

MODERATE DEVIATIONS FOR STATIONARY SEQUENCES OF HILBERT-VALUED BOUNDED RANDOM VARIABLES

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ABSTRACT. In this paper, we derive the Moderate Deviation Principle for stationary sequences of bounded random variables with values in a Hilbert space. The conditions obtained are expressed in terms of martingale-type conditions. The main tools are martingale approximations and a new Hoeffding inequality for non-adapted sequences of Hilbert-valued random variables. Applications to Cramér-Von Mises statistics, functions of linear processes and stable Markov chains are given.

1. INTRODUCTION

Let \mathbb{H} be a separable Hilbert space with norm $\|\cdot\|_{\mathbb{H}}$ generated by an inner product, $\langle \cdot, \cdot \rangle_{\mathbb{H}}$, and $(e_l)_{l \geq 1}$ be an orthonormal basis of \mathbb{H} .

For the stationary sequence $(X_i)_{i \in \mathbb{Z}}$, of centered random variables with values in \mathbb{H} , define the partial sums and the normalized process $\{Z_n(t) : t \in [0, 1]\}$ by

$$S_n = \sum_{j=1}^n X_j \quad \text{and} \quad Z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i + \frac{1}{\sqrt{n}}(nt - [nt])X_{[nt]+1},$$

$[\cdot]$ denoting the integer part.

In this paper, we are concerned with the Moderate Deviation Principle, for the process $Z_n(\cdot)$, considered as an element of $C_{\mathbb{H}}([0, 1])$, the set of all continuous functions from $[0, 1]$ to \mathbb{H} . This is a separable Banach space under the sup-norm $\|x\|_{\infty} = \sup\{\|x(t)\|_{\mathbb{H}} : t \in [0, 1]\}$. More generally, we say that a family of random variables $\{Z_n, n > 0\}$ satisfies the Moderate Deviation Principle (MDP) in E , a separable metric space, with speed $a_n \rightarrow 0$, and good rate function $I(\cdot)$, if the level sets $\{x, I(x) \leq \alpha\}$ are compact for all $\alpha < \infty$, and for all Borel sets Γ of E ,

$$\begin{aligned} -\inf\{I(x); x \in \overset{\circ}{\Gamma}\} &\leq \liminf_{n \rightarrow \infty} a_n \log \mathbb{P}(\sqrt{a_n}Z_n \in \Gamma) \\ &\leq \limsup_{n \rightarrow \infty} a_n \log \mathbb{P}(\sqrt{a_n}Z_n \in \Gamma) \leq -\inf\{I(x); x \in \bar{\Gamma}\}. \end{aligned} \tag{1.1}$$

From now, we assume that the stationary sequence $(X_i)_{i \in \mathbb{Z}}$ is given by $X_i = X_0 \circ T^i$, where $T : \Omega \rightarrow \Omega$ is a bijective bimeasurable transformation preserving the probability \mathbb{P} on (Ω, \mathcal{A}) . For a subfield \mathcal{F}_0 satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, let $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. By $\|X\|_{\mathbb{H}}$, we denote the $\mathbf{L}_{\mathbb{H}}^{\infty}$ -norm, that is the smallest u such that $\mathbb{P}(\|X\|_{\mathbb{H}} > u) = 0$.

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When $\mathbb{H} = \mathbb{R}$, Dedecker, Merlevède, Peligrad and Utev [6] have recently proved (see their Theorem 1), by using a martingale approximation approach, that:

Theorem 1.1. *Assume that $\|X_0\|_\infty < \infty$, and that X_0 is \mathcal{F}_0 -measurable. In addition, assume that*

$$\sum_{n=1}^{\infty} \frac{\|\mathbb{E}(S_n | \mathcal{F}_0)\|_\infty}{n^{3/2}} < \infty,$$

and that there exists $\sigma^2 \geq 0$ with

$$\lim_{n \rightarrow \infty} \left\| \mathbb{E} \left(\frac{S_n^2}{n} \middle| \mathcal{F}_0 \right) - \sigma^2 \right\|_\infty = 0.$$

Then, for all positive sequences $a_n \rightarrow 0$ and $na_n \rightarrow \infty$, the normalized process $Z_n(\cdot)$ satisfies the MDP in $C_{\mathbb{R}}([0, 1])$, with the good rate function $I_\sigma(\cdot)$ defined by

$$I_\sigma(h) = \frac{1}{2\sigma^2} \int_0^1 (h'(u))^2 du$$

if simultaneously $\sigma > 0$, $h(0) = 0$ and h is absolutely continuous, and $I_\sigma(h) = \infty$ otherwise.

The first aim of this paper is to extend the above result to random variables taking their values in a real and separable Hilbert space \mathbb{H} . Indeed, having asymptotic results concerning dependent random variables with values in \mathbb{H} allows for instance, to derive the corresponding asymptotic results for statistics of the type $\int_0^1 |\mathbb{F}_n(t) - \mathbb{F}(t)|^2 \mu(dt)$ where $\mathbb{F}(\cdot)$ is the cumulative distribution function of a strictly stationary sequence of real random variables $(Y_i)_{i \in \mathbb{Z}}$ and $\mathbb{F}_n(\cdot)$ is the corresponding empirical distribution function (see Section 3.4).

On an other hand, since Theorem 1.1 is stated for adapted sequences, the second aim of this paper is to extend this result to non-adapted sequences.

To extend Theorem 1.1 to non-adapted sequences of Hilbert-valued random variables, we use a similar martingale approach as done for instance in Volný [22] for the central limit theorem. In infinite dimensional cases, the authors have essentially considered i.i.d or triangular arrays of i.i.d random variables (see for instance de Acosta [1], Borovkov and Mogulskii [2] [3], Ledoux [13], ...). However for dependent sequences with values in functional spaces, there are few results available in the literature. Since our approach is based on martingale approximation, we first extend Puhalskii [19] results for \mathbb{R}^d -valued martingale differences sequences to the \mathbb{H} -valued case (see Section 4.2). In Section 2.1, we derive a Hoffding inequality for a sequence of non-adapted Hilbert-valued random variables. Section 4 is dedicated to the proofs.

2. MAIN RESULTS

We begin with some notations,

Notation 2.1. For any real $p \geq 1$, denote by $\mathbf{L}_{\mathbb{H}}^p$ the space of \mathbb{H} -valued random variables X such that $\|X\|_{\mathbf{L}_{\mathbb{H}}^p}^p = \mathbb{E}(\|X\|_{\mathbb{H}}^p)$ is finite. For example, $\mathbf{L}_{\mathbb{H}}^1([0, 1])$ is the space of \mathbb{H} -valued Bochner integrable functions on $[0, 1]$.

2.1. A Hoeffding inequality.

Firstly, we start by establishing a maximal inequality, which is obtained through a generalization of the ideas in Peligrad, Utev and Wu [17].

Theorem 2.2. *Assume that $\| \|X_0\|_{\mathbb{H}} \|_{\infty} < \infty$. For any $x > 0$, we have*

$$\mathbb{P}\left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}} \geq x\right) \leq 2\sqrt{e} \exp\left(-\frac{x^2}{4n(\| \|X_0\|_{\mathbb{H}} \|_{\infty} + C\Delta)^2}\right), \tag{2.1}$$

for some constant $C > 0$ and

$$\Delta = \sum_{j=1}^n \frac{1}{j^{3/2}} (\| \mathbb{E}(S_j | \mathcal{F}_0) \|_{\mathbb{H}} \|_{\infty} + \| \|S_j - \mathbb{E}(S_j | \mathcal{F}_j) \|_{\mathbb{H}} \|_{\infty}).$$

2.2. The Moderate Deviation Principle.

Before establishing our main result, we need more definitions.

Definition 2.3. A nonnegative self-adjoint operator Γ on \mathbb{H} will be called an $\mathcal{S}(\mathbb{H})$ -operator, if it has finite trace, i.e, for some (and therefore every) orthonormal basis $(e_l)_{l \geq 1}$ of \mathbb{H} , $\sum_{l \geq 1} \langle \Gamma e_l, e_l \rangle_{\mathbb{H}} < \infty$.

Let

$$\begin{aligned} \mathcal{AC}_0([0, 1]) &= \{ \phi \in C_{\mathbb{H}}([0, 1]) : \text{there exists } g \in \mathbf{L}_{\mathbb{H}}^1([0, 1]) \\ &\quad \text{such that } \phi(t) = \int_0^t g(s) ds \text{ for } t \in [0, 1] \}. \end{aligned}$$

Now, we give the extension of Theorem 1.1.

Theorem 2.4. *Assume that $\| \|X_0\|_{\mathbb{H}} \|_{\infty} < \infty$. Moreover, assume that*

$$\sum_{n \geq 1} \frac{1}{n^{3/2}} \| \mathbb{E}(S_n | \mathcal{F}_0) \|_{\mathbb{H}} \|_{\infty} < \infty \quad \text{and} \quad \sum_{n \geq 1} \frac{1}{n^{3/2}} \| \|S_n - \mathbb{E}(S_n | \mathcal{F}_n) \|_{\mathbb{H}} \|_{\infty} < \infty, \tag{2.2}$$

and that there exists $Q \in \mathcal{S}(\mathbb{H})$ such that

i. for all k, l in \mathbb{N}^* ,

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \mathbb{E}(\langle S_n, e_k \rangle_{\mathbb{H}} \langle S_n, e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) - \langle Q e_k, e_l \rangle_{\mathbb{H}} \right\|_{\infty} = 0, \tag{2.3}$$

ii.

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \mathbb{E}(\|S_n\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \text{Tr}(Q) \right\|_{\infty} = 0. \tag{2.4}$$

Then, for all positive sequences a_n with $a_n \rightarrow 0$ and $na_n \rightarrow \infty$, the process $Z_n(\cdot)$ satisfies the functional MDP in $C_{\mathbb{H}}([0, 1])$ with the good rate function,

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\phi'(t)) dt & \text{if } \phi \in \mathcal{AC}_0([0, 1]) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.5}$$

where Λ^* is given by:

$$\Lambda^*(x) = \sup_{y \in \mathbb{H}} \left(\langle y, x \rangle_{\mathbb{H}} - \frac{1}{2} \langle y, Qy \rangle_{\mathbb{H}} \right). \tag{2.6}$$

As an immediate consequence, we have:

Corollary 2.5. *Under the same notations and assumptions of Theorem 2.4, we have that, for all positive sequences a_n with $a_n \rightarrow 0$ and $na_n \rightarrow \infty$, $n^{-1/2}S_n$ satisfies the MDP in \mathbb{H} with the good rate function, Λ^* defined in (2.6).*

Since $\text{Tr}(Q) < \infty$, Q is a compact operator. If $x \in Q(\mathbb{H})$, then there is $z \in \mathbb{H}$, such that $x = Qz$. Hence, the rate function is

$$\forall x \in Q(\mathbb{H}), \Lambda^*(x) = \frac{1}{2} \langle z, Qz \rangle_{\mathbb{H}} = \frac{1}{2} \langle z, x \rangle_{\mathbb{H}}.$$

If $x \notin Q(\mathbb{H})$, we have $\Lambda^*(x) = +\infty$. In particular, if Q is injective, $(\lambda_i)_{i \geq 1}$ are its eigenvalues, and $(f_i)_{i \geq 0}$ the associated eigenvectors, we can simplify the rate function,

$$\forall x \in Q(\mathbb{H}), \Lambda^*(x) = \frac{1}{2} \sum_{i \geq 1} \frac{1}{\lambda_i} \langle x, f_i \rangle_{\mathbb{H}}^2.$$

The following corollary gives simplified conditions for the MDP.

Corollary 2.6. *Assume that $\|X_0\|_{\mathbb{H}} < \infty$. Moreover, assume that*

$$\sum_{n \geq 1} \frac{1}{\sqrt{n}} \|\mathbb{E}(X_n | \mathcal{F}_0)\|_{\mathbb{H}} < \infty \quad \text{and} \quad \sum_{n \geq 1} \frac{1}{\sqrt{n}} \|X_{-n} - \mathbb{E}(X_{-n} | \mathcal{F}_0)\|_{\mathbb{H}} < \infty, \quad (2.7)$$

and that for all i, j in \mathbb{N}^* ,

1. for all k, l in \mathbb{N}^* ,

$$\lim_{n \rightarrow \infty} \|\mathbb{E}(\langle X_i, e_k \rangle_{\mathbb{H}} \langle X_j, e_l \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle X_i, e_k \rangle_{\mathbb{H}}) \mathbb{E}(\langle X_j, e_l \rangle_{\mathbb{H}})\|_{\infty} = 0, \quad (2.8)$$

2.

$$\lim_{n \rightarrow \infty} \|\mathbb{E}(\langle X_i, X_j \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle X_i, X_j \rangle_{\mathbb{H}})\|_{\infty} = 0. \quad (2.9)$$

Then, the conclusion of Theorem 2.4 holds, with Q defined by

$$\text{for all } k, l \geq 1, \quad \langle e_k, Qe_l \rangle_{\mathbb{H}} = \sum_{p \in \mathbb{Z}} \mathbb{E}(\langle X_0, e_k \rangle_{\mathbb{H}} \langle X_p, e_l \rangle_{\mathbb{H}}).$$

2.3. Functional law of the iterated logarithm.

Throughout this section, let $\beta(n) = \sqrt{2n \log \log n}$, $n \geq 3$. Let $\tilde{S}_n(\cdot)$ be the process $\{\tilde{S}_n(t) = \sum_{i=1}^{[nt]} X_i + (nt - [nt])X_{[nt]+1} : t \in [0, 1]\}$.

Theorem 2.7. *Assume that $\|X_0\|_{\mathbb{H}} < \infty$. Assume in addition that (2.2), (2.3) and (2.4) hold. Then, with probability 1, the following sequence*

$$\left\{ \xi_n(\cdot) = \frac{\tilde{S}_n(\cdot)}{\beta(n)} \right\}_{n \geq 1}$$

is relatively compact in $C_{\mathbb{H}}([0, 1])$ and the set of its limit points is precisely the compact set

$$\mathcal{K} = \{\phi \in C_{\mathbb{H}}([0, 1]), \text{ such that } 2I(\phi) \leq 1\}.$$

Proof of Theorem 2.7. It can be proved by the arguments of the proof of Theorem 3.1 in Hu and Lee [11] (see also Deuschel and Stroock [9]). \square

3. APPLICATIONS

3.1. ϕ -mixing sequences.

Recall that if Y is a random variable with values in a Polish space \mathcal{Y} and if \mathcal{F} is a σ -field, the ϕ -mixing coefficient between \mathcal{F} and $\sigma(Y)$ is defined by

$$\phi(\mathcal{F}, \sigma(Y)) = \sup_{A \in \mathcal{B}(\mathcal{Y})} \|\mathbb{P}_{Y|\mathcal{F}}(A) - \mathbb{P}_Y(A)\|_\infty.$$

For the sequence $(X_i)_{i \in \mathbb{Z}}$, let

$$\phi_1(n) = \phi(\mathcal{F}_0, \sigma(X_n)) \text{ and } \phi_2(n) = \sup_{i > j \geq n} \phi(\mathcal{F}_0, \sigma(X_i, X_j)).$$

Proposition 3.1. *Assume that $\|X_0\|_{\mathbb{H}} < \infty$ and X_0 is \mathcal{F}_0 -measurable. Then, for all $x \geq 0$, we have*

$$\mathbb{P}\left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}} \geq x\right) \leq 2\sqrt{e} \exp\left(-\frac{x^2}{4n\|X_0\|_{\mathbb{H}}^2(1 + 6C \sum_{j \geq 1} j^{-1/2} \phi_1(j))^2}\right), \tag{3.1}$$

for the same positive constant C defined in Theorem 2.2.

Proof of Proposition 3.1. Applying triangle inequality and changing the order of summation, observe that

$$\sum_{n \geq 1} \frac{1}{n^{3/2}} \|\mathbb{E}(S_n | \mathcal{F}_0)\|_{\mathbb{H}} \leq 3 \sum_{n \geq 1} \frac{1}{\sqrt{n}} \|\mathbb{E}(X_n | \mathcal{F}_0)\|_{\mathbb{H}}.$$

Since $\mathbb{E}(X_0) = 0$, we have

$$\|\mathbb{E}(X_n | \mathcal{F}_0)\|_{\mathbb{H}} \leq 2\|X_0\|_{\mathbb{H}} \phi_1(n).$$

□

Next, we have a Moderate Deviation Principle.

Proposition 3.2. *Assume that $\|X_0\|_{\mathbb{H}} < \infty$ and X_0 is \mathcal{F}_0 -measurable. If*

$$\sum_{n \geq 1} \frac{1}{\sqrt{n}} \phi_1(n) < \infty \text{ and } \phi_2(n) \xrightarrow[n \rightarrow \infty]{} 0,$$

then the conclusion of Theorem 2.4 holds.

3.2. Functions of linear processes.

In this section, we shall focus on functions of \mathbb{H} -valued linear processes,

$$X_k = f\left(\sum_{i \in \mathbb{Z}} c_i(\varepsilon_{k-i})\right) - \mathbb{E}\left(f\left(\sum_{i \in \mathbb{Z}} c_i(\varepsilon_{k-i})\right)\right), \tag{3.2}$$

where $f : \mathbb{H} \rightarrow \mathbb{H}$, $(c_i)_{i \in \mathbb{Z}}$ are linear operators from \mathbb{H} to \mathbb{H} and $(\varepsilon_i)_{i \in \mathbb{Z}}$ is a sequence of i.i.d \mathbb{H} -valued random variables such that $\|\varepsilon_0\|_{\mathbb{H}} < \infty$.

The sequence $\{X_k\}_{k \geq 1}$ defined by (3.2) is a natural extension of the multivariate linear processes. These types of processes with values in functional spaces also facilitate the study of estimating and forecasting problems for several classes of continuous time processes (see Bosq [4]).

We denote by $\|\cdot\|_{L(\mathbb{H})}$, the operator norm. We shall give sufficient conditions for the Moderate Deviation Principle in terms of the regularity of the function f .

Let $\delta(\varepsilon_0) = 2 \inf\{\| \varepsilon_0 - x \|_{\mathbb{H}}, x \in \mathbb{H}\}$ and define the modulus of continuity of f by

$$w_f(h) = \sup_{\|t\|_{\mathbb{H}} \leq h, x \in \mathbb{H}} \|f(x+t) - f(x)\|_{\mathbb{H}}.$$

Proposition 3.3. *Assume that*

$$\sum_{i \in \mathbb{Z}} \|c_i\|_{L(\mathbb{H})} < \infty,$$

and that X_k is defined as in (3.2). If moreover

$$\sum_{n \geq 1} \frac{1}{\sqrt{n}} w_f\left(\delta(\varepsilon_0) \sum_{k \geq n} \|c_k\|_{L(\mathbb{H})}\right) + \sum_{n \geq 1} \frac{1}{\sqrt{n}} w_f\left(\delta(\varepsilon_0) \sum_{k \leq -n} \|c_k\|_{L(\mathbb{H})}\right) < \infty, \quad (3.3)$$

then the conclusion of Theorem 2.4 holds.

In particular, if $\|c_i\|_{L(\mathbb{H})} = O(\rho^{|i|})$, $0 < \rho < 1$, the condition (3.3) is equivalent to

$$\int_0^1 \frac{w_f(t)}{t \sqrt{|\log t|}} dt < \infty. \quad (3.4)$$

For example, if $w_f(t) \leq D |\log t|^{-\gamma}$ for some $D > 0$ and some $\gamma > 1/2$, then (3.4) holds.

Remark: Under a Cramer type condition, Mas and Menneveau [15] were interested in the MDP for the asymptotic behavior of the empirical mean of $\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$, where for all $n \geq 1$, X_n is an autoregressive process: $X_n = \sum_{j=0}^{\infty} \rho^j (\varepsilon_{n-j})$. Here, $(\varepsilon_k)_{k \in \mathbb{Z}}$ is a sequence of i.i.d Hilbert-valued centered random variables satisfying a Cramer type condition and ρ is a bounded Hilbert linear operator, satisfying $\sum_{j=0}^{\infty} \|\rho^j\|_{L(\mathbb{H})} < \infty$. They gave also the MDP for the difference between the empirical and theoretical covariance operators.

3.3. Stable Markov chains.

Let $(Y_n)_{n \geq 0}$ be a stationary Markov chain of \mathbb{H} -valued bounded random variables. Denote by μ the law of Y_0 and by K its transition kernel.

Let

$$X_k = f(Y_k) - \mathbb{E}(f(Y_k)). \quad (3.5)$$

For all Lipschitz functions $g : \mathbb{H} \rightarrow \mathbb{H}$, let

$$\text{Lip}(g) = \sup_{x, y \in \mathbb{H}} \frac{\|g(x) - g(y)\|_{\mathbb{H}}}{\|x - y\|_{\mathbb{H}}}.$$

We write $K(g)$ and $K^n(g)$ respectively for the functions $y \mapsto \int g(z) K(y, dz)$ and $y \mapsto \int g(z) K^n(y, dz) = \mathbb{E}(g(Y_n) | Y_0 = y)$.

Proposition 3.4. *Assume that the transition kernel K satisfies $\text{Lip}(K^n(g)) \leq C \rho^n \text{Lip}(g)$ for some $\rho < 1$ and any Lipschitz function g . If f is Lipschitz and X_i is defined by (3.5), then the normalized process $Z_n(\cdot)$ satisfies the MDP in $C_{\mathbb{H}}([0, 1])$.*

Let $(Y_n)_{n \geq 0}$ satisfying the equation $Y_n = F(Y_{n-1}, \xi_n)$ for some measurable map F and some sequence $(\xi_i)_{i \geq 0}$ of i.i.d random variables independent of Y_0 .

Corollary 3.5. *Assume that for all $x, y \in \mathbb{H}$, $\|F(x, \xi_1) - F(y, \xi_1)\|_{\mathbb{H}} \|\mathbf{1}_{\mathbb{R}, \mathbb{P}_{\xi_1}}\|_{\mathbf{L}^1} \leq \rho \|x - y\|_{\mathbb{H}}$ with $\rho < 1$. If f is Lipschitz and X_i is defined by (3.5), then the normalized process $Z_n(\cdot)$ satisfies the MDP in $C_{\mathbb{H}}([0, 1])$.*

Proof of Corollary 3.5. The condition: for all $x, y \in \mathbb{H}$, $\|F(x, \xi_1) - F(y, \xi_1)\|_{\mathbb{H}} \|\mathbf{1}_{\mathbb{R}, \mathbb{P}_{\xi_1}}\|_{\mathbf{L}^1} \leq \rho \|x - y\|_{\mathbb{H}}$ with $\rho < 1$, implies that, for all f Lipschitz function,

$$\text{Lip}(K(f)) \leq \rho \text{Lip}(f).$$

Indeed, for all y in \mathbb{H} , we get

$$K(f)(y) = \mathbb{E}(f(Y_1) \mid Y_0 = y) = \mathbb{E}(f(F(y, \xi_1)) \mid Y_0 = y) = \int f(F(y, z)) \mathbb{P}_{\xi_1}(dz).$$

Hence, for all $x, y \in \mathbb{H}$, we derive

$$\begin{aligned} |K(f)(x) - K(f)(y)| &\leq \int |f(F(x, z)) - f(F(y, z))| \mathbb{P}_{\xi_1}(dz) \\ &\leq \text{Lip}(f) \|F(x, \xi_1) - F(y, \xi_1)\|_{\mathbb{H}} \|\mathbf{1}_{\mathbb{R}, \mathbb{P}_{\xi_1}}\|_{\mathbf{L}^1} \\ &\leq \rho \text{Lip}(f) \|x - y\|_{\mathbb{H}}. \end{aligned}$$

It follows that

$$\text{Lip}(K(f)) \leq \rho \text{Lip}(f),$$

so that

$$\text{Lip}(K^n(f)) \leq \rho^n \text{Lip}(f).$$

We conclude by applying Proposition 3.4. □

3.4. Moderate Deviation Principle for the empirical distribution function in \mathbf{L}^2 .

Let $Y = (Y_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables with common distribution function \mathbb{F} . Set $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$. We denote \mathbb{F}_n , the empirical distribution function of Y :

$$\forall t \in \mathbb{R}, \mathbb{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{Y_i \leq t}.$$

Note that for any probability measure μ on \mathbb{R} , the random variable $X_i = \{t \mapsto \mathbf{1}_{Y_i \leq t} - \mathbb{F}(t) : t \in \mathbb{R}\}$ may be viewed as a random variable with values in the Hilbert space $\mathbb{H} := \mathbf{L}^2(\mathbb{R}, \mu)$. Hence to derive the MDP for $n(\mathbb{F}_n - \mathbb{F})$ we shall apply Corollary 2.6 to the random variables $(X_i)_{i \geq 1}$. With this aim, we first recall the following dependance coefficients from Dedecker and Prieur [7].

Definition 3.6. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let \mathcal{F} be a sub σ -algebra of \mathcal{A} . Let $Y = (Y_1, \dots, Y_k)$ be a random variable with values in \mathbb{R}^k . Let \mathbb{P}_Y be the distribution of Y and let $\mathbb{P}_{Y|\mathcal{F}}$ be a conditional distribution of Y given \mathcal{F} . For $1 \leq i \leq k$ and t in \mathbb{R} , let $g_{t,i}(x) = \mathbf{1}_{x \leq t} - \mathbb{P}(Y_i \leq t)$. Define the random variable

$$b(\mathcal{F}, Y_1, \dots, Y_k) = \sup_{(t_1, \dots, t_k) \in \mathbb{R}^k} \left| \int \prod_{i=1}^k g_{t_i, i}(x_i) \mathbb{P}_{Y|\mathcal{F}}(dx) - \int \prod_{i=1}^k g_{t_i, i}(x_i) \mathbb{P}_Y(dx) \right|$$

with $\mathbb{P}_{Y|\mathcal{F}}(dx) = \mathbb{P}_{Y|\mathcal{F}}(dx_1, \dots, dx_k)$ and $\mathbb{P}_Y(dx) = \mathbb{P}_Y(dx_1, \dots, dx_k)$.

For the stationary sequence $(Y_i)_{i \in \mathbb{Z}}$, define the coefficient $\tilde{\phi}_k$, for any integer $k \geq 1$, by

$$\tilde{\phi}_k(n) = \max_{1 \leq l \leq k} \sup_{i_1 > \dots > i_l \geq n} \|b(\mathcal{F}_0, Y_{i_1}, \dots, Y_{i_l})\|_\infty.$$

Proposition 3.7. *If*

$$\sum_{n \geq 1} \frac{1}{\sqrt{n}} \tilde{\phi}_2(n) < \infty, \quad (3.6)$$

then $\{\sqrt{n}(\mathbb{F}_n(t) - \mathbb{F}(t)), t \in \mathbb{R}\}$ satisfies the MDP in $\mathbf{L}^2(\mathbb{R}, \mu)$ with the good rate function,

$$\forall f \in \mathbf{L}^2(\mathbb{R}, \mu), I(f) = \sup_{g \in \mathbf{L}^2(\mathbb{R}, \mu)} \left(\langle f, g \rangle_{\mathbf{L}^2(\mathbb{R}, \mu)} - \frac{1}{2} \langle g, Qg \rangle_{\mathbf{L}^2(\mathbb{R}, \mu)} \right), \quad (3.7)$$

where Q is defined as follows, for all (f, g) in $\mathbf{L}^2(\mathbb{R}, \mu) \times \mathbf{L}^2(\mathbb{R}, \mu)$,

$$Q(f, g) = \int_{\mathbb{R}^2} f(s)g(t)C(s, t) \mu(dt)\mu(ds)$$

with

$$C(s, t) = \mathbb{F}(t \wedge s) - \mathbb{F}(t)\mathbb{F}(s) + 2 \sum_{k \geq 1} (\mathbb{P}(Y_0 \leq t, Y_k \leq s) - \mathbb{F}(t)\mathbb{F}(s)).$$

If we use the contraction principle in Dembo and Zeitouni [8], with the continuous function $f : x \mapsto \|x\|_{\mathbf{L}^2(\mathbb{R}, \mu)}$, the Cramér-Von Mises statistics,

$$\sqrt{n} \left(\int_{\mathbb{R}} (\mathbb{F}_n(t) - \mathbb{F}(t))^2 \mu(dt) \right)^{1/2}$$

satisfies the MDP in \mathbb{R} with the good rate function,

$$\forall y \geq 0, I'(y) = \frac{1}{2} \frac{1}{\nu} y^2,$$

where $\nu = \max_k \lambda_k$, the λ_k 's are the eigenvalues of the covariance function Q .

Now we suppose that $Y_i \in [0, 1]$ and $\mu(dt) = dt$. Always, by the contraction principle in Dembo and Zeitouni [8] with the continuous function,

$$\begin{aligned} u : \mathbf{L}^2([0, 1], \mu) &\longrightarrow \mathbb{R}^+ \\ g &\longmapsto \|g\|_{\mathbf{L}^1([0, 1], \mu)}, \end{aligned}$$

we prove that the Kantorovitch distance $\sqrt{n} \|\mathbb{F}_n - \mathbb{F}\|_{\mathbf{L}^1([0, 1], \mu)}$ satisfies the MDP in \mathbb{R}^+ , with the good rate function,

$$\forall y \in \mathbb{R}^+, J(y) = \inf \{ I(f), f \in \mathbf{L}^2([0, 1], \mu), y = \|f\|_{\mathbf{L}^1([0, 1], \mu)} \}.$$

We deduce from the proof of Theorem in Ledoux [13] page 274, that

$$\forall y \geq 0, J(y) = \frac{1}{2} \frac{y^2}{\sigma(Z)^2},$$

where Z is a random variable with covariance function Q defined in Proposition 3.7 and $\sigma(Z) = \sup_{\|g\|_\infty \leq 1} \left(\mathbb{E} \left(\int_0^1 g(t) Z(t) dt \right)^2 \right)^{1/2}$.

By using a remark (8.22) in Ledoux and Talagrand [14] page 216, we also have

$$\limsup_{n \rightarrow \infty} \frac{\sqrt{n} \|\mathbb{F}_n - \mathbb{F}\|_{\mathbf{L}^1([0,1],\mu)}}{\sqrt{2n \log \log n}} = \sigma(Z).$$

Remark: (3.6) is satisfied for a large class of dependent sequences. For instance, it is verified for the class of expanding maps as considered in Dedecker and Prieur [7].

4. PROOFS

4.1. Hoeffding inequality's proof.

4.1.1. Technical propositions.

The proofs of the following lemma and propositions use the same ideas as in the proof of Theorem 1 in Peligrad, Utev and Wu [17] (see also Mackey and Tyran-Kamińska [21]).

Lemma 4.1. *Let $\{Z_k\}_{k \in \mathbb{Z}}$ be a stationary sequence of martingale differences with values in \mathbb{H} . For all integer $n \geq 1$ and real $p \geq 1$, we have*

$$\mathbb{E} \left(\max_{1 \leq i \leq n} \|Z_1 + \dots + Z_i\|_{\mathbb{H}}^{2p} \right) \leq 2^{p+1} \Gamma(p+1) n^p \|\|Z_1\|_{\mathbb{H}}\|_{\infty}^{2p}. \tag{4.1}$$

Proof of Lemma 4.1. We note here, $S_i = \sum_{k=1}^i Z_k$. Applying an inequality given in Pinelis [18] (Theorem 3.5) and using stationarity, we have

$$\begin{aligned} \mathbb{E} \left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}}^{2p} \right) &= \int_0^\infty (2p) z^{2p-1} \mathbb{P} \left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}} \geq z \right) dz \\ &\leq 2 \int_0^\infty (2p) z^{2p-1} \exp \left(- \frac{z^2}{2n \|\|Z_1\|_{\mathbb{H}}\|_{\infty}^2} \right) dz. \end{aligned}$$

By the change of variable $u = \frac{z}{\sqrt{n} \|\|Z_1\|_{\mathbb{H}}\|_{\infty}}$,

$$\int_0^\infty z^{2p-1} \exp \left(- \frac{z^2}{2(\sqrt{n} \|\|Z_1\|_{\mathbb{H}}\|_{\infty})^2} \right) dz = n^p \|\|Z_1\|_{\mathbb{H}}\|_{\infty}^{2p} \int_0^\infty u^{2p-1} \exp \left(- \frac{u^2}{2} \right) du.$$

Next

$$\int_0^\infty u^{2p-1} \exp \left(- \frac{u^2}{2} \right) du = 2^{p-1} \Gamma(p).$$

Therefore, we conclude that

$$\mathbb{E} \left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}}^{2p} \right) \leq 2^{p+1} \Gamma(p+1) n^p \|\|Z_1\|_{\mathbb{H}}\|_{\infty}^{2p}.$$

□

The next proposition is a generalization of Lemma 4.1 to an adapted stationary sequence.

Proposition 4.2. *Let n, q be integers such that $n \geq 1$, $2^{q-1} \leq n < 2^q$. Assume that $\| \|Z_0\|_{\mathbb{H}} \|_{\infty} < \infty$, and Z_0 is \mathcal{F}_0 -measurable. Let $Z_i = (Z_0 \circ T^i)_{i \in \mathbb{Z}}$. Then, for all real $p \geq 1$,*

$$\mathbb{E} \left(\max_{1 \leq i \leq n} \left\| \sum_{j=1}^i Z_j \right\|_{\mathbb{H}}^{2p} \right)^{1/2p} \leq (2^{p+1} \Gamma(p+1))^{1/2p} \sqrt{n} \{ \| \|Z_1 - \mathbb{E}(Z_1 | \mathcal{F}_0) \|_{\mathbb{H}} \|_{\infty} + \frac{5}{\sqrt{2}} \Delta_q \} \quad (4.2)$$

where

$$\Delta_q = \sum_{j=0}^{q-1} \frac{1}{2^{j/2}} \| \| \mathbb{E}(S_{2^j} | \mathcal{F}_0) \|_{\mathbb{H}} \|_{\infty}.$$

Proof of Proposition 4.2. The proof is almost identical to the proof of the corresponding facts in Theorem 1 of Peligrad, Utev and Wu [17] if we replace everywhere the absolute value $|\cdot|$ by $\|\cdot\|_{\mathbb{H}}$, and C_p is here, $C_p = 2^{p+1} \Gamma(p+1)$. Also, we work with the $\mathbf{L}_{\mathbb{H}}^{2p}$ -norm. Consequently, we give here, only the crucial inequalities. We note $K = 5/\sqrt{2}$. By triangle inequality, notice that

$$\max_{1 \leq i \leq n} \| S_i \|_{\mathbb{H}} \leq \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i (Z_k - \mathbb{E}(Z_k | \mathcal{F}_{k-1})) \right\|_{\mathbb{H}} + \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i \mathbb{E}(Z_k | \mathcal{F}_{k-1}) \right\|_{\mathbb{H}}. \quad (4.3)$$

By the inequality for martingale differences (4.1), we get

$$\left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i (Z_k - \mathbb{E}(Z_k | \mathcal{F}_{k-1})) \right\|_{\mathbb{H}} \right\|_{2p} \leq C_p^{1/2p} \sqrt{n} \| \|Z_1 - \mathbb{E}(Z_1 | \mathcal{F}_0) \|_{\mathbb{H}} \|_{\infty}. \quad (4.4)$$

Moreover, if we start by writing, $n = 2m$ or $n = 2m + 1$, we have

$$\begin{aligned} \left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i \mathbb{E}(Z_k | \mathcal{F}_{k-1}) \right\|_{\mathbb{H}} \right\|_{2p} &\leq \left\| \max_{1 \leq l \leq m} \left\| \sum_{k=1}^{2l} \mathbb{E}(Z_k | \mathcal{F}_{k-1}) \right\|_{\mathbb{H}} \right\|_{2p} \\ &\quad + \left\| \max_{0 \leq l \leq m} \| \mathbb{E}(Z_{2l+1} | \mathcal{F}_{2l}) \|_{\mathbb{H}} \right\|_{2p}, \end{aligned} \quad (4.5)$$

and

$$\left\| \max_{0 \leq l \leq m} \| \mathbb{E}(Z_{2l+1} | \mathcal{F}_{2l}) \|_{\mathbb{H}} \right\|_{2p} \leq (m+1)^{1/2p} \| \| \mathbb{E}(Z_1 | \mathcal{F}_0) \|_{\mathbb{H}} \|_{2p}. \quad (4.6)$$

For the first term of (4.5), we proceed as in the proof of Peligrad, Utev and Wu [17], to get:

$$\begin{aligned} \left\| \max_{1 \leq l \leq m} \left\| \sum_{k=1}^{2l} \mathbb{E}(Z_k | \mathcal{F}_{k-1}) \right\|_{\mathbb{H}} \right\|_{2p} &\leq C_p^{1/2p} \sqrt{m} \{ 4 \| \| \mathbb{E}(Z_1 | \mathcal{F}_0) \|_{\mathbb{H}} \|_{\infty} \\ &\quad + K \sqrt{2} (\Delta_q - \| \| \mathbb{E}(Z_1 | \mathcal{F}_0) \|_{\mathbb{H}} \|_{\infty}) \}. \end{aligned} \quad (4.7)$$

Consequently, combining (4.3)-(4.7), we obtain the bound (4.2). \square

The next proposition is the main tool allowing us to extend Proposition 4.2 to non-adapted stationary sequences of \mathbb{H} -valued random variables.

Proposition 4.3. *Let n, q be integers such that $n \geq 1, 2^{q-1} \leq n < 2^q$. Assume that $\|Z_0\|_{\mathbb{H}} < \infty$, and $\mathbb{E}(Z_0 | \mathcal{F}_{-1}) = 0$. Then, for all real $p \geq 1$,*

$$\mathbb{E} \left(\max_{1 \leq i \leq n} \left\| \sum_{k=1}^i Z_k \right\|_{\mathbb{H}}^{2p} \right)^{1/2p} \leq (2^{p+1} \Gamma(p+1))^{1/2p} \sqrt{n} \left\{ \| \mathbb{E}(Z_0 | \mathcal{F}_0) \|_{\mathbb{H}} + \frac{2}{\sqrt{2}} \Delta'_q \right\} \tag{4.8}$$

where

$$\Delta'_q = \sum_{j=0}^{q-1} \frac{1}{2^{j/2}} \| S_{2^j} - \mathbb{E}(S_{2^j} | \mathcal{F}_{2^j}) \|_{\mathbb{H}}.$$

Proof of Proposition 4.3. Here also, the proof is widely inspired by the proof of Theorem 1 in Peligrad, Utev and Wu [17] and we note always $C_p = 2^{p+1} \Gamma(p+1)$. We prove (4.8) by induction on n . For $n = 1, q = 1$, we clearly have

$$\|Z_1\|_{\mathbb{H}} \leq \| \mathbb{E}(Z_1 | \mathcal{F}_1) \|_{\mathbb{H}} + \Delta'_1.$$

Then, assume that the inequality holds for all $n < 2^{q-1}$. Fix n such that, $2^{q-1} \leq n < 2^q$. By triangle inequality, we obtain that

$$\max_{1 \leq i \leq n} \left\| \sum_{k=1}^i Z_k \right\|_{\mathbb{H}} \leq \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i \mathbb{E}(Z_k | \mathcal{F}_k) \right\|_{\mathbb{H}} + \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i (Z_k - \mathbb{E}(Z_k | \mathcal{F}_k)) \right\|_{\mathbb{H}}. \tag{4.9}$$

Since $\mathbb{E}(Z_0 | \mathcal{F}_{-1}) = 0$, we can use the inequality (4.1) for martingale differences,

$$\left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i \mathbb{E}(Z_k | \mathcal{F}_k) \right\|_{\mathbb{H}} \right\|_{2p} \leq C_p^{1/2p} \sqrt{n} \| \mathbb{E}(Z_0 | \mathcal{F}_0) \|_{\mathbb{H}}. \tag{4.10}$$

Now, as in Peligrad, Utev and Wu [17], we write $n = 2m$ or $n = 2m + 1$, for the second term in the right-hand side in (4.9),

$$\begin{aligned} \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i (Z_k - \mathbb{E}(Z_k | \mathcal{F}_k)) \right\|_{\mathbb{H}} &\leq \max_{1 \leq l \leq m} \left\| \sum_{k=1}^{2l} (Z_k - \mathbb{E}(Z_k | \mathcal{F}_k)) \right\|_{\mathbb{H}} \\ &\quad + \max_{0 \leq l \leq m} \| Z_{2l+1} - \mathbb{E}(Z_{2l+1} | \mathcal{F}_{2l+1}) \|_{\mathbb{H}}, \end{aligned} \tag{4.11}$$

and

$$\max_{0 \leq l \leq m} \| Z_{2l+1} - \mathbb{E}(Z_{2l+1} | \mathcal{F}_{2l+1}) \|_{\mathbb{H}}^{2p} \leq \sum_{l=0}^m \| Z_{2l+1} - \mathbb{E}(Z_{2l+1} | \mathcal{F}_{2l+1}) \|_{\mathbb{H}}^{2p}. \tag{4.12}$$

For the first term in the right-hand side in (4.11), we apply the induction hypothesis to the stationary sequence, $Y_0 = Z_0 - \mathbb{E}(Z_0 | \mathcal{F}_0) + Z_{-1} - \mathbb{E}(Z_{-1} | \mathcal{F}_{-1})$, for all j in \mathbb{Z} , $Y_j = Y_0 \circ T^{2j}$, the sigma algebra $\mathcal{G}_0 = \mathcal{F}_0$, and the operator T^2 . Notice that the new filtration becomes $\{\mathcal{G}_i : i \in \mathbb{Z}\}$ where $\mathcal{G}_i = \mathcal{F}_{2i}$. Whence, we have

$$\left\| \max_{1 \leq l \leq m} \left\| \sum_{k=1}^{2l} (Z_k - \mathbb{E}(Z_k | \mathcal{F}_k)) \right\|_{\mathbb{H}} \right\|_{2p} = \left\| \max_{1 \leq l \leq m} \left\| \sum_{k=1}^l Y_0 \circ T^{2k} \right\|_{\mathbb{H}} \right\|_{2p}.$$

Since $m < 2^{q-1}$ and $\mathbb{E}(Y_0 | \mathcal{G}_{-1}) = 0$, we obtain by the induction hypothesis,

$$\left\| \max_{1 \leq l \leq m} \left\| \sum_{k=1}^l Y_0 \circ T^{2k} \right\|_{\mathbb{H}} \right\|_{2p} \leq C_p^{1/2p} \sqrt{m} (\|\mathbb{E}(Y_0 | \mathcal{G}_0)\|_{\mathbb{H}} + \frac{2}{\sqrt{2}} \Delta'_{q-1}(Y)). \quad (4.13)$$

But, $\|\mathbb{E}(Y_0 | \mathcal{G}_0)\|_{\mathbb{H}} \leq \|Z_0 - \mathbb{E}(Z_0 | \mathcal{F}_0)\|_{\mathbb{H}}$ and rewriting,

$$\begin{aligned} \Delta'_{q-1}(Y) &= \sum_{j=0}^{q-2} \frac{1}{2^{j/2}} \left\| \sum_{k=1}^{2^j} Y_k - \mathbb{E} \left(\sum_{k=1}^{2^j} Y_k \middle| \mathcal{G}_{2^j} \right) \right\|_{\mathbb{H}} \\ &= \sum_{j=0}^{q-2} \frac{1}{2^{j/2}} \|S_{2^{j+1}} - \mathbb{E}(S_{2^{j+1}} | \mathcal{F}_{2^{j+1}})\|_{\mathbb{H}} \\ &= \sqrt{2} (\Delta'_q - \|Z_0 - \mathbb{E}(Z_0 | \mathcal{F}_0)\|_{\mathbb{H}}), \end{aligned} \quad (4.14)$$

we derive that

$$\left\| \max_{1 \leq l \leq m} \left\| \sum_{k=1}^l Y_0 \circ T^{2k} \right\|_{\mathbb{H}} \right\|_{2p} \leq C_p^{1/2p} \sqrt{m} (\|Z_0 - \mathbb{E}(Z_0 | \mathcal{F}_0)\|_{\mathbb{H}} + 2\Delta'_q - 2\|Z_0 - \mathbb{E}(Z_0 | \mathcal{F}_0)\|_{\mathbb{H}}).$$

Consequently, we conclude that

$$\begin{aligned} \left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i Z_k \right\|_{\mathbb{H}} \right\|_{2p} &\leq C_p^{1/2p} (\sqrt{n} \|\mathbb{E}(Z_0 | \mathcal{F}_0)\|_{\mathbb{H}} + \sqrt{2m} \frac{2}{\sqrt{2}} \Delta'_q) \\ &\leq C_p^{1/2p} \sqrt{n} (\|\mathbb{E}(Z_0 | \mathcal{F}_0)\|_{\mathbb{H}} + \frac{2}{\sqrt{2}} \Delta'_q). \end{aligned}$$

□

Now we give, the main proposition.

Proposition 4.4. *Assume that $\|X_0\|_{\mathbb{H}} < \infty$, then, for all $p \geq 1$,*

$$\begin{aligned} \mathbb{E} \left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}}^{2p} \right) &\leq 2^{p+1} \Gamma(p+1) n^p \left(\|X_0\|_{\mathbb{H}} + D \sum_{j=1}^n j^{-3/2} \|\mathbb{E}(S_j | \mathcal{F}_0)\|_{\mathbb{H}} \right. \\ &\quad \left. + D' \sum_{j=1}^n j^{-3/2} \|S_j - \mathbb{E}(S_j | \mathcal{F}_j)\|_{\mathbb{H}} \right)^{2p} \end{aligned} \quad (4.15)$$

where

$$7D = 40\sqrt{2} + 27 \quad \text{and} \quad 7D' = 24\sqrt{2} + 12.$$

Proof of Proposition 4.4. We set $K = \frac{5}{\sqrt{2}}$, $K' = \frac{2}{\sqrt{2}}$ and $C_p = 2^{p+1} \Gamma(p+1)$. Let n and q be integers such that $n \geq 1$ and $2^{q-1} \leq n < 2^q$. Let

$$\begin{aligned} \delta_n &= \sum_{j=1}^n j^{-3/2} \|\mathbb{E}(S_j | \mathcal{F}_0)\|_{\mathbb{H}}, & \delta'_n &= \sum_{j=1}^n j^{-3/2} \|S_j - \mathbb{E}(S_j | \mathcal{F}_j)\|_{\mathbb{H}}, \\ \Delta_q &= \sum_{j=0}^{q-1} 2^{-j/2} \|\mathbb{E}(S_{2^j} | \mathcal{F}_0)\|_{\mathbb{H}}, & \Delta'_q &= \sum_{j=0}^{q-1} 2^{-j/2} \|S_{2^j} - \mathbb{E}(S_{2^j} | \mathcal{F}_{2^j})\|_{\mathbb{H}}. \end{aligned}$$

We shall prove a slightly stronger inequality,

$$\left\| \mathbb{E} \left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}} \right) \right\|_{2p} \leq C_p^{1/2p} \sqrt{n} (\|X_1 - \mathbb{E}(X_1 | \mathcal{F}_0)\|_{\mathbb{H}} + K\Delta_q + K'\Delta'_q). \quad (4.16)$$

Note first that $V_n = \|\|\mathbb{E}(S_n | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty}$ is a sub-additive sequence as proved by Peligrad and Utev [16] in Lemma 2.6 (replace the \mathbf{L}^2 -norm by the $\mathbf{L}_{\mathbb{H}}^{\infty}$ -norm). The sequence $(V_n)_{n \geq 0}$ verifies for all i, j in \mathbb{N}^* ,

$$V_{i+j} \leq V_i + V_j.$$

Whence, using Lemma 5.1 in Appendix with $\tilde{C}_1 = \tilde{C}_2 = 1$, we get

$$\Delta_q \leq \left(\frac{4\sqrt{2}}{7} + \frac{16}{7} \right) \delta_n.$$

On an other hand, the sequence $V'_n = \|\|S_n - \mathbb{E}(S_n | \mathcal{F}_n)\|_{\mathbb{H}}\|_{\infty}$ verifies for all i, j in \mathbb{N}^* ,

$$\begin{aligned} V'_{i+j} &\leq \|\|S_{i+j} - \mathbb{E}(S_{i+j} | \mathcal{F}_{i+j})\|_{\mathbb{H}}\|_{\infty} \\ &\leq \|\|S_i - \mathbb{E}(S_i | \mathcal{F}_i)\|_{\mathbb{H}}\|_{\infty} + \|\|S_{i+j} - S_i - \mathbb{E}(S_{i+j} - S_i | \mathcal{F}_{i+j})\|_{\mathbb{H}}\|_{\infty} \\ &\quad + \|\|\mathbb{E}(S_i | \mathcal{F}_i) - \mathbb{E}(S_i | \mathcal{F}_{i+j})\|_{\mathbb{H}}\|_{\infty} \\ &\leq \|\|S_i - \mathbb{E}(S_i | \mathcal{F}_i)\|_{\mathbb{H}}\|_{\infty} + \|\|S_j - \mathbb{E}(S_j | \mathcal{F}_j)\|_{\mathbb{H}}\|_{\infty} \\ &\quad + \|\|\mathbb{E}(S_i - \mathbb{E}(S_i | \mathcal{F}_i) | \mathcal{F}_{i+j})\|_{\mathbb{H}}\|_{\infty} \\ &\leq 2V'_i + V'_j. \end{aligned}$$

Whence, using Lemma 5.1 in Appendix with $\tilde{C}_1 = 2$ and $\tilde{C}_2 = 1$, we have

$$\Delta'_q \leq \left(\frac{6\sqrt{2}}{7} + \frac{24}{7} \right) \delta'_n.$$

Setting $k_1 = \frac{4\sqrt{2}}{7} + \frac{16}{7}$ and $k_2 = \frac{6\sqrt{2}}{7} + \frac{24}{7}$, we get

$$\|\|X_1 - \mathbb{E}(X_1 | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} + K\Delta_q + K'\Delta'_q \leq \|\|X_1\|_{\mathbb{H}}\|_{\infty} + (Kk_1 + 1)\delta_n + K'k_2\delta'_n.$$

Since (4.16) implies (4.15), it remains to prove (4.16).

By triangle inequality, we obtain that

$$\begin{aligned} \left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i X_k \right\|_{\mathbb{H}} \right\|_{2p} &\leq \left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i (X_k - \mathbb{E}(X_k | \mathcal{F}_k)) \right\|_{\mathbb{H}} \right\|_{2p} \\ &\quad + \left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i \mathbb{E}(X_k | \mathcal{F}_k) \right\|_{\mathbb{H}} \right\|_{2p}. \end{aligned} \quad (4.17)$$

Applying Proposition 4.2, we derive

$$\left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i \mathbb{E}(X_k | \mathcal{F}_k) \right\|_{\mathbb{H}} \right\|_{2p} \leq C_p^{1/2p} \sqrt{n} (\|\|\mathbb{E}(X_1 | \mathcal{F}_1) - \mathbb{E}(X_1 | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} + K\Delta_q^*) \quad (4.18)$$

where

$$\Delta_q^* = \sum_{j=0}^{q-1} 2^{-j/2} \left\| \left\| \mathbb{E} \left(\sum_{k=1}^{2^j} \mathbb{E}(X_k | \mathcal{F}_k) \middle| \mathcal{F}_0 \right) \right\|_{\mathbb{H}} \right\|_{\infty} = \Delta_q.$$

On the other hand, Proposition 4.3 gives

$$\left\| \max_{1 \leq i \leq n} \left\| \sum_{k=1}^i (X_k - \mathbb{E}(X_k | \mathcal{F}_k)) \right\|_{\mathbb{H}} \right\|_{2p} \leq C_p^{1/2p} \sqrt{n} \{ \|\mathbb{E}(X_0 - \mathbb{E}(X_0 | \mathcal{F}_0) | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} + K' \Delta_q^{*'} \} \quad (4.19)$$

where

$$\Delta_q^{*'} = \sum_{j=0}^{q-1} 2^{-j/2} \left\| \left\| \sum_{k=1}^{2^j} (\{X_k - \mathbb{E}(X_k | \mathcal{F}_k)\} - \mathbb{E}(X_k - \mathbb{E}(X_k | \mathcal{F}_k) | \mathcal{F}_{2^j})) \right\|_{\mathbb{H}} \right\|_{\infty} = \Delta'_q.$$

Combining (4.18) and (4.19) in (4.17), (4.16) follows. \square

4.1.2. Proof of Theorem 2.2.

Now, assume that p is an integer, and $p \geq 1$.

Let

$$B = \|X_0\|_{\mathbb{H}}\|_{\infty} + \left(D \sum_{j=1}^n \frac{1}{j^{3/2}} \|\mathbb{E}(S_j | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} + D' \sum_{j=1}^n \frac{1}{j^{3/2}} \|S_j - \mathbb{E}(S_j | \mathcal{F}_j)\|_{\mathbb{H}}\|_{\infty} \right)$$

for some constants $D > 0$ and $D' > 0$ defined in Proposition 4.4. We can use the approach of the proof of Theorem 2.4 in Rio [20], because

$$\begin{aligned} \mathbb{E} \left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}}^{2p} \right) &\leq 2^{p+1} p! B^{2p} n^p \\ &\leq 2(2p-1)!! (2nB^2)^p. \end{aligned}$$

Consequently, if we use the notation of the proof of Theorem 2.4 in Rio [20], the constant A is here,

$$A = \frac{x^2}{4nB^2},$$

and with the estimation given in Rio [20] (page 42),

$$\mathbb{P} \left(\max_{1 \leq i \leq n} \|S_i\|_{\mathbb{H}} \geq x \right) \leq 2\sqrt{e} \exp(-A).$$

Taking $C = \max\{D, D'\}$, we obtain exactly Theorem 2.2. \square

4.2. MDP for martingale differences.

Our main proposition is a generalization of a result of Theorem 3.1 in Puhalskii [19] to \mathbb{H} -valued random variables.

Proposition 4.5. *Let a_n be a sequence of positive numbers satisfying $a_n \rightarrow 0$ and $na_n \xrightarrow{n \rightarrow \infty} +\infty$. Let k_n be an increasing sequence of integers going to infinity and $\{d_{j,n}\}_{1 \leq j \leq k_n}$ be a triangular array of martingale differences, with values in \mathbb{H} , such that*

$$\forall 1 \leq j \leq k_n, \|d_{j,n}\|_{\mathbf{L}_{\mathbb{H}}^{\infty}} \leq \beta_n \sqrt{na_n} \quad \text{with } \beta_n \xrightarrow{n \rightarrow \infty} 0. \quad (4.20)$$

Assume that, there exists $Q \in \mathcal{S}(\mathbb{H})$ such that:

i. for all k, l in \mathbb{N}^* and $\delta > 0$,

$$\limsup_{n \rightarrow \infty} a_n \log \mathbb{P} \left(\left| \frac{1}{n} \sum_{j=1}^{k_n} \mathbb{E}(\langle d_{j,n}, e_k \rangle_{\mathbb{H}} \langle d_{j,n}, e_l \rangle_{\mathbb{H}} | \mathcal{F}_{j-1,n}) - \langle Qe_k, e_l \rangle_{\mathbb{H}} \right| > \delta \right) = -\infty, \quad (4.21)$$

ii. for all $\delta > 0$,

$$\limsup_{n \rightarrow \infty} a_n \log \mathbb{P} \left(\left| \frac{1}{n} \sum_{j=1}^{k_n} \mathbb{E}(\|d_{j,n}\|_{\mathbb{H}}^2 | \mathcal{F}_{j-1,n}) - \text{Tr}(Q) \right| > \delta \right) = -\infty. \quad (4.22)$$

Then $\{W_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[k_n t]} d_{j,n} + \frac{1}{\sqrt{n}}(k_n t - [k_n t])d_{[k_n t]+1,n} : t \in [0, 1]\}$ satisfies the MDP in $C_{\mathbb{H}}([0, 1])$ with speed a_n and the good rate function,

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\phi'(t)) dt & \text{if } \phi \in \mathcal{AC}_0([0, 1]) \\ \infty & \text{otherwise} \end{cases} \quad (4.23)$$

where Λ^* is defined by

$$\Lambda^*(x) = \sup_{y \in \mathbb{H}} \left(\langle y, x \rangle_{\mathbb{H}} - \frac{1}{2} \langle y, Qy \rangle_{\mathbb{H}} \right). \quad (4.24)$$

Proof of Proposition 4.5.

Firstly, we need some notations.

Notation 4.6. For all integer $m \geq 1$, let P^m be the projection on the first m components of the orthonormal basis, $(e_i)_{1 \leq i \leq m}$, in \mathbb{H} then

$$d_{j,n}^m = P^m(d_{j,n}), \quad r_{j,n}^m = (I - P^m)d_{j,n}.$$

where I is the identity operator.

Let $\{d_{j,n}\}_{1 \leq j \leq k_n}$ be a \mathbb{H} -valued triangular array of martingale differences. We start by proving that $\{d_{j,n}^m\}_{1 \leq j \leq k_n}$, which is a \mathbb{R}^m -valued triangular array of martingale differences satisfies the conditions of Theorem 3.1 of Pulhalskii [19] (see also Djellout [10], Proposition 1).

The conditions (4.20) and (4.21) imply conditions *i*) and *ii*) of Proposition 1 in Djellout [10].

Consequently, $\{W_n^m(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[k_n t]} d_{j,n}^m + \frac{1}{\sqrt{n}}(k_n t - [k_n t])d_{[k_n t]+1,n}^m : t \in [0, 1]\}$ satisfies the MDP, with the good rate function, $I_m(\cdot)$,

$$I_m(\phi) = \begin{cases} \int_0^1 \Lambda_m^*(\phi'(t)) dt & \text{if } \phi \in \mathcal{AC}_0([0, 1]) \\ \infty & \text{otherwise,} \end{cases}$$

where Λ_m^* is:

$$\forall x \in \mathbb{H}, \quad \Lambda_m^*(x) = \sup_{y \in \mathbb{H}} \left(\langle P^m y, P^m x \rangle_{\mathbb{H}} - \frac{1}{2} \langle P^m y, QP^m y \rangle_{\mathbb{H}} \right).$$

By using Theorem 4.2.13 in Dembo and Zeitouni [8], it remains to prove, that for any $\eta > 0$,

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} a_n \log \mathbb{P} \left(\max_{1 \leq j \leq k_n} \sqrt{\frac{a_n}{n}} \left\| \sum_{k=1}^j r_{k,n}^m \right\|_{\mathbb{H}} > \eta \right) = -\infty.$$

Notice that, for all $\eta > 0$,

$$\begin{aligned} & a_n \log \mathbb{P} \left(\max_{1 \leq j \leq k_n} \sqrt{\frac{a_n}{n}} \left\| \sum_{k=1}^j r_{k,n}^m \right\|_{\mathbb{H}} > \eta \right) \\ & \leq a_n \log \left(\mathbb{P} \left(\left\{ \max_{1 \leq j \leq k_n} \sqrt{\frac{a_n}{n}} \left\| \sum_{k=1}^j r_{k,n}^m \right\|_{\mathbb{H}} > \eta \right\} \cap \left\{ \left| \frac{1}{n} \sum_{k=1}^{k_n} \mathbb{E}(\|r_{k,n}^m\|_{\mathbb{H}}^2 \mid \mathcal{F}_{k-1,n}) - \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}} \right| \leq \varepsilon \right\} \right) \right. \\ & \quad \left. + \mathbb{P} \left(\left| \frac{1}{n} \sum_{k=1}^{k_n} \mathbb{E}(\|r_{k,n}^m\|_{\mathbb{H}}^2 \mid \mathcal{F}_{k-1,n}) - \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}} \right| > \varepsilon \right) \right), \end{aligned}$$

where $\varepsilon > 0$.

With the notations

$$\begin{aligned} A(n, m, \eta, \varepsilon) & := \mathbb{P} \left(\left\{ \max_{1 \leq j \leq k_n} \sqrt{\frac{a_n}{n}} \left\| \sum_{k=1}^j r_{k,n}^m \right\|_{\mathbb{H}} > \eta \right\} \right. \\ & \quad \left. \cap \left\{ \left| \frac{1}{n} \sum_{k=1}^{k_n} \mathbb{E}(\|r_{k,n}^m\|_{\mathbb{H}}^2 \mid \mathcal{F}_{k-1,n}) - \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}} \right| \leq \varepsilon \right\} \right), \end{aligned}$$

and

$$B(n, m, \varepsilon) := \mathbb{P} \left(\left| \frac{1}{n} \sum_{j=1}^{k_n} \mathbb{E}(\|r_{j,n}^m\|_{\mathbb{H}}^2 \mid \mathcal{F}_{j-1,n}) - \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}} \right| > \varepsilon \right),$$

we derive

$$a_n \log \mathbb{P} \left(\max_{1 \leq j \leq k_n} \sqrt{\frac{a_n}{n}} \left\| \sum_{k=1}^j r_{k,n}^m \right\|_{\mathbb{H}} > \eta \right) \leq a_n \log \{A(n, m, \eta, \varepsilon) + B(n, m, \varepsilon)\}.$$

Now notice

$$\begin{aligned} & a_n \log(B(n, m, \varepsilon)) \\ & \leq a_n \log \left\{ \mathbb{P} \left(\left| \frac{1}{n} \sum_{j=1}^{k_n} \mathbb{E}(\|d_{j,n}\|_{\mathbb{H}}^2 \mid \mathcal{F}_{j-1,n}) - \text{Tr}(Q) \right| > \frac{\varepsilon}{2} \right) \right. \\ & \quad \left. + \mathbb{P} \left(\left| \frac{1}{n} \sum_{j=1}^{k_n} \mathbb{E}(\|d_{j,n}^m\|_{\mathbb{H}}^2 \mid \mathcal{F}_{j-1,n}) - \sum_{p=1}^m \langle Q e_p, e_p \rangle_{\mathbb{H}} \right| > \frac{\varepsilon}{2} \right) \right\}. \end{aligned}$$

Using (4.21) and (4.22), it follows

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} a_n \log(B(n, m, \varepsilon)) = -\infty.$$

With the notations

$$C(n, m, \eta) := \left\{ \max_{1 \leq j \leq k_n} \sqrt{\frac{a_n}{n}} \left\| \sum_{k=1}^j r_{k,n}^m \right\|_{\mathbb{H}} > \eta \right\},$$

and

$$D(n, m, \varepsilon) := \left\{ \left| \frac{1}{n} \sum_{j=1}^{k_n} \mathbb{E}(\|r_{j,n}^m\|_{\mathbb{H}}^2 \mid \mathcal{F}_{j-1,n}) - \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}} \right| \leq \varepsilon \right\},$$

applying Theorem 5.1 (inequality (5.2)), in Kallenberg and Sztencel [12] or Theorem 3.4 in Pinelis [18] to the martingale difference,

$$U_i = r_{i,n}^m \mathbf{1}_{\{\sum_{j=1}^i \mathbb{E}(\|r_{j,n}^m\|_{\mathbb{H}}^2 \mid \mathcal{F}_{j-1,n}) \leq n(\varepsilon + \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}})\}},$$

we obtain that

$$\begin{aligned} & a_n \log \mathbb{P}(C(n, m, \eta) \cap D(n, m, \varepsilon)) \\ & \leq a_n \log(2) + a_n \log \left(\exp \left(- \frac{\eta^2 n}{2a_n(n\varepsilon + n \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}}) + \frac{2}{3} a_n \beta_n \sqrt{na_n} \sqrt{n} \frac{\eta}{\sqrt{a_n}}} \right) \right) \\ & \leq a_n \log(2) - \frac{\eta^2}{2\varepsilon + 2 \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}} + \frac{2}{3} \beta_n \eta}. \end{aligned}$$

Since Q has a finite trace, it follows

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} - \frac{\eta^2}{2\varepsilon + 2 \sum_{p=m+1}^{\infty} \langle Q e_p, e_p \rangle_{\mathbb{H}} + \frac{2}{3} \beta_n \eta} = -\infty.$$

Consequently, we conclude by Theorem 4.2.13, in Dembo and Zeitouni [8] that $\{n^{-1/2} \sum_{j=1}^{[k_n t]} d_{j,n} + \frac{1}{\sqrt{n}}(k_n t - [k_n t])d_{[k_n t]+1,n} : t \in [0, 1]\}$ satisfies the MDP in $C_{\mathbb{H}}([0, 1])$. The rate function is the same that the i.i.d gaussian random variable with mean 0 and covariance Q , therefore equal to

$$\forall x \in \mathbb{H}, \Lambda^*(x) = \sup_{y \in \mathbb{H}} \left(\langle y, x \rangle_{\mathbb{H}} - \frac{1}{2} \langle y, Qy \rangle_{\mathbb{H}} \right).$$

□

4.3. MDP for stationary sequences.

4.3.1. Proof of Theorem 2.4.

The proof of Theorem 2.4 uses the same arguments as in the proof of Theorem 1 in Dedecker, Merlevède, Peligrad and Utev [6], but for a \mathbb{H} -valued non-adapted sequences.

Let $m_n = o(\sqrt{na_n})$, and $k_n = [n/m_n]$ (where, as before, $[x]$ denotes the integer part of x).

We divide the variables in blocks of size m_n and make the sums in each block,

$$X_{i,m_n} = \sum_{j=(i-1)m_n+1}^{im_n} X_j, \quad i \geq 1.$$

Then, we construct the martingales,

$$\begin{aligned} M_{k_n}^{(m_n)} &= \sum_{i=1}^{k_n} (\mathbb{E}(X_{i,m_n} | \mathcal{F}_{im_n}) - \mathbb{E}(X_{i,m_n} | \mathcal{F}_{(i-1)m_n})) \\ &:= \sum_{i=1}^{k_n} D_{i,m_n}, \end{aligned}$$

and we define the process $\{M_{k_n}^{(m_n)}(t) : t \in [0, 1]\}$ by

$$M_{k_n}^{(m_n)}(t) := M_{[k_n t]}^{(m_n)} + \frac{1}{\sqrt{n}}(k_n t - [k_n t])D_{[k_n t]+1,m_n}.$$

Now, we shall use Proposition 4.5, applied with $d_{j,n} = D_{j,m_n}$, and verify the conditions (4.21) and (4.22).

We start by proving (4.21). By stationarity, it is enough to prove that, for all $k, l \geq 1$,

$$\limsup_{n \rightarrow \infty} \left\| \frac{1}{m_n} \mathbb{E}(\langle D_{1,m_n}, e_k \rangle_{\mathbb{H}} \langle D_{1,m_n}, e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) - \langle Qe_k, e_l \rangle_{\mathbb{H}} \right\|_{\infty} = 0. \quad (4.25)$$

But, we notice

$$\begin{aligned} &\mathbb{E}(\langle D_{1,m_n}, e_k \rangle_{\mathbb{H}} \langle D_{1,m_n}, e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) \\ &= \mathbb{E}(\langle \mathbb{E}(X_{1,m_n} | \mathcal{F}_{m_n}), e_k \rangle_{\mathbb{H}} \langle \mathbb{E}(X_{1,m_n} | \mathcal{F}_{m_n}), e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) \\ &\quad - \langle \mathbb{E}(X_{1,m_n} | \mathcal{F}_0), e_k \rangle_{\mathbb{H}} \langle \mathbb{E}(X_{1,m_n} | \mathcal{F}_0), e_l \rangle_{\mathbb{H}}, \end{aligned}$$

thus

$$\begin{aligned} &\left\| \mathbb{E}(\langle D_{1,m_n}, e_k \rangle_{\mathbb{H}} \langle D_{1,m_n}, e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) - \langle Qe_k, e_l \rangle_{\mathbb{H}} \right\|_{\infty} \\ &\leq \left\| \mathbb{E}(\langle \mathbb{E}(S_{m_n} | \mathcal{F}_{m_n}), e_k \rangle_{\mathbb{H}} \langle \mathbb{E}(S_{m_n} | \mathcal{F}_{m_n}), e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) - \langle Qe_k, e_l \rangle_{\mathbb{H}} \right\|_{\infty} \\ &\quad + \left\| \langle \mathbb{E}(S_{m_n} | \mathcal{F}_0), e_k \rangle_{\mathbb{H}} \langle \mathbb{E}(S_{m_n} | \mathcal{F}_0), e_l \rangle_{\mathbb{H}} \right\|_{\infty}. \end{aligned}$$

By triangle inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left\| \frac{1}{m_n} \mathbb{E}(\langle \mathbb{E}(S_{m_n} | \mathcal{F}_{m_n}), e_k \rangle_{\mathbb{H}} \langle \mathbb{E}(S_{m_n} | \mathcal{F}_{m_n}), e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) - \langle Qe_k, e_l \rangle_{\mathbb{H}} \right\|_{\infty} \\ &\leq \frac{1}{m_n} \left\| \mathbb{E}(S_{m_n} | \mathcal{F}_{m_n}) - S_{m_n} \right\|_{\mathbb{H}}^2 \\ &\quad + \frac{2}{m_n} \left\| \mathbb{E}(S_{m_n} | \mathcal{F}_{m_n}) - S_{m_n} \right\|_{\mathbb{H}} \left\| \sqrt{\mathbb{E}(\|S_{m_n}\|_{\mathbb{H}}^2 | \mathcal{F}_0)} \right\|_{\infty} \\ &\quad + \left\| \frac{1}{m_n} \mathbb{E}(\langle S_{m_n}, e_k \rangle_{\mathbb{H}} \langle S_{m_n}, e_l \rangle_{\mathbb{H}} | \mathcal{F}_0) - \langle Qe_k, e_l \rangle_{\mathbb{H}} \right\|_{\infty}. \end{aligned}$$

By using Lemma 5.1 in Appendix, and the hypothesis (2.3), we deduce (4.25).

Now, to prove (4.22), by stationarity, we have to verify

$$\limsup_{n \rightarrow \infty} \left\| \frac{1}{m_n} \mathbb{E}(\|D_{1,m_n}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \text{Tr}(Q) \right\|_{\infty} = 0. \quad (4.26)$$

Notice that

$$\mathbb{E}(\|D_{1,m_n}\|_{\mathbb{H}}^2 | \mathcal{F}_0) \leq \mathbb{E}(\|S_{m_n}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \|\mathbb{E}(S_{m_n} | \mathcal{F}_0)\|_{\mathbb{H}}^2,$$

thus

$$\left\| \frac{1}{m_n} \mathbb{E}(\|D_{1,m_n}\|_{\mathbb{H}}^2 \mid \mathcal{F}_0) - \text{Tr}(Q) \right\|_{\infty} \leq \left\| \frac{1}{m_n} \mathbb{E}(\|S_{m_n}\|_{\mathbb{H}}^2 \mid \mathcal{F}_0) - \text{Tr}(Q) \right\|_{\infty} + \frac{1}{m_n} \|\mathbb{E}(S_{m_n} \mid \mathcal{F}_0)\|_{\mathbb{H}}^2\|_{\infty}.$$

By using Lemma 5.1 in Appendix, and the hypothesis (2.4), we deduce (4.26).

To finish the proof, it remains to prove, that for all $\delta > 0$,

$$\limsup_{n \rightarrow \infty} a_n \log \mathbb{P} \left(\sqrt{\frac{a_n}{n}} \sup_{t \in [0,1]} \|S_{[nt]} - M_{[k_n t]}^{(m_n)}\|_{\mathbb{H}} \geq \delta \right) = -\infty \quad (4.27)$$

and

$$\limsup_{n \rightarrow \infty} a_n \log \mathbb{P} \left(\sqrt{\frac{a_n}{n}} \sup_{t \in [0,1]} \|(k_n t - [k_n t])D_{[k_n t]+1, m_n} - (nt - [nt])X_{[nt]+1}\|_{\mathbb{H}} \geq \delta \right) = -\infty. \quad (4.28)$$

(4.28) holds since $m_n = o(\sqrt{a_n n})$ and the random variables are bounded. We turn now to the proof of (4.27). Notice that

$$\begin{aligned} \sup_{t \in [0,1]} \|S_{[nt]} - M_{[k_n t]}^{(m_n)}\|_{\mathbb{H}} &\leq \sup_{t \in [0,1]} \left\| \sum_{i=[k_n t]m_n+1}^{[nt]} X_i \right\|_{\mathbb{H}} + \sup_{t \in [0,1]} \left\| \sum_{i=1}^{[k_n t]} (X_{i, m_n} - \mathbb{E}(X_{i, m_n} \mid \mathcal{F}_{im_n})) \right\|_{\mathbb{H}} \\ &\quad + \sup_{t \in [0,1]} \left\| \sum_{i=1}^{[k_n t]} \mathbb{E}(X_{i, m_n} \mid \mathcal{F}_{(i-1)m_n}) \right\|_{\mathbb{H}} \\ &\leq o(\sqrt{na_n}) + \max_{1 \leq j \leq k_n} \left\| \sum_{i=1}^j (X_{i, m_n} - \mathbb{E}(X_{i, m_n} \mid \mathcal{F}_{im_n})) \right\|_{\mathbb{H}} \\ &\quad + \max_{1 \leq j \leq k_n} \left\| \sum_{i=1}^j \mathbb{E}(X_{i, m_n} \mid \mathcal{F}_{(i-1)m_n}) \right\|_{\mathbb{H}}. \end{aligned} \quad (4.29)$$

For the last term of the right-hand side in (4.29), we use the same arguments as in the proof of Theorem 1 in Dedecker, Merlevède, Peligrad and Utev [6], so we give only the proof of the non-adapted term, i.e. the second term of the right-hand side of inequality (4.29).

We apply Theorem 2.2 to the stationary sequence, $Y_{0, m_n} = X_{0, m_n} - \mathbb{E}(X_{0, m_n} \mid \mathcal{F}_0)$, and $Y_{i, m_n} = Y_{0, m_n} \circ T^{im_n}$. Notice that the new filtration becomes $\{\mathcal{G}_i, i \in \mathbb{Z}\}$ where $\mathcal{G}_0 = \mathcal{F}_0$, and $\mathcal{G}_i = T^{-(im_n)}(\mathcal{G}_0)$.

Consequently, we have

$$\begin{aligned} &a_n \log \mathbb{P} \left(\sqrt{\frac{a_n}{n}} \max_{1 \leq j \leq k_n} \left\| \sum_{i=1}^j (X_{i, m_n} - \mathbb{E}(X_{i, m_n} \mid \mathcal{F}_{im_n})) \right\|_{\mathbb{H}} \geq \delta \right) \\ &= a_n \log \mathbb{P} \left(\sqrt{\frac{a_n}{n}} \max_{1 \leq j \leq k_n} \left\| \sum_{i=1}^j Y_{i, m_n} \right\|_{\mathbb{H}} \geq \delta \right) \\ &\leq a_n \log(2\sqrt{e}) - \frac{\delta^2}{4 \frac{1}{m_n} E(n, \delta)^2}, \end{aligned} \quad (4.30)$$

where

$$E(n, \delta) := \|\|S_{m_n} - \mathbb{E}(S_{m_n} | \mathcal{F}_{m_n})\|_{\mathbb{H}}\|_{\infty} + C \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \|\|S_{jm_n} - \mathbb{E}(S_{jm_n} | \mathcal{F}_{jm_n})\|_{\mathbb{H}}\|_{\infty},$$

with C is the positive constant defined in Theorem 2.2.

We conclude by using Lemma 5.1 in Appendix, and the inequality (4.30) converges to 0, when $n \rightarrow \infty$. \square

4.4. Proof of Corollary 2.6.

The proof of Corollary 2.6 uses the same arguments as in the proof of Corollary 2 in Dedecker, Merlevède, Peligrad and Utev [6] but for a non-adapted stationary \mathbb{H} -valued sequence.

By triangle inequality and changing the order of summation, (2.7) implies (2.2).

4.4.1. A technical lemma.

Lemma 4.7. *Assume that $\|\|X_0\|_{\mathbb{H}}\|_{\infty} < \infty$. Let n be a dyadic integer, $n = 2^q$. Then*

$$\begin{aligned} \|\mathbb{E}(\|S_n\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty} &\leq n(\|\mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty} + \frac{1}{2}\Delta_q + \frac{1}{2}\Delta'_q)^2 \quad (4.31) \\ &\leq n\Delta_{\infty}^2 \end{aligned}$$

where Δ_q, Δ'_q are, respectively, defined as in Proposition 4.2 and in Proposition 4.3 and

$$\Delta_{\infty} = \|\mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty} + \frac{1}{2}\Delta_q + \frac{1}{2}\Delta'_q.$$

Proof of Lemma 4.7. As in the proof of Proposition 2.1 in Peligrad and Utev [16], we prove Lemma 4.7 by induction on q .

Obviously, (4.31) is true for $q = 0$. Assume now, that (4.31) holds for all dyadic integers $n \leq 2^{q-1}$.

Writing $S_{2^q} = S_{2^{q-1}} + S_{2^q} - S_{2^{q-1}}$, notice that

$$\|S_{2^q}\|_{\mathbb{H}}^2 = \|S_{2^{q-1}}\|_{\mathbb{H}}^2 + \|S_{2^q} - S_{2^{q-1}}\|_{\mathbb{H}}^2 + 2 \langle S_{2^{q-1}}, S_{2^q} - S_{2^{q-1}} \rangle_{\mathbb{H}}.$$

By stationarity, we have

$$\begin{aligned} &\|\mathbb{E}(\|S_{2^q}\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty} \\ &\leq 2\|\mathbb{E}(\|S_{2^{q-1}}\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty} + 2\|\mathbb{E}(\langle S_{2^{q-1}} - \mathbb{E}(S_{2^{q-1}} | \mathcal{F}_{2^{q-1}}), S_{2^q} - S_{2^{q-1}} \rangle_{\mathbb{H}} | \mathcal{F}_0)\|_{\infty} \\ &\quad + 2\|\mathbb{E}(\langle \mathbb{E}(S_{2^{q-1}} | \mathcal{F}_{2^{q-1}}), S_{2^q} - S_{2^{q-1}} \rangle_{\mathbb{H}} | \mathcal{F}_0)\|_{\infty}. \end{aligned} \quad (4.32)$$

The last term in (4.32) can be treated as in the proof of the corresponding facts in Proposition 2.1 of Peligrad and Utev [16], if we replace everywhere the product in \mathbb{R} by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$, and the \mathbf{L}^2 -norm $\|x\|$ by the infinite norm. Consequently, we derive

$$\begin{aligned} &\|\mathbb{E}(\langle \mathbb{E}(S_{2^{q-1}} | \mathcal{F}_{2^{q-1}}), S_{2^q} - S_{2^{q-1}} \rangle_{\mathbb{H}} | \mathcal{F}_0)\|_{\infty} \\ &\leq \sqrt{\|\mathbb{E}(\|S_{2^{q-1}}\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty}} 2^{(q-1)/2} (\Delta_q - \Delta_{q-1}). \end{aligned} \quad (4.33)$$

In the same way, since $\| \|S_{2^{q-1}} - \mathbb{E}(S_{2^{q-1}} | \mathcal{F}_{2^{q-1}}) \|_{\mathbb{H}} \|_{\infty} = 2^{(q-1)/2}(\Delta'_q - \Delta'_{q-1})$, we have

$$\begin{aligned}
 & \| \mathbb{E}(\langle S_{2^{q-1}} - \mathbb{E}(S_{2^{q-1}} | \mathcal{F}_{2^{q-1}}), S_{2^q} - S_{2^{q-1}} \rangle_{\mathbb{H}} | \mathcal{F}_0) \|_{\infty} \\
 & \leq \sqrt{\| \mathbb{E}(\|S_{2^{q-1}} - \mathbb{E}(S_{2^{q-1}} | \mathcal{F}_{2^{q-1}})\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty}} \sqrt{\| \mathbb{E}(\|S_{2^q} - S_{2^{q-1}}\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty}} \\
 & \leq \| \|S_{2^{q-1}} - \mathbb{E}(S_{2^{q-1}} | \mathcal{F}_{2^{q-1}}) \|_{\mathbb{H}} \|_{\infty} \sqrt{\| \mathbb{E}(\|S_{2^q} - S_{2^{q-1}}\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty}} \\
 & \leq \sqrt{\| \mathbb{E}(\|S_{2^{q-1}}\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty}} 2^{(q-1)/2}(\Delta'_q - \Delta'_{q-1}). \tag{4.34}
 \end{aligned}$$

By induction and combining (4.33) and (4.34), we conclude that

$$\begin{aligned}
 & \| \mathbb{E}(\|S_{2^q}\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty} \\
 & \leq 2 \times 2^{q-1} (\| \mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty} + \frac{1}{2}\Delta_{q-1} + \frac{1}{2}\Delta'_{q-1})^2 \\
 & \quad + 2 \times 2^{(q-1)/2} (\| \mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty} + \frac{1}{2}\Delta_{q-1} + \frac{1}{2}\Delta'_{q-1}) \times 2^{(q-1)/2}(\Delta_q - \Delta_{q-1}) \\
 & \quad + 2 \times 2^{(q-1)/2} (\| \mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty} + \frac{1}{2}\Delta_{q-1} + \frac{1}{2}\Delta'_{q-1}) \times 2^{(q-1)/2}(\Delta'_q - \Delta'_{q-1}) \\
 & \leq 2^q (\| \mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty} + \frac{1}{2}\Delta_q + \frac{1}{2}\Delta'_q)^2 \\
 & \leq n (\| \mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0) \|_{\infty} + \frac{1}{2}\Delta_q + \frac{1}{2}\Delta'_q)^2.
 \end{aligned}$$

□

4.4.2. Proof of Corollary 2.6.

The proof splits in two parts, and uses the same arguments as in the proof of Lemma 28 in Dedecker, Merlevède, Peligrad and Utev [6].

Lemma 4.8. *Assume that $\| \|X_0 \|_{\mathbb{H}} \|_{\infty} < \infty$.*

i. *Under (2.2) and (2.8), (2.3) holds.*

ii. *Under (2.2) and (2.9), (2.4) holds.*

Proof of Lemma 4.8. The proofs of (i) and (ii) are quite similarly, so here we prove only (ii).

Firstly, as in the proof of Lemma 28 in Dedecker, Merlevède, Peligrad and Utev [6], we prove by diadic recurrence (2.4). Let $S_{a,b} = S_b - S_a$. Denote, for any t integer,

$$A_{t,k} = \| \mathbb{E}(\|S_t\|_{\mathbb{H}}^2 | \mathcal{F}_{-k}) - \mathbb{E}(\|S_t\|_{\mathbb{H}}^2) \|_{\infty}.$$

By stationarity, we have

$$\begin{aligned}
 A_{2t,k} & = \| \mathbb{E}(\|S_{2t}\|_{\mathbb{H}}^2 | \mathcal{F}_{-k}) - \mathbb{E}(\|S_{2t}\|_{\mathbb{H}}^2) \|_{\infty} \\
 & \leq 2 \| \mathbb{E}(\|S_t\|_{\mathbb{H}}^2 | \mathcal{F}_{-k}) - \mathbb{E}(\|S_t\|_{\mathbb{H}}^2) \|_{\infty} \\
 & \quad + 2 \| \mathbb{E}(\langle S_t, S_{t,2t} \rangle_{\mathbb{H}} | \mathcal{F}_{-k}) \|_{\infty} + 2 \| \mathbb{E}(\langle S_t, S_{t,2t} \rangle_{\mathbb{H}}) \|_{\infty}.
 \end{aligned}$$

Moreover by Cauchy-Schwarz inequality and Lemma 4.7, we get that

$$\begin{aligned} A_{2t,k} &\leq 2A_{t,k} + 4\sqrt{\|\mathbb{E}(\|S_t\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty}} \|\mathbb{E}(S_t | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} \\ &\quad + 4\|\|S_t - \mathbb{E}(S_t | \mathcal{F}_t)\|_{\mathbb{H}}\|_{\infty} \sqrt{\|\mathbb{E}(\|S_t\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty}} \\ &\leq 2A_{t,k} + 4t^{1/2}\Delta_{\infty} \{ \|\| \mathbb{E}(S_t | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} + \|\| S_t - \mathbb{E}(S_t | \mathcal{F}_t)\|_{\mathbb{H}}\|_{\infty} \}. \end{aligned}$$

With the notation

$$B_{r,k} = 2^{-r} \|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_{-k}) - \mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2)\|_{\infty} = 2^{-r} A_{2^r,k},$$

by recurrence, for all $r \geq m$ and all $k > 0$, we derive

$$\begin{aligned} B_{r,k} &\leq B_{r-1,k} + 2^{\frac{-r+3}{2}} \Delta_{\infty} \{ \|\| S_{2^{r-1}} - \mathbb{E}(S_{2^{r-1}} | \mathcal{F}_{2^{r-1}})\|_{\mathbb{H}}\|_{\infty} + \|\| \mathbb{E}(S_{2^{r-1}} | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} \} \\ &\leq B_{m,k} + 2\Delta_{\infty} \left\{ \sum_{j=m}^r 2^{-j/2} \|\| S_{2^j} - \mathbb{E}(S_{2^j} | \mathcal{F}_{2^j})\|_{\mathbb{H}}\|_{\infty} + \sum_{j=m}^r 2^{-j/2} \|\| \mathbb{E}(S_{2^j} | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} \right\} \\ &\leq B_{m,k} + 2\Delta_{\infty} \{ \Delta_{m,\infty} + \Delta'_{m,\infty} \}, \end{aligned}$$

where

$$\Delta_{m,\infty} = \sum_{j=m}^{\infty} 2^{-j/2} \|\| \mathbb{E}(S_{2^j} | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} \text{ and } \Delta'_{m,\infty} = \sum_{j=m}^{\infty} 2^{-j/2} \|\| S_{2^j} - \mathbb{E}(S_{2^j} | \mathcal{F}_{2^j})\|_{\mathbb{H}}\|_{\infty}.$$

By stationarity and triangle inequality,

$$\begin{aligned} &\|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2)\|_{\infty} \\ &\leq \|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 - \|S_{k,k+2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty} + \|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_{-k}) - \mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2)\|_{\infty}, \end{aligned}$$

we have, for all integer $r \geq m+1$,

$$\begin{aligned} &2^{-r} \|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2)\|_{\infty} \\ &\leq B_{m,k} + 2\Delta_{\infty} (\Delta_{m,\infty} + \Delta'_{m,\infty}) + 2^{-r/2+2} k \|\mathbb{E}(\|X_1\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty}^{1/2} \Delta_{\infty}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\limsup_{r \rightarrow \infty} 2^{-r} \|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2)\|_{\infty} \\ &\leq B_{m,k} + 2\Delta_{\infty} (\Delta_{m,\infty} + \Delta'_{m,\infty}). \end{aligned}$$

Letting $k \rightarrow \infty$, and using condition (2.9), it follows that $\lim_{k \rightarrow \infty} B_{m,k} = 0$. Next letting $m \rightarrow \infty$, and using condition (2.2), we then derive that

$$\lim_{r \rightarrow \infty} 2^{-r} \|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2)\|_{\infty} = 0. \quad (4.35)$$

To finish the proof, we use the dyadic expansion $n = \sum_{k=0}^{r-1} 2^k a_k$, where $a_{r-1} = 1$ and $a_k \in \{0, 1\}$, as the proof of Proposition 2.1 in Peligrad and Utev [16] in order to treat the whole sequence S_n , for $2^{r-1} \leq n < 2^r$. We then use the following representation,

$$S_n = \sum_{j=0}^{r-1} T_{2^j} a_j \text{ where } T_{2^j} = \sum_{i=n_{j-1}+1}^{n_j} X_i, \quad n_j = \sum_{k=0}^j 2^k a_k, \quad n_{-1} = 0.$$

Notice that

$$\begin{aligned} & \frac{1}{n} \|\mathbb{E}(\|S_n\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \mathbb{E}(\|S_n\|_{\mathbb{H}}^2)\|_{\infty} \\ & \leq \frac{1}{n} \sum_{j=0}^{r-1} a_j \|\mathbb{E}(\|S_{2^j}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \mathbb{E}(\|S_{2^j}\|_{\mathbb{H}}^2)\|_{\infty} \\ & \quad + \frac{1}{n} \sum_{i \neq j=0}^{r-1} a_i a_j \|\mathbb{E}(\langle T_{2^i}, T_{2^j} \rangle_{\mathbb{H}} | \mathcal{F}_0) - \mathbb{E}(\langle T_{2^i}, T_{2^j} \rangle_{\mathbb{H}})\|_{\infty}. \end{aligned} \quad (4.36)$$

For the first term of the right-hand side in (4.36), we treat it as a diadic integer,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{r-1} a_j \|\mathbb{E}(\|S_{2^j}\|_{\mathbb{H}}^2 | \mathcal{F}_0) - \mathbb{E}(\|S_{2^j}\|_{\mathbb{H}}^2)\|_{\infty} = 0.$$

Suppose that $i < j < r$, we then have

$$\begin{aligned} & \|\mathbb{E}(\langle T_{2^j}, T_{2^i} \rangle_{\mathbb{H}} | \mathcal{F}_0) - \mathbb{E}(\langle T_{2^j}, T_{2^i} \rangle_{\mathbb{H}})\|_{\infty} \\ & \leq \|\mathbb{E}(\langle T_{2^j} - \mathbb{E}(T_{2^j} | \mathcal{F}_{n_i}), T_{2^i} \rangle_{\mathbb{H}} | \mathcal{F}_0) - \mathbb{E}(\langle T_{2^j} - \mathbb{E}(T_{2^j} | \mathcal{F}_{n_i}), T_{2^i} \rangle_{\mathbb{H}})\|_{\infty} \\ & \quad + \|\mathbb{E}(\langle \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i}), T_{2^j} \rangle_{\mathbb{H}} | \mathcal{F}_0) - \mathbb{E}(\langle \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i}), T_{2^j} \rangle_{\mathbb{H}})\|_{\infty}. \end{aligned} \quad (4.37)$$

For the first term in the right-hand side in (4.37), we have by Cauchy-Schwarz inequality,

$$\begin{aligned} & \sum_{i=0}^{r-2} \sum_{j=i+1}^{r-1} \|\mathbb{E}(\langle T_{2^j} - \mathbb{E}(T_{2^j} | \mathcal{F}_{n_i}), T_{2^i} \rangle_{\mathbb{H}} | \mathcal{F}_0)\|_{\infty} \\ & \leq \sum_{i=0}^{r-2} \sum_{j=i+1}^{r-1} \|\|T_{2^j} - \mathbb{E}(T_{2^j} | \mathcal{F}_{n_i})\|_{\mathbb{H}}\|_{\infty} \|\mathbb{E}(\|T_{2^i}\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty}^{1/2}. \end{aligned}$$

By (4.35),

$$\|\mathbb{E}(\|S_{2^r}\|_{\mathbb{H}}^2 | \mathcal{F}_0)\|_{\infty}^{1/2} = O(2^{r/2}).$$

Hence, we get

$$\begin{aligned} & \sum_{i=0}^{r-2} \sum_{j=i+1}^{r-1} \|\mathbb{E}(\langle T_{2^j} - \mathbb{E}(T_{2^j} | \mathcal{F}_{n_i}), T_{2^i} \rangle_{\mathbb{H}} | \mathcal{F}_0)\|_{\infty} \\ & \leq C_1 \sum_{i=0}^{r-2} \|\|T_{2^i} - \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i})\|_{\mathbb{H}}\|_{\infty} \sum_{j=i+1}^{r-1} 2^{j/2} \\ & \leq C_2 \sum_{i=0}^{r-2} \|\|T_{2^i} - \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i})\|_{\mathbb{H}}\|_{\infty} 2^{r/2}, \end{aligned}$$

where C_1 and C_2 are positive constants.

Therefore, for all $2^{r-1} \leq n < 2^r$, we obtain

$$\begin{aligned} & \frac{1}{n} \sum_{i=0}^{r-2} \sum_{j=i+1}^{r-1} \|\mathbb{E}(\langle T_{2^i} - \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i}), T_{2^j} \rangle_{\mathbb{H}} | \mathcal{F}_0)\|_{\infty} \\ & \leq C_2 2^{-r/2+1} \sum_{i=0}^{r-2} 2^{i/2} \frac{\|\|S_{2^i} - \mathbb{E}(S_{2^i} | \mathcal{F}_{2^i})\|_{\mathbb{H}}\|_{\infty}}{2^{i/2}}. \end{aligned}$$

As

$$\sum_{i=0}^{\infty} \frac{\|\|S_{2^i} - \mathbb{E}(S_{2^i} | \mathcal{F}_{2^i})\|_{\mathbb{H}}\|_{\infty}}{2^{i/2}} < \infty,$$

we conclude by Kronecker lemma that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{r-2} \sum_{j=i+1}^{r-1} \|\mathbb{E}(\langle T_{2^i} - \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i}), T_{2^j} \rangle_{\mathbb{H}} | \mathcal{F}_0)\|_{\infty} = 0. \quad (4.38)$$

For the second term of the right-hand side in (4.37), we use the same arguments as in the proof of Proposition 2.1 in Peligrad and Utev [16] by replacing the product in \mathbb{R} , by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the \mathbf{L}^2 -norm by the infinite norm. Consequently, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq i < j \leq r-1} \|\mathbb{E}(\langle \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i}), T_{2^j} \rangle_{\mathbb{H}} | \mathcal{F}_0) - \mathbb{E}(\langle \mathbb{E}(T_{2^i} | \mathcal{F}_{n_i}), T_{2^j} \rangle_{\mathbb{H}})\|_{\infty} = 0. \quad (4.39)$$

Combining (4.38) and (4.39), we conclude

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i \neq j=0}^{r-1} a_i a_j \|\mathbb{E}(\langle T_{2^i}, T_{2^j} \rangle_{\mathbb{H}} | \mathcal{F}_0) - \mathbb{E}(\langle T_{2^i}, T_{2^j} \rangle_{\mathbb{H}})\|_{\infty} = 0.$$

This proves (2.4). \square

4.5. Proof of Proposition 3.3.

Let ε' be an independent copy of ε , and denote by $\mathbb{E}_{\varepsilon}(\cdot)$ the conditional expectation with respect to ε . Define

$$Y_n = \sum_{i < n} c_i(\varepsilon_{n-i}), Y'_n = \sum_{i < n} c_i(\varepsilon'_{n-i}), Z_n = \sum_{i \geq n} c_i(\varepsilon_{n-i}), Z'_n = \sum_{i \geq n} c_i(\varepsilon'_{n-i}).$$

Then, taking $\mathcal{F}_l = \sigma(\varepsilon_i, i \leq l)$, we have

$$\begin{aligned} \|\|\mathbb{E}(X_n | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} &= \|\|\mathbb{E}_{\varepsilon}[f(Y'_n + Z_n) - f(Y'_n + Z'_n)]\|_{\mathbb{H}}\|_{\infty} \\ &\leq w_f(\|\|\varepsilon_0 - \varepsilon'_0\|_{\mathbb{H}}\|_{\infty} \sum_{k \geq n} \|c_k\|_{L(\mathbb{H})}) \end{aligned}$$

and

$$\begin{aligned} \|\|X_{-n} - \mathbb{E}(X_{-n} | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} &= \|\|\mathbb{E}_{\varepsilon}(f(Y_{-n} + Z_{-n}) - f(Z_{-n} + Y'_{-n}))\|_{\mathbb{H}}\|_{\infty} \\ &\leq w_f(\|\|\varepsilon_0 - \varepsilon'_0\|_{\mathbb{H}}\|_{\infty} \sum_{k \leq -n} \|c_k\|_{L(\mathbb{H})}). \end{aligned}$$

Then the condition (2.7) is satisfied as soon as (3.3) holds.

As the proof of (2.8) is quite similar of the proof of (2.9), we only prove (2.9).

We have for all integer $p \geq 0$,

$$\begin{aligned} & \|\mathbb{E}(\langle X_i, X_{i+p} \rangle_{\mathbb{H}} | \mathcal{F}_0) - \mathbb{E}(\langle X_i, X_{i+p} \rangle_{\mathbb{H}})\|_{\infty} \\ & \leq C \left\{ w_f \left(\|\varepsilon_0 - \varepsilon'_0\|_{\mathbb{H}} \sum_{k \geq i} \|c_k\|_{L(\mathbb{H})} \right) \right. \\ & \quad \left. + w_f \left(\|\varepsilon_0 - \varepsilon'_0\|_{\mathbb{H}} \sum_{k \geq i+p} \|c_k\|_{L(\mathbb{H})} \right) \right\}, \end{aligned} \quad (4.40)$$

where C is a constant.

By (3.3) and Corollary 2.6, Proposition 3.3 holds. \square

4.6. Proof of Proposition 3.4.

Firstly, we give a technical lemma,

Lemma 4.9. *If $\text{Lip}(K^n(f)) \leq C\rho^n \text{Lip}(f)$, then*

$$\|\mathbb{E}(f(Y_k) | Y_0) - \mathbb{E}(f(Y_k))\|_{\infty} \leq 2\|Y_0\|_{\mathbb{H}} C\rho^k \text{Lip}(f). \quad (4.41)$$

Proof of Lemma 4.9. As

$$\mathbb{E}(f(Y_k) | Y_0 = y) - \mathbb{E}(f(Y_k)) = \int (K^k(f)(y) - K^k(f)(z))\mu(dz),$$

we deduce

$$\begin{aligned} \|\mathbb{E}(f(Y_k) | Y_0 = y) - \mathbb{E}(f(Y_k))\|_{\mathbb{H}}\|_{\infty} & \leq \left\| \int \|K^k(f)(y) - K^k(f)(z)\|_{\mathbb{H}}\mu(dz) \right\|_{\infty} \\ & \leq \text{Lip}(K^k(f)) \left\| \int \|y - z\|_{\mathbb{H}}\mu(dz) \right\|_{\infty} \\ & \leq C\rho^k \text{Lip}(f) \left\| \int \|y - z\|_{\mathbb{H}}\mu(dz) \right\|_{\infty}. \end{aligned} \quad (4.42)$$

Observe that

$$\left\| \int \|y - z\|_{\mathbb{H}}\mu(dz) \right\|_{\infty} \leq \left\| \int (\|y\|_{\mathbb{H}} + \|z\|_{\mathbb{H}})\mu(dz) \right\|_{\infty} \leq 2\|Y_0\|_{\mathbb{H}}. \quad (4.43)$$

Consequently, combining (4.42) and (4.43), we have

$$\|\mathbb{E}(f(Y_k) | Y_0 = y) - \mathbb{E}(f(Y_k))\|_{\mathbb{H}}\|_{\infty} \leq 2\|Y_0\|_{\mathbb{H}} C\rho^k \text{Lip}(f). \quad \square$$

Proof of Proposition 3.4. We apply Corollary 2.6 to the following random variables,

$$X_k = f(Y_k) - \mathbb{E}(f(Y_k)), \quad \forall k \geq 0.$$

Since $(Y_n)_{n \geq 0}$ is a Markov chain, we have to prove that

$$\sum_{k \geq 1} \frac{1}{\sqrt{k}} \|\mathbb{E}(X_k | \mathcal{F}_0)\|_{\mathbb{H}}\|_{\infty} < \infty.$$

By Lemma 4.9, we derive

$$\sum_{k \geq 1} \frac{1}{\sqrt{k}} \|\mathbb{E}(X_k | \mathcal{F}_0)\|_{\mathbb{H}} \leq 2 \|Y_0\|_{\mathbb{H}} CLip(f) \sum_{k \geq 1} \frac{1}{\sqrt{k}} \rho^k < \infty.$$

The proof of (2.8) is quite similar of the proof of (2.9), so we only detail (2.9). If $k > l$, by triangle inequality

$$\begin{aligned} & \|\mathbb{E}(\langle X_k, X_l \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle X_k, X_l \rangle_{\mathbb{H}})\|_{\infty} \\ & \leq \|\mathbb{E}(\langle f(Y_k), f(Y_l) \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle f(Y_k), f(Y_l) \rangle_{\mathbb{H}})\|_{\infty} \\ & \quad + \|\mathbb{E}(\langle f(Y_k), \mathbb{E}(f(Y_l)) \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle f(Y_k), \mathbb{E}(f(Y_l)) \rangle_{\mathbb{H}})\|_{\infty} \\ & \quad + \|\mathbb{E}(\langle \mathbb{E}(f(Y_k)), f(Y_l) \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle \mathbb{E}(f(Y_k)), f(Y_l) \rangle_{\mathbb{H}})\|_{\infty}. \end{aligned}$$

Using Lemma 4.9, we get

$$\begin{aligned} & \|\mathbb{E}(\langle f(Y_k), \mathbb{E}(f(Y_l)) \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle f(Y_k), \mathbb{E}(f(Y_l)) \rangle_{\mathbb{H}})\|_{\infty} \\ & \leq \|\mathbb{E}(f(Y_l))\|_{\mathbb{H}} \|\mathbb{E}(f(Y_k) | \mathcal{F}_{-n}) - \mathbb{E}(f(Y_k))\|_{\mathbb{H}} \\ & \leq 2C \|\mathbb{E}(f(Y_0))\|_{\mathbb{H}} \|Y_0\|_{\mathbb{H}} Lip(f) \rho^{k+n} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

and for $k > l$,

$$\begin{aligned} & \|\mathbb{E}(\langle f(Y_k), f(Y_l) \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle f(Y_k), f(Y_l) \rangle_{\mathbb{H}})\|_{\infty} \\ & = \|\mathbb{E}(\langle \mathbb{E}(f(Y_k) | \mathcal{F}_l), f(Y_l) \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle \mathbb{E}(f(Y_k) | \mathcal{F}_l), f(Y_l) \rangle_{\mathbb{H}})\|_{\infty} \\ & = \|\mathbb{E}(\langle K^{k-l}(f)(Y_l), f(Y_l) \rangle_{\mathbb{H}} | \mathcal{F}_{-n}) - \mathbb{E}(\langle K^{k-l}(f)(Y_l), f(Y_l) \rangle_{\mathbb{H}})\|_{\infty} \\ & = \|K^{l+n}(\langle K^{k-l}(f)(\cdot), f(\cdot) \rangle_{\mathbb{H}})(Y_{-n}) - \mu(K^{l+n}(\langle K^{k-l}(f)(\cdot), f(\cdot) \rangle_{\mathbb{H}})(Y_{-n}))\|_{\infty} \\ & \leq 2C \|Y_0\|_{\mathbb{H}} Lip(\langle K^{k-l}(f)(\cdot), f(\cdot) \rangle_{\mathbb{H}}) \rho^{l+n} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Hence (2.9) holds. \square

4.7. Proof of Proposition 3.7.

We apply Corollary 2.6 to the random variables $X_i = \{t \mapsto \mathbf{1}_{Y_i \leq t} - \mathbb{F}(t) : t \in \mathbb{R}\}$. Since

$$\sum_{n \geq 1} \frac{1}{\sqrt{n}} \tilde{\phi}_2(n) < \infty \implies \sum_{n \geq 1} \frac{1}{\sqrt{n}} \tilde{\phi}_1(n) < \infty,$$

the condition (2.7) holds.

As the proofs of (2.8) and (2.9) are quite similar, we only detail the proof of (2.9).

By Fubini, we have, for any $i < j$,

$$\begin{aligned} & \left\| \mathbb{E}(\langle X_i, X_j \rangle_{\mathbf{L}^2(\mathbb{R}, \mu)} \mid \mathcal{F}_0) - \mathbb{E}(\langle X_i, X_j \rangle_{\mathbf{L}^2(\mathbb{R}, \mu)}) \right\|_\infty \\ = & \left\| \mathbb{E} \left(\int (\mathbf{1}_{Y_i \leq t} - \mathbb{F}(t)) (\mathbf{1}_{Y_j \leq t} - \mathbb{F}(t)) \mu(dt) \mid \mathcal{F}_0 \right) \right. \\ & \left. - \mathbb{E} \left(\int (\mathbf{1}_{Y_i \leq t} - \mathbb{F}(t)) (\mathbf{1}_{Y_j \leq t} - \mathbb{F}(t)) \mu(dt) \right) \right\|_\infty \\ \leq & \int \left\| \mathbb{E}((\mathbf{1}_{Y_i \leq t} - \mathbb{F}(t)) (\mathbf{1}_{Y_j \leq t} - \mathbb{F}(t)) \mid \mathcal{F}_0) - \mathbb{E}((\mathbf{1}_{Y_i \leq t} - \mathbb{F}(t)) (\mathbf{1}_{Y_j \leq t} - \mathbb{F}(t))) \right\|_\infty \mu(dt) \\ \leq & \|b(\mathcal{F}_0, Y_i, Y_j)\|_\infty \leq \tilde{\phi}_2(i). \end{aligned}$$

Since $\sum_{n \geq 1} n^{-1/2} \tilde{\phi}_2(i) < \infty$, $\tilde{\phi}_2(i) \xrightarrow{i \rightarrow \infty} 0$, all the conditions of Corollary 2.6 are true.

From Dedecker and Merlevède [5], the $\mathbf{L}^2(\mathbb{R}, \mu)$ -valued random variable $\sqrt{n}(\mathbb{F}_n - \mathbb{F})$ converges stably to a zero mean $\mathbf{L}^2(\mathbb{R}, \mu)$ -valued gaussian random variable \mathbb{G} , with covariance function Q , given in Proposition 3.7.

We deduce that $\sqrt{n}(\mathbb{F}_n - \mathbb{F})$ satisfies the MDP in $\mathbf{L}^2(\mathbb{R}, \mu)$, with the good rate function

$$\forall f \in \mathbf{L}^2(\mathbb{R}, \mu), I(f) = \sup_{g \in \mathbf{L}^2(\mathbb{R}, \mu)} \left(\langle f, g \rangle_{\mathbf{L}^2(\mathbb{R}, \mu)} - \frac{1}{2} \langle g, Qg \rangle_{\mathbf{L}^2(\mathbb{R}, \mu)} \right).$$

□

5. APPENDIX

Lemma 5.1. *Let $(U_j)_{j \geq 0}$ be a sequence of positive reals such that $U_0 = 0$ and $U_{i+j} \leq \tilde{C}_1 U_i + \tilde{C}_2 U_j$. Let $C = \tilde{C}_1 + \tilde{C}_2$. Then,*

1. *For n , and r integers such that $n \geq 1$, $2^{r-1} \leq n < 2^r$, and $p \geq 1$,*

$$\sum_{j=0}^{r-1} \frac{1}{2^{j(p-1)}} U_{2^j} \leq \frac{C}{(1 - 2^{-p})} \sum_{k=1}^{n-1} \frac{1}{k^p} U_k. \tag{5.1}$$

2. *If $\sum_{k=1}^\infty k^{-p} U_k < \infty$ for a $p > 1$, then*

$$\frac{1}{m^{p-1}} \sum_{j \geq 1} \frac{1}{j^p} U_{jm} \xrightarrow{m \rightarrow \infty} 0. \tag{5.2}$$

In particular,

$$\frac{U_m}{m^{p-1}} \xrightarrow{m \rightarrow \infty} 0.$$

Proof of Lemma 5.1. Firstly, we prove (5.1). With this aim, we note that

$$U_T \leq \tilde{C}_1 U_k + \tilde{C}_2 U_{T-k},$$

so

$$(T + 1)U_T \leq C \sum_{k=0}^T U_k.$$

Thus, for $n \geq 2^{r-1}$, we get

$$\sum_{j=0}^{r-1} \frac{U_{2^j}}{2^{j(p-1)}} \leq C \sum_{k=1}^{2^{r-1}} U_k \sum_{j:2^j \geq k} \frac{1}{(2^j + 1)2^{j(p-1)}} \leq \frac{C}{(1 - 2^{-p})} \sum_{k=1}^n \frac{U_k}{k^p}.$$

To prove (5.2), we write

$$\frac{1}{m^{p-1}} \sum_{j=1}^{\infty} \frac{U_{jm}}{j^p} \leq C \sum_{k=1}^{\infty} U_k \sum_{j: jm \geq k} \frac{1}{j^p(jm + 1)m^{p-1}} \leq C c_p \sum_{k=1}^{\infty} \frac{U_k}{(k + m)^p} \xrightarrow{m \rightarrow \infty} 0,$$

by the Fatou lemma since $U_k(k + m)^{-p} \downarrow 0$ as $m \rightarrow \infty$. Here, c_p is a positive constant depending on p . \square

REFERENCES

- [1] de Acosta, A. (1992). Moderate deviations and associated Laplace approximations for sums of independent random vectors. *Trans. Amer. Math. Soc.* **329**, No. 1, 357-375.
- [2] Borovkov, A.A. and Mogulskii, A.A. (1978). Probabilities of large deviations in topological spaces I. *Sib. Math. J.* **19**, 697-709.
- [3] Borovkov, A.A. and Mogulskii, A.A. (1980). Probabilities of large deviations in topological spaces II. *Sib. Math. J.* **21**, 12-26.
- [4] Bosq, D. (2000). Linear processes in function spaces. Theory and applications. *Lecture Notes in Statistics.* **149**.
- [5] Dedecker, J. and Merlevède, F. (2003). The conditional central limit theorem in Hilbert spaces. *Stochastic Process. Appl.* **108**, No. 2, 229-262.
- [6] Dedecker, J., Merlevède, F., Peligrad, M. and Utev, S. (2007). Moderate deviations for stationary sequences of bounded random variables. *To appear in Ann. Inst. H. Poincaré*. Prépublication LPMA, No. 1183.
- [7] Dedecker, J. and Prieur, C. (2007). An empirical central limit theorem for dependent sequences. *Stochastic Process. Appl.* **117**, No. 1, 121-142.
- [8] Dembo, A. and Zeitouni, O. (1998). Large Deviations Techniques and Applications. 2nd edition, Springer New York.
- [9] Deuschel, J.D. and Stroock, D.W. (1989). Large deviations. Pure and Applied Mathematics, 137. Academic Press, Inc., Boston, MA.
- [10] Djellout, H. (2002). Moderate Deviations for Martingale Differences and applications to ϕ -mixing sequences. *Stoch. Stoch. Rep.* **73**, No. 1-2, 37-63.
- [11] Hu, Y. and Lee, T.Y. (2003). Moderate deviation principles for trajectories of sums of independent Banach space valued random variables. *Trans. Amer. Math. Soc.* **355**, No. 8, 3047-3064.
- [12] Kallenberg, O. and Sztencel, R. (1991). Some dimension-free features of vector-valued martingales. *Probab. Theory Related Fields.* **88**, 215-247.
- [13] Ledoux, M. (1992). Sur les déviations modérées des sommes de variables aléatoires vectorielles indépendantes de même loi. *Ann. Inst. H. Poincaré.* **28**, No. 2, 267-280.
- [14] Ledoux, M. and Talagrand, M. (1991). Probability in Banach spaces. Isoperimetry and processes. *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Folge. Band 23. Springer-Verlag, Berlin.
- [15] Mas, A. and Menneteau, L. (2003). Large and moderate deviations for infinite-dimensional autoregressive processes. *Journal of Multivariate Analysis.* **87**, 241-260.
- [16] Peligrad, M. and Utev, S. (2005). A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.* **33**, No. 2, 798-815.
- [17] Peligrad, M. Utev, S. and Wu, W.B. (2007). A maximal \mathbf{L}_p -inequality for stationary sequences and its applications. *Proc. Amer. Math. Soc.* **135**, 541-550.

- [18] Pinelis, I. (1994) Optimum bounds for the distributions of martingales in Banach spaces. *Ann. Probab.* **22**, 1679-1706.
- [19] Puhalskii, A. (1994). Large deviations of semimartingales via convergence of the predictable characteristics. *Stoch. Stoch. Rep.* **49**, No. 1-2, 27-85.
- [20] Rio, E. (2000). Théorie asymptotique des processus aléatoires faiblement dépendants. Collection Mathématiques and Applications. **31**. Springer, Berlin.
- [21] Tyran-Kamińska, M. and Mackey, M. (2007). Central limit theorem for non-invertible measure preserving maps. *Colloq. Math. to appear.* **110**, No. 1, 167-191.
- [22] Volný, D. (2007). A nonadapted version of the invariance principle of Peligrad and Utev. *C. R. Acad. Sci. Paris, Ser. I* **345**, 167-169.

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