ and operators $\tilde{\Lambda}^{d,\nu}_{(j)}(\zeta)$, which are the building blocks of the polynomial symbols $\widetilde{S}^{d,\nu}_{(k),N_0}(\zeta)$ and operators $\widetilde{\mathcal{L}}^{d,\nu}_{(k)}(\zeta)$. These polynomial symbols or operators have arguments $(u_1,\dots,u_j)$ belonging to $C^\infty(\mathbb{S}^1,\mathbb{R}^2)^j$. It will be convenient to identify $C^\infty(\mathbb{S}^1,\mathbb{R}^2)$ to $C^\infty(\mathbb{S}^1,\mathbb{C})$, and so to consider symbols or operators which are functions of arguments in $C^\infty(\mathbb{S}^1,\mathbb{C})^j$. We introduce a special notation for them.

Definition 3.4.2. — (i) Let $d \in \mathbb{R}$ (resp. $d \in \mathbb{R}_+$), $\nu,\zeta \in \mathbb{R}_+$, $j \in \mathbb{N}^*$, $N_0 \in \mathbb{N}$. One denotes by ${}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$ (resp. ${}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu}_{(j)}(\zeta)$) the space of all $\mathbb{C}$ j-linear maps $(u_1,\dots,u_j) \to ((x,n) \to a(u_1,\dots,u_j;x,n))$ (resp. $(u_1,\dots,u_j) \to M(u_1,\dots,u_j)$) defined on $C^\infty(\mathbb{S}^1,\mathbb{C})^j$, with values in $C^\infty(\mathbb{S}^1\times\mathbb{Z},\mathbb{C})$ (resp. with values in $\mathcal{L}(L^2(\mathbb{S}^1;\mathbb{C}),L^2(\mathbb{S}^1,\mathbb{C}))$) satisfying conditions (2.1.24), (2.1.25) and (2.1.26) (resp. satisfying estimate (2.1.40) for any $\sigma' \geq \nu+2+\max(\zeta,\frac{d}{3})$, with $\frac{(k+j-1)!}{(j+1)!}c(j)B^j$ replaced by an arbitrary constant) for any $u_1,\dots,u_j \in C^\infty(\mathbb{S}^1;\mathbb{C})$.

(ii) We denote by ${}^{\mathbb{C}}\widetilde{S}^{d,\nu}_{(k),N_0}(\zeta)$ (resp. ${}^{\mathbb{C}}\widetilde{\mathcal{L}}^{d,\nu}_{(k)}(\zeta)$) the space of finite sums of form (2.1.29) (resp. (2.1.41)) with $a_j \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$ (resp. $M_j \in {}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu}_{(j)}(\zeta)$).

Let j be an even integer, $\ell = \frac{j}{2}, b \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$. Define

$$b'(u_1,\dots,u_j;x,n) = {\sum_{n'}}' \int_{\mathbb{S}^1} b(\Pi_{n'}U';x,n)\frac{dx}{2\pi} + {\sum_{n'}}' \int_{\mathbb{S}^1} b(\Pi_{n'}U';x-y,n)e^{-2iny}\frac{dy}{2\pi} \tag{3.4.7}$$

where $\Pi_{n'}U' = (\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)$, and where the sum ${\sum}'$ is taken over all indices $n' = (n_1,\dots,n_j) \in \mathbb{Z}^j$ such that there is a bijection $\theta' : \{1,\dots,\ell\} \to \{\ell+1,\dots,j\}$ so that $|n_{\theta'(\ell')}| = |n_{\ell'}|$ for any $1 \leq \ell' \leq \ell$. Then $b' \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$. Actually, integrations by parts show that the last term in (3.4.7) belongs to ${}^{\mathbb{C}}\widetilde{\Sigma}^{d-N,\nu+N}_{(j),N_0}(\zeta)$ for any N. We set

$$b''(u_1,\dots,u_j;x,n) = (b-b')(u_1,\dots,u_j;x,n). \tag{3.4.8}$$

Note that, denoting by $\mathcal{F}$ the x Fourier transform,

(3.4.9)
$$2\pi\mathcal{F}[\mathrm{Op}_\chi[b'(\Pi_{n'}U';\cdot)]\Pi_{n_{j+1}}u_{j+1}](n_0) = \hat{b}'_\chi(\Pi_{n'}U';n_0-n_{j+1},n_{j+1})\hat{u}_{j+1}(n_{j+1})$$
$$= [\delta(n_0-n_{j+1})\hat{b}_\chi(\Pi_{n'}U';0,n_{j+1}) + \delta(n_0+n_{j+1})\hat{b}_\chi(\Pi_{n'}U';-2n_{j+1},n_{j+1})]\hat{u}_{j+1}(n_{j+1})$$

so that

$$\mathrm{Op}_\chi[b'(U';\cdot)]u_{j+1} = \sum_{\substack{n_0,n_{j+1}\\ |n_0|=|n_{j+1}|}} \sum_{n'}{}' \Pi_{n_0}\mathrm{Op}_\chi[b(\Pi_{n'}U';\cdot)]\Pi_{n_{j+1}}u_{j+1}. \tag{3.4.10}$$

By the support condition (2.1.24), if $b(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j;\cdot,n) \not\equiv 0$, we have $|n_1|,\dots,|n_j| \le \frac{1}{4}|n|$. This shows that the conditions on $(n_0,\dots,n_{j+1})$ in the sum in (3.4.10) is equivalent to

(3.4.11) There is a bijection $\theta : \{0,\dots,\ell\} \to \{\ell+1,\dots,j+1\}$ such that $|n_{\theta(\ell')}| = |n_{\ell'}|$ for any $\ell' \in \{0,\dots,\ell\}$.

Consequently, we may write as well (3.4.10) as

$$\mathrm{Op}_\chi[b'(U';\cdot)]u_{j+1} = \sum_{n}{}' \Pi_{n_0}\mathrm{Op}_\chi[b(\Pi_{n'}U';\cdot)]\Pi_{n_{j+1}}u_{j+1}$$

where $\sum'$ means the sum over all $n = (n_0,\dots,n_{j+1})$ satisfying (3.4.11).

If $\omega = (\omega_0,\dots,\omega_{j+1}) \in \{-1,1\}^{j+2}$, if $(u_1,\dots,u_j) \to A(u_1,\dots,u_j)$ is a j-linear map with values in the space of linear maps from $C^\infty(\mathbb{S}^1,\mathbb{C})$ to $C^\infty(\mathbb{S}^1,\mathbb{C})$, if $\Lambda_m = \sqrt{-\Delta+m^2}$, we set

$$\begin{aligned}L_\omega(A)(u_1,\dots,u_j) = \omega_0\Lambda_m A(u_1,\dots,u_j) + \sum_{\ell'=1}^{j}\omega_j A(u_1,\dots,\Lambda_m u_{\ell'},\dots,u_j)\\ +\omega_{j+1}A(u_1,\dots,u_j)\Lambda_m.\end{aligned} \tag{3.4.12}$$

We shall use the following lemma.

Lemma 3.4.3. — *Define*

$$F_m^{(\omega)}(n_0,\dots,n_{j+1}) = \sum_{\ell'=0}^{j+1}\omega_\ell\sqrt{m^2+n_\ell^2}. \tag{3.4.13}$$

(i) Assume $\omega_0\omega_{j+1} = 1$. Then for any $m \in]0,+\infty[$ there is $c_0 > 0$ and for any $\gamma \in \mathbb{N}$, there is $C > 0$, such that for any $(h,n_1,\dots,n_{j+1}) \in \mathbb{Z}^{j+2}$ with

$$1+|n'| \overset{\text{def}}{=} 1+\max(|n_1|,\dots,|n_j|) < c_0|n_{j+1}|$$

and $|h| < \frac{1}{2}\langle n_{j+1}\rangle$,

$$|\partial^\gamma_{n_{j+1}}[F_m^{(\omega)}(h+n_{j+1},n_1,\dots,n_{j+1})]^{-1}| \le C\langle n_{j+1}\rangle^{-1-\gamma}. \tag{3.4.14}$$

(ii) Assume $\omega_0\omega_{j+1} = -1$ *and* $\#\{\ell'; \omega_{\ell'} = -1\} \neq \#\{\ell'; \omega_{\ell'} = 1\}$*. Then for any* $m \in]0, +\infty[-\mathcal{N}$*, for any* $\gamma \in \mathbb{N}$*, there is* $C > 0$ *such that for any* $(h, n_1, \ldots, n_{j+1}) \in \mathbb{Z}^{j+2}$ *with* $|n'| < \frac{1}{4}|n_{j+1}|$ *and* $|h| < \frac{1}{2}|n_{j+1}|$

$$|\partial^{\gamma}_{n_{j+1}}[F^{(\omega)}_m(h + n_{j+1}, n_1, \ldots, n_{j+1})]^{-1}| \leq C\langle h\rangle^{\gamma}\langle n'\rangle^{(\gamma+1)N_1}\langle n_{j+1}\rangle^{-\gamma}. \tag{3.4.15}$$

(iii) Assume $\omega_0\omega_{j+1} = -1$ *and* $\#\{\ell'; \omega_{\ell'} = -1\} = \#\{\ell'; \omega_{\ell'} = 1\}$*.Then for any* $m \in]0, +\infty[-\mathcal{N}$*, for any* $\gamma \in \mathbb{N}$*, there is* $C > 0$ *such that for any* $(h, n_1, \ldots, n_{j+1}) \in \mathbb{Z}^{j+2}$ *with* $|n'| < \frac{1}{4}|n_{j+1}|$*,* $|h| < \frac{1}{2}|n_{j+1}|$ *and* $(h + n_{j+1}, n_1, \ldots, n_{j+1}) \notin Z(\omega)$*, where*

$$\begin{aligned} Z(\omega) = \{(n_0, \ldots, n_{j+1}) \in \mathbb{Z}^{j+2};\ &\text{there is a bijection } \theta : \{\ell; \omega_\ell = 1\} \to \{\ell; \omega_\ell = -1\} \\ &\text{with } |n_{\theta(\ell)}| = |n_\ell| \text{ for any } \ell \text{ with } \omega_\ell = 1\}. \end{aligned}$$

one has

$$|\partial^{\gamma}_{n_{j+1}}[F^{(\omega)}_m(h + n_{j+1}, n_1, \ldots, n_{j+1})]^{-1}| \leq C\langle h\rangle^{\gamma}\langle n'\rangle^{(\gamma+1)N_1}\langle n_{j+1}\rangle^{-\gamma}. \tag{3.4.16}$$

Proof. — We prove (ii). Since $\omega_0\omega_{j+1} = -1$ we may write $F^{(\omega)}_m(n_0, \ldots, n_{j+1})$ as the sum of a term depending only on $n' = (n_1, \ldots, n_j)$ and of a quantity given up to sign by

$$(n_0 - n_{j+1})\int_0^1 [m^2 + (tn_0 + (1-t)n_{j+1})^2]^{-1/2}(tn_0 + (1-t)n_{j+1})dt.$$

This implies that for any fixed m, any $\gamma \geq 1$, any $(h, n_1, \ldots, n_{j+1})$ as in the statement

$$|\partial^{\gamma}_{n_{j+1}}[F^{(\omega)}_m(h + n_{j+1}, n_1, \ldots, n_{j+1})]| \leq C_\gamma\langle h\rangle\langle n_{j+1}\rangle^{-\gamma}. \tag{3.4.17}$$

From this we deduce by induction that $\partial^{\gamma}_{n_{j+1}}[F^{(\omega)}_m(h + n_{j+1}, n_1, \ldots, n_{j+1})]^{-1}$ may be written as a linear combination of quantities of form

$$\frac{\Gamma^{\gamma}_{\gamma'}(h, n', n_{j+1})}{H^{\gamma}_0(h, n', n_{j+1})\cdots H^{\gamma}_{\gamma'}(h, n', n_{j+1})} \tag{3.4.18}$$

where $0 \leq \gamma' \leq \gamma$ and $\Gamma^{\gamma}_{\gamma'}$, H^{γ}_{ℓ} satisfy

$$\begin{aligned} &|\partial^{\alpha}_{n_{j+1}}\Gamma^{\gamma}_{\gamma'}| \leq C_\alpha\langle h\rangle^{\gamma}\langle n_{j+1}\rangle^{-\gamma-\alpha} \\ &|H^{\gamma}_{\ell}(h, n', n_{j+1})| \geq c_\gamma\langle n'\rangle^{-N_1} \\ &|\partial^{\alpha}_{n_{j+1}}H^{\gamma}_{\ell}(h, n', n_{j+1})| \leq C_{\alpha,\gamma}\langle h\rangle\langle n_{j+1}\rangle^{-\alpha}, \alpha > 0. \end{aligned} \tag{3.4.19}$$

Actually, at the first step of the induction, $\Gamma^0_0 = 1$, $H^0_0 = F^{(\omega)}_m(n_{j+1} + h, n', n_{j+1})$ and the second and third inequalities (3.4.19) are just (3.4.3) and (3.4.17). Estimate (3.4.15) follows from (3.4.18), (3.4.19).

Let us prove (i). In this case, $\omega_0\omega_{j+1} = 1$, so that the square roots involving the largest arguments are affected of the same sign. Consequently, if the constant c_0 of the statement is small enough

$$|F^{(\omega)}_m(n_{j+1} + h, n_1, \ldots, n_{j+1})| \geq c\langle n_{j+1}\rangle.$$

Moreover

$$|\partial^\gamma_{n_{j+1}} F_m^{(\omega)}(n_{j+1}+h, n_1, \dots, n_{j+1})| \le C_\gamma \langle n_{j+1}\rangle^{1-\gamma}.$$

These inequalities imply (3.4.14).

Finally, let us show that (iii) holds true. We may apply the proof of statement (ii) if we are able to show that the lower bound of H^γ_ℓ in (3.4.19) still holds. The functions $H^\gamma_\ell(h, n', n_{j+1})$ equal either $F_m^{(\omega)}(n_{j+1}+h, n_1, \dots, n_{j+1})$, or a translate of such a function obtained replacing n_{j+1} by $n_{j+1}+\lambda$. Up to a change of notations, inequality (3.4.3) shows that the lower bound of the second line of (3.4.19) holds true for those (h, n', n_{j+1}) satisfying the assumptions of the statement (since, changing notations, we may reduce to the case when $Z(\omega)$ is given by (3.4.4)). The proof of (ii) applies then without any change and brings (3.4.16). □

Proposition 3.4.4. — *Let $m \in]0, +\infty[$ be outside the exceptional subset $\mathcal{N}$ of proposition 3.4.1. Let $j \in \mathbb{N}^*, d \in \mathbb{R}_+, N_0 \in \mathbb{N}, \nu, \zeta \in \mathbb{R}_+, (\omega_0, \dots, \omega_{j+1}) \in \{-1, 1\}^{j+2}$.*

(i) Assume $\omega_0\omega_{j+1} = 1$. Let $b \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d+1,\nu}_{(j),N_0}(\zeta)$. There is $a \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu+2}_{(j),N_0}(\zeta)$ such that

$$L_\omega(\mathrm{Op}_\chi[a(u_1, \dots, u_j; \cdot)]) - \mathrm{Op}_\chi[b(u_1, \dots, u_j; \cdot)] \tag{3.4.20}$$

belongs to ${}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu+2}_{(j)}(\zeta)$.

(ii) Assume that $\omega_0\omega_{j+1} = -1$ and that $\#\{\ell; \omega_\ell = 1\} \ne \#\{\ell; \omega_\ell = -1\}$. Then if $N_0 \ge 2(N_1+1)$ (where N_1 is the exponent in (3.4.3)), for any $b \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$, there is $a \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu+\zeta+N_1+2}_{(j),N_0}(\zeta)$ such that

$$L_\omega(\mathrm{Op}_\chi[a(u_1, \dots, u_j; \cdot)]) = \mathrm{Op}_\chi[b(u_1, \dots, u_j; \cdot)]. \tag{3.4.21}$$

(iii) Assume that $\omega_0 = 1, \omega_{j+1} = -1$, that j is even and $\omega_1 = \cdots = \omega_{j/2} = 1, \omega_{j/2+1} = \cdots = \omega_j = -1$. Then if $N_0 \ge 2(N_1+1)$ for any $b \in\in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu}_{(j),N_0}(\zeta)$, there is $a \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu+\zeta+N_1+2}_{(j),N_0}(\zeta)$ such that

$$L_\omega(\mathrm{Op}_\chi[a(u_1, \dots, u_j; \cdot)]) = \mathrm{Op}_\chi[b''(u_1, \dots, u_j; \cdot)] \tag{3.4.22}$$

where b'' is defined by (3.4.8).

Proof. — (i) Let $\chi_1 \in C_0^\infty(\mathbb{R}), \chi_1 \equiv 1$ close to zero and decompose $b = b_1 + b_2$ where

$$b_1(u_1, \dots, u_j; x, n) = \sum_{n_1} \cdots \sum_{n_j} \chi_1\Big(\frac{\max(|n_1|, \dots, |n_j|)}{\langle n\rangle}\Big) b(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; x, n).$$

If we apply (2.1.39) to $a = b_2, N = 2$, and use that if $b_2(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; x, n) \not\equiv 0$ there is an index ℓ for which $|n_\ell| \ge c\langle n\rangle$, we see that $\mathrm{Op}_\chi[b_2(u_1, \dots, u_j; \cdot)]$ defines an element of ${}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu+2}_{(j)}(\zeta)$. Consequently, we just have to find a solving $L_\omega(\mathrm{Op}_\chi(a)) = \mathrm{Op}_\chi(b_1)$. Writing from now on b instead of b_1 i.e. assuming that if $b(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; x, n)$ is not zero, then $|n_1| + \cdots + |n_j| \le c\langle n\rangle$ for some given positive constant c, we have to find a so that, for any $n_0, \dots, n_{j+1}$

$$\begin{aligned}\Pi_{n_0} L_\omega[\mathrm{Op}_\chi[a(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; \cdot)]]\Pi_{n_{j+1}}u_{j+1}\\ = \Pi_{n_0}\mathrm{Op}_\chi[b(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; \cdot)]]\Pi_{n_{j+1}}u_{j+1}.\end{aligned} \tag{3.4.23}$$

If we use the definition (3.4.12) of L_ω and $\Lambda_m\Pi_n = \sqrt{m^2+n^2}\Pi_n$, we may write this equality

$$\begin{aligned}F_m^{(\omega)}(n_0, \dots, n_{j+1})\hat{a}_\chi(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; n_0 - n_{j+1}, n_{j+1})\\ = \hat{b}_\chi(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; n_0 - n_{j+1}, n_{j+1}).\end{aligned} \tag{3.4.24}$$

We solve (3.4.24) defining a by

$$\begin{aligned}a(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; x, n_{j+1}) =\\ \frac{1}{2\pi}\sum_h \tilde{\chi}\Big(\frac{h}{\langle n_{j+1}\rangle}\Big)\int_{\mathbb{S}^1} e^{ihy}F_m^{(\omega)}(h+n_{j+1}, n_1, \dots, n_{j+1})^{-1}\\ \times b(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; x-y, n_{j+1})dy\end{aligned} \tag{3.4.25}$$

where $\tilde{\chi} \in C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[)$, $\tilde{\chi} \equiv 1$ close to $[-\frac{1}{4}, \frac{1}{4}]$. We estimate

$$\partial_x^\alpha \partial_{n_{j+1}}^\beta a(\Pi_{n_1}u_1, \dots, \Pi_{n_j}u_j; x, n_{j+1})$$

from (3.4.25), using the Leibniz formula (2.1.10), estimate (3.4.14) and performing two integrations by parts of $L = (1+h^2)^{-1}(1+h\cdot D_y)$ to gain a $\langle h\rangle^{-2}$ factor. We obtain estimates of type (2.1.25), (2.1.26) with ν replaced by $\nu+2$. Since (2.1.24) is also trivially satisfied, we obtain that $a \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu+2}_{(j),N_0}(\zeta)$.

(ii) Let us define again a from b by (3.4.25). We make act $\partial_x^\alpha\partial_{n_{j+1}}^\beta$ on a, using the Leibniz formula (2.1.10). We get a sum of contributions with β' $\partial_{n_{j+1}}$-derivatives falling on $\tilde{\chi}(h/\langle n_{j+1}\rangle)(F_m^{(\omega)})^{-1}$ and β'' $\partial_{n_{j+1}}$-derivatives falling on b, with $\beta'+\beta'' = \beta$. We perform $\beta'+2$ integrations by parts using the same vector field as above, to get a $\langle h\rangle^{-2}$ factor to make converge the series. Using (2.1.25) and (3.4.15) we obtain a bound in terms of the sum for $\beta'+\beta''=\beta$ of

$$\langle n_{j+1}\rangle^{d-\beta+(\alpha+\beta'+2+\nu+N_0\beta''-\sigma')_+}\langle n_1\rangle^{(\beta'+1)N_1}\prod_{\ell=1}^j \langle n_\ell\rangle^{\sigma'}\|\Pi_{n_\ell}u_\ell\|_{L^2} \tag{3.4.26}$$

for any $\sigma' \geq \nu+\zeta+2$, if n_1 is the index such that $|n_1| = \max(|n_1|, \dots, |n_j|)$. We want, to get the conclusion, find a bound in

$$\langle n_{j+1}\rangle^{d-\beta+(\alpha+2\beta'(1+N_1)+\beta''N_0+2+\nu+\zeta+N_1-\sigma)_+}\prod_{\ell=1}^j \langle n_\ell\rangle^{\sigma}\|\Pi_{n_\ell}u_\ell\|_{L^2} \tag{3.4.27}$$

for any $\sigma \geq \nu+\zeta+2$. If $\sigma \geq \beta'(1+N_1)+\nu+\zeta+2$, (3.4.26) applied to $\sigma' = \sigma-\beta'(1+N_1)$ implies (3.4.27). If $\sigma \leq \beta'(1+N_1)+\nu+\zeta+2$, (3.4.26) with $\sigma'=\sigma$ implies (3.4.27). Assuming $N_0 \geq 2(1+N_1)$, we obtain estimate (2.1.25) for the symbol a, with ν replaced by $\nu+\zeta+N_1+2$.

If we estimate $\partial_x^\alpha \partial_n^{\beta''} b$ using (2.1.26), we get instead of (3.4.26) the bound

$$\langle n_{j+1}\rangle^{d-\beta+\alpha+\beta'+\beta'' N_0+2+\nu+\sigma'} \langle n'\rangle^{(\beta'+1)N_1} \prod_{\substack{1\leq \ell'\leq \ell\\ \ell'\neq \ell}} \langle n'_\ell\rangle^{\sigma'} \|\Pi_{n_{\ell'}} u_{\ell'}\|_{L^2} \langle n_\ell\rangle^{-\sigma'} \|\Pi_{n_\ell} u_\ell\|_{L^2}$$

which implies a bound of type (2.1.26) for a, with ν replaced by $\nu + N_1 + 2$, using that $\langle n'\rangle \leq C\langle n_{j+1}\rangle$ and $N_0 \geq N_1 + 1$. Since moreover the support condition (2.1.24) is satisfied by a by construction, we get that $a \in {}^{\mathbb{C}}\widetilde{\Sigma}^{d,\nu+\zeta+N_1+2}_{(j),N_0}(\zeta)$.

(iii) We define a by (3.4.25) with b replaced by b''. By (3.4.10), (3.4.11), we have

$$\begin{aligned} &F_m^{(\omega)}(n_0,\dots,n_{j+1})\hat{a}_\chi(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j; n_0 - n_{j+1}, n_{j+1}) \\ &= \mathbf{1}_{\{(n_0,\dots,n_{j+1})\notin Z(\omega)\}}\hat{b}_\chi(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j; n_0 - n_{j+1}, n_{j+1}) \end{aligned}$$

so that in (3.4.25) with b replaced by b'' we may insert in the integral the cut-off $\mathbf{1}_{\{(h+n_{j+1},n_1,\dots,n_{j+1})\notin Z(\omega)\}}$.

The rest of the proof is similar to the case (ii) above, using estimate (3.4.16) instead of (3.4.15). This concludes the proof. □

We conclude this section by an analogous of the preceding proposition for remainder operators. Let $d \geq 0, \nu, \zeta \in \mathbb{R}_+$. When $M \in {}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu}_{(j)}(\zeta), \omega \in \{-1,1\}^{j+2}$ with j even and when $\#\{\ell; \omega_\ell = 1\} = \#\{\ell; \omega_\ell = -1\}$, we decompose $M = M' + M''$ with

$$M'(u_1,\dots,u_j) = \sum_{n_0,\dots,n_{j+1}}{}' \Pi_{n_0} M(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}$$

where $\sum'$ stands for the sum on those indices for which (3.4.11) holds true.

Proposition 3.4.5. — *Let $m \in]0,+\infty[$ be outside the exceptional subset $\mathcal{N}$ of proposition 3.4.1.*

(i) When j is odd or j is even and $\#\{\ell; \omega_\ell = 1\} \neq \#\{\ell; \omega_\ell = -1\}$, there is for any M in ${}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu}_{(j)}(\zeta)$ an element $\widetilde{M} \in {}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu+N_1}_{(j)}(\zeta)$ such that $L_\omega(\widetilde{M}) = M$.

(ii) When j is even and $\#\{\ell; \omega_\ell = 1\} = \#\{\ell; \omega_\ell = -1\}$, there is for any M in ${}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu}_{(j)}(\zeta)$ an element $\widetilde{M} \in {}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu+N_1}_{(j)}(\zeta)$ such that $L_\omega(\widetilde{M}) = M''$.

Proof. — (i) The equation to be solved may be written

$$\Pi_{n_0} L_\omega(\widetilde{M})(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}} = \Pi_{n_0} M(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}$$

or equivalently

(3.4.28)
$$F_m^{(\omega)}(n_0,\dots,n_{j+1})\Pi_{n_0}\widetilde{M}(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}} = \Pi_{n_0} M(\Pi_{n_1}u_1,\dots,\Pi_{n_j}u_j)\Pi_{n_{j+1}}.$$

If ℓ is such that $|n_\ell| = \max(|n_0|,\dots,|n_{j+1}|)$, we have by (3.4.3)

$$|F_m^{(\omega)}(n_0,\dots,n_{j+1})| \geq c\mu(n_0,\dots,n_{j+1})^{-N_1} \geq c(1+|n_\ell|)^{-N_1}.$$

If we use estimate (2.1.40) for the right hand side of (3.4.28), we deduce from this that $\widetilde{M}$ satisfies the estimates of an element of ${}^{\mathbb{C}}\tilde{\Lambda}^{d,\nu+N_1}_{(j)}(\zeta)$.

(ii) The proof is similar, using that on the support of

$$\Pi_{n_0} M''(\Pi_{n_1} u_1, \dots, \Pi_{n_j} u_j)\Pi_{n_{j+1}},$$

estimate (3.4.3) holds true. □

3.5. Structure of the Hamiltonian

In this section, we shall express the Hamiltonian given by (1.2.8) using the classes of operators introduced in section 2.1.

Proposition 3.5.1. — *Let G be the Hamiltonian (1.2.8). One may find $\nu > 0$ and:*

- *A symbol $e(u;\cdot)$ in $\widetilde{S}^{1,\nu}_{(1),0}(0)$ satisfying $\overline{e(u;\cdot)}^{\vee} = e(u;\cdot)$,*
- *An element $M \in \widetilde{\mathcal{L}}^{1,\nu}_{(1)}(0)$,*

such that if we denote $E(u;x,n) = \begin{bmatrix} 0 & 0 \\ 0 & e(u;x,n) \end{bmatrix}$, we may write

$$G(u) = \frac{1}{2}\langle \Lambda_m u, u\rangle + \frac{1}{2}\langle \mathrm{Op}_\chi[E(u;\cdot)]u, u\rangle + \frac{1}{2}\langle M(u)u, u\rangle. \tag{3.5.1}$$

Before starting the proof, we study some multi-linear expressions. Consider a collection of $j+2 \geq 3$ constant coefficient operators

$$Q_\ell = \Lambda_m^{-1/2} \text{ or } Q_\ell = \Lambda_m^{-1/2}\partial_x, 0 \leq \ell \leq j+1 \tag{3.5.2}$$

of order $-1/2$ or $1/2$. Let $a \in C^\infty(\mathbb{S}^1;\mathbb{R})$. For any function u_ℓ in $C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ denote $u_\ell = \begin{bmatrix} u^1_\ell \\ u^2_\ell \end{bmatrix}$ and set $v_\ell = u^2_\ell \in C^\infty(\mathbb{S}^1;\mathbb{R})$. Consider

$$\int_{\mathbb{S}^1} a(x)(Q_0 v_0)\cdots(Q_{j+1}v_{j+1})dx. \tag{3.5.3}$$

Lemma 3.5.2. — *Let $\chi \in C_0^\infty(]-1,1[)$, χ even, $\chi \equiv 1$ close to zero,* $\operatorname{Supp}\chi$ *small enough. One may find $\nu > 0$ and for any i, i' with $0 \leq i < i' \leq j+1$ symbols*

$$a^i_{i'}(u;x,n) \text{ in } \widetilde{\Sigma}^{\frac{1}{2},\nu}_{(j),0}(0) \tag{3.5.4}$$

and remainder operators

$$M^i_{i'}(u) \in \tilde{\Lambda}^{1,\nu}_{(j)}(0) \tag{3.5.5}$$

such that (3.5.3) may be written

$$\begin{aligned} &\sum_{0\leq i<i'\leq j+1} \int (Q_i v_i)(x)\mathrm{Op}_\chi[a^i_{i'}(u_0,\dots,\widehat{u_i},\dots,\widehat{u_{i'}},\dots,u_{j+1};\cdot)v_{i'}dx \\ &\quad + \sum_{0\leq i<i'\leq j+1}\int u_i(x)[M^i_{i'}(u_0,\dots,\widehat{u_i},\dots,\widehat{u_{i'}},\dots,u_{j+1})u_{i'}]dx \end{aligned} \tag{3.5.6}$$

Proof. — We decompose $v_\ell = \sum \Pi_{n_\ell} v_\ell$ and write $Q_\ell \Pi_{n_\ell} v_\ell = b_\ell(n_\ell)\Pi_{n_\ell} v_\ell$ with

$$b_\ell(n_\ell) = (m^2 + n_\ell^2)^{-1/2} \text{ or } b_\ell(n_\ell) = \frac{in_\ell}{\sqrt{m^2+n_\ell^2}}. \tag{3.5.7}$$

We may write (3.5.3) as

$$\frac{1}{(2\pi)^{j+2}} \sum_{n_0} \cdots \sum_{n_{j+1}} \hat{a}(-n_0 - \cdots - n_{j+1}) \prod_{\ell=0}^{j+1} b_\ell(n_\ell)\hat{v}_\ell(n_\ell). \tag{3.5.8}$$

Let $\chi_1 \in C_0^\infty(\mathbb{R})$, χ_1 even, $\chi_1 \equiv 1$ close to zero with $\operatorname{Supp}\chi_1$ much smaller than $\operatorname{Supp}\chi$. Define for $0 \leq i < i' \leq j+1$

$$\Phi^i_{i'}\big((n_\ell)_{\ell\neq i}\big) = \chi_1\big(\max_{\ell\neq i,i'}(|n_\ell|)/\langle n_{i'}\rangle\big). \tag{3.5.9}$$

Decompose (3.5.8) as

$$\sum_{0\leq i<i'\leq j+1} I^i_{i'} + I'' \tag{3.5.10}$$

with
(3.5.11)

$$I^i_{i'} = \frac{1}{(2\pi)^{j+2}} \sum_{n_0} \cdots \sum_{n_{j+1}} \hat{a}(-n_0 - \cdots - n_{j+1}) \prod_{\ell=0}^{j+1} b_\ell(n_\ell)\hat{v}_\ell(n_\ell)\Phi^i_{i'}$$

$$I'' = \frac{1}{(2\pi)^{j+2}} \sum_{n_0} \cdots \sum_{n_{j+1}} \hat{a}(-n_0 - \cdots - n_{j+1}) \prod_{\ell=0}^{j+1} b_\ell(n_\ell)\hat{v}_\ell(n_\ell)\big(1 - \sum_{i,i';i<i'} \Phi^i_{i'}\big)$$

• **Contribution of I''**

We write I'' as $\int v_0(x) M(v_1,\dots,v_j)v_{j+1}dx$ with

$$\begin{aligned} M(v_1,\dots,v_j)v_{j+1} &= \frac{1}{(2\pi)^{j+2}} \sum_{n_0} \cdots \sum_{n_{j+1}} e^{-in_0x}\hat{a}(-n_0 - \cdots - n_{j+1}) \\ &\quad \times\big(1 - \sum_{i,i';i<i'} \Phi^i_{i'}\big) b_0(n_0) \prod_{\ell=1}^{j+1} b_\ell(n_\ell)\hat{v}_\ell(n_\ell) \end{aligned} \tag{3.5.12}$$

so that

$$\begin{aligned} &\|\Pi_{n_0} M(\Pi_{n_1}v_1,\dots,\Pi_{n_j}v_j)\Pi_{n_{j+1}}\|_{\mathcal{L}(L^2)} \\ &\leq |\hat{a}(n_0 - \cdots - n_{j+1})| \Big|\big(1 - \sum_{i,i';i<i'} \Phi^i_{i'}\big)\Big| \prod_{\ell=0}^{j+1} |b_\ell(n_\ell)| \prod_{\ell=1}^{j} \|\Pi_{n_\ell}v_\ell\|_{L^2}. \end{aligned} \tag{3.5.13}$$

We may bound the right hand side by the product of $C\prod_{\ell=0}^{j+1}\langle n_\ell\rangle^\sigma \prod_{\ell=1}^{j}\|\Pi_{n_\ell}v_\ell\|_{L^2}$ times

$$|\hat{a}(n_0 - \cdots - n_{j+1})| \Big|\big(1 - \sum_{i,i';i<i'} \Phi^i_{i'}\big)\Big| \prod_{\ell=0}^{j+1} \langle n_\ell\rangle^{-\sigma+\frac{1}{2}} \tag{3.5.14}$$

as each b_ℓ is a symbol of order at most 1/2. To prove that M defined by (3.5.12) may be written as an element of $\tilde{\Lambda}^{1,\nu}_{(j)}(0)$ for some ν, we just need to bound (3.5.14) by $C\langle n_\ell\rangle^{-3\sigma+\nu+1}$ for any $\ell = 0,\dots,j+2$. If one among $|n_0|,\dots,|n_{j+1}|$ is much larger than any other one, the rapid decay of $\hat{a}$ brings the wanted estimate. If not, and if $i_0 < i'_0$ are those two indices for which $|n_{i_0}|$ and $|n_{i'_0}|$ are the largest two among $|n_0|,\dots,|n_{j+1}|$, we may assume that $C^{-1}|n_{i_0}| \leq |n_{i'_0}| \leq C|n_{i_0}|$ for some constant $C > 0$. If there is another index $\ell_0 \neq i_0$, $\ell_0 \neq i'_0$ and a positive constant $c > 0$ such that $|n_{\ell_0}| \geq c|n_{i_0}|$, (3.5.14) has again the wanted estimate as $\langle n_{i_0}\rangle^{-\sigma}\langle n_{i'_0}\rangle^{-\sigma}\langle n_{\ell_0}\rangle^{-\sigma} \leq C\langle n_{i_0}\rangle^{-3\sigma}$. On the other hand, if for any $\ell \neq i_0, i'_0$, $|n_\ell|$ is much smaller than $|n_{i_0}| \sim |n_{i'_0}|$ then $\Phi^{i_0}_{i'_0}(n_0,\dots,\widehat{n_{i_0}},\dots,n_{j+1}) = 1$ and $\Phi^{i}_{i'}(n_0,\dots,\widehat{n_i},\dots,n_{j+1}) = 0$ for any $(i,i') \neq (i_0,i'_0)$, so that the cut-off in (3.5.14) vanishes. This shows that I'' contributes to the last sum in (3.5.6).

• **Contribution of $I^i_{i'}$**

We take, to simplify notations, $i = 0, i' = j+1$, set $n' = (n_1,\dots,n_{j+1})$ and write $\Phi(n')$ instead of $\Phi^0_{j+1}(n')$. We decompose

$$I^0_{j+1} = I(1) + I(2)$$

where

(3.5.15)
$$I(1) = \frac{1}{(2\pi)^{j+2}}\sum_{n_0}\cdots\sum_{n_{j+1}}\hat{a}(-n_0-\cdots-n_{j+1})\Phi(n')\chi\Big(\frac{n_0+n_{j+1}}{\langle n_{j+1}\rangle}\Big)\prod_{\ell=0}^{j+1} b_\ell(n_\ell)\hat{v}_\ell(n_\ell).$$

We may write $I(2) = \int v_0(x)M(v_1,\dots,v_j)\cdot v_{j+1}dx$ with

(3.5.16)
$$\begin{aligned} M(v_1,\dots,v_j)\cdot v_{j+1} = &\sum_{n_0}\cdots\sum_{n_{j+1}}\hat{a}(-n_0-\cdots-n_{j+1})e^{-in_0x}\\ &\times\Phi(n')\Big(1-\chi\Big(\frac{n_0+n_{j+1}}{\langle n_{j+1}\rangle}\Big)\Big)b_0(n_0)\prod_{\ell=1}^{j+1} b_\ell(n_\ell)\hat{v}_\ell(n_\ell).\end{aligned}$$

We thus get for M a bound of type (3.5.13) except that $(1-\sum\Phi^i_{i'})$ has to be replaced by $\Phi(n')\Big(1-\chi\Big(\frac{n_0+n_{j+1}}{\langle n_{j+1}\rangle}\Big)\Big)$. To show that M may be written as an element of $\tilde{\Lambda}^{1,\nu}_{(j)}(0)$, we just need to bound

(3.5.17)
$$|\hat{a}(n_0-\cdots-n_{j+1})|\Phi(n')\Big(1-\chi\Big(\frac{n_0-n_{j+1}}{\langle n_{j+1}\rangle}\Big)\Big)\prod_{\ell=0}^{j+1}\langle n_\ell\rangle^{-\sigma+\frac{1}{2}}$$

by $C\langle n_\ell\rangle^{-3\sigma+\nu+1}$ for any ℓ and some ν. By definition of Φ, on its support $|n_\ell| < c_1\langle n_{j+1}\rangle, \ell = 1,\dots,j$ for some small $c_1 > 0$ depending on $\operatorname{Supp}\chi_1$. If $|n_0| \gg |n_{j+1}|$ or $|n_{j+1}| \gg |n_0|$, the $|\hat{a}|$ factor in (3.5.17) gives the wanted estimate. If on the contrary $C^{-1}|n_0| \leq |n_{j+1}| \leq C|n_0|$ for some constant $C > 0$, and if we use that because of the $(1-\chi)$ cut-off, we may assume that $|n_0 - n_{j+1}| \geq c\langle n_{j+1}\rangle$ for some small $c > 0$ much larger than c_1, we get again from the $|\hat{a}|$ factor a bound in $\langle n_{j+1}\rangle^{-N} \sim$

$\langle \max(|n_0|, \dots, |n_{j+1}|)\rangle^{-N}$ for any N. This implies the wanted upper bound, and shows that I_2 contributes to the second sum in (3.5.6).

We are left with studying quantity (3.5.15). Let us define

$$\begin{aligned}a^0_{j+1}(v_1, \dots, v_j; x, n) = \frac{1}{(2\pi)^j} \sum_{n_1} \cdots \sum_{n_j} e^{ix(n_1+\cdots+n_j)} a(x)\\ \times \Phi(n_1, \dots, n_j, n) \prod_{\ell=1}^{j} b_\ell(n_\ell)\hat{v}_\ell(n_\ell) b_{j+1}(n).\end{aligned} \tag{3.5.18}$$

Then a^0_{j+1} satisfies (2.1.24), (2.1.25), (2.1.26) for $N_0 = 0, \zeta = 0$, some ν and $d = \frac{1}{2}$ so that $a^0_{j+1} \in \widetilde{\Sigma}^{\frac{1}{2},\nu}_{(j),0}(0)$. Moreover

$$\hat{a}^0_{j+1}(v_1, \dots, v_j; n_0, n) = \frac{1}{(2\pi)^j} \sum_{n_1} \cdots \sum_{n_j} \hat{a}(n_0 - n_1 - \cdots - n_j)\Phi(n_1, \dots, n_j, n) \times \prod_{\ell=1}^{j} b_\ell(n_\ell)\hat{v}_\ell(n_\ell) b_{j+1}(n)$$

so that if $w_0 = b_0(D)v_0$

$$\begin{aligned}&\langle w_0, \mathrm{Op}_\chi[a^0_{j+1}(\Pi_{n_1}v_1, \dots, \Pi_{n_j}v_j; \cdot)]v_{j+1}\rangle\\ &\quad = \frac{1}{(2\pi)^2} \sum_{n_0} \sum_{n_{j+1}} \hat{w}(n_0)\chi\Big(\frac{n_0 + n_{j+1}}{\langle n_{j+1}\rangle}\Big)\\ &\qquad\qquad \times \hat{a}^0_{j+1}(\Pi_{n_1}v_1, \dots, \Pi_{n_j}v_j; -n_0 - n_{j+1}, n_{j+1})\hat{v}_{j+1}(n_{j+1})\\ &\qquad\qquad = I(1).\end{aligned}$$

This shows that $I(1)$ may be written as a contribution to the first sum in (3.5.6) and concludes the proof of the lemma. □

Proof of proposition 3.5.1. — According to (1.2.8), $G(u)$ is the sum of $\frac{1}{2}\langle \Lambda_m u, u\rangle$ and of quantities of form (3.5.3) with $v_\ell = u^2$ the second component of u. By lemma 3.5.2, these quantities may be written as a contribution to the last term in (3.5.1) and to expressions of form

$$\int u^2 Q[\mathrm{Op}_\chi[\tilde{e}(u; \cdot)]]u^2 dx \tag{3.5.19}$$

where Q is a constant coefficients operator of order 1/2, and where $\tilde{e} \in \widetilde{S}^{\frac{1}{2},\nu}_{(j),0}(0)$ for some ν. By theorem 2.3.1, (3.5.19) may be written as contributions to the last two terms of (3.5.1), replacing eventually ν by some larger value. □

CHAPTER 4

SYMPLECTIC REDUCTIONS

The goal of this chapter is to construct an almost symplectic change of variables in a neighborhood of zero in $H^s(\mathbb{S}^1;\mathbb{R}^2)$ such that a Hamiltonian of form $\langle \mathrm{Op}_\chi[E(u;\cdot)]u,u\rangle$, where E is a 2×2 matrix of symbols of order one, be transformed, up to remainders, into $\langle \mathrm{Op}_\chi[E'(u;\cdot)]u,u\rangle$ where the matrix E' is a linear combination of I, J with coefficients symbols of order one.

4.1. Symplectic diagonalization of principal symbol

Let $B>0, \nu>0, \sigma\in\mathbb{R}, \sigma\geq \nu+2$ be given. Let $D.$ be a $(\sigma+\nu+1,1)$-conveniently increasing sequence. Let κ be a positive integer. We set

$$\lambda_0(u;x,n)=\lambda_0(n)\stackrel{\text{def}}{=}\sqrt{m^2+n^2},\ \mu_0(u;x,n)\equiv 0$$

and assume given for $1\leq k\leq \kappa-1$ elements λ_k,μ_k of $S^{1,\nu}_{(k),0}(\sigma,0,B,D.)$, such that

$$\overline{\lambda_k(u;x,n)}^{\vee}=\lambda_k(u;x,n),\ \overline{\mu_k(u;x,n)}^{\vee}=\mu_k(u;x,n) \tag{4.1.1}$$

and that

$$\lambda_k(u;x,n)-\lambda_k^{\vee}(u;x,n),\ \mu_k(u;x,n)+\mu_k^{\vee}(u;x,n) \tag{4.1.2}$$

belong to $S^{0,\nu+1}_{(k),0}(\sigma,0,B,D.)$. Let Ω be an element of $S^{1,\nu}_{(\kappa),0}(\sigma,0,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$ satisfying

(4.1.3)

$$\overline{\Omega(u;x,n)}^{\vee}=\Omega(u;x,n),\ \text{and}\ {}^t\Omega^{\vee}(u;x,n)-\Omega(u;x,n)\in S^{0,\nu+1}_{(\kappa),0}(\sigma,0,B,D.)\otimes\mathcal{M}_2(\mathbb{R}).$$

Since for any matrix valued symbol A, $\overline{\mathrm{Op}_\chi(A)u}=\mathrm{Op}_\chi(\overline{A}^{\vee})\bar{u}$, condition (4.1.1) and the first condition (4.1.3) imply that $\mathrm{Op}_\chi(\lambda_kI+\mu_kJ)$ and $\mathrm{Op}_\chi(\Omega)$ send real valued functions to real valued functions. Condition (4.1.2) and the second condition (4.1.3) imply in view of proposition 2.3.3 that these operators are self-adjoint at leading order. According to proposition 2.1.13 (i), if $s_0>\nu+\frac{5}{2}$ is fixed, there is $r>0$ such that if u belongs to the ball $B_{s_0}(r)$ of center 0 and radius r in $H^{s_0}(\mathbb{S}^1;\mathbb{R}^2)$, then $\mathrm{Op}_\chi[\lambda_k(u;\cdot)I+\mu_k(u;\cdot)J]u$ and $\mathrm{Op}_\chi[\Omega(u;\cdot)]u$ are well defined and belong to $H^{s_0-1}(\mathbb{S}^1;\mathbb{R}^2)$. This allows us to consider for u in such a ball

$$G'(u)=\frac{1}{2}\sum_{k=0}^{\kappa-1}\langle \mathrm{Op}_\chi[\lambda_k(u;\cdot)I+\mu_k(u;\cdot)J]u,u\rangle+\frac{1}{2}\langle \mathrm{Op}_\chi[\Omega(u;\cdot)]u,u\rangle. \tag{4.1.4}$$

In this section, we want to "diagonalize" the Ω contribution, i.e. replace Ω by a matrix which is a linear combination of I and J, up to lower order terms. Moreover, we want to do that in an approximately symplectic way.

Proposition 4.1.1. — *There are a constant $B'>B$ and a symbol q belonging to $S^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$ satisfying $\bar{q}^\vee=q$ such that if we set*

$$a'(u;x,n)=\sum_{k=0}^{\kappa-1}(\lambda_k(u;x,n)I+\mu_k(u;x,n)J)+\Omega(u;x,n) \tag{4.1.5}$$

and $p(u;x,n)=I+q(u;x,n)$ the following properties hold:

(i) ${}^tp^\vee(u;x,n)Jp(u;x,n)-J\in S^{-1,\nu}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$.

(ii) There are scalar symbols $\lambda_\kappa(u;x,n),\mu_\kappa(u;x,n)$ in $S^{1,\nu}_{(\kappa),0}(\sigma,0,B',D.)$ such that

$$\begin{gathered}\overline{\lambda_\kappa(u;x,n)}^\vee=\lambda_\kappa(u;x,n),\ \overline{\mu_\kappa(u;x,n)}^\vee=\mu_\kappa(u;x,n)\\ \lambda_\kappa-\lambda_\kappa^\vee,\ \mu_\kappa+\mu_\kappa^\vee \text{ belong to } S^{0,\nu+1}_{(\kappa),0}(\sigma,0,B',D.)\end{gathered} \tag{4.1.6}$$

and

$$\begin{aligned}{}^tp^\vee(u;x,n)a'(u;x,n)p(u;x,n)-\sum_{k=0}^{\kappa}&(\lambda_k(u;x,n)I+\mu_k(u;x,n)J)\\ &\in S^{0,\nu+1}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R}).\end{aligned} \tag{4.1.7}$$

Before starting the proof, let us comment on the meaning of the proposition. If we set

$$I'=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ J'=\begin{bmatrix}0&1\\1&0\end{bmatrix} \tag{4.1.8}$$

and decompose the matrix Ω in (4.1.5) as

$$\Omega(u;x,n)=b_1(u;x,n)I+b_2(u;x,n)J+b_1'(u;x,n)I'+b_2'(u;x,n)J' \tag{4.1.9}$$

where b_1,b_1',b_2,b_2' are scalar symbols of order 1, formula (4.1.7) asserts that using p, we may transform Ω in a matrix for which b_1',b_2' are of order zero. Moreover, (i) means that $\mathrm{Op}_\chi[p(u;\cdot)]$ will be a linear symplectic transformation (up to a remainder of order -1).

Let us define some notation. Since $\lambda_0(n) = \sqrt{m^2+n^2}$ is invertible, we may set

$$
\begin{aligned}
l_k(u;x,n) &= \lambda_0(n)^{-1}\lambda_k(u;x,n), k=1,\dots,\kappa-1\\
\mathfrak{m}_k(u;x,n) &= \lambda_0(n)^{-1}\mu_k(u;x,n), k=1,\dots,\kappa-1\\
l_\kappa(u;x,n) &= \lambda_0(n)^{-1}b_1(u;x,n)\\
l'(u;x,n) &= \lambda_0(n)^{-1}b_1'(u;x,n)\\
\mathfrak{m}_\kappa(u;x,n) &= \lambda_0(n)^{-1}b_2(u;x,n)\\
\mathfrak{m}'(u;x,n) &= \lambda_0(n)^{-1}b_2'(u;x,n)\\
l(u;x,n) &= \sum_{k=1}^{\kappa} l_k(u;x,n)\\
\mathfrak{m}(u;x,n) &= \sum_{k=1}^{\kappa}\mathfrak{m}_k(u;x,n).
\end{aligned}
\tag{4.1.10}
$$

By construction, $l,\mathfrak{m}$ belong to $S^{0,\nu}_{(1),0}(\sigma,0,B,D.)$, $l',\mathfrak{m}'$ belong to $S^{0,\nu}_{(\kappa),0}(\sigma,0,B,D.)$. Moreover, $l=\bar{l}^\vee, \mathfrak{m}=\bar{\mathfrak{m}}^\vee, l'=\bar{l}'^\vee, \mathfrak{m}'=\bar{\mathfrak{m}}'^\vee$ and $l-l^\vee, \mathfrak{m}+\mathfrak{m}^\vee$, (resp. $l'-l'^\vee, \mathfrak{m}'-\mathfrak{m}'^\vee$) are in $S^{-1,\nu+1}_{(1),0}(\sigma,0,B,D.)$ (resp. $S^{-1,\nu+1}_{(\kappa),0}(\sigma,0,B,D.)$) by (4.1.1), (4.1.2), (4.1.3). According to (4.1.5), (4.1.9) and (4.1.10), we may write

$$a'(u;x,n) = \lambda_0(n)[(1+l(u;x,n))I+\mathfrak{m}(u;x,n)J+l'(u;x,n)I'+\mathfrak{m}'(u;x,n)J']. \tag{4.1.11}$$

Set

$$K = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix},\; K^{-1} = i^tJ^tKJ = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix} \tag{4.1.12}$$

and define

$$S(u;x,n) = KJa'K^{-1} = i\lambda_0\begin{bmatrix}1+l+i\mathfrak{m} & l'+i\mathfrak{m}'\\ -(l'-i\mathfrak{m}') & -(1+l)+i\mathfrak{m}\end{bmatrix}. \tag{4.1.13}$$

The proof of proposition 4.1.1 will rely on the diagonalization of $S(u;x,n)$.

Lemma 4.1.2. — *There is a constant B', depending on B and on the quantities $\mathfrak{N}^{0,\nu}_{(1),0}(\sigma,0,B,D.;l)$, $\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B,D.;l')$, $\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B,D.;\mathfrak{m}')$ and there are symbols $\lambda_\kappa,\mu_\kappa \in S^{1,\nu}_{(\kappa),0}(\sigma,0,B',D.)$, satisfying conditions (4.1.6), and a matrix of symbols $\tilde{q}\in S^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$, satisfying*

$$
\begin{aligned}
\overline{K^{-1}\tilde{q}^\vee K} - K^{-1}\tilde{q}K &\in S^{-1,\nu+1}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})\\
{}^tJ^t(I+\tilde{q})^\vee J(I+\tilde{q}) - I &\in S^{-1,\nu+1}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})
\end{aligned}
\tag{4.1.14}
$$

such that

$${}^tJ^t(I+\tilde{q})^\vee JS(I+\tilde{q}) - i\begin{bmatrix}(\sum_0^\kappa\lambda_k)+i(\sum_1^\kappa\mu_k) & 0\\ 0 & -(\sum_0^\kappa\lambda_k)+i(\sum_1^\kappa\mu_k)\end{bmatrix} \tag{4.1.15}$$

belongs to $S^{0,\nu+1}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$.

Proof. — Define

$$\delta(u;x,n)=\sqrt{1-\frac{l'^2+\mathfrak{m}'^2}{(1+l)^2}}-1. \tag{4.1.16}$$

Since l belongs to $S^{0,\nu}_{(1),0}(\sigma,0,B,D.)$ and $l',\mathfrak{m}'$ belong to $S^{0,\nu}_{(\kappa),0}(\sigma,0,B,D.)$, we may consider them as elements of $S^{0,\nu}_{(1),0}(\sigma,0,B'',D.)$ and $S^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.)$ respectively for any $B''>B$. If B'' is large enough, we may make $\mathfrak{N}^{0,\nu}_{(1),0}(\sigma,0,B'',D.;l)$, $\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.;l')$, $\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.;\mathfrak{m}')$ arbitrarily small, so that assumptions of proposition 2.4.1 will be satisfied with B replaced by B''. This proposition implies that $\delta\in S^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.)$ with $B'=2B''$. Moreover, $\delta=\bar{\delta}^\vee$ and $\delta-\delta^\vee$ belongs to $S^{-1,\nu+1}_{(\kappa),0}(\sigma,0,B',D.)$. The eigenvalues of the matrix

$$\begin{bmatrix} 1+l+i\mathfrak{m} & l'+i\mathfrak{m}' \\ -(l'-i\mathfrak{m}') & -(1+l)+i\mathfrak{m}\end{bmatrix} \tag{4.1.17}$$

are $\pm(1+l)(1+\delta)+i\mathfrak{m}$. Define $\tilde{q}$ by

$$(I+\tilde{q}(u;x,n))=\Big(1-\frac{l'^2+\mathfrak{m}'^2}{(1+l)^2(2+\delta)^2}\Big)^{-1/2}\begin{bmatrix} 1 & -\frac{l'+i\mathfrak{m}'}{(1+l)(2+\delta)} \\ -\frac{l'-i\mathfrak{m}'}{(1+l)(2+\delta)} & 1\end{bmatrix}. \tag{4.1.18}$$

Applying again proposition 2.4.1, we see that $\tilde{q}$ belongs to $S^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$, eventually with a new (larger) value of B'. The inverse matrix is

$$(I+\tilde{q}(u;x,n))^{-1}={}^tJ^t(\mathrm{Id}+\tilde{q}(u;x,n))J. \tag{4.1.19}$$

Moreover since $l'-l'^\vee$, $\mathfrak{m}'-\mathfrak{m}'^\vee$, $l-l^\vee$ are of order -1, $\tilde{q}-\tilde{q}^\vee\in S^{-1,\nu+1}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$. Since the eigenvectors of (4.1.17) associated to the eigenvalues $(1+l)(1+\delta)+i\mathfrak{m}$ and $-(1+l)(1+\delta)+i\mathfrak{m}$ are collinear respectively to

$$\begin{bmatrix} 1 \\ -\frac{l'-i\mathfrak{m}'}{(1+l)(2+\delta)}\end{bmatrix} \text{ and } \begin{bmatrix} -\frac{l'+i\mathfrak{m}'}{(1+l)(2+\delta)} \\ 1\end{bmatrix},$$

$(I+\tilde{q})$ diagonalizes (4.1.13), so taking (4.1.19) into account

$$\begin{aligned} &{}^tJ^t(I+\tilde{q}(u;x,n))JS(u:x,n)(I+\tilde{q}(u;x,n)) \\ &\qquad = i\lambda_0\begin{bmatrix}(1+l)(1+\delta)+i\mathfrak{m} & 0 \\ 0 & -(1+l)(1+\delta)+i\mathfrak{m}\end{bmatrix}. \end{aligned} \tag{4.1.20}$$

By (4.1.10)

$$\pm\lambda_0(1+l)(1+\delta)+i\lambda_0\mathfrak{m}=\pm\Big(\sum_{k=0}^{\kappa-1}\lambda_k+b_1\Big)(1+\delta)+i\Big(\sum_{k=1}^{\kappa-1}\mu_k+b_2\Big)$$

may be written since $\delta \in S^{0,\nu}_{(\kappa),0}(\sigma, 0, B', D.)$, $b_1, b_2 \in S^{1,\nu}_{(\kappa),0}(\sigma, 0, B', D.)$, and using (i) of theorem 2.3.1 as

$$\pm(\sum_{k=0}^{\kappa} \lambda_k) + i(\sum_{k=1}^{\kappa} \mu_k)$$

with $\lambda_\kappa, \mu_\kappa \in S^{1,\nu}_{(\kappa),0}(\sigma, 0, B', D.)$. Since $\delta = \bar{\delta}^\vee$, $b_1 = \bar{b}_1^\vee$, $b_2 = \bar{b}_2^\vee$, $\delta - \delta^\vee$ (resp. $b_1 - b_1^\vee, b_2 + b_2^\vee$) is of order -1 (resp. of order 0), conditions (4.1.6) are satisfied by $\lambda_\kappa, \mu_\kappa$. Since $\tilde{q} - \tilde{q}^\vee$ is of order -1, (4.1.19) and (4.1.20) imply the second relation (4.1.14) and (4.1.15). By a direct computation, $\overline{K^{-1}\tilde{q}K} = K^{-1}\tilde{q}K$. Since $\tilde{q} - \tilde{q}^\vee$ is of order -1, this implies the first relation (4.1.14). The proof is complete. □

Proof of Proposition 4.1.1. — We set

$$q_1(u; x, n) = K^{-1}\tilde{q}(u; x, n)K,\ q(u; x, n) = \frac{1}{2}[q_1(u; x, n) + \bar{q}_1^\vee(u; x, n)]. \tag{4.1.21}$$

By the first relation (4.1.14), $q - q_1$ belongs to $S^{-1,\nu+1}_{(\kappa),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R})$ and by construction q is an element of $S^{0,\nu}_{(\kappa),0}(\sigma, 0, B, D.) \otimes \mathcal{M}_2(\mathbb{R})$ satisfying $q = \bar{q}^\vee$. We set $p = I + q$ and show that (i) of proposition 4.1.1 holds. By (4.1.21) and the second relation (4.1.12)

$$p(u; x, n) - i^t J^t KJ(1 + \tilde{q}(u; x, n))K \in S^{-1,\nu+1}_{(\kappa),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R}). \tag{4.1.22}$$

Together with the second relation (4.1.14) and (4.1.12), this implies that

$${}^t p^\vee Jp - J \in S^{-1,\nu+1}_{(\kappa),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R})$$

i.e. (i) of proposition 4.1.1 is satisfied. If we use (4.1.22), the definition (4.1.13) of S in terms of a' and the second equality (4.1.12), we get that

$${}^t p^\vee a' p + i^t KJ[{}^t J^t (I + \tilde{q})^\vee JS(I + \tilde{q})]K$$

belongs to $S^{0,\nu+1}_{(\kappa),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R})$. Using (4.1.15) and the definition of K, we obtain (4.1.7). This concludes the proof of the proposition. □

4.2. Symplectic change of coordinates

Our goal is to define from the symbol $p = I + q$ constructed in proposition 4.1.1 an almost symplectic change of variables near the origin in $H^s(\mathbb{S}^1; \mathbb{R}^2)$ for s large enough.

Proposition 4.2.1. — *Let $\sigma > 0, \nu > 0, B > 0$ be given with $\sigma - \nu$ large enough and let $D.$ be the $(\sigma + \nu + 1, 1)$-conveniently increasing sequence fixed at the beginning of section 4.1. Let $B' > B$ be the constant given in the statement of proposition 4.1.1. There are $B'' > B'$, $\rho_0 > 0, s_0 > 0$ and an element $r \in S^{0,\nu}_{(\kappa),0}(\sigma, 0, B'', D.)$ such that, if we set for $v \in B_{s_0}(\rho_0)$*

$$\psi(v) = (\mathrm{Id} + \mathrm{Op}_\chi[r(v; \cdot)])v, \tag{4.2.1}$$

then ψ is for any $s \geq s_0$ a C^1 diffeomorphism from a neighborhood U_s of 0 in $H^s(\mathbb{S}^1;\mathbb{R}^2)$ to a neighborhood W_s of 0 in the same space, satisfying the equality

$$q(\psi(v);x,n) = r(v;x,n). \tag{4.2.2}$$

Moreover, for any $v \in U_s$, $\psi'(v)$ extends as an element of $\mathcal{L}(H^{-s},H^{-s})$. In addition, ψ is almost symplectic in the following sense: for any $\sigma+1 > s \geq s_0+1$, there is $C>0$ such that for any $v\in U_s$, ${}^t\partial\psi(v)J\psi(v) - J$ extends as a bounded linear map from $H^{s-1}(\mathbb{S}^1;\mathbb{R}^2)$ to $H^s(\mathbb{S}^1;\mathbb{R}^2)$ with the bounds

$$\|{}^t\partial\psi(v)J\partial\psi(v) - J\|_{\mathcal{L}(H^{s-1},H^s)} \leq C\|v\|^{\kappa}_{H^s}. \tag{4.2.3}$$

Remark. — The gain of one derivative in (4.2.3) above will be essential when applying this proposition to our quasi-linear problem (which loses one derivative).

Let us first construct r through a fixed point argument.

Lemma 4.2.2. — *Let $q \in S^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$ be the symbol constructed in proposition 4.1.1. There is a constant $B''>B'$ and a symbol $r \in S^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.)\otimes\mathcal{M}_2(\mathbb{R})$ such that*

$$q(v+\mathrm{Op}_\chi[r(v;\cdot)]v;x,n) = r(v;x,n). \tag{4.2.4}$$

Proof. — Recall that elements of $S^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.)$ are formal series of homogeneous terms, so that (4.2.4) is an equality between formal series. Decompose $q(v;x,n) = \sum_{i\geq\kappa} q_i(\underbrace{v,\dots,v}_{i},x,n)$ with $q_i \in \Sigma^{0,\nu}_{(\kappa,i),0}(\sigma,0,B',D.)\otimes\mathcal{M}_2(\mathbb{R})$ and look for r as

$$r(v;x,n) = \sum_{j\geq\kappa} r_j(\underbrace{v,\dots,v}_{j},x,n)$$

with $r_j \in \Sigma^{0,\nu}_{(\kappa,j),0}(\sigma,0,B'',D.)\otimes\mathcal{M}_2(\mathbb{R})$. We shall define

$$\begin{aligned} q_{<i}(v;x,n) &= \sum_{\kappa\leq i'<i} q_{i'}(v,\dots,v;x,n)\\ r_{<j}(v;x,n) &= \sum_{\kappa\leq j'<j} r_{j'}(v,\dots,v;x,n). \end{aligned}$$

We construct the r_j's by induction. We first set $r_\kappa = q_\kappa$. By definition of $\mathfrak{N}^{0,\nu}_{(\kappa),0}(\cdot)$ we have, since $\kappa\geq 1$

$$\begin{aligned} \mathfrak{N}^{0,\nu}_{(\kappa,\kappa),0}(\sigma,0,B'',D.;r_\kappa) &\leq \mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.;q)\\ &\leq \frac{B'}{B''}\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.;q) \end{aligned} \tag{4.2.5}$$

If B'' is large enough, we may assume that the right hand side of (4.2.5) is smaller than 1. Assume next that $r_\kappa,\dots,r_{j-1}$ have been constructed such that

$$\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.;r_{<j}) \leq 1. \tag{4.2.6}$$

Remark that the term homogeneous of degree j in the left hand side of (4.2.4) depends only on $r_\kappa, \dots, r_{j-1}$, so that, equating terms of homogeneous degree j in (4.2.4) is equivalent to taking the term homogeneous of degree j in

$$q(v + \mathrm{Op}_\chi[r_{<j}(v;\cdot)]v; x, n).$$

We define r_j to be this term of degree j. By proposition 2.2.1, we know that $r_j(u_1,\dots,u_j;\cdot)$ is in $\Sigma^{0,\nu}_{(\kappa,j),0}(\sigma,0,B'',D.)\otimes\mathcal{M}_2(\mathbb{R})$, or equivalently that $r_{<j+1}$ is in $S^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.)\otimes\mathcal{M}_2(\mathbb{R})$, and by (2.2.7)

$$\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.;r_{<j+1}) \le C\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.;q) \tag{4.2.7}$$

with a constant C depending only on $\mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B'',D.;r_{<j})$. The induction assumption (4.2.6) shows that C is independent of j, and using the last inequality in (4.2.5), and assuming that B'' is taken large enough in function of $C, B', \mathfrak{N}^{0,\nu}_{(\kappa),0}(\sigma,0,B',D.;q)$, we obtain that the left hand side of (4.2.7) is smaller than 1. We have performed the induction hypothesis (4.2.6) at step $j+1$. This concludes the proof. □

Proof of proposition 4.2.1. — We define $\psi(v)$ by (4.2.1). Note that this is meaningful if $v \in B_{s_0}(\rho_0)$ for some large enough s_0 and small enough ρ_0. Actually, if $s_0 > \nu + \frac{5}{2}$, (i) of proposition 2.1.13 shows that for $\|v\|_{H^{s_0}}$ small enough and $s \ge s_0$

$$\|\mathrm{Op}_\chi[r(v;\cdot)]v\|_{H^s} \le C_s\|v\|^\kappa_{H^{s_0}}\|v\|_{H^s}. \tag{4.2.8}$$

Together with the implicit function theorem, this shows moreover that ψ is a local diffeomorphism from a neighborhood of zero in $H^s(\mathbb{S}^1;\mathbb{R}^2)$ to a neighborhood of zero in $H^s(\mathbb{S}^1;\mathbb{R}^2)$, for any $s \ge s_0$. Equality (4.2.2) follows from (4.2.4) and the definition of ψ. Let us show that (4.2.3) holds when $s \ge s_0+1$. By (4.2.1) the differential of ψ acting on a tangent vector V is given by

$$\begin{aligned}\partial\psi(v)\cdot V &= (\mathrm{Id} + \mathrm{Op}_\chi[r(v;\cdot)])V + \mathrm{Op}_\chi[\partial_v r(v;\cdot)\cdot V]v\\ &= (\mathrm{Id} + \mathrm{Op}_\chi[q(\psi(v);\cdot)])V + R(v)\cdot V\end{aligned} \tag{4.2.9}$$

where we used (4.2.2) and defined

$$R(v)\cdot V = \mathrm{Op}_\chi[\partial_v r(v;\cdot)\cdot V]v. \tag{4.2.10}$$

From (i) of proposition 2.1.13, we have

$$\|R(v)\cdot V\|_{H^s} \le C\|v\|^{\kappa-1}_{H^{s_0}}\|V\|_{H^{s_0}}\|v\|_{H^s}. \tag{4.2.11}$$

From estimate (2.1.46), we deduce

$$\|R(v)\cdot V\|_{H^{-s_0}} \le C\|v\|^{\kappa-1}_{H^{s_0}}\|V\|_{H^{-s}}\|v\|_{H^s}. \tag{4.2.12}$$

This implies together with (i) of proposition 2.1.13 that $\psi'(v)$ extends as an element of $\mathcal{L}(H^{-s},H^{-s})$ if $s \ge s_0$ large enough. Moreover, by duality

$$\|{}^tR(v)\|_{\mathcal{L}(H^{s_0},H^s)} \le C\|v\|^{\kappa-1}_{H^{s_0}}\|v\|_{H^s}. \tag{4.2.13}$$

Let us compute

$$\begin{aligned}{}^t\partial\psi(v)J\psi(v) = {}^t(\mathrm{Id}+\mathrm{Op}_\chi[q(\psi(v);\cdot)])J(\mathrm{Id}+\mathrm{Op}_\chi[q(\psi(v);\cdot)])\\ + {}^tR(v)J(\mathrm{Id}+\mathrm{Op}_\chi[q(\psi(v);\cdot)])\\ + {}^t(\mathrm{Id}+\mathrm{Op}_\chi[q(\psi(v);\cdot)])JR(v)\\ + {}^tR(v)JR(v).\end{aligned} \tag{4.2.14}$$

Since $(\mathrm{Id}+\mathrm{Op}_\chi[q(\psi(v);\cdot)])$ is bounded on any Sobolev space, (4.2.11) and (4.2.13) imply that the last three terms in (4.2.14) are bounded operators from H^{s-1} to H^s (actually from H^{s_0} to H^s) if $s\geq s_0+1$, with operator norm smaller than $C\|v\|^\kappa_{H^s}$.

We apply to the first term in the right hand side of (4.2.14) (ii) of theorem 2.3.1, proposition 2.3.3 and (ii), (iii) of proposition 3.1.1. This allows us to write, since $p=I+q$

$$\begin{aligned}{}^t(\mathrm{Id}{+}\mathrm{Op}_\chi[q(u;\cdot)])J(\mathrm{Id}+\mathrm{Op}_\chi[q(u;\cdot)])\\ = \mathrm{Op}_\chi[{}^tp^\vee(u;\cdot)Jp(u;\cdot)]+\mathrm{Op}_\chi[e(u;\cdot)]+M(u)\end{aligned} \tag{4.2.15}$$

with $e\in S^{-1,\nu'}_{(\kappa),0}(\sigma,0,B'',\tilde{D}.)\otimes\mathcal{M}_2(\mathbb{R})$ for some $\nu'\geq\nu$, some new sequence $\tilde{D}.$ and $M\in\mathcal{L}^{0,\nu'}_{(\kappa)}(\sigma,0,B'')$. By (i) of proposition 4.1.1 and (i) and (iii) of proposition 2.1.13 (in which we take in (2.1.47) $\sigma'=s-\frac{3}{2}-\delta$), we obtain if $s_0+1\leq s<\sigma+1$ that (4.2.15) may be written $J+S(u)$ where $S(u)$ is a bounded operator from H^{s-1} to H^s, with operator norm bounded from above by $C\|u\|^\kappa_{H^s}$. Setting $u=\psi(v)$, we get the conclusion of the proposition. □

We end this section stating a corollary of proposition 4.1.1 and 4.2.1 that will be needed in the last chapter.

Corollary 4.2.3. — *Let $G'(u)$ be given by (4.1.4) and let ψ be the local diffeomorphism constructed in proposition 4.2.1. There are symbols*

$$\tilde{\lambda}_\kappa(v;x,n),\tilde{\mu}_\kappa(v;x,n)\ \text{in}\ S^{1,\nu}_{(\kappa),0}(\sigma,0,B'',D.) \tag{4.2.16}$$

for some $B''>B$, satisfying

$$\bar{\tilde{\lambda}}^\vee_\kappa=\tilde{\lambda}_\kappa,\bar{\tilde{\mu}}^\vee_\kappa=\tilde{\mu}_\kappa \tag{4.2.17}$$

and there are $s_0>0$, $\rho_0>0$ and a map $v\to L(v)$, defined on $B_{s_0}(\rho_0)$, C^1 on a neighborhood of zero in $H^s(\mathbb{S}^1;\mathbb{R}^2)$, with values in $\mathbb{R}$, with $\nabla L(u)\in H^s$ for any $s\in[s_0+1,\sigma+1[$, satisfying

$$\|\nabla L(u)\|_{H^s}\leq C\|u\|^{\kappa+1}_{H^s}, \tag{4.2.18}$$

such that for any $v \in B_{s_0}(\rho_0)$

$$G'(\psi(v)) = \frac{1}{2}\sum_{k=0}^{\kappa-1}\langle \mathrm{Op}_\chi[\lambda_k(v;\cdot)I + \mu_k(v;\cdot)J]v, v\rangle + \frac{1}{2}\langle \mathrm{Op}_\chi[\tilde{\lambda}_\kappa(v;\cdot)I + \tilde{\mu}_\kappa(v;\cdot)J]v, v\rangle + L(v). \tag{4.2.19}$$

Moreover, ψ *satisfies*

$$\|\partial\psi(v)J^t\partial\psi(v) - J\|_{\mathcal{L}(H^{s-1},H^s)} \leq C\|v\|^\kappa_{H^s} \tag{4.2.20}$$

for any $s \in [s_0+1, \sigma+1[$*, any* v *in an* H^s *neighborhood of zero.*

Remark. — The above corollary states that if we set $u = \psi(v)$ in (4.1.4), the matrix-valued symbol Ω may be replaced by a new symbol, which is a combination of I, J with coefficients scalar symbols of order 1. The remainder $L(v)$ has by (4.2.18) a gradient belonging to H^s when v is in H^s, while the gradient of the duality brackets in (4.2.19) is only in H^{s-1}. In that way, we can say that the change of variables ψ diagonalizes the principal part of the Hamiltonian, removing the components of Ω on I' and J' in a decomposition of type (4.1.9).

Proof. — By (4.2.1) and (4.2.2)

$$\psi(v) = \mathrm{Op}_\chi[p(\psi(v);\cdot)]v \tag{4.2.21}$$

with $p = I + q$. We plug (4.2.21) in (4.1.4), which gives using notation (4.1.5)

$$G'(\psi(v)) = \frac{1}{2}\langle {}^t\mathrm{Op}_\chi[p(\psi(v);\cdot)]\mathrm{Op}_\chi[a'(\psi(v);\cdot)]\mathrm{Op}_\chi[p(\psi(v);\cdot)]v, v\rangle. \tag{4.2.22}$$

By (4.1.7) and the theorems of symbolic calculus (theorem 2.3.1, proposition 2.3.3 and proposition 3.1.1) we may write

$${}^t\mathrm{Op}_\chi[p(u;\cdot)]\mathrm{Op}_\chi[a'(u;\cdot)]\mathrm{Op}_\chi[p(u;\cdot)] = \mathrm{Op}_\chi[{}^tp^\vee(u;\cdot)a'(u;\cdot)p(u;\cdot)] + \mathrm{Op}_\chi[e(u;\cdot)] + M(u) \tag{4.2.23}$$

where $e(u;\cdot) \in S^{0,\nu'}_{(\kappa),0}(\sigma, 0, B'', \tilde{D}.) \otimes \mathcal{M}_2(\mathbb{R})$ for some $\nu' > \nu$, $\sigma \geq \nu' + 2$, and some new sequence $\tilde{D}.$, and where $M \in \mathcal{L}^{1,\nu'}_{(\kappa)}(\sigma, 0, B'')$. Define $\tilde{L}(u, v) = \langle \mathrm{Op}_\chi[e(u;\cdot)]v, v\rangle + \langle M(u)v, v\rangle$. It follows from (2.1.44) and (2.1.48) that $\partial_v\tilde{L}(u, v)$ belongs to $\mathcal{L}(H^{-s}, \mathbb{R})$ if $u, v \in H^s$ and s is large enough. The same is true for $\partial_u\tilde{L}(u, v)$ by (2.1.46) and (2.1.48). Consequently, since we have seen in proposition 4.2.1 that $\psi'(v)$ is in $\mathcal{L}(H^{-s}, H^{-s})$, we see that $L(v) = \tilde{L}(\psi(v), v)$ satisfies (4.2.18). We deduce from that that the contribution of e, M in (4.2.23) to (4.2.22) give the last term in (4.2.19). By (4.1.7), the first term in the right hand side of (4.2.23) brings to (4.2.22) a contribution of form $L(v)$ (coming from the remainder in (4.1.7)) and the main term

$$\frac{1}{2}\sum_{k=0}^{\kappa}\langle \mathrm{Op}_\chi[\lambda_k(\psi(v);\cdot)I + \mu_k(\psi(v);\cdot)J]v, v\rangle.$$

Note that for any $k = 1, \dots, \kappa - 1$

$$\begin{aligned}\lambda_k(\psi(v); x, n) &= \lambda_k((I + \mathrm{Op}_\chi[r(v; \cdot)])v; x, n) \\ &= \lambda_k(v; x, n) + \tilde{\lambda}_k(v; x, n)\end{aligned}$$

with $\tilde{\lambda}_k \in S^{1,\nu}_{(\kappa),0}(\sigma, 0, B'', D.)$ by proposition 2.2.1. Since $\lambda_\kappa(\psi(v); \cdot)$ is also in such a class of symbols by the same proposition, and since similar properties hold true for μ_k, we obtain (4.2.19). Finally, property (4.2.20) follows from (4.2.3) and the fact that $\psi'(v)$ is invertible from H^s to H^s and from H^{s-1} to H^{s-1} for any $s \in [s_0 + 1, \sigma + 1[$ with s_0 large enough. □

CHAPTER 5

PROOF OF ALMOST GLOBAL EXISTENCE

The aim of this chapter is to combine the results obtained so far to prove theorem 1.1.1. We shall do that constructing a function Θ_s, defined on a neighborhood of zero in the phase space $H^s(\mathbb{S}^1;\mathbb{R}^2)$, equivalent to the square of the H^s Sobolev norm, and such that $\Theta_s(u(t,\cdot))$ will be uniformly controlled on a long time interval when u is a solution to (1.2.9). We shall construct Θ_s in several steps, using composition by (almost) symplectic transformations.

5.1. Composition with symplectic transformations

We discuss here several composition formulas. We consider a small neighborhood of zero in $H^s(\mathbb{S}^1;\mathbb{R}^2)$, namely $B_s(\rho)$ for some $\rho > 0$ small enough. Let us recall that if $F : B_s(\rho) \to \mathbb{R}$ is a C^1 function such that for any $u \in B_s(\rho)$, $\partial F(u) \in \mathcal{L}(H^s,\mathbb{R})$ extends as an element of $\mathcal{L}(H^{-s},\mathbb{R})$, we may consider the gradient $\nabla F(u)$ and the Hamiltonian vector field $X_F(u)$ as elements of $H^s(\mathbb{S}^1;\mathbb{R}^2)$. If we assume moreover that $u \to X_F(u)$ is C^1 on $B_s(\rho)$ with values in H^s, we may solve locally the differential equation

$$\begin{aligned}&\dot{\Phi}(\tau,u) = X_F(\Phi(\tau,u))\\&\Phi(0,u) = u.\end{aligned} \tag{5.1.1}$$

Let us remark that if F is C^2 on $B_s(\rho)$, then for any τ, $D\Phi(\tau,u)$ which is *a priori* an element of $\mathcal{L}(H^s,H^s)$, extends as an element of $\mathcal{L}(H^{-s},H^{-s})$. Actually $D\Phi$ solves the ordinary differential equation

$$\begin{aligned}&\dot{\widehat{D\Phi}}(\tau,u) = (DX_F)(\Phi(\tau,u))D\Phi(\tau,u)\\&D\Phi(0,u) = \mathrm{Id}\end{aligned}$$

so that we just need to show that $DX_F(u) = JD\nabla F(u)$ is a continuous function of u, with values in $\mathcal{L}(H^{-s},H^{-s})$. Note that the definition of the gradient, namely

$\int_{\mathbb{S}^1} \nabla F(u) \cdot V dx = DF(u) \cdot V$ for any $V \in C^\infty(\mathbb{S}^1;\mathbb{R})$, implies for any $W \in C^\infty(\mathbb{S}^1;\mathbb{R})$

$$\begin{aligned}\int_{\mathbb{S}^1} \big(D(\nabla F(u)) \cdot W\big) \cdot V dx &= D^2F(u)(W,V)\\ &= D^2F(u)(V,W)\\ &= \int_{\mathbb{S}^1} \big(D(\nabla F(u)) \cdot V\big) \cdot W dx.\end{aligned}$$

We want to see that the left hand side extends continuously to $W \in H^{-s}$ and $V \in H^s$. This follows from the fact that such an extension holds for the right hand side, as $D(\nabla F(u)) \in \mathcal{L}(H^s, H^s)$, since we assume that $u \to X_F(u)$ is C^1 on $B_s(\rho)$.

If moreover $F(0) = 0$, $\partial F(0) = 0$, for ρ small enough, the solution of (5.1.1) is defined up to time $\tau = 1$ and $\chi_F(u) = \Phi(1,u)$ is a canonical transformation from $B_s(\rho)$ to a neighborhood of zero in H^s, satisfying $\chi_F(0) = 0$. If Θ and G are two functions on a neighborhood of zero in $H^s(\mathbb{S}^1;\mathbb{R}^2)$, we get for $u \in B_s(\rho)$ for small enough ρ the usual equality

$$\{\Theta \circ \chi_F, G\}(u) = \{\Theta, G \circ \chi_F^{-1}\}(\chi_F(u)). \tag{5.1.2}$$

If we assume that G is a C^1 function on $B_s(\rho)$ such that, for any $k \in \mathbb{N}^*$ $\mathrm{Ad}^k F \cdot G = \{F, \mathrm{Ad}^{k-1} F \cdot G\}$ is also C^1 on $B_s(\rho)$, we have

$$\frac{d^k}{dt^k} G(\Phi(t,u)) = (-1)^k (\mathrm{Ad}^k F \cdot G)(\Phi(t,u))$$

for any $k \in \mathbb{N}$, so that

$$G \circ \chi_F^{-1}(u) = \sum_{k=0}^{N} \frac{\mathrm{Ad}^k F}{k!} \cdot G(u) + \frac{1}{N!} \int_0^1 (1-\tau)^N (\mathrm{Ad}^{N+1} F \cdot G)(\Phi(-\tau,u)) d\tau. \tag{5.1.3}$$

If we have moreover an estimate of type $|\mathrm{Ad}^k F \cdot G| \leq C k! A^k \|u\|_{H^s}^k$ for some constants $C > 0, A > 0$, then for ρ small enough, we shall get

$$G \circ \chi_F^{-1}(u) = \sum_{k=0}^{+\infty} \frac{\mathrm{Ad}^k F}{k!} \cdot G(u). \tag{5.1.4}$$

The above formula will apply when F is given by an expression $\langle \mathrm{Op}_\chi[a(u;\cdot)]u, u\rangle$, with a symbol of order zero. Nevertheless, we shall have to consider also expressions of that form involving symbols of order 1. In that case, $\nabla F(u)$ or $X_F(u)$ belong only to H^{s-1} when $u \in H^s$. Consequently, we cannot consider (5.1.1) as an ordinary differential equation. To avoid the resolution of (5.1.1) in that case, we shall use instead of (5.1.2) a formula of the same type, up to a *finite order* of homogeneity, and use special assumptions on Θ, G, F to be able to write convenient substitute to (5.1.3)

Remind that we defined in definition 3.3.3 the class $\mathcal{H}'^{d,\nu}_{(k),N_0}(\zeta)$ of functions on $H^{s_0}(\mathbb{S}^1;\mathbb{R}^2)$ for $s_0 > \nu + \frac{5}{2} + \max(\zeta, \frac{d}{3}), s_0 \geq \frac{d}{2}$. By proposition 3.3.4, if

$k_1, k_2 \in \mathbb{N}^*$, $F_1 \in \mathcal{H}'^{d,\nu}_{(k_1),N_0}(\zeta)$, $F_2 \in \mathcal{H}'^{1,\nu}_{(k_2),N_0}(\zeta)$, their Poisson bracket $\{F_1, F_2\}$ is in $\mathcal{H}'^{d,\nu'}_{(k_1+k_2),N_0}(\tilde{\zeta})$ for some $\nu' \geq \nu$ depending only on ν, N_0, and where $\tilde{\zeta} = \max(\zeta, \frac{d+1}{3})$.

We shall denote by $\mathcal{H}'^{d,\nu}_{(0),N_0}(\zeta)$ the space of functions of form

$$\alpha\langle \Lambda_m^d u, u\rangle + F(u) \tag{5.1.5}$$

where $\alpha \in \mathbb{R}$, $F \in \mathcal{H}'^{d,\nu}_{(1),N_0}(\zeta)$. Proposition 3.3.4 extends to the case when $F_1 \in \mathcal{H}'^{d,\nu}_{(0),N_0}(\zeta)$, $F_2 \in \mathcal{H}'^{1,\nu}_{(k),N_0}(\zeta)$ $(k \in \mathbb{N}^*)$ and shows that $\{F_1, F_2\}$ is in $\mathcal{H}'^{1,\nu'}_{(k),N_0}(\tilde{\zeta})$ for some $\nu' \geq \nu$.

From now on, we fix a large integer κ. We introduce truncated Poisson brackets.

Definition 5.1.1. — Let F (resp. G) be an element of $\mathcal{H}'^{1,\nu}_{(1),N_0}(\zeta)$ (resp. $\mathcal{H}'^{d,\nu}_{(0),N_0}(\zeta)$) with $d \in \mathbb{N}^*$, $\nu > 0, N_0 \in \mathbb{N}^*$. Decompose F and G as sums of homogeneous terms and assume that all components of order larger or equal to κ vanish,

$$F(u) = \sum_{k=1}^{\kappa-1} F_k(u),\ G(u) = \sum_{k=0}^{\kappa-1} G_k(u). \tag{5.1.6}$$

We define

$$\{F, G\}_\kappa = \sum_{\substack{\ell+\ell' \leq \kappa-1 \\ \ell \geq 1, \ell' \geq 0}} \{F_\ell, G_{\ell'}\}. \tag{5.1.7}$$

We obtain an element of $\mathcal{H}'^{d,\nu'}_{(1),N_0}(\tilde{\zeta})$ for some $\nu' \geq \nu$. We set by induction

$$\begin{aligned} &\mathrm{Ad}_\kappa F \cdot G = \{F, G\}_\kappa \\ &\mathrm{Ad}_\kappa{}^j F \cdot G = \mathrm{Ad}_\kappa F \cdot (\mathrm{Ad}_\kappa{}^{j-1} F) \cdot G. \end{aligned} \tag{5.1.8}$$

We have for some increasing sequence ν_j depending only on ν, N_0 and for $\zeta_j = \max(\zeta, \frac{d+j}{3})$

$$\mathrm{Ad}_\kappa{}^j F \cdot G \in \mathcal{H}'^{d,\nu_j}_{(j),N_0}(\zeta_j). \tag{5.1.9}$$

Finally, we define

$$\exp[T\mathrm{Ad}_\kappa F] \cdot G = \sum_{j=0}^{+\infty} \frac{T^j}{j!} \mathrm{Ad}_\kappa{}^j \cdot F. \tag{5.1.10}$$

Note that by (5.1.9) and the truncation in definition (5.1.7), the coefficients of T^j vanish when $j \geq \kappa$.

Lemma 5.1.2. — *Let $s \in \mathbb{N}^*$, $N_0 \in \mathbb{N}$, $\Theta_s^0(u) = \frac{1}{2}\langle \Lambda_m^s u, \Lambda_m^s u\rangle$ element of $\mathcal{H}'^{2s,0}_{(0),0}(0)$. Let $G \in \mathcal{H}'^{1,+\infty}_{(0),N_0}(0) \overset{\text{def}}{=} \cup_{\nu>0} \mathcal{H}'^{1,\nu}_{(0),N_0}(0)$ and let $H \in \mathcal{H}'^{1,+\infty}_{(1),N_0}(0)$. Assume that G and*

H have no component homogeneous of order greater than or equal to κ. We have the equality

$$\{\exp(T\mathrm{Ad}_\kappa H)\Theta_s^0, G\}_\kappa = \exp(T\mathrm{Ad}_\kappa H)\cdot\{\Theta_s^0, \exp(-T\mathrm{Ad}_\kappa H)\}_\kappa. \tag{5.1.11}$$

Remark that for fixed κ, the functions in the preceding formula are well defined when $u \in H^s$ with s large enough: the regularity condition of definition 3.3.3 of the class $\mathcal{H}'^{d,\nu_j}_{(j),N_0}(\zeta_j)$, namely

$$s > \nu + \frac{5}{2} + \max(\zeta_j, \frac{d}{3}) = \nu + \frac{5}{2} + \max(\zeta, \frac{d+j}{3})$$

is satisfied for any $j = 1, \dots, \kappa$ when $d = 2s$ and s is large enough relatively to κ, ν.

Proof. — Since (5.1.11) is an equality between polynomials in T, we just need to check that all T derivatives coincide at $T = 0$. Note first that

$$\frac{d}{dT}\{\exp(T\mathrm{Ad}_\kappa H)\cdot\Theta_s^0, G\}_\kappa = \{\exp(T\mathrm{Ad}_\kappa H)\mathrm{Ad}_\kappa H\cdot\Theta_s^0, G\}_\kappa$$

and that

$$\begin{aligned}&\frac{d}{dT}[\exp(T\mathrm{Ad}_\kappa H)\cdot\{\Theta_s^0, \exp(-T\mathrm{Ad}_\kappa H)\cdot G\}_\kappa]\\ &= \exp(T\mathrm{Ad}_\kappa H)[\mathrm{Ad}_\kappa H\cdot\{\Theta_s^0, \exp(-T\mathrm{Ad}_\kappa H)\cdot G\}_\kappa\\ &\qquad\qquad -\{\Theta_s^0, \mathrm{Ad}_\kappa H\cdot\exp(-T\mathrm{Ad}_\kappa H)\cdot G\}_\kappa]\\ &= \exp(T\mathrm{Ad}_\kappa H)\{\mathrm{Ad}_\kappa H\cdot\Theta_s^0, \exp(-T\mathrm{Ad}_\kappa H)\cdot G\}_\kappa\end{aligned}$$

using the Jacobi identity

$$\{\{F_1, F_2\}, F_3\} + \{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = 0.$$

Iterating the above two inequalities, we get for any $j \in \mathbb{N}$

$$\begin{aligned}&\frac{d^j}{dT^j}\{\exp(T\mathrm{Ad}_\kappa H)\cdot\Theta_s^0, G\}_\kappa = \{\exp(T\mathrm{Ad}_\kappa H)\mathrm{Ad}_\kappa{}^j H\cdot\Theta_s^0, G\}_\kappa\\ &\frac{d^j}{dT^j}[\exp(T\mathrm{Ad}_\kappa H)\cdot\{\Theta_s^0, \exp(-T\mathrm{Ad}_\kappa H)\cdot G\}_\kappa]\\ &\qquad = \exp(T\mathrm{Ad}_\kappa H)\{\mathrm{Ad}_\kappa{}^j H\cdot\Theta_s^0, \exp(-T\mathrm{Ad}_\kappa H)\cdot G\}_\kappa.\end{aligned} \tag{5.1.12}$$

This shows that the two quantities (5.1.12) coincide at $T = 0$ and concludes the proof. □

To write a formula similar to (5.1.2), we introduce if Θ_s^0, G, H are as in the statement of the preceding lemma, the notations

$$\begin{aligned}\Theta_s^0\circ\chi_H^\kappa(u) &\stackrel{\text{def}}{=} \exp(\mathrm{Ad}_\kappa H)\cdot\Theta_s^0(u)\\ G\circ(\chi_H^\kappa)^{-1}(u) &= \exp(-\mathrm{Ad}_\kappa H)\cdot G(u)\end{aligned} \tag{5.1.13}$$

so that (5.1.11) may be written at $T = 1$

$$\{\Theta_s^0\circ\chi_H^\kappa, G\}_\kappa = \{\Theta_s^0, G\circ(\chi_H^\kappa)^{-1}\}_\kappa\circ\chi_H^\kappa. \tag{5.1.14}$$

We shall deduce theorem 1.1.1 from the following result.

Theorem 5.1.3. — *There is a large enough* $s_0 \in \mathbb{N}$ *and* $N_0 \in \mathbb{N}$ *and for any* $s \geq s_0$ *there are* $\rho_0 > 0$ *and*

- *A* C^1 *map* $F : B_s(\rho_0) \to \mathbb{R}$, *such that* $u \to \nabla F(u)$ *is* C^1 *from* $B_s(\rho_0)$ *to* $H^s(\mathbb{S}^1; \mathbb{R}^2)$ *and* $F(0) = 0, \partial F(0) = 0, \partial\nabla F(0) = 0$,
- *A diffeomorphism* ψ *from* $B_s(\rho_0)$ *to a neighborhood of 0 in* $H^s(\mathbb{S}^1; \mathbb{R}^2)$ *with* $\psi(0) = 0$,
- *An element* $H \in \mathcal{H}'^{1,+\infty}_{(1),N_0}(0)$,

such that if we set

$$\Theta_s(u) = (\Theta_s^0 \circ \chi_H^\kappa) \circ \psi^{-1} \circ \chi_F(u), \tag{5.1.15}$$

any solution u *of (1.2.9) satisfies, as long as it exists and stays in* $B_s(\rho_0)$,

$$\Big|\frac{d}{dt}\Theta_s(u(t,\cdot))\Big| \leq C\|u(t,\cdot)\|_{H^s}^{\kappa+2} \tag{5.1.16}$$

with a uniform constant $C > 0$.

Remark. — In (5.1.15) note that we use on the one hand the notation χ_F to denote the canonical transformation defined after (5.1.1) from a C^1 map on H^s such that $u \to \nabla F(u)$ is also a C^1 map from H^s to H^s, and on the other hand the notation χ_H^κ defined by (5.1.13). We could not give a meaning to χ_H as a map from a neighborhood of zero in H^s to H^s solving an equation of form (5.1.1). Nevertheless, notation (5.1.13) is perfectly meaningful since it involves only elements of classes $\mathcal{H}'^{d,\nu}_{(k),N_0}(0)$ for which the stability property with gain of one derivative of proposition 3.3.4 (i) holds.

Let us show that theorem 5.1.3 implies theorem 1.1.1. It is enough to show that if the solution of (1.2.9) exists over some interval $[0, T]$ and satisfies for $t \in [0, T]$, $u(t, \cdot) \in B_s(\rho_0)$ with a large enough s, then for any $t \in [0, T]$

$$\|u(t,\cdot)\|_{H^s}^2 \leq C\big[\|u(0,\cdot)\|_{H^s}^2 + \int_0^t \|u(\tau,\cdot)\|_{H^s}^{\kappa+2} d\tau\big] \tag{5.1.17}$$

with a uniform $C > 0$. Actually, since $\|u(0,\cdot)\|_{H^s} \leq A\epsilon$ for some $A > 0$, a standard continuation argument allows one to deduce from (5.1.17) that there is $c > 0$ and $A' > A$ such that if $T < c\epsilon^{-\kappa}$ and $\epsilon > 0$ is small enough, $\|u(t,\cdot)\|_{H^s} \leq A'\epsilon$ for any $t \in [0, T]$. This allows one to extend the solution up to a time of magnitude $c\epsilon^{-\kappa}$.

Let us deduce (5.1.17) from (5.1.16). By this inequality, as long as $u(t, \cdot)$ stays in $B_s(\rho_0)$ and $t \in [0, T]$,

$$\Theta_s(u(t,\cdot)) \leq \Theta_s(u(0,\cdot)) + C\int_0^t \|u(\tau,\cdot)\|_{H^s}^{\kappa+2} d\tau.$$

We just have to find some $K > 0$ such that for any $u \in B_s(\rho_0)$

$$K^{-1}\|u\|_{H^s}^2 \leq \Theta_s(u) \leq K\|u\|_{H^s}^2. \tag{5.1.18}$$

Since χ_F and ψ are C^1 local diffeomorphisms sending 0 to 0, it is enough to get such an estimate for $\Theta_s^0 \circ \chi_H^\kappa$. By (5.1.13), (5.1.10) and (5.1.9), $\Theta_s^0 \circ \chi_H^\kappa - \Theta_s^0$ belongs to $\mathcal{H}'^{2s,\nu_\kappa}_{(1),N_0}(\zeta_\kappa)$. Definition 3.3.3 of that space and proposition 2.1.13 (in the special case of polynomial symbols) show that

$$|(\Theta_s^0 \circ \chi_H^\kappa - \Theta_s^0)(u)| \leq C\|u\|^3_{H^s}$$

if s is large enough and $u \in B_s(\rho_0)$. Estimate (5.1.18) follows from that.

We have reduced ourselves to the proof of theorem 5.1.3. In the following three sections we shall construct successively maps F, ψ, H involved in (5.1.15).

5.2. First reduction: elimination of low degree non diagonal terms

Let u be a solution of (1.2.9), smooth enough and defined on some interval $[0, T]$. Then

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) &= D\Theta_s(u(t,\cdot)) \cdot X_G(u(t,\cdot)) \\ &= \{\Theta_s, G\}(u(t,\cdot)) \\ &= \{(\Theta_s^0 \circ \chi_H^\kappa) \circ \psi^{-1}, G \circ \chi_F^{-1}\}(\chi_F(u(t,\cdot)))\end{aligned} \tag{5.2.1}$$

using (5.1.15) and (5.1.2). The aim of this section is to construct F in order to simplify $G \circ \chi_F^{-1}$ up to a given degree of homogeneity κ. By proposition 3.5.1 we may write, using notation (3.3.1),

$$\begin{aligned}G(u) = &\frac{1}{2}\langle \Lambda_m u, u\rangle + \frac{1}{4}\langle \mathrm{Op}_\chi[e(u;\cdot)]Iu, u\rangle \\ &- \frac{1}{4}\langle \mathrm{Op}_\chi[e(u;\cdot)]I'u, u\rangle + \frac{1}{2}\langle M(u)u, u\rangle\end{aligned} \tag{5.2.2}$$

where $e \in \widetilde{S}^{1,\nu}_{(1),0}(0), M \in \widetilde{\mathcal{L}}^{1,\nu}_{(1)}(0)$ for some $\nu > 0$, e verifying $\bar{e}^\vee = e$. We want to choose F in such a way that $G \circ \chi_F^{-1}$ will be given by a similar expression where all contributions in I' (or J') up to order $\kappa+1$ will be removed. In that way, $G \circ \chi_F^{-1}$ will be the sum of $\frac{1}{2}\langle \Lambda_m u, u\rangle$, of an element of $\mathcal{H}'^{1,\nu}_{(1),0}(0)$ for some new value of ν, and of contributions vanishing at least at order $\kappa+2$ at zero. We shall first compute $G \circ \chi_F^{-1}$ for any given F with a convenient structure and then, in a second step, choose F in order to eliminate all bad terms in the expansion brought by the first step. Remind that we denote by $B_s(\rho)$ the open ball of center 0, radius $\rho > 0$ in $H^s(\mathbb{S}^1; \mathbb{R}^2)$.

Proposition 5.2.1. — *One may find $\nu > 0$, symbols $\alpha, \beta \in \widetilde{S}^{0,\nu}_{(1),0}(0)$ satisfying $\bar{\alpha}^\vee = \alpha$, $\bar{\beta}^\vee = \beta$, an element $\underline{G}' \in \mathcal{H}'^{1,\nu}_{(1),0}(0)$, a large enough number $s_0 > 0$ and, for any $\sigma > s_0$, a constant $B > 0$, a $(\nu + \sigma + 1, 1)$-conveniently increasing sequence $D_.$, an element $\tilde{g}_\kappa \in S^{1,\nu}_{(\kappa),0}(\sigma, 0, B, D_.) \otimes \mathcal{M}_2(\mathbb{R})$ verifying $\bar{\tilde{g}}_\kappa^\vee = \tilde{g}_\kappa$, a C^1 function $u \to L(u)$*

defined on $B_{s_0}(\rho)$ *for some* $\rho_0 > 0$*, satisfying for any* $s \in [s_0, \sigma[$

$$\begin{aligned} &\nabla L(u) \in H^s(\mathbb{S}^1;\mathbb{R}^2) \text{ if } u \in B_s(\rho) \text{ for a small enough } \rho > 0 \\ &\|\nabla L(u)\|_{H^s} \leq C\|u\|_{H^s}^{\kappa+1} \end{aligned} \tag{5.2.3}$$

such that if we set

$$F(u) = \langle \mathrm{Op}_\chi[\alpha(u;\cdot)I' + \beta(u;\cdot)J']u, u\rangle \tag{5.2.4}$$

we have

$$G \circ \chi_F^{-1}(u) = \frac{1}{2}\langle \Lambda_m u, u\rangle + \underline{G}'(u) + \langle \mathrm{Op}_\chi[\tilde{g}_\kappa(u;\cdot)]u, u\rangle + L(u). \tag{5.2.5}$$

Let us note that the map F defined by (5.2.4) satisfies $\nabla F(u) \in H^s$ if $u \in H^s$, $s \in [s_0, \sigma[$ i.e. that $\partial F(u)$ extends as an element of $\mathcal{L}(H^{-s}, \mathbb{R})$. This follows from (i) and (ii) of proposition 2.1.13 if s_0 is large enough (see (2.1.44) and (2.1.46)). Moreover, since F is polynomial in u, these estimates show that $u \to \nabla F(u)$ and $u \to X_F(u)$ are C^1 maps from $H^s(\mathbb{S}^1;\mathbb{R}^2)$ to $H^s(\mathbb{S}^1;\mathbb{R}^2)$. We may thus consider the flow $\Phi(\tau, u)$ of (5.1.1), and for $u \in B_s(\rho)$ with ρ small enough, define

$$\chi_F(u) = \Phi(1, u),\ \chi_F^{-1}(u) = \Phi(-1, u). \tag{5.2.6}$$

As mentioned before the statement of the proposition, the first step of the proof will be the computation of $G \circ \chi_F^{-1}$ for any given F of form (5.2.4).

Lemma 5.2.2. — *Let* $\nu_0 > 0$*,* $\alpha, \beta \in \widetilde{S}^{0,\nu_0}_{(1),0}(0)$ *be given with* $\bar{\alpha}^\vee = \alpha$*,* $\bar{\beta}^\vee = \beta$*. One may find* $s_0 > 0, \rho_{s_0} > 0, \nu \geq \nu_0$ *and for any* $\sigma > s_0$ *a constant* $B > 0$ *and a* $(\nu+1+\sigma, 1)$*-conveniently increasing sequence* $D.$*, a symbol* $\tilde{g}_\kappa \in S^{1,\nu}_{(\kappa),0}(\sigma, 0, B, D.) \otimes \mathcal{M}_2(\mathbb{R})$*, and a* C^1 *function* $u \to L(u)$ *defined on* $B_{s_0}(\rho_{s_0})$*, satisfying (5.2.3) such that*

$$G \circ \chi_F^{-1}(u) = \sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} \cdot G + \langle \mathrm{Op}_\chi[\tilde{g}_\kappa(u;\cdot)]u, u\rangle + L(u). \tag{5.2.7}$$

Proof. — Let us show first that we may find $s_0 > 0, \rho_0 > 0, \nu \geq \nu_0$ and for any $\sigma > s_0$ a constant $B' > 0$, a $(\sigma+\nu+1, 1)$-conveniently increasing sequence $D.$, a constant $C > 0$ and

- A sequence $(g_k)_{k\geq\kappa}$ of elements of $S^{1,\nu}_{(k),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R})$ satisfying $\mathfrak{N}^{1,\nu}_{(k),0}(\sigma, 0, B', D.; g_k) \leq 1$,
- A sequence $(L_k)_{k\geq\kappa}$ of C^1-functions on $B_{s_0}(\rho_{s_0})$, such that for any $s \in [s_0, \sigma[$ there is $\rho_s > 0$, $C_s > 0$ so that for any $u \in B_s(\rho_s)$, $\nabla L_k(u) \in H^s$ and $\|\nabla L_k(u)\|_{H^s} \leq C_s C^k k! \|u\|_{H^s}^{k+1}$,

such that for any $K \geq \kappa$

$$G \circ \chi_F^{-1}(u) = \sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} \cdot G(u) + \sum_{k=\kappa}^{K} \frac{1}{k!} \langle \mathrm{Op}_\chi[g_k(u;\cdot)]u, u\rangle + \sum_{k=\kappa}^{K+1} \frac{1}{(k-1)!} L_k(u) + \int_0^1 \frac{(1-\tau)^K}{K!} \langle \mathrm{Op}_\chi[g_{K+1}(w;\cdot)]w, w\rangle|_{w=\Phi(-\tau,u)} d\tau. \tag{5.2.8}$$

We prove (5.2.8) by induction on K. By (5.1.3) with $N = \kappa - 1$

$$G \circ \chi_F^{-1}(u) = \sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} \cdot G(u) + \int_0^1 \frac{(1-\tau)^{\kappa-1}}{(\kappa-1)!} (\mathrm{Ad}^\kappa F \cdot G)(\Phi(-\tau, u)) d\tau. \tag{5.2.9}$$

The definition (5.2.4) of F shows that F belongs to the class $\mathcal{H}^{0,\nu_0}_{(1),0}(0)$ of definition 3.3.3, and $G \in \mathcal{H}^{1,\nu_0}_{(1),0}(0)$ if ν_0 is large enough. Proposition 3.3.4 (ii) implies that

$$\mathrm{Ad}^\kappa F \cdot G \in \mathcal{H}^{1,\nu-\frac{1}{3}}_{(\kappa),0}(1/3) \subset \mathcal{H}^{1,\nu}_{(\kappa),0}(0)$$

for some $\nu \geq \nu_0$ i.e. we may write

$$\mathrm{Ad}^\kappa F \cdot G = \langle \mathrm{Op}_\chi[g_\kappa(u;\cdot)]u, u\rangle + \langle M_\kappa(u)u, u\rangle \tag{5.2.10}$$

with $g_\kappa \in \widetilde{S}^{1,\nu}_{(\kappa),0}(0) \otimes \mathcal{M}_2(\mathbb{R})$, $M_\kappa \in \widetilde{\mathcal{L}}^{1,\nu}_{(\kappa)}(0)$. Let $L^1_\kappa(u) = \langle M_\kappa(u)u, u\rangle$. By estimates (2.1.47) and (2.1.48) of proposition 2.1.13, if $s \geq s_0$ large enough, $\nabla L^1_\kappa(u)$ belongs to $H^s(\mathbb{S}^1; \mathbb{R}^2)$ when $u \in H^s(\mathbb{S}^1; \mathbb{R}^2)$, and

$$\|\nabla L^1_\kappa(u)\|_{H^s} \leq C\|u\|^{\kappa+1}_{H^s}.$$

If we set $L_\kappa(u) = \int_0^1 (1-\tau)^{\kappa-1} \langle M_\kappa(\cdot)\cdot, \cdot\rangle(\Phi(-\tau, u)) d\tau$, L_κ verifies similar properties since $D\Phi(-\tau, u) \in \mathcal{L}(H^{-s}, H^{-s})$ as seen at the beginning of section 5.1. Let $\sigma > s_0$ and choose a $(\nu+1+\sigma, 1)$-conveniently increasing sequence $D.$ and a positive constant B' such that $g_\kappa \in S^{1,\nu}_{(\kappa),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R})$, $\alpha, \beta \in S^{0,\nu}_{(1),0}(\sigma, 0, B', D.)$ with

$$\mathfrak{N}^{0,\nu}_{(1),0}(\sigma, 0, B', D.; \alpha I' + \beta J') \leq 1, \mathfrak{N}^{1,\nu}_{(\kappa),0}(\sigma, 0, B', D.; g_\kappa) \leq 1. \tag{5.2.11}$$

(Note that taking B' large enough, we may always make the left hand side of the preceding inequalities as small as we want for given α, β, g_κ). It follows from (5.2.9), (5.2.10) and the definition of L_κ that (5.2.8) with $K = \kappa - 1$ holds true.

Let us show that (5.2.8) at rank K implies (5.2.8) at rank $K + 1$. Integrating by parts the integral in (5.2.8), we get

$$\frac{1}{(K+1)!} \langle \mathrm{Op}_\chi[g_{K+1}(u;\cdot)]u, u\rangle + \int_0^1 \frac{(1-\tau)^{K+1}}{(K+1)!} \{F, \langle \mathrm{Op}_\chi[g_{K+1}(w;\cdot)]w, w\rangle\}|_{w=\Phi(-\tau,u)} d\tau. \tag{5.2.12}$$

Taking definition (5.2.4) of F into account, we may apply to the Poisson bracket in the above integral proposition 3.3.1. This allows us to write this bracket as

$$\langle \mathrm{Op}_\chi[g_{K+2}(w;\cdot)]w,w\rangle + \langle \mathrm{Op}_\chi[e_{K+2}(w;\cdot)]w,w\rangle + \langle M_{K+2}(w)w,w\rangle \tag{5.2.13}$$

where $g_{K+2} \in S^{1,\nu}_{(K+2),0}(\sigma,0,B',D.) \otimes \mathcal{M}_2(\mathbb{R})$, and where for some $\tilde{\nu} \geq \nu$ and some new sequence $\tilde{D}.$ (independents of the induction step)

$$e_{K+2} \in S^{0,\tilde{\nu}}_{(K+2),0}(\sigma,0,B',\tilde{D}.) \otimes \mathcal{M}_2(\mathbb{R}), M_{K+2} \in \mathcal{L}^{1,\tilde{\nu}}_{(K+2)}(\sigma,0,B').$$

Moreover, by (3.3.4), (5.2.11) and the induction hypothesis

$$\mathfrak{N}^{1,\nu}_{(K+2)}(\sigma,0,B',D.;g_{K+2}) \leq 1$$

and by (3.3.5)

$$\mathfrak{N}^{0,\tilde{\nu}}_{(K+2)}(\sigma,0,B',\tilde{D}.;e_{K+2}) \leq C_0$$
$$\mathfrak{N}^{1,\tilde{\nu}}_{(K+2)}(\sigma,0,B';M_{K+2}) \leq C_0.$$

The first term in (5.2.13) gives, when plugged in the integral (5.2.12), the last term in (5.2.8), at order $K+1$. Set

$$L^1_{K+2}(u) = \langle \mathrm{Op}_\chi[e_{K+2}(u;\cdot)]u,u\rangle + \langle M_{K+2}(u)u,u\rangle \tag{5.2.14}$$

By estimates (2.1.44), (2.1.46), (2.1.47), (2.1.48) of proposition 2.1.13, L^1_{K+2} is a C^1 function of u on $B_s(\rho_s)$ (for $\rho_s > 0$ independent of K) such that $u \to \nabla L^1_{K+2}(u)$ is in $H^s(\mathbb{S}^1;\mathbb{R}^2)$ with an estimate

$$\|\nabla L^1_{K+2}(u)\|_{H^s} \leq C(s)(\tilde{C}B')^K(K+1)!\|u\|^{K+3}_{H^s}.$$

If we set

$$L_{K+2}(u) = \int_0^1 (1-\tau)^{K+1} L^1_{K+2}(\Phi(-\tau,u))d\tau$$

it obeys similar estimates, since we have seen after formula (5.1.1) that $D\Phi(-\tau,u)$ extends as an element of $\mathcal{L}(H^{-s},H^{-s})$ so that $\nabla(L^1_{K+2}(\Phi(-\tau,u)))$ is in H^s. We have proved (5.2.8) at order $K+1$.

To finish the proof of lemma 5.2.2, we still have to make K go to $+\infty$ in (5.2.8). We just need to prove that for some $B > B'$

- There is a symbol $\tilde{g}_\kappa \in S^{1,\nu}_{(\kappa),0}(\sigma,0,B,D.) \otimes \mathcal{M}_2(\mathbb{R})$ such that

$$\tilde{g}_\kappa(u;x,n) = \sum_{k=\kappa}^{+\infty} \frac{1}{k!} g_k(u;x,n), \tag{5.2.15}$$

- The function $L(u) = \sum_{k=\kappa}^{+\infty} \frac{1}{(k-1)!} L_k(u)$ satisfies (5.2.3),
- The integral

$$\int_0^1 \frac{(1-\tau)^K}{K!} \langle \mathrm{Op}_\chi[g_{K+1}(w;\cdot)]w,w\rangle|_{w=\Phi(-\tau,u)} d\tau \tag{5.2.16}$$

goes to zero when K goes to $+\infty$ and u remains in $B_{s_0}(\rho_{s_0})$.

Let us prove (5.2.15). Since $g_k \in S^{1,\nu}_{(k),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R})$, we decompose using definition 2.1.5

$$g_k(u; x, n) = \sum_{j \geq k} g_{k,j}(u, \dots, u; x, n)$$

with $g_{k,j} \in \Sigma^{1,\nu}_{(k,j),0}(\sigma, 0, B', D.) \otimes \mathcal{M}_2(\mathbb{R})$. Then $\tilde{g}_\kappa(u; x, n) = \sum_{j \geq \kappa} \tilde{g}_{\kappa,j}(u, \dots, u; x, n)$ with

$$\tilde{g}_{\kappa,j}(u_1, \dots, u_j; x, n) = \sum_{k=\kappa}^{j} \frac{1}{k!} g_{k,j}(u_1, \dots, u_j; x, n).$$

We need to check estimates (2.1.20) and (2.1.21) i.e. we have to evaluate

$$\sum_{k=\kappa}^{j} \frac{(k + j - 1)!}{(j + 1)!k!} \leq \frac{2^{2j}}{j + 1} \leq 2^{2j} \frac{(\kappa + j - 1)!}{(j + 1)!}.$$

We thus obtain for $\tilde{g}_{\kappa,j}$ estimates of type (2.1.20), (2.1.21) with a new constant $B = 4B'$.

We must next verify that $L(u)$ satisfies (5.2.3). This follows from the bounds $\|\nabla L_k(u)\|_{H^s} \leq C_s C^k k! \|u\|^{k+1}_{H^s}$ satisfied by each L_k if $\|u\|_{H^s} \leq \rho_s$ small enough.

Finally, by (i) of proposition 2.1.13,

$$|\langle \mathrm{Op}_\chi[g_{K+1}(u; \cdot)]u, u\rangle| \leq C(\tilde{C}B)^{K+1} \|u\|^{K+3}_{H^{s_0}} K!$$

which shows that (5.2.16) goes to zero when K goes to infinity if $\|u\|_{H^{s_0}} < \rho_{s_0}$ small enough. This concludes the proof of the lemma. □

Proof of proposition 5.2.1. — The last two terms in (5.2.7) contribute to the last two terms in (5.2.5), for any F of form (5.2.4). We have to show that we may find such a F so that the sum in the right hand side of (5.2.7) may be written $\frac{1}{2}\langle \Lambda_m u, u\rangle + \underline{G}'(u)$ with $\underline{G}'(u) \in \mathcal{H}'^{1,\nu}_{(1),0}(0)$ for some ν, up to remainders contributing to the last two terms in (5.2.5). Let us write

$$\sum_{k=0}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} \cdot G = G + \{F, G_0\} + \{F, G - G_0\} + \sum_{k=2}^{\kappa-1} \frac{\mathrm{Ad}^k F}{k!} G \tag{5.2.17}$$

with $G_0(u) = \frac{1}{2}\langle \Lambda_m u, u\rangle$. Since $G - G_0$ vanishes at least at order three at zero, the contribution to $\{F, G - G_0\}$ homogeneous of degree k depends only on $F_{k'}$, $k' < k$. The same is true for the last sum in (5.2.17). Consequently the expression may be written

$$G_0 + \sum_{k=1}^{\kappa-1} [G_k + \{F_k, G_0\} + H_k] + \sum_{k \geq \kappa} [G_k + H_k] \tag{5.2.18}$$

where the last sum is finite and where H_k is homogeneous of degree $k + 2$ and may be expressed using iterated brackets of $F_{k'}$, $k' < k$, and $G_{k'}$. Consequently, by proposition 3.3.4 (ii), H_k belongs to $\mathcal{H}^{1,\nu_0'}_{(k),0}(0)$ for some ν_0'. Moreover, the expression $\sum_{k \geq \kappa} [G_k + H_k]$ belongs to $\mathcal{H}^{1,\nu_0'}_{(\kappa),0}(0)$, so may be incorporated to the last two terms in

(5.2.5), reasoning as in the study of (5.2.14), if the constants ν, B of the statement of the proposition are taken large enough. For $1 \leq k \leq \kappa - 1$ write, using decomposition (4.1.9) of any matrix

$$G_k + H_k = G'_k + G''_k$$

with $G'_k \in \mathcal{H}'^{1,\nu'_0}_{(k),0}(0)$ homogeneous of degree $k+2$ and

$$G''_k(u) = \frac{1}{2}\langle \mathrm{Op}_\chi[\underline{\alpha}_k I' + \underline{\beta}_k J']u, u\rangle \tag{5.2.19}$$

where $\underline{\alpha}_k, \underline{\beta}_k \in \widetilde{S}^{1,\nu'_0}_{(k),0}(0)$ satisfy $\bar{\underline{\alpha}}^\vee_k = \underline{\alpha}_k, \bar{\underline{\beta}}^\vee_k = \underline{\beta}_k$ and are homogeneous of degree k. To reduce expression (5.2.18) to (5.2.5), we have to construct F_k so that $\{F_k, G_0\} + G''_k$ may be written as a term $\langle M_k(u)u, u\rangle$ with $M_k \in \widetilde{\mathcal{L}}^{1,\nu'_0}_{(k)}(0)$ (for a new value of ν'_0), that may be incorporated to G'_k. In other words, we are left with proving the following lemma: □

Lemma 5.2.3. — *Let $\underline{\alpha}_k, \underline{\beta}_k$ be as above. There are $\alpha_k, \beta_k \in \widetilde{S}^{0,\nu'_0+2}_{(k),0}(0)$, satisfying $\bar{\alpha}^\vee_k = \alpha_k, \bar{\beta}^\vee_k = \beta_k$ and $M_k \in \widetilde{\mathcal{L}}^{1,\nu'_0+2}_{(k)}(0)$ so that*

$$\begin{aligned}&\{\langle \mathrm{Op}_\chi[\alpha_k(u;\cdot)I' + \beta_k(u;\cdot)J']u, u\rangle, G_0\}\\ &\quad = \langle \mathrm{Op}_\chi[\underline{\alpha}_k(u;\cdot)I' + \underline{\beta}_k(u;\cdot)J']u, u\rangle + \langle M_k(u)u, u\rangle.\end{aligned} \tag{5.2.20}$$

Proof. — In the proof, we omit the subscripts k in $\underline{\alpha}, \underline{\beta}, \alpha, \beta, M$. Let us take complex coordinates $(w, \overline{w})$ related to the real coordinates $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ of $u \in H^s(\mathbb{S}^1; \mathbb{R}^2)$ through

$$\begin{bmatrix} w \\ \overline{w} \end{bmatrix} = K \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{5.2.21}$$

Since $\overline{\mathrm{Op}_\chi[\underline{\alpha}(u;\cdot)]} = \mathrm{Op}_\chi[\underline{\alpha}(u;\cdot)]$, $\overline{\mathrm{Op}_\chi[\underline{\beta}(u;\cdot)]} = \mathrm{Op}_\chi[\underline{\beta}(u;\cdot)]$ we have, denoting

$$\underline{\gamma}(w, \overline{w}; \cdot) = \underline{\alpha}\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}; \cdot\Big) - i\underline{\beta}\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}; \cdot\Big) \tag{5.2.22}$$

the equality

$$\frac{1}{2}\langle \mathrm{Op}_\chi[\underline{\alpha}(u;\cdot)I' + \underline{\beta}(u;\cdot)J']u, u\rangle = \mathrm{Re}\int_{\mathbb{S}^1}[\mathrm{Op}_\chi[\underline{\gamma}(w, \overline{w}; \cdot)]w]w\,dx. \tag{5.2.23}$$

Since

$$G_0(u) = \frac{1}{2}\langle \Lambda_m u, u\rangle = \int_{\mathbb{S}^1}(\Lambda_m w)\overline{w}\,dx$$

we look for a symbol $\gamma(w, \overline{w}; \cdot)$ in ${}^{\mathbb{C}}\widetilde{S}^{0,\nu'_0+2}_{(k),0}(0)$ such that

$$\Big\{\int\big(\mathrm{Op}_\chi[\gamma(w, \overline{w}; \cdot)]w\big)w\,dx, G_0\Big\} - \int\big(\mathrm{Op}_\chi[\underline{\gamma}(w, \overline{w}; \cdot)]w\big)w\,dx \tag{5.2.24}$$

equals some remainder. Let us decompose

$$\underline{\gamma}(w,\overline{w};\cdot) = \sum_{\ell=0}^{k} \underline{\gamma}_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell};\cdot)$$

with $\underline{\gamma}_\ell \in {}^{\mathbb{C}}\widetilde{\Sigma}^{0,\nu_0'}_{(k),0}(0)$. We look for γ as

$$\gamma(w,\overline{w};\cdot) = \sum_{\ell=0}^{k} \gamma_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell};\cdot). \tag{5.2.25}$$

with $\gamma_\ell \in {}^{\mathbb{C}}\widetilde{\Sigma}^{0,\nu_0'+2}_{(k),0}(0)$. Using expression (1.2.14) for the Poisson bracket in complex coordinates, we may write the first term in (5.2.24) as

$$i\sum_{\ell=0}^{k}\int_{\mathbb{S}^1} L_\ell[\mathrm{Op}_\chi\gamma_\ell](\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell})w\cdot w dx \tag{5.2.26}$$

where $L_\ell(\cdot)$ is defined by (3.4.12) with $\omega_0 = 1$, $\omega_1 = \cdots = \omega_\ell = -1, \omega_{\ell+1} = \cdots = \omega_{k+1} = 1$. By (i) of proposition 3.4.4, we may find $\gamma_\ell \in {}^{\mathbb{C}}\widetilde{\Sigma}^{0,\nu_0'+2}_{(k),0}(0)$ and $M_\ell \in {}^{\mathbb{C}}\tilde{\Lambda}^{1,\nu_0'+2}_{(k)}(0)$ such that (5.2.26) equals

$$\sum_{\ell=0}^{k}\int \mathrm{Op}_\chi[\underline{\gamma}_\ell(\overline{w},\dots,\overline{w},w,\dots,w;\cdot)]w\cdot w dx + \sum_{\ell=0}^{k}\int [M_\ell(\overline{w},\dots,\overline{w},w,\dots,w)w]w dx.$$

If we define γ by (5.2.25), we get that (5.2.24) equals $\int \widetilde{M}(w,\overline{w})w\cdot w dx$ with

$$\widetilde{M}(w,\overline{w}) = \sum_{\ell=0}^{k} M_\ell(\overline{w},\dots,\overline{w},w,\dots,w).$$

Let us define

$$\alpha(u;x,n) = \frac{1}{2}\big[\gamma(Ku;x,n) + \overline{\gamma(Ku;x,-n)}\big]$$
$$\beta(u;x,n) = -\frac{1}{2i}\big[\gamma(Ku;x,n) - \overline{\gamma(Ku;x,-n)}\big]$$

We obtain elements of $\widetilde{S}^{0,\nu_0'+2}_{(k),0}(0)$ satisfying $\bar{\alpha}^\vee = \alpha, \bar{\beta}^\vee = \beta$ such that

$$\mathrm{Re}\int_{\mathbb{S}^1}\mathrm{Op}_\chi[\gamma(w,\overline{w};\cdot)]w\cdot w dx = \frac{1}{2}\langle \mathrm{Op}_\chi[\alpha I' + \beta J']u,u\rangle.$$

Taking the real part of (5.2.24) and using (5.2.23) we have proved

$$\frac{1}{2}\{\langle \mathrm{Op}_\chi[\alpha(u;\cdot)I' + \beta(u;\cdot)J']u,u\rangle, G_0\} = \frac{1}{2}\langle \mathrm{Op}_\chi[\underline{\alpha}(u;\cdot)I' + \underline{\beta}(u;\cdot)J']u,u\rangle + \mathrm{Re}\int (\widetilde{M}(Ku)w)w dx.$$

Writing the last term as $\langle M(u)u,u\rangle$ for some $M \in \widetilde{\mathcal{L}}^{1,\nu_0'+2}_{(k)}(0)$ we obtain (5.2.20). This concludes the proof. □

5.3. Second reduction: elimination of higher order non diagonal part

The construction of F performed in section 5.1 allowed us by (5.2.1) and proposition 5.2.1 to write

$$\frac{d}{dt}\Theta_s(u(t;\cdot)) = \{(\Theta_s^0 \circ \chi_H^\kappa)\circ\psi^{-1}, G\circ\chi_F^{-1}\}(\chi_F(u(t,\cdot))) \tag{5.3.1}$$

with

$$G\circ\chi_F^{-1}(u) = G_0(u) + \underline{G}'(u) + \langle \mathrm{Op}_\chi[\tilde g_\kappa(u;\cdot)]u,u\rangle + L(u) \tag{5.3.2}$$

where $\underline{G}' \in \mathcal{H}'^{1,\nu}_{(1),0}(0)$, and is the sum of homogeneous terms of order $k=1,\dots,k-1$, $\tilde g_\kappa \in S^{1,\nu}_{(\kappa),0}(\sigma,0,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$ and L satisfies (5.2.3). The goal of this section is to choose ψ in (5.3.1) in order to eliminate the non diagonal components of $\tilde g_\kappa$ i.e. those along I' and J'. In other words, we want to do with $\tilde g_\kappa$ what we did in the preceding section for components of lower degree of homogeneity, except that we do not want to get as remainders symbols of order one, homogeneous of degree $\kappa+1$, but a symbol of order *zero*, homogeneous of degree κ.

By definition of $\mathcal{H}'^{1,\nu}_{(1),0}(0)$, we may find $\lambda(u;\cdot)$, $\mu(u;\cdot)$ in $\widetilde{S}^{1,\nu}_{(1),0}(0)$ satisfying $\bar\lambda^\vee=\lambda$, $\bar\mu^\vee=\mu$ and $M\in\widetilde{\mathcal{L}}^{1,\nu}_{(1)}(0)$ such that

$$\underline{G}'(u) = \frac12\langle \mathrm{Op}_\chi(\lambda(u;\cdot)I+\mu(u;\cdot)J)u,u\rangle + \frac12\langle M(u)u,u\rangle. \tag{5.3.3}$$

Note that in the duality bracket, we may always replace $\mathrm{Op}_\chi(\lambda I+\mu J)$ by

$$\frac12[\mathrm{Op}_\chi(\lambda I+\mu J) + {}^t\mathrm{Op}_\chi(\lambda I+\mu J)]$$

so that, by proposition 2.3.3, and up to a modification of ν and M, we may assume that $\lambda^\vee-\lambda, \mu^\vee+\mu$ belong to $\widetilde{S}^{0,\nu}_{(1),0}(0)$. In the same way, we may in (5.3.2) replace $\tilde g_\kappa$ by a symbol $\frac12\Omega(u;\cdot)\in S^{1,\nu}_{(\kappa),0}(\sigma,0,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$, satisfying ${}^t\Omega^\vee-\Omega\in S^{0,\nu+1}_{(\kappa),0}(\sigma,0,B,D.)\otimes\mathcal{M}_2(\mathbb{R})$ (for a new value of $\nu, D.$), up to a modification of L in (5.3.2). Decomposing λ,μ,M as sums of homogeneous contributions λ_k,μ_k,M_k, $k=1,\dots,\kappa-1$ we write

$$G_0(u) + \underline{G}'(u) + \langle\mathrm{Op}_\chi[\tilde g_\kappa(u;\cdot)]u,u\rangle = G'(u) + \tilde G'(u) \tag{5.3.4}$$

with

$$\begin{aligned} G'(u) &= \frac12\sum_{k=0}^{\kappa-1}\langle \mathrm{Op}_\chi(\lambda_k(u;\cdot)I+\mu_k(u;\cdot)J)u,u\rangle + \frac12\langle\mathrm{Op}_\chi[\Omega(u;\cdot)]u,u\rangle \\ \tilde G'(u) &= \frac12\sum_{k=1}^{\kappa-1}\langle M_k(u)u,u\rangle \end{aligned} \tag{5.3.5}$$

and conditions (4.1.1), (4.1.2), (4.1.3) are satisfied. Consider ψ the local diffeomorphism constructed in proposition 4.2.1, and let us apply corollary 4.2.3. We write the right hand side of (5.3.1) evaluated at $w = \chi_F(u)$, according to (5.3.2), (5.3.4), (5.3.5)

(5.3.6)
$$\begin{aligned}\{(\Theta_s^0 \circ \chi_H^\kappa) \circ \psi^{-1}, (G' + \tilde{G}') \circ \psi \circ \psi^{-1}\}(w) + \{(\Theta_s^0 \circ \chi_H^\kappa) \circ \psi^{-1}, L\}(w)\\ = \partial[\Theta_s^0 \circ \chi_H^\kappa](\psi^{-1}(w)) \circ \partial\psi^{-1}(w) \circ J \circ {}^t(\partial\psi)^{-1}(w) \cdot \nabla[(G' + \tilde{G}') \circ \psi](\psi^{-1}(w))\\ + \partial[\Theta_s^0 \circ \chi_H^\kappa](\psi^{-1}(w)) \circ \partial\psi^{-1}(w) \circ J \cdot \nabla L(w).\end{aligned}$$

By (4.2.20), $J = \partial\psi(v)J^t(\partial\psi(v)) + R_1(v)$ where $R_1(v)$ is a map sending H^{s-1} to H^s, with norm $O(\|v\|_{H^s}^\kappa)$. Plugging this into the first term in the right hand side of (5.3.6), we get setting $\tilde{R}_1(v) = \partial\psi^{-1}(\psi(v))R_1(v)^t(\partial\psi)^{-1}(\psi(v))$

$$\begin{aligned}&\{(\Theta_s^0 \circ \chi_H^\kappa), (G' + \tilde{G}') \circ \psi\}(\psi^{-1}(w))\\ &+ \partial[\Theta_s^0 \circ \chi_H^\kappa](\psi^{-1}(w))\tilde{R}_1(\psi^{-1}(w))\big(\nabla[(G' + \tilde{G}') \circ \psi]\big)(\psi^{-1}(w)).\end{aligned} \tag{5.3.7}$$

By assumption, $\Theta_s^0 \in {\mathcal{H}'}^{2s,0}_{(0),0}(0), H \in {\mathcal{H}'}^{1,\nu_0}_{(1),N_0}(0)$ for some $\nu_0 > 0$, some $N_0 \geq 0$. Consequently (5.1.13), (5.1.10), (5.1.9) imply that $\Theta_s^0 \circ \chi_H^\kappa$ will belong to ${\mathcal{H}'}^{2s,\nu_0'}_{(0),N_0}(\frac{2s+\kappa-1}{3}) \subset {\mathcal{H}'}^{2s,\nu_0'+\frac{\kappa-1}{3}}_{(0),N_0}(\frac{2s}{3})$ (for a new value ν_0' of ν_0). By lemma 3.3.7 $\partial[\Theta_s^0 \circ \chi_H^\kappa]$ belongs to $\mathcal{L}(H^s, \mathbb{R})$ and $\nabla[(G' + \tilde{G}') \circ \psi]$ belongs to $H^{s-1}(\mathbb{S}^1; \mathbb{R}^2)$. Since R_1 gains one derivative, we see that the last term in (5.3.7) belongs to $H^s(\mathbb{S}^1; \mathbb{R}^2)$ and has H^s-norm $O(\|w\|_{H^s}^{\kappa+2})$. A similar property holds for the last term in (5.3.6), so that (5.3.1) may be written

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) = \{\Theta_s^0 \circ \chi_H^\kappa, (G' + \tilde{G}') \circ \psi\}(\psi^{-1}(\chi_F(u(t,\cdot))))\\ + O(\|u(t,\cdot)\|_{H^s}^{\kappa+2})\end{aligned} \tag{5.3.8}$$

when u remains in some small ball $B_s(\rho_s)$.

We express in the above formula $G' \circ \psi$ using (4.2.19). Moreover

$$\tilde{G}' \circ \psi(v) = \frac{1}{2}\sum_{k=1}^{\kappa-1} \langle M_k(\psi(v))\psi(v), \psi(v)\rangle.$$

By definition $\Phi(v) = \psi(v) - v$ satisfies $\|\Phi(v)\|_{H^s} \leq C\|v\|_{H^s}^{\kappa+1}$ and $\partial\Phi(v)$ extends as an element of $\mathcal{L}(H^{-s}, H^{-s})$ with $\|\partial\Phi(v)\|_{\mathcal{L}(H^{-s},H^{-s})} \leq C\|v\|_{H^s}^\kappa$. It follows from this and from the remark after the proof of lemma 3.3.7 that

$$\tilde{G}' \circ \psi(v) = \tilde{G}'(v) + L(v)$$

where L satisfies again (5.2.3). Consequently, we may write the right hand side of (5.3.8) as

$$\{\Theta_s^0 \circ \chi_H^\kappa, G_1' + \tilde{G}'\}(\psi^{-1}(\chi_F(u(t,\cdot))) + O(\|u(t,\cdot)\|_{H^s}^{\kappa+2})$$

with

$$G_1'(v) = \frac{1}{2}\sum_{k=0}^{\kappa-1}\langle \mathrm{Op}_\chi[\lambda_k(v;\cdot)I + \mu_k(v;\cdot)J]v, v\rangle + \frac{1}{2}\langle \mathrm{Op}_\chi[\tilde{\lambda}_\kappa(v;\cdot)I + \tilde{\mu}_\kappa(v;\cdot)J]v, v\rangle \tag{5.3.9}$$

and

$$\tilde{G}'(v) = \frac{1}{2}\sum_{k=1}^{\kappa-1}\langle M_k(v)v, v\rangle. \tag{5.3.10}$$

Moreover, up to a modification of the remainder, we may always assume

$$^t(\tilde{\lambda}_\kappa(v;\cdot)I + \tilde{\mu}_\kappa(v;\cdot)J)^\vee - (\tilde{\lambda}_\kappa(v;\cdot)I + \tilde{\mu}_\kappa(v;\cdot)J) \in S^{0,\nu+1}_{(\kappa),0}(\sigma, 0, B, D.) \otimes \mathcal{M}_2(\mathbb{R}).$$

Summarizing the above results, we may state:

Proposition 5.3.1. — *There are $\nu > 0, s_0 > 0$ and for any $\sigma > s_0$ a constant $B > 0$, a $(\nu+\sigma+1, 1)$-conveniently increasing sequence $D.$, elements $\lambda_k(v;\cdot)$, $\mu_k(v;\cdot)$ in $\widetilde{S}^{1,\nu}_{(k),0}(0)$, $k = 1, \dots, \kappa-1$, $\tilde{\lambda}_\kappa$, $\tilde{\mu}_\kappa$ in $S^{1,\nu}_{(\kappa),0}(\sigma, 0, B, D.)$ satisfying conditions (4.1.1), (4.1.2), (4.2.17) and*

$$\tilde{\lambda}_\kappa(u;x,n) - \tilde{\lambda}^\vee_\kappa(u;x,n), \tilde{\mu}_\kappa(u;x,n) + \tilde{\mu}^\vee_\kappa(u;x,n) \in S^{0,\nu+1}_{(\kappa),0}(\sigma, 0, B, D.), \tag{5.3.11}$$

such that for any $s \in [s_0, \sigma[$ there is a local diffeomorphism ψ defined on a neighborhood of zero $B_s(\rho_s)$ in $H^s(\mathbb{S}^1;\mathbb{R}^2)$ satisfying the following: For any $H \in \mathcal{H}'^{1,\nu_0}_{(1),N_0}(0)$

$$\frac{d}{dt}\Theta_s(u(t,\cdot)) = \{\Theta^0_s \circ \chi^\kappa_H, G_1' + \tilde{G}'\}(\psi^{-1} \circ \chi_F(u(t,\cdot))) + O(\|u(t,\cdot)\|^{\kappa+2}_{H^s}) \tag{5.3.12}$$

as long as $u(t,\cdot)$ exists and stays in a small enough neighborhood of zero in H^s.

5.4. Third reduction: elimination of low degree diagonal terms

This last section will be devoted to the proof of the following:

Proposition 5.4.1. — *Let G_1', $\tilde{G}'$ be given respectively by (5.3.9), (5.3.10). Set*

$$\tilde{G}_1'(u) = \frac{1}{2}\sum_{k=0}^{\kappa-1}\langle \mathrm{Op}_\chi[\lambda_k(v;\cdot)I + \mu_k(v;\cdot)J]v, v\rangle. \tag{5.4.1}$$

There are $\nu_0 > 0, N_0 \in \mathbb{R}_+, s_0 > 0$ and $H \in \mathcal{H}'^{1,\nu_0}_{(1),N_0}(0)$ such that

$$\{\Theta^0_s \circ \chi^\kappa_H, \tilde{G}_1' + \tilde{G}'\}_\kappa(v) = 0 \tag{5.4.2}$$

for any $v \in H^s(\mathbb{S}^1;\mathbb{R}^2), s \geq s_0$.

Before starting the proof, let us make some preparations. Remind that the function Θ^0_s belongs to the space $\mathcal{H}'^{2s,0}_{(0),0}(0)$ defined by (5.1.5). Let us prove:

Lemma 5.4.2. — *Let $H \in \mathcal{H}'^{1,\nu_0}_{(1),N_0}(0)$. Let $\nu \in \mathbb{R}_+, s_0 > 0, B > 0, D.$ be as in the statement of proposition 5.3.1. Then for any $s \in [s_0, \sigma[$*

$$\{\Theta^0_s \circ \chi^\kappa_H, \langle \mathrm{Op}_\chi[\tilde{\lambda}_\kappa(u;\cdot)I + \tilde{\mu}_\kappa(u;\cdot)J]u, u\rangle\} = O(\|u\|^{\kappa+2}_{H^s}). \tag{5.4.3}$$

Proof. — We note first that if we are given $d_1, d_2 \in \mathbb{N}^*, k_2 \in \mathbb{N}^*, \nu > 0, \sigma \geq \nu + \frac{d_1+d_2}{3} + 2N_0 + 8$ and λ_2, μ_2 in $S^{d_2,\nu}_{(k_2),N_0}(\sigma, 0, B, D.)$, $M_2 \in \mathcal{L}^{d_2,\nu}_{(k_2)}(\sigma, 0, B)$, satisfying $\bar{\lambda}_2^\vee = \lambda_2$, $\bar{\mu}_2^\vee = \mu_2$ the bracket

$$\langle \frac{1}{2}\langle \Lambda^{d_1}_m u, u\rangle, \frac{1}{2}\langle \mathrm{Op}_\chi[\lambda_2(u;\cdot)I + \mu_2(u;\cdot)J]u, u\rangle + \langle M_2(u)u, u\rangle\rangle \tag{5.4.4}$$

may be written as

$$\frac{1}{2}\langle \mathrm{Op}_\chi[\lambda(u;\cdot)I + \mu(u;\cdot)J]u, u\rangle + \frac{1}{2}\langle M(u)u, u\rangle \tag{5.4.5}$$

for $\lambda, \nu \in S^{d_1+d_2-1,\nu'}_{(k_1+k_2),N_0}(\sigma, \tilde{\zeta}, \tilde{B}, \tilde{D}.)$, $M \in \mathcal{L}^{d_1+d_2,\nu'}_{(k_1+k_2)}(\sigma, 0, \tilde{B})$, with a new value ν' of ν (independent of d_1, d_2), a new constant $\tilde{B}$, a new sequence $\tilde{D}.$ and $\tilde{\zeta} = \frac{d_1+d_2}{3}$. Actually, this is a version of lemma 3.3.6, applying when the left half of bracket (3.3.17) is given in terms of a symbol vanishing at order 0 at $u = 0$ instead of some order $k_1 \geq 1$. The only place in the proof of lemma 3.3.6 (and in the proofs of the results used to demonstrate it) where the fact that $k_1 > 0$ is needed is when applying inequality (2.1.16). Actually, this inequality allows one to gain one negative power of $j' + 1$ and $j'' + 1$. When studying a bracket of form (5.4.4), we have $j' = k' = 0$, $j'' \geq k'' = k_2$, and we can gain $\frac{1}{j''+1}$ writing in estimates of form (2.1.20), (2.1.25) $B'' \leq \frac{1}{j''+1}(2B)^{j''}$ i.e. replacing B by $\tilde{B} = 2B$. This allows one to get an expression of form (5.4.5) for (5.4.4).

We have seen when obtaining (5.3.8) that $\Theta^0_s \circ \chi^\kappa_H \in \mathcal{H}'^{2s,\nu_0}_{(0),N_0}(\frac{2s}{3})$ for some ν_0, so that function may be written as a multiple of $\langle \Lambda^{2s}u, u\rangle$ plus an element of $\mathcal{H}'^{2s,\nu_0}_{(1),N_0}(\frac{2s}{3})$. The contribution of the $\langle \Lambda^{2s}u, u\rangle$ term to (5.4.3) is an expression of form (5.4.4) with $d_1 = 2s, d_2 = 1$, and so may be written as (5.4.5), with symbols $\lambda, \mu \in S^{2s,\nu'}_{(\kappa),N_0}(\sigma, \tilde{\zeta}, \tilde{B}, \tilde{D}.)$ for some ν' independent of s, $\tilde{\zeta} = \frac{2s+1}{3}$, $M \in \mathcal{L}^{2s+1,\nu'}_{(\kappa)}(\sigma, 0, \tilde{B})$. The contribution of the component of $\Theta^0_s \circ \chi^\kappa_H$ belonging to $\mathcal{H}'^{2s,\nu_0}_{(1),N_0}(\frac{2s}{3})$ to the Poisson bracket (5.4.3) may be treated applying lemma 3.3.6, and gives contributions of the same type.

If $s \geq s_0$ large enough, and $s < \sigma$, it follows from (2.1.44) and (2.1.47) that (5.4.5) is $O(\|u\|^{\kappa+2}_{H^s})$. This is the wanted conclusion. □

Before proving proposition 5.4.1, let us show that together with the preceding lemma it implies theorem 5.1.3. According to proposition 5.3.1, inequality (5.1.16) will follow if we prove that H may be chosen so that $\{\Theta^0_s \circ \chi^\kappa_H, G'_1 + \tilde{G}'\}(v) = O(\|v\|^{\kappa+2}_{H^s})$. By lemma 5.4.2, such a bound holds for $\{\Theta^0_s \circ \chi^\kappa_H, G'_1 - \tilde{G}'_1\}(v)$. We may thus prove that $\{\Theta^0_s \circ \chi^\kappa_H, \tilde{G}'_1 + \tilde{G}'\}(v) = O(\|v\|^{\kappa+2}_{H^s})$. If H is given by proposition 5.4.1, (5.4.2) holds, so that we just have to check that

$$\{\Theta_s^0 \circ \chi_H^\kappa, \tilde{G}_1' + \tilde{G}'\} - \{\Theta_s^0 \circ \chi_H^\kappa, \tilde{G}_1' + \tilde{G}'\}_\kappa = O(\|v\|_{H^s}^{\kappa+2}). \tag{5.4.6}$$

The left hand side of (5.4.6) is made of those contributions to $\{\Theta_s^0 \circ \chi_H^\kappa, \tilde{G}_1' + \tilde{G}'\}$ which are homogeneous of degree $k+2$ with $k \geq \kappa$ according to definition (5.1.7) of the truncated bracket. As we have seen in the proof of the preceding lemma, the first argument in the above bracket is in $\mathcal{H}'^{2s,\nu_0}_{(0),N_0}(\frac{2s}{3})$ for some ν_0. Moreover, $\tilde{G}_1' + \tilde{G}'$ defines an element of $\mathcal{H}'^{1,\nu}_{(0),0}(0)$ for some ν. By (i) of proposition 3.3.4 (and the extension of that result to components of order zero discussed in the proof of lemma 5.4.2), (5.4.6) is a finite sum of elements of $\mathcal{H}'^{2s,\nu'}_{(k),0}(\frac{2s+1}{3})$ for some ν' and for $k \geq \kappa$. We just have to apply (2.1.44), (2.1.47) to get (5.4.6).

To conclude the proof of our main theorem, we still need to prove proposition 5.4.1.

Proof of proposition 5.4.1. — We decompose $\tilde{G}_1' + \tilde{G}'$ as a sum of homogeneous terms

$$\tilde{G}_1' + \tilde{G}' = \sum_{k=0}^{\kappa-1} Q_k(v) = Q(v) \tag{5.4.7}$$

with $Q_0(v) = \frac{1}{2}\langle \Lambda_m v, v\rangle$ and for $1 \leq k \leq \kappa - 1$

$$Q_k(v) = \frac{1}{2}\langle \mathrm{Op}_\chi[\lambda_k(v;\cdot)I + \mu_k(v;\cdot)J]v, v\rangle + \frac{1}{2}\langle M_k(v)v, v\rangle \tag{5.4.8}$$

so that $Q_k \in \mathcal{H}'^{1,\nu}_{(k),0}(0)$ for some $\nu > 0$. According to (5.1.14)

$$\{\Theta_s^0 \circ \chi_H^\kappa, Q\}_\kappa = \{\Theta_s^0, Q \circ (\chi_H^\kappa)^{-1}\}_\kappa \circ \chi_H^\kappa. \tag{5.4.9}$$

We shall construct $H \in \mathcal{H}'^{1,\nu_0}_{(1),N_0}(0)$ for some ν_0, so that $\{\Theta_s^0, Q \circ (\chi_H^\kappa)^{-1}\}_\kappa$ is zero. This will give the wanted conclusion. By the second relation (5.1.13) and (5.1.10)

$$Q \circ (\chi_H^\kappa)^{-1}(v) = \sum_{k=0}^{\kappa-1}\sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \mathrm{Ad}_\kappa{}^j H \cdot Q_k \tag{5.4.10}$$

(where the j sum is actually finite). We look for H as $H = \sum_{\ell=1}^{\kappa-1} H_\ell$ with $H_\ell \in \mathcal{H}'^{1,\nu_\ell}_{(\ell),N_0}(0)$ for some increasing ν_ℓ, $\ell = 1, \dots, \kappa-1$, H_ℓ homogeneous of degree ℓ. By (i) of proposition 3.3.4

$$\{H_{\ell_1}, \{H_{\ell_2}, \cdots, \{H_{\ell_p}, Q_k\}\} \cdots\}$$

belongs to $\mathcal{H}'^{1,\nu'_\ell}_{(\ell),N_0}(0)$ for some ν'_ℓ, with $\ell = \ell_1 + \cdots + \ell_p + k$ (we used again that $\mathcal{H}'^{1,\nu'}_{(\ell),N_0}(\tilde{\zeta}) \subset \mathcal{H}'^{1,\nu'+\tilde{\zeta}}_{(\ell),N_0}(0)$). Consequently the contribution homogeneous of degree k, $1 \leq k \leq \kappa - 1$ in (5.4.10) may be written

$$Q_k - \{H_k, Q_0\} + K_k \tag{5.4.11}$$

where $K_k \in \mathcal{H}'^{1,\nu'_k}_{(\ell),N_0}(0)$ for some increasing ν'_k, $1 \leq k \leq \kappa - 1$, K_k depending only on $H_1, \dots, H_{k-1}$. To solve the equation

$$\{\Theta_s^0, Q \circ (\chi_H^\kappa)^{-1}\}_\kappa = 0$$

we just need to construct recursively H_k, $k = 1, \dots, \kappa - 1$ so that, by (5.4.10), (5.4.11)

$$\{\Theta_s^0, Q_k + K_k - \{H_k, Q_0\}\} = 0. \tag{5.4.12}$$

By definition of $\mathcal{H}'^{1,\nu'_k}_{(k),N_0}(0)$, and the fact that Q_k, H_k are homogeneous of degree k, we may write

$$(Q_k + K_k)(v) = \frac{1}{2}\langle \mathrm{Op}_\chi[\lambda_k(v;\cdot)I + \mu_k(v;\cdot)J]v, v\rangle + \frac{1}{2}\langle M_k(v)v, v\rangle \tag{5.4.13}$$

with $\lambda_k, \mu_k \in \widetilde{S}^{1,\nu'_k}_{(k),N_0}(0)$ with $\bar{\lambda}_k^\vee = \lambda_k$, $\bar{\mu}_k^\vee = \mu_k$, $M_k \in \widetilde{\mathcal{L}}^{1,\nu'_k}_{(k)}(0)$, λ_k, μ_k, M_k being homogeneous of degree k. The proof of proposition 5.4.1 will be complete as soon as we shall have solved (5.4.12). This is the aim of next lemma. □

Lemma 5.4.3. — *There is $N_0 \in \mathbb{N}$ and there are symbols $\tilde{\lambda}_k, \tilde{\mu}_k \in \widetilde{S}^{1,\nu'_k+N_0}_{(k),N_0}(0)$ and operators $\widetilde{M}_k \in \widetilde{\mathcal{L}}^{1,\nu'_k+N_0}_{(k)}(0)$ homogeneous of degree k, with $\bar{\tilde{\lambda}}_k^\vee = \tilde{\lambda}_k, \bar{\tilde{\mu}}_k^\vee = \tilde{\mu}_k$ such that*

$$\begin{aligned} &\frac{1}{2}\langle \mathrm{Op}_\chi[\lambda_k(v;\cdot)I + \mu_k(v;\cdot)J]v, v\rangle + \frac{1}{2}\langle M_k(v)v, v\rangle \\ &\quad - \frac{1}{2}\{\langle \mathrm{Op}_\chi[\tilde{\lambda}_k(v;\cdot)I + \tilde{\mu}_k(v;\cdot)J]v, v\rangle + \langle \widetilde{M}_k(v)v, v\rangle, Q_0\} \end{aligned} \tag{5.4.14}$$

Poisson commutes with Θ_s^0.

Proof. — We shall prove lemma 5.4.3 using the same complex coordinates system as in section 5.2, namely

$$\begin{bmatrix} w \\ \overline{w} \end{bmatrix} = K \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

We do not write the index k all along the proof. Define

$$\gamma(w, \overline{w}; x, n) = \lambda\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}; x, n\Big) + i\mu\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}; x, n\Big) \tag{5.4.15}$$

so that, since $\lambda = \bar{\lambda}^\vee, \mu = \bar{\mu}^\vee$,

$$\frac{1}{2}\langle \mathrm{Op}_\chi[\lambda(v;\cdot)I + \mu(v;\cdot)J]v, v\rangle = \mathrm{Re}\int_{\mathbb{S}^1}[\mathrm{Op}_\chi[\gamma(w, \overline{w};\cdot)]w]\overline{w}dx. \tag{5.4.16}$$

Decompose

$$M(v) = M_1(v)I + M_2(v)J + \underline{M}_1(v)I' + \underline{M}_2(v)J'$$

where $M_i(v), \underline{M}_i(v)$ are operators acting from $H^s(\mathbb{S}^1;\mathbb{R})$ to itself. We define

$$\begin{aligned} \Gamma(w, \overline{w}) &= M_1\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big) + iM_2\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big) \\ \underline{\Gamma}(w, \overline{w}) &= \underline{M}_1\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big) - i\underline{M}_2\Big(K^{-1}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big) \end{aligned} \tag{5.4.17}$$

so that

$$\frac{1}{2}\langle M(v)v,v\rangle = \mathrm{Re}\int_{\mathbb{S}^1}[\Gamma(w,\overline{w})w]\overline{w}dx + \mathrm{Re}\int_{\mathbb{S}^1}[\underline{\Gamma}(w,\overline{w})w]wdx. \tag{5.4.18}$$

We shall look for a symbol $\tilde{\gamma}(w,\overline{w};\cdot)$ and for operators $\tilde{\Gamma}(w,\overline{w}), \underline{\tilde{\Gamma}}(w,\overline{w})$ so that

$$\begin{aligned}&\mathrm{Re}\Big[\int_{\mathbb{S}^1}[\mathrm{Op}_\chi(\gamma)w]\overline{w}dx + \int_{\mathbb{S}^1}[\Gamma(w,\overline{w})w]\overline{w}dx + \int_{\mathbb{S}^1}[\underline{\Gamma}(w,\overline{w})w]wdx\\ &-\Big\{\int_{\mathbb{S}^1}[\mathrm{Op}_\chi(\tilde{\gamma})w]\overline{w}dx + \int_{\mathbb{S}^1}[\tilde{\Gamma}(w,\overline{w})w]\overline{w}dx + \int_{\mathbb{S}^1}[\underline{\tilde{\Gamma}}(w,\overline{w})w]w, Q_0\Big\}\Big]\end{aligned} \tag{5.4.19}$$

Poisson commutes with $\Theta_s^0(w,\overline{w}) = \int_{\mathbb{S}^1}(\Lambda_m^{2s}w)\overline{w}dx$. We decompose

$$\gamma(w,\overline{w};\cdot) = \sum_{\ell=0}^{k}\gamma_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell};\cdot)$$

$$\Gamma(w,\overline{w};\cdot) = \sum_{\ell=0}^{k}\Gamma_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell})$$

$$\underline{\Gamma}(w,\overline{w};\cdot) = \sum_{\ell=-1}^{k-1}\underline{\Gamma}_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell+1},\underbrace{w,\dots,w}_{k-\ell-1})$$

with $\gamma_\ell \in {}^{\mathbb{C}}\widetilde{\Sigma}^{1,\nu}_{(k),N_0}(0)$, $\Gamma_\ell, \underline{\Gamma}_\ell \in {}^{\mathbb{C}}\widetilde{\mathcal{L}}^{1,\nu}_{(k)}(0)$. When k is odd or k is even and $\ell \neq \frac{k}{2}$, we set $\gamma''_\ell = \gamma_\ell$. When k is even and $\ell = \frac{k}{2}$ we decompose

$$\gamma_\ell(w_1,\dots,w_k;\cdot) = \gamma'_\ell(w_1,\dots,w_k;\cdot) + \gamma''_\ell(w_1,\dots,w_k;\cdot) \tag{5.4.20}$$

according to (3.4.7), (3.4.8). By (3.4.10), (3.4.11) and (1.2.14)

$$\begin{aligned}&\Big\{\int_{\mathbb{S}^1}(\mathrm{Op}_\chi[\gamma'_\ell(\overline{w},\cdots,w;\cdot)]w)\overline{w}dx, \int_{\mathbb{S}^1}[\Lambda_m^{2s}w]\overline{w}dx\Big\}\\ &= i{\sum}' q(n_0,\dots,n_{k+1})\\ &\quad\times\int_{\mathbb{S}^1}[\Pi_{n_0}\mathrm{Op}_\chi[\gamma_\ell(\Pi_{n_1}\overline{w},\dots,\Pi_{n_\ell}\overline{w},\Pi_{n_{\ell+1}}w,\dots,\Pi_{n_k}w;\cdot)]\Pi_{n_{k+1}}w]\overline{w}dx\end{aligned} \tag{5.4.21}$$

where ${\sum}'$ denotes the sum over those $(n_0,\dots,n_{k+1})$ such that there is a bijection $\theta : \{0,\dots,\ell\} \to \{\ell+1,\dots,k+1\}$ with $|n_{\theta(j)}| = |n_j|$ for $j = 0,\dots,\ell$ and where

$$q(n_0,\dots,n_{k+1}) = -\sum_{j=0}^{\ell}(m^2+n_j^2)^s + \sum_{j=\ell+1}^{k+1}(m^2+n_j^2)^s.$$

By definition of ${\sum}'$, this quantity vanishes on the summation, so that (5.4.21) is identically zero, and since we want to find $\tilde{\gamma}, \tilde{\Gamma}, \underline{\tilde{\Gamma}}$ such that (5.4.19) is equal to quantities that Poisson commute to Θ_s^0, we may in the left hand side of (5.4.19) replace γ

by

$$\gamma''(w,\overline{w};\cdot) = \sum_{\ell=0}^{k} \gamma''_\ell(\overline{w},\dots,\overline{w},w,\dots,w;\cdot). \tag{5.4.22}$$

We decompose in the same way Γ_ℓ, $\underline{\Gamma}_\ell$. When k is odd or when k is even and $\ell \neq \frac{k}{2}$ we set $\Gamma''_\ell = \Gamma_\ell$, $\underline{\Gamma}''_\ell = \underline{\Gamma}_\ell$. When k is even and $\ell = \frac{k}{2}$, we write $\Gamma_\ell = \Gamma'_\ell + \Gamma''_\ell$, $\underline{\Gamma}_\ell = \underline{\Gamma}'_\ell + \underline{\Gamma}''_\ell$ with

$$\Gamma'_\ell(w_1,\dots,w_\ell) = {\sum}' \Pi_{n_0}\Gamma'_\ell(\Pi_{n_1}w_1,\dots,\Pi_{n_k}w_k)\Pi_{n_{k+1}}$$

resp.

$$\underline{\Gamma}'_\ell(w_1,\dots,w_\ell) = {\sum}' \Pi_{n_0}\underline{\Gamma}'_\ell(\Pi_{n_1}w_1,\dots,\Pi_{n_k}w_k)\Pi_{n_{k+1}},$$

where ${\sum}'$ is the sum for those $n_0,\dots,n_{k+1}$ such that there is a bijection $\theta : \{0,\dots,\ell\} \to \{\ell+1,\dots,k+1\}$ (resp. $\theta : \{1,\dots,\ell+1\} \to \{0,\ell+2,\dots,k+1\}$) with $|n_{\theta(j)}| = |n_j|$ for any $j \in \{0,\dots,\ell\}$ (resp. $j \in \{1,\dots,\ell+1\}$). As above,

$$\Big\{\int_{\mathbb{S}^1} [\Gamma'_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell})w]\overline{w}dx, \Theta_s^0\Big\} \equiv 0$$

$$\Big\{\int_{\mathbb{S}^1} [\underline{\Gamma}'_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell+1},\underbrace{w,\dots,w}_{k-\ell-1})w]wdx, \Theta_s^0\Big\} \equiv 0.$$

Consequently we may replace in (5.4.19) Γ (resp. $\underline{\Gamma}$) by $\Gamma'' = \sum_{\ell=0}^k \Gamma''_\ell$ (resp. $\underline{\Gamma}'' = \sum_{\ell=-1}^{k-1} \underline{\Gamma}''_\ell$). We have in this way reduced ourselves to finding $\tilde{\gamma}_\ell$, $\widetilde{\Gamma}_\ell$, $\widetilde{\underline{\Gamma}}_\ell$ such that

$$\begin{aligned}\int_{\mathbb{S}^1} [(\mathrm{Op}_\chi\gamma''_\ell)w]\overline{w}dx + \int_{\mathbb{S}^1} [\Gamma''_\ell w]\overline{w}dx + \int_{\mathbb{S}^1} [\underline{\Gamma}''_\ell w]wdx \\ = \Big\{\int_{\mathbb{S}^1} [(\mathrm{Op}_\chi\tilde{\gamma}_\ell)w]\overline{w}dx + \int_{\mathbb{S}^1} [\widetilde{\Gamma}_\ell w]\overline{w}dx + \int_{\mathbb{S}^1} [\widetilde{\underline{\Gamma}}_\ell w]wdx, Q_0\Big\}\end{aligned} \tag{5.4.23}$$

where in these expressions $\gamma''_\ell, \Gamma''_\ell, \tilde{\gamma}_\ell, \widetilde{\Gamma}_\ell$ (resp. $\underline{\Gamma}''_\ell, \widetilde{\underline{\Gamma}}_\ell$) are computed at the argument $(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell})$ (resp. $(\underbrace{\overline{w},\dots,\overline{w}}_{\ell+1},\underbrace{w,\dots,w}_{k-\ell-1})$). Let us define $L_\ell[\mathrm{Op}_\chi(\tilde{\gamma}_\ell)]$ and $L_\ell(\widetilde{\Gamma}_\ell)$ by (3.4.12) with $\omega = (\omega_0,\dots,\omega_{k+1})$ given by $\omega_0 = \omega_1 = \dots = \omega_\ell = -1$, $\omega_{\ell+1} = \dots = \omega_{k+1} = 1$ and $L_\ell(\widetilde{\underline{\Gamma}}_\ell)$ by (3.4.12) with $\omega_1 = \dots = \omega_{\ell+1} = -1$, $\omega_0 = \omega_{\ell+2} = \dots = \omega_{k+1} = 1$. To solve (5.4.23), we remark that since $Q_0(w,\overline{w}) = \int_{\mathbb{S}^1}(\Lambda_m w)\overline{w}dx$, we have by (1.2.14)

$$\Big\{\int_{\mathbb{S}^1} [(\mathrm{Op}_\chi(\tilde{\gamma}_\ell))w]\overline{w}dx, Q_0\Big\} = i\int_{\mathbb{S}^1} [L_\ell(\mathrm{Op}_\chi(\tilde{\gamma}_\ell))w]\overline{w}dx$$

$$\Big\{\int_{\mathbb{S}^1} [\widetilde{\Gamma}_\ell w]\overline{w}dx, Q_0\Big\} = i\int_{\mathbb{S}^1} [L_\ell(\widetilde{\Gamma}_\ell)w]\overline{w}dx$$

$$\Big\{\int_{\mathbb{S}^1} [\widetilde{\underline{\Gamma}}_\ell w]wdx, Q_0\Big\} = i\int_{\mathbb{S}^1} [L_\ell(\widetilde{\underline{\Gamma}}_\ell)w]wdx$$

so that we need to find $\tilde{\gamma}_\ell \in {}^{\mathbb{C}}\widetilde{\Sigma}^{1,\nu+N_0}_{(k),N_0}(0)$, $\widetilde{\Gamma}_\ell \in {}^{\mathbb{C}}\tilde{\Lambda}^{1,\nu+N_0}_{(k)}(0)$, $\widetilde{\underline{\Gamma}}_\ell \in {}^{\mathbb{C}}\tilde{\Lambda}^{1,\nu+N_0}_{(k)}(0)$ such that

$$\begin{aligned} iL_\ell[\mathrm{Op}_\chi(\tilde{\gamma}_\ell)] &= \mathrm{Op}_\chi(\gamma''_\ell) \\ iL_\ell(\widetilde{\Gamma}_\ell) = \Gamma''_\ell, iL_\ell(\widetilde{\underline{\Gamma}}_\ell) &= \underline{\Gamma}''_\ell. \end{aligned} \tag{5.4.24}$$

By (ii) and (iii) of proposition 3.4.4, we may solve the first equation (5.4.24) if we assume that m is outside the exceptional subset $\mathcal{N}$ of the statement of that proposition. We get a symbol $\tilde{\gamma}_\ell$ if we assume that N_0 has been taken larger than $2(N_1+1)$. To solve the equation involving $\widetilde{\Gamma}_\ell, \widetilde{\underline{\Gamma}}_\ell$ we use proposition 3.4.5. We set next

$$\tilde{\gamma}(w,\overline{w};\cdot) = \sum_{\ell=0}^{k} \tilde{\gamma}_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell},\cdot)$$

$$\tilde{\Gamma}(w,\overline{w}) = \sum_{\ell=0}^{k} \tilde{\Gamma}_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell},\underbrace{w,\dots,w}_{k-\ell})$$

$$\underline{\tilde{\Gamma}}(w,\overline{w}) = \sum_{\ell=-1}^{k-1} \underline{\tilde{\Gamma}}_\ell(\underbrace{\overline{w},\dots,\overline{w}}_{\ell+1},\underbrace{w,\dots,w}_{k-\ell-1}).$$

Let us define

$$\begin{aligned} \tilde{\lambda}(u;x,n) &= \frac{1}{2}[\tilde{\gamma}(Ku;x,n) + \overline{\tilde{\gamma}(Ku;x,-n)}] \\ \tilde{\mu}(u;x,n) &= \frac{1}{2i}[\tilde{\gamma}(Ku;x,n) - \overline{\tilde{\gamma}(Ku;x,-n)}] \end{aligned}$$

so that

$$\mathrm{Re}\int_{\mathbb{S}^1} [\mathrm{Op}_\chi[\tilde{\gamma}(w,\overline{w};\cdot)]w]\overline{w}dx = \frac{1}{2}\langle \mathrm{Op}_\chi[\tilde{\lambda}I + \tilde{\mu}J]u,u\rangle.$$

In the same way, if we set

$$\begin{aligned} \widetilde{M}_1 &= \frac{1}{2}[\widetilde{\Gamma}(Ku) + \overline{\widetilde{\Gamma}(Ku)}] \\ \widetilde{M}_2 &= \frac{1}{2i}[\widetilde{\Gamma}(Ku) - \overline{\widetilde{\Gamma}(Ku)}], \end{aligned}$$

we get

$$\mathrm{Re}\int_{\mathbb{S}^1} [\widetilde{\Gamma}(w,\overline{w})w]\overline{w}dx = \frac{1}{2}\langle (\widetilde{M}_1(u)I + \widetilde{M}_2(u)J)u,u\rangle.$$

Analogously, setting

$$\begin{aligned} \widetilde{\underline{M}}_1(u) &= \frac{1}{2}[\widetilde{\underline{\Gamma}}(Ku) + \overline{\widetilde{\underline{\Gamma}}(Ku)}] \\ \widetilde{\underline{M}}_2(u) &= -\frac{1}{2i}[\widetilde{\underline{\Gamma}}(Ku) - \overline{\widetilde{\underline{\Gamma}}(Ku)}], \end{aligned}$$

we get

$$\mathrm{Re}\int_{\mathbb{S}^1} [\widetilde{\underline{\Gamma}}(w,\overline{w})w]wdx = \frac{1}{2}\langle (\widetilde{\underline{M}}_1(u)I' + \widetilde{\underline{M}}_2(u)J')u,u\rangle.$$

Finally, if $\widetilde{M}(u) = \widetilde{M}_1(u)I + \widetilde{M}_2(u)J + \underline{\widetilde{M}}_1(u)I' + \underline{\widetilde{M}}_2(u)J'$, we see that (5.4.19) implies the conclusion (5.4.14). This concludes the proof of the lemma. □

BIBLIOGRAPHY

[1] D. Bambusi: *Birkhoff normal form for some nonlinear PDEs*, Comm. Math. Phys. 234 (2003), no. 2, 253–285.

[2] D. Bambusi, J.-M. Delort, B. Grébert and J. Szeftel: *Almost global existence for Hamiltonian semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Comm. Pure Appl. Math. 60 (2007), no. 11, 1665–1690.

[3] D. Bambusi and B. Grébert: *Birkhoff normal form for partial differential equations with tame modulus*, Duke Math. J. 135 (2006), no. 3, 507–567.

[4] J.-M. Bony: *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 2, 209–246.

[5] J. Bourgain: *Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations*, Geom. Funct. Anal. 6 (1996), no. 2, 201–230.

[6] J. Bourgain: *Green's function estimates for lattice Schrödinger operators and applications*. Annals of Mathematics Studies, 158. Princeton University Press, Princeton, NJ, (2005), x+173 pp.

[7] W. Craig: *Problèmes de petits diviseurs dans les équations aux dérivées partielles*. Panoramas et Synthèses, 9. Société Mathématique de France, Paris, (2000), viii+120 pp.

[8] W. Craig and C. E. Wayne: *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure Appli. Math., 46 (1993), 1409–1498.

[9] J.-M. Delort: *Existence globale et comportement asymptotique pour l´équation de Klein-Gordon quasilinéaire à données petites en dimension 1*, Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 1, 1–61. Erratum: Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 2, 335–345.

[10] J.-M. Delort: *Long-time Sobolev stability for small solutions of quasi-linear Klein-Gordon equations on the circle*. To appear, Trans. Amer. Math. Soc.

[11] J.-M. Delort and J. Szeftel: *Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres*, Internat. Math. Res. Notices (2004), no. 37, 1897–1966.

[12] J.-M. Delort and J. Szeftel: *Long-time existence for semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Amer. J. Math. 128 (2006), no. 5, 1187–1218.

[13] H. Eliasson and S. Kuksin: *KAM For the non-linear Schrödinger equation*, to appear, Annals of Mathematics.

[14] B. Grébert: *Birkhoff normal form and Hamiltonian PDEs*. Partial differential equations and applications, 1–46, Sémin. Congr., 15, Soc. Math. France, Paris, 2007.

[15] S. Klainerman : *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space–time dimensions*, Comm. Pure Appl. Math. 38, (1985) 631–641.

[16] S. Kuksin: *Nearly integrable infinite-dimensional Hamiltonian systems*. Lecture Notes in Mathematics, 1556. Springer-Verlag, Berlin, 1993. xxviii+101 pp.

[17] S. Kuksin: *Analysis of Hamiltonian PDEs*. Oxford Lecture Series in Mathematics and its Applications, 19. Oxford University Press, Oxford, 2000. xii+212 pp.

[18] T. Ozawa, K. Tsutaya and Y. Tsutsumi: *Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions*. Math. Z. 222 (1996), no. 3, 341–362.

[19] J. Shatah: *Normal forms and quadratic nonlinear Klein-Gordon equations*. Comm. Pure Appl. Math. 38 (1985), no. 5, 685–696.

[20] C. E. Wayne: *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Comm. Math. Phys., 127 (1990), 479–528.

INDEX