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Expansion of the propagation of chaos for Bird and Nanbu systems.

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Abstract

The Bird and Nanbu systems are particle systems used to approximate the solution of Boltzmann mollified equation. In particular, they have the propagation of chaos property. Following [GM94], we use coupling techniques and results on branching processes to write an expansion of the error in the propagation of chaos in terms of the number of particles, for slightly more general systems than the ones cited above. As explained in [DMPR] and [DMPR09], this result will lead to the proof of the convergence of U-statistics for these systems.

Keywords: interacting particle systems, Boltzmann equation, nonlinear diffusion with jumps, random graphs and trees, coupling, propagation of chaos, Monte Carlo algorithms.

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1 Introduction

In a recent work ([DMPR]), we showed a expansion of the propagation of chaos for a Feynman-Kac particle system. This particle system is approximating a particular Feynman-Kac measure, in the sense that the empirical measure associated to the system converges to the Feynman-Kac measure when the number of particles N goes to ∞. What is called propagation of chaos is the property of the particle system that q particles, amongst the total of N particles, looked upon at a fixed time, are asymptotically independent when $N \to +\infty$ (q is fixed) and their law is converging to the Feynman-Kac law. In [DMPR], we writed a expansion in powers of N of the difference between the law of q independent particles, each of them of the Feynman-Kac law, and the law of q particles coming from the particle system. One can also call this expansion a functional representation like in [DMPR]; in this paper, we call it an expansion of the error in propagation of chaos. In the setting of [DMPR], the time is discrete. In a forthcoming paper ([DMPR09], we wish to extend the result of [DMPR] to the case where the time is continuous, still in the Feynman-Kac framework, and we wish to show central-limit theorems for U-statistics of these systems of particles. The proof of the central-limit theorems for U-statistics relies only on the exploitation of the expansion described above.

We wish here to establish a similar expansion for particle systems approximating the solution of mollified Boltzmann equation, namely Bird and Nanbu systems. We refer mainly

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to [GM97] and take into account models described in (2.5), (2.6) of [GM97] (a similar description can be found in [GM99], Section 3). An other reference paper on the subject is [GM94]. The two main points of interest of this paper are: it provides a sequel to the estimates on propagation of chaos of [GM97], [GM99] and it allows to apply the results of [DMPR09] to Bird and Nanbu systems.

In Section 2, we will recall the definitions of Bird and Nanbu models, as can be found in [GM97] and will give an equivalent definition, useful to our purposes. In Section 3, we will state and prove our main theorem about the expansion of the error in propagation of chaos (Theorem 3.1). The proof relies on estimates on population growth found in [AN72] and on coupling ideas. In Section 4, we prove what is called a Wick-type formula in [DMPR], this formula will be useful to prove central-theorems for U-statistics in [DMPR09].

2 Definition of the model

2.1 Bird and Nanbu models

In all the following, we deal with particles evolving in $\mathbb{R}^d$. We set the mappings $e_i : h \in \mathbb{R}^d \mapsto e_i(h) = (0, \dots, 0, h_i, 0, \dots, 0) \in \mathbb{R}^{d\times N}$ $(1 \le i \le N)$. We have a kernel $\hat{\mu}(v, w, dh, dk)$ on $\mathbb{R}^{2d}$ which is symmetrical (that is $\hat{\mu}(v, w, dh, dk) = \hat{\mu}(w, v, dk, dh)$). We set $\mu(v, w, dh)$ to be the marginal $\hat{\mu}(v, w, dh \times \mathbb{R}^d)$. We suppose $\sup_{x,a} \hat{\mu}(x, a, \mathbb{R}^d \times \mathbb{R}^d) \le \Lambda < \infty$. We are also given a Markov generator L on $\mathbb{R}^d$ such that $\mathcal{D}(L) \subset L^\infty(\mathbb{R}^d)$. The kernel $\hat{\mu}$ and the generator L might have specific features coming from physical considerations, the coordinates in $\mathbb{R}^d$ might represent the position and speed of molecules but these considerations have no effect on our proof. This is why we claim to have a proof for systems more general than Bird and Nanbu systems.

We deal here with the Nanbu model and the Bird model. These particle models are defined in (2.5) and (2.6) of [GM97], by the mean of integrals over Poisson processes. We give here an equivalent definition.

Definition 2.1. *The particle system described in [GM97] is denoted by*

$$(\bar{Z}_t^N)_{t\ge 0} = (\bar{Z}_t^{N,i})_{t\ge 0, 1\le i\le N} \ .$$

It is a process of N particles in $\mathbb{R}^d$ and can be summarized by the following.

1. *Particles $(\bar{Z}_0^i)_{1\le i\le N}$ in $\mathbb{R}^d$ are drawn i.i.d. at time 0 according to a law $\tilde{P}_0$.*

2. *Between jump times, the particles evolve independently from each other according to L.*

3. *We have a collection $(N_{i,j})_{1\le i<j\le N}$ of independant Poisson processes of parameter $\Lambda/(N-1)$. For $i > j$, we set $N_{i,j} = N_{j,i}$. If $N_{i,j}$ has a jump at time t, we say there is an <u>interaction</u> between particles i and j and we take a uniform variable U on $[0, 1]$, independant of all the other variables, if $U \le \frac{\hat{\mu}(\bar{Z}_{t-}^{N,i}, \bar{Z}_{t-}^{N,j}, \mathbb{R}^{2d})}{\Lambda}$ then the system undergoes a <u>jump</u>:*

   - *In the Nanbu model:*

$$\bar{Z}_t^N = \begin{cases} \bar{Z}_{t-}^N + e_i(H) & \text{with proba. } 1/2 \\ \bar{Z}_{t-}^N + e_j(H) & \text{with proba. } 1/2 \end{cases} \tag{2.1}$$

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with

$$H \sim \frac{\mu(\bar{Z}^{N,i}_{t-}, \bar{Z}^{N,j}_{t-}, .)}{\hat{\mu}(\bar{Z}^{N,i}_{t-}, \bar{Z}^{N,j}_{t-}, \mathbb{R}^{2d})}$$

(independently of all the other variables).

- *In the Bird model:* $\bar{Z}^n_t = \bar{Z}^n_{t-} + e_i(H) + e_j(K)$ *with*

$$(H,K) \sim \frac{\mu(\bar{Z}^{N,i}_{t-}, \bar{Z}^{N,j}_{t-}, ., .)}{\hat{\mu}(\bar{Z}^{N,i}_{t-}, \bar{Z}^{N,j}_{t-}, \mathbb{R}^{2d})}$$

(independently of all the other variables).

Theorem 3.1 of [GM97] implies that there is propagation of chaos for this system. This theorem says:

$$\|\mathcal{L}(\bar{Z}^{N,1}_t, \dots, \bar{Z}^{N,q}_t) - \mathcal{L}(\bar{Z}^{N,1}_t)^{\otimes q}\|_{TV} \leq 2q(q-1)\frac{\Lambda t + \Lambda^2 t^2}{N-1} \ ,$$

and

$$\|\mathcal{L}(\bar{Z}^{N,1}_t) - \tilde{P}_t\|_{TV} \leq 6\frac{e^{\Lambda t}-1}{N+1} \ ,$$

where $(\tilde{P}_t)$ is solution of (with $\tilde{P}_0$ fixed)

$$\begin{aligned}&\forall \phi \in \mathcal{D}(L),\\ &\partial_t\langle \phi, \tilde{P}_t\rangle - \langle L\phi, \tilde{P}_t\rangle\\ &= \left\langle \int \frac{1}{2}(\phi(z+h) - \phi(z) + \phi(a+k) - \phi(a))\hat{\mu}(z,a,dh,dk), \tilde{P}_t(dz)\tilde{P}_t(da) \right\rangle\end{aligned}$$

We can deduce propagation of chaos from the previous results that is $\forall t$, $\forall F$ bounded measurable,

$$|\mathcal{L}(\bar{Z}^{N,1}_t, \dots, \bar{Z}^{N,q}_t)(F) - \tilde{P}^{\otimes q}_t(F)| \leq \left(2q(q-1)\frac{\Lambda t + \Lambda^2 t^2}{N-1} + 6\frac{e^{\Lambda t}-1}{N+1}\right)\|F\|_\infty \ .$$

In Theorem 3.1, we will go further than the above bound by writing an expansion of the left hand side term above in powers of N. We will use techniques introduced in [GM97]. The main point is that we want to look at the processes backward in time.

2.2 Backward point of view

From now on, we will work with a fixed time horizon $T > 0$. For any $j \in \mathbb{N}^*$, we set $[j] = \{1, \dots, j\}$. For $\lambda > 0$, we call $\mathcal{E}(\lambda)$ the exponential law of parameter λ.

We start in $s = 0$ with $C^i_0 = \{i\}$, $\forall i \in [q]$. We set $\forall i$, $K^i_0 = \#C^i_0$. For $1 \leq i < j \leq q$, we define processes $(N^{i,j}_s = N^{j,i}_s)_{s\geq 0}$, $(C^i_s)_{s\geq 0}$, $(K^i_s)_{s\geq 0}$ (respectively in $\mathbb{N}$, $\mathcal{P}([N])$, $\mathbb{N}^*$)by the following. For all $s \in [0,T]$, we set

$$K_s = \#(C^1_s \cup \dots \cup C^q_s) \ .$$

The processes $(N^{i,j}), (C^i), (K^i)$ are piecewise constant and make jumps. We define the jump times recursively by (taking $(U_k)_{1\leq i\leq q, 1\leq k}, (V_k)_{1\leq i\leq q,1\leq k}$ i.i.d. $\sim \mathcal{E}(1)$) $T_0 = 0$ and (always with the convention $\inf \emptyset = +\infty$)

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$$\begin{aligned} T'_k &= \inf\left\{T_{k-1}\le s\le T : \int_{T_{k-1}}^{s}\frac{\Lambda K_u(N-K_u)_+}{N-1}du\ge U_k\right\} \\ T''_k &= \inf\left\{T_{k-1}\le s\le T : \int_{T_{k-1}}^{s}\frac{\Lambda K_u(K_u-1)}{2(N-1)}du\ge V_k\right\} \\ T_k &= \inf(T'_k,T''_k)\ . \end{aligned}$$

In T_k:

- If $T_k = T'_k$, we take $r(k)$ uniformly in $C^1_{T_k-}\cup\dots\cup C^q_{T_k-}$ and $j(k)$ uniformly in $[N]\backslash(C^1_{T_k-}\cup\dots\cup C^q_{T_k-})$. Suppose that $r(k)\in C^i_{T_k-}$, we then perform the jumps: $C^i_{T_k}=C^i_{T_k-}\cup\{j(k)\}$, $K^i_{T_k}=K^i_{T_k-}+1$ (at any time $K^i_t=\#C^i_t$), $N_{r(k),j(k)}(T_k)=N_{r(k),j(k)}(T_k-)+1$. So, in short, we have added particle $j(k)$, which was in no set $C^{i'}$, to the set C^i.

  Notice that the $(\dots)_+$ in the definition of T'_k above forbids to be in the situation where we would be looking for $j(k)$ in $\emptyset$.

- If $T_k = T''_k$, we take $r(k)$ uniformly in $C^1_{T_k-}\cup\dots\cup C^q_{T_k-}$ and $j(k)$ uniformly in $C^1_{T_k-}\cup\dots\cup C^q_{T_k-}\backslash\{r(k)\}$. Suppose $r(k)\in C^i_{T_k-}$ and $j(k)\in C^{i'}_{T_k-}$, we perform the jumps: $C^i_{T_k}=C^i_{T_k-}\cup\{j(k)\}$, $C^{i'}_{T_k}=C^{i'}_{T_k-}\cup\{r(k)\}$, $K^{r(k)}_{T_k}=(K^{r(k)}_{T_k-}+1)\wedge N$, $K^{j(k)}_{T_k}=(K^{j(k)}_{T_k-}+1)\wedge N$, $N_{r(k),j(k)}(T_k)=N_{r(k),j(k)}(T_k-)+1$. So, in short, we have added $r(k)$ to the set $C^{i'}_{T_k-}$ and we have added $j(k)$ to the set $C^i_{T_k-}$.

This whole construction is analogous to the construction of the interaction graph found in [GM97], p. 122.

We now define, for a fixed time horizon $T\ge 0$, an auxiliary process $(Z^N_s)_{0\le s\le T}=(Z^{N,i}_s)_{0\le s\le T,1\le i\le N}$ of N particles in $\mathbb{R}^d$.

Definition 2.2. *The interaction times of the $(Z^{N,i}_s)_{1\le s\le T,1\le i\le q}$ are $\{T-T_k, k\ge 1, T_k\le T\}$. (We say that the jump times (T_k) are defined backward in time.)*

- $Z^{N,1}_0,\dots,Z^{N,N}_0$ *are i.i.d.* $\sim\tilde{P}_0$
- *Between the jump times $(T_k)_{k\ge 1}$, the $Z^{N,i}$'s evolve independently from each other according to the Markov generator L.*
- *At a jump time $T-T_k$ which is a jump time of $N^{i,j}$, (Z^N) undergoes an interaction having the same law as in Definition 2.1, (3.).*

Definition 2.3. *For all $t\ge 0$, we set*

$$L_t=\#\{k\in\mathbb{N}:T_k\le t, T_k=T''_k\}\ .$$

We call this quantity the number of loops on $[0,t]$.

Example 2.4. *Take $q=2$. Suppose for example, that the only jumps of the $N^{i,j}$'s occurring in $[0,T]$ are*

$$\Delta N^{2,3}(2T/3)=1\ ,\ \Delta N^{1,2}(T/3)=1$$

then for $s\in[0,2T/3[$, $K_s=2$ and for $s\in[2T/3,T]$, $K_s=3$. We then have $L_T=1$.

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We have to keep in mind the following lemma throughout the whole paper.

Lemma 2.5. *1. If $Y_1, \dots, Y_k$ are respectively of law $\mathcal{E}(\lambda_1), \dots, \mathcal{E}(\lambda_k)$ ($\lambda_1,\dots,\lambda_k > 0$, $k \in \mathbb{N}^*$) then*

$$\inf(Y_1, \dots, Y_k) \sim \mathcal{E}(\lambda_1 + \dots + \lambda_k) \ ,$$
$$\mathbb{P}(Y_1 = \inf(Y_1, \dots, Y_k)) = \frac{\lambda_1}{\lambda_1 + \dots + \lambda_k} \ .$$

2. For $Y \sim \mathcal{E}(\lambda)$ ($\lambda > 0$), $\mathcal{L}(Y - t | Y \geq t) = \mathcal{E}(\lambda)$ (for any $t \geq 0$).

3. If $Y_1, Y_2, \dots$ i.i.d. $\sim \mathcal{E}(\lambda)$ ($\lambda > 0$), $U_1, U_2, \dots$ are i.i.d. $\sim \mathcal{U}([0,1])$, then $\forall k \in \mathbb{N}^*$,

$$\mathcal{L}(\{Y_1, Y_1 + Y_2, \dots, Y_1 + \dots + Y_k\} | Y_1 + \dots + Y_k < T \leq Y_1 + \dots + Y_{k+1}) \\ = \mathcal{L}(\{U_1, \dots, U_k\})$$

We then have:

Lemma 2.6. *For all $T \geq 0$, $(Z_T^1, \dots, Z_T^q) \stackrel{law}{=} (\bar{Z}_T^1, \dots, \bar{Z}_T^q)$.*

The system $(Z_s^N)_{0 \leq s \leq Y}$ is of use in Section 4 but is also useful to understand the next auxiliary process, which we use in Section 3. We now define, for a fixed time horizon $T \geq 0$, the auxiliary process $(\hat{Z}_s^N)_{0 \leq s \leq T} = (\hat{Z}^{N,i}_{0 \leq s \leq T})_{1 \leq i \leq N, 0 \leq s \leq T}$.

We start in $s = 0$, with $\hat{C}_0 = [q]$, $\hat{K}_0 = q$, $\hat{L}_0 = 0$. We define processes $(\hat{C}_s)_{s \geq 0}$, $(\hat{K})_{s \geq 0}$, $(\hat{L}_s)_{s \geq 0}$ (respectively in $\mathcal{P}([N])$, $\mathbb{N}^*$, $\mathbb{N}^*$). These processes are piecewise constant and make jumps. We take $(\hat{U}_k)_{k \geq 0}$ to be i.i.d. $\sim \mathcal{E}(1)$ and $(\hat{A}_k)$ to be i.i.d. $\sim \mathcal{U}([0,1])$. We define recursively the jump times $(\hat{T}_k)_{k \geq 0}$ by $\hat{T}_0 = 0$ and

$$\hat{T}_{k+1} = \inf \left\{ s \geq \hat{T}_k : \int_{\hat{T}_k}^s \frac{(2(N - \hat{K}_u)_+ + \hat{K}_u - 1)}{2(N-1)} \Lambda \hat{K}_u du \geq \hat{U}_k \right\}$$

In $\hat{T}_k$:

- If $\hat{A}_k \leq \frac{2(N - \hat{K}_u)_+}{2(N - \hat{K}_u)_+ + \hat{K}_u - 1}$ then we perform the following jump: $\hat{K}_{\hat{T}_k} = \hat{K}_{\hat{T}_k -} + 1$, $\hat{L}_{\hat{T}_k} = \hat{L}_{\hat{T}_k -}$, we choose $\hat{i}(k)$ uniformly in $[N] \backslash \hat{C}_{\hat{T}_k -}$ and $\hat{C}_{\hat{T}_k} = \hat{C}_{\hat{T}_k -} \cup \{\hat{i}(k)\}$.
- If $\hat{A}_k > \frac{2(N - \hat{K}_u)_+}{2(N - \hat{K}_u)_+ + \hat{K}_u - 1}$ then we perform the following jump: $\hat{K}_{\hat{T}_k} = \hat{K}_{\hat{T}_k -}$, $\hat{L}_{\hat{T}_k} = \hat{L}_{\hat{T}_k -} + 1$, $\hat{C}_{\hat{T}_k} = \hat{C}_{\hat{T}_k -}$.

Definition 2.7. *The interaction times of the $(\hat{Z}_s^{N,i})_{1 \leq s \leq T, 1 \leq i \leq q}$ are $\{T - \hat{T}_k, k \geq 1, \hat{T}_k \leq T\}$. (This is why we say that the jump times $(\hat{T}_k)$ are defined backward in time.)*

- *$\hat{Z}_0^{N,1}, \dots, \hat{Z}_0^{N,N}$ are i.i.d. $\sim \tilde{P}_0$*
- *Between the jump times $(\hat{T}_k)_{k \geq 1}$, the $\hat{Z}^{N,i}$'s evolve independently from each other according to the Markov generator L.*
- *At a jump time $T - \hat{T}_k$, $(\hat{Z}^N)$ undergoes an interaction having the same law as in Definition 2.1, (3.), with i, j replaced by $\hat{i}(k), \hat{j}(k)$.*

Keeping in mind Lemma 2.5, we get:

Lemma 2.8. *For all $T \geq 0$,*

$$(Z_T^1, \dots, Z_T^q) \stackrel{law}{=} (\hat{Z}_T^1, \dots, \hat{Z}_T^q) \ ,$$
$$(K_t, L_t)_{0 \leq t \leq T} \stackrel{law}{=} (\hat{K}_t, \hat{L}_t)_{0 \leq t \leq T} \ .$$

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3 Expansion of the propagation of chaos

We define for any $N, q \in \mathbb{N}^*$, $q \le N$:

$$\langle q, N\rangle = \{a : [q] \to [N], a \text{ injective }\}\ , (N)_q = \#\langle q, N\rangle = \frac{N!}{(N-q)!}\ .$$

Let us set

$$\eta_t^N = \frac{1}{N} \sum_{1\le i\le N} \delta_{\bar{Z}_t^{N,i}}\ ,$$

$$(\eta_t^N)^{\odot q} = \frac{1}{(N)_q} \sum_{a\in\langle q,N\rangle} \delta_{(\bar{Z}_t^{N,a(1)},\dots,\bar{Z}_t^{N,a(q)})}\ .$$

Notice that for all function F, $\mathbb{E}(F(\bar{Z}_t^{N,1},\dots,\bar{Z}_t^{N,q})) = \mathbb{E}((\eta_t^N)^{\odot q}(F))$.

Theorem 3.1. *Set $\alpha = e^{-\Lambda T}$. For all $q \ge 1$, for any $F \in C_b^+(\mathbb{R}^{qd})$, $\forall T \ge 0$, $\forall l_0 \ge 1$,*

$$\mathbb{E}((\eta_T^N)^{\odot q}(F)) = \tilde{P}_T^{\otimes q}(F) + \sum_{1\le l\le l_0} \left[\frac{1}{(N-1)^l}\Delta_{q,T}^{N,l}(F)\right] + \frac{1}{(N-1)^{l_0+1}}\overline{\Delta}_{q,T}^{N,l_0+1}(F), \quad (3.1)$$

where the $\Delta_{q,T}^{N,l}$, $\overline{\Delta}_{q,T}^{N,l_0+1}$ are nonnegative measures uniformly bounded in N defined by, $\forall F \in C_b^+(\mathbb{R}^{qd})$ (the set of continuous bounded nonnegative functions on $\mathbb{R}^{qd}$),

$$\Delta_{q,T}^{N,l}(F) = \mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})|L_T = l)\mathbb{P}(L_T = l)(N-1)^l$$

$$\overline{\Delta}_{q,T}^{N,l}(F) = \mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})|L_T \ge l)\mathbb{P}(L_T \ge l)(N-1)^l\ .$$

We further have the following bounds

$$\begin{aligned}&\sup(\Delta_{q,T}^{N,l}(F), \overline{\Delta}_{q,T}^{N,l}(F))\\ &\quad\le q(1-\alpha)^{1/q-1}\frac{(2l+1)!}{(1-(1-\alpha)^{1/q})^{2l+2}} \times \|F\|_infty\\ &\qquad+ \frac{q}{(1-(1-\alpha)^{1/q})}\sup_{N\ge 1}\left((1-\alpha)^{\lfloor N-1\rfloor/q-1}(N-1)^l\right)\|F\|_\infty\ .\end{aligned}$$

Proof. According to [GM97] (section 3.4, p. 124) or, equivalently [GM94] (section 5),

$$\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})|L_T = 0) = \tilde{P}_T(F)\ .$$

We have, $\forall l_0$:

$$\begin{aligned}\mathbb{E}(F(\bar{Z}_T^{N,1},\dots,\bar{Z}_T^{N,q})) \quad = \quad & \mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})|L_T = 0)\mathbb{P}(L_T = 0)\\ &+ \sum_{l=1}^{l_0}[\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})|L_T = l)\mathbb{P}(L_T = l)]\\ &+ \mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})|L_T \ge l_0+1)\mathbb{P}(L_T \ge l_0+1).\end{aligned}$$

It is sufficient for the proof of (3.1) to show that $\mathbb{P}(L_T \ge l)$ is of order $1/N^l$, $\forall l \in \mathbb{N}^*$.

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We define piecewise constant processes $(\hat{K}'_s)_{s\geq 0}$, $(\hat{L}'_s)_{s\geq 0}$ (in $\mathbb{N}$) such that $\hat{K}'_0 = q$, $\hat{L}'_0 = 0$. Their jump times are $(\hat{T}'_k)_{k\geq 0}$ defined recursively by $\hat{T}'_0 = 0$ and

$$\hat{T}'_k = \inf\left\{ s : \int_{\hat{T}'_{k-1}}^{s} \Lambda \hat{K}'_u du \geq \hat{U}_k \right\} .$$

In $\hat{T}'_k$:

- If $\hat{A}_k \leq \frac{(N-\hat{K}'_{\hat{T}'_k-})_+}{N-1}$, then we perform the following jump: $\hat{K}'_{\hat{T}'_k} = \hat{K}'_{\hat{T}'_k-} + 1$, $\hat{L}'_{\hat{T}'_k} = \hat{L}'_{\hat{T}'_k-}$.
- If $\hat{A}_k > \frac{(N-\hat{K}'_{\hat{T}'_k-})_+}{N-1}$, then we perform the following jump: $\hat{K}'_{\hat{T}'_k} = \hat{K}'_{\hat{T}'_k-} + 1$, $\hat{L}'_{\hat{T}'_k} = \hat{L}'_{\hat{T}'_k-} + 1$.

Notice that we use there the same variables $\hat{U}_k$'s and $\hat{A}_k$'s coming from the definition of $(\hat{Z}^N)$. We have for all $t \geq 0$, $\forall\omega$,

$$\begin{array}{rcl} \hat{K}_t(\omega) & \leq & \hat{K}'_t(\omega) \\ \hat{L}_t(\omega) & \leq & \hat{L}'_t(\omega) \\ \hat{L}'_t(\omega) & \leq & \hat{K}'_t(\omega) . \end{array} \tag{3.2}$$

The process $(\hat{K}'_s)_{s\geq 0}$ is equal in law to the sum of q independant Yule processes $Y_s^{(1)}$, ..., $Y_s^{(q)}$ (see [AN72], p. 102-109, p. 109 for the law of the Yule process). We have $\mathbb{P}(Y_s^{(1)} = k) = e^{-s\Lambda}(1 - e^{-s\Lambda})^{k-1}$ and so:

$$\begin{aligned} \mathbb{P}(\hat{K}'_t = k) &= \mathbb{P}(Y_t^{(1)} + \cdots + Y_t^{(q)} = k) \\ &\leq \sum_{i=1}^{q} \mathbb{P}(Y_t^{(i)} \geq \lceil k/q \rceil) \\ &\leq q(1 - e^{-t\Lambda})^{k/q-1} . \end{aligned} \tag{3.3}$$

Notice that $\forall t \in [0, T]$

$$\mathbb{P}(\Delta \hat{L}'_t = 1 | \Delta \hat{K}'_t = 1, (\hat{K}'_t)_{t\geq 0}) = 1 - \frac{(N - \hat{K}'_{t-})_+}{N-1} \leq \frac{\hat{K}'_T}{N-1} .$$

We decompose

$$\begin{array}{rcl} \mathbb{P}(L_T = l) & \leq & \mathbb{P}(L_T \geq l) \\ \text{(by Lem. 2.8)} & = & \mathbb{P}(\hat{L}_T \geq l) \\ & \leq & \mathbb{P}(\hat{L}'_T \geq l) \\ & = & \mathbb{P}(\hat{K}'_T \geq \lfloor \sqrt{N-1} \rfloor) + \mathbb{P}(\ \hat{L}'_T \geq l, \hat{K}'_T \leq \lfloor \sqrt{N-1} \rfloor) , \end{array}$$

and we compute

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$$\begin{aligned}
&\mathbb{P}(\hat{L}'_T \geq l, \hat{K}'_T \leq \lfloor\sqrt{N-1}\rfloor) \\
&= \sum_{l\leq r\leq\lfloor\sqrt{N-1}\rfloor} \mathbb{P}(\hat{L}'_T = r, \hat{K}'_T \leq \lfloor\sqrt{N-1}\rfloor) \\
&= \sum_{l\leq r\leq\lfloor\sqrt{N-1}\rfloor} \sum_{r\leq k\leq\lfloor\sqrt{N-1}\rfloor} \mathbb{P}(\hat{L}'_T = r | \hat{K}'_t = k)\mathbb{P}(\hat{K}'_T = k) \\
&= \sum_{l\leq r\leq\lfloor\sqrt{N-1}\rfloor} \sum_{r\leq k\leq\lfloor\sqrt{N-1}\rfloor} C_k^r \left(\frac{k}{N-1}\right)^r q(1-\alpha)^{k/q-1} \\
&\leq \sum_{l\leq k\leq\lfloor\sqrt{N-1}\rfloor} \sum_{l\leq r\leq k} \frac{k^{2r}}{r!(N-1)^r} q(1-\alpha)^{k/q-1} \ .
\end{aligned}$$

As, for $k \leq \lfloor\sqrt{N-1}\rfloor$, $\frac{k^2}{N-1} \leq 1$, we get

$$\begin{aligned}
\mathbb{P}(\hat{L}'_T \geq l, \hat{K}'_T \leq \lfloor\sqrt{N-1}\rfloor) &\leq \sum_{l\leq k\leq\lfloor\sqrt{N-1}\rfloor} k\frac{k^{2l}}{l!(N-1)^l} q(1-\alpha)^{k/q-1} \\
&\leq \frac{q}{l!(N-1)^l} \sum_{l\leq k} k(k+1)(k+2)\dots(k+2l) \\
&\qquad \times ((1-\alpha)^{1/q})^{k-1}(1-\alpha)^{1/q-1} \\
&\leq \frac{q(1-\alpha)^{1/q-1}}{(N-1)^l} \frac{(2l+1)!}{(1-(1-\alpha)^{1/q})^{2l+2}} \ .
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbb{P}(\hat{K}_T \geq \lfloor N-1\rfloor) &\leq \sum_{k\geq\lfloor N-1\rfloor} q(1-\alpha)^{k/q-1} \\
&= \frac{q(1-\alpha)^{\lfloor N-1\rfloor/q-1}}{(1-(1-\alpha)^{1/q})} \\
&\leq \frac{q}{(1-(1-\alpha)^{1/q})} \sup_{N\geq 1} \left((1-\alpha)^{\lfloor N-1\rfloor/q-1}(N-1)^l\right) \frac{1}{(N-1)^l} \ .
\end{aligned}$$

□

4 Wick formula

We now define an auxiliary system $(\tilde{Z}^i_t)_{0\leq t\leq T, i\geq 1}$ with an infinite number of particles (for a fixed time horizon $T \geq 0$). We start in $s = 0$ with $\tilde{C}^i_0 = \{i\}$, $\forall i \in [q]$. We set $\forall i$, $\tilde{K}^i_0 = \#\tilde{C}^i_0$. For $1 \leq i < j \leq N$, we define processes $(\tilde{N}^{i,j}_s = \tilde{N}^{j,i}_s)_{s\geq 0}$, $(\tilde{C}^i_s)_{s\geq 0}$, $(\tilde{K}^i_s)_{s\geq 0}$ respectively in $\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathbb{N}$, by the following. The processes $(\tilde{N}^{i,j}), (\tilde{C}^i), (\tilde{K}^i)$ are piecewise constant. We set

$$\tilde{K}_s = \#(\tilde{C}^1_s + \dots + \tilde{C}^q_s) \ .$$

We define the jump times recursively by (taking $(\tilde{U}_k)_{1\leq i\leq q, 1\leq k}$ i.i.d. $\sim \mathcal{E}(1)$), $\tilde{T}_0 = 0$ and

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$$\begin{aligned}\tilde{T}'_k &= \inf\left\{\tilde{T}_{k-1} \le s \le T : \int_{\tilde{T}_{k-1}}^{s} \Lambda\tilde{K}_u - \frac{\Lambda K_u(N-K_u)_+}{N-1} du \ge \tilde{U}_k\right\}\\ \tilde{T}_k &= \inf(\tilde{T}'_k, \inf\{T_l : T_l > \tilde{T}_{k-1}\})\end{aligned}$$

(recall that the process (K_t) is defined in subsection 2.2). Notice that $\{T_k, k \ge 0\} \subset \{\tilde{T}_k, k \ge 0\}$. In $\tilde{T}_k$:

- If $\tilde{T}_k \notin \{T''_l, l \ge 1\}$, we take $\tilde{r}(k)$ uniformly in $\tilde{C}^1_{T_k-} \cup \dots \cup \tilde{C}^q_{T_k-}$ and $\tilde{j}(k)$ uniformly in $\mathbb{N}^*\backslash(\tilde{C}^1_{T_k-} \cup \dots \cup \tilde{C}^q_{T_k-})$. Suppose $\tilde{r}(k) \in \tilde{C}^i_{T_k-}$, we perform the jumps: $\tilde{C}^i_{T_k} = \tilde{C}^i_{T_k-} \cup \{\tilde{j}(k)\}$, $\tilde{K}^i_{T_k} = \tilde{K}^i_{T_k-} + 1$ (at any time s and any index i, we will have $\tilde{K}^i_s = \#\tilde{C}^i_s$), $\tilde{N}^{\tilde{r}(k),\tilde{j}(k)}_{T_k} = \tilde{N}^{\tilde{r}(k),\tilde{j}(k)}_{T_k-} + 1$.
- If $\tilde{T}_k \in \{T''_l, l \ge 1\}$, we take $\tilde{r}(k)$ uniformly in $\tilde{C}^1_{T_k-} \cup \dots \cup \tilde{C}^q_{T_k-}$ and $\tilde{j}(k)$ uniformly in $\tilde{C}^1_{T_k-} \cup \dots \cup \tilde{C}^q_{T_k-} \backslash \{\tilde{r}(k)\}$. Suppose $\tilde{r}(k) \in \tilde{C}^i_{T_k-}$, $\tilde{j}(k) \in \tilde{C}^{i'}_{T_k-}$, we perform the jumps: $\tilde{C}^i_{T_k} = \tilde{C}^i_{T_k-} \cup \{\tilde{j}(k)\}$, $\tilde{K}^i_{T_k} = \tilde{K}^i_{T_k-} + 1$, $\tilde{C}^{i'}_{T_k} = \tilde{C}^{i'}_{T_k-} \cup \{\tilde{r}(k)\}$, $\tilde{K}^{i'}_{T_k} = \tilde{K}^{i'}_{T_k-} + 1$, $\tilde{N}^{\tilde{r}(k),\tilde{j}(k)}_{T_k} = \tilde{N}^{\tilde{r}(k),\tilde{j}(k)}_{T_k-} + 1$

The following lemma is a consequence of Lemma 2.5.

Lemma 4.1. *1. The process $(\tilde{K}_s)_{s\ge0}$ is piecewise constant, has jumps of size 1 and satisfies $\forall 0 \le s \le t$*

$$\mathbb{P}(\tilde{K}_t = \tilde{K}_s | \tilde{K}_s) = \exp(-\Lambda \tilde{K}_s) \ .$$

And so it has the same law as $(\tilde{K}'_s)_{s\ge0}$.

2. *For all t, $\tilde{K}_t \ge K_t$, a.s.*
3. *If $T_1 = \tilde{T}_1, \dots, T_k = \tilde{T}_k$ then $\tilde{K}_{T_k} = K_{T_k}$.*

In other word, knowing that t is a jump time of $(\tilde{K}_s)$, the time to the next jump time is of law $\mathcal{E}(\Lambda\tilde{K}_t)$.

Definition 4.2. *The interaction times of the $\tilde{Z}^i$ are $\{T - \tilde{T}_k, k \ge 1\}$ (we say they are defined backward in time).*

- *The $(\tilde{Z}^i_0)$ are i.i.d. $\sim \tilde{P}_0$.*
- *Between the jump times, the $\tilde{Z}^i$ evolve independently from each other according to the Markov generator L.*
- *At a jump time $T - \tilde{T}_k$, $(\tilde{Z})$ undergo a jump like in Definition 2.1, (3), with i, j replaced by $\tilde{r}(k), \tilde{j}(k)$.*

We define ($\forall t \ge 0$)

$$\begin{aligned}G &= \{\forall k \ge 1 \text{ such that } \tilde{T}_k \le T,\ \tilde{T}_k = T_k\}\\ \mathcal{K}_t &= \{(K^i_s)_{1\le s\le t}, i \in [q]\}\\ \tilde{\mathcal{K}}_t &= \{(\tilde{K}^i_s)_{1\le s\le t}, i \in [q]\}\end{aligned}$$

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and, for q even, we set $\{T_{k_1} \leq T_{k_2} \leq \dots\} = \{T_k : k \geq 1, T_k = T''_k\}$, $\{\tilde{T}_{\tilde{k}_1} \leq \tilde{T}_{\tilde{k}_2} \leq \dots\} = \{\tilde{T}_k : k \geq 1, \exists l, \tilde{T}_k = T''_l\}$

$$\begin{aligned} A &= \{\#\{k \geq 1 : T_k = T''_k, T_k < T\} = q/2\} \\ &\quad \cap \{r(k_1) \in C^1_{T_{k_1}-}, j(k_1) \in C^2_{T_{k_1}-}, \dots, r(k_{q/2}) \in C^{q/2-1}_{T_{k_{q/2}}-}, j(k_{q/2}) \in C^{q/2}_{T_{k_{q/2}}-}\} \\ \tilde{A} &= \{\#\{k \geq 1 : \exists l, \tilde{T}_k = T''_l, \tilde{T}_k < T\} = q/2\} \\ &\quad \cap \{\tilde{r}(\tilde{k}_1) \in \tilde{C}^1_{\tilde{T}_{\tilde{k}_1}-}, \tilde{j}(\tilde{k}_1) \in \tilde{C}^2_{\tilde{T}_{\tilde{k}_1}-}, \dots, \tilde{r}(\tilde{k}_{q/2}) \in \tilde{C}^{q/2-1}_{\tilde{T}_{\tilde{k}_{q/2}}-}, \tilde{j}(\tilde{k}_{q/2}) \in \tilde{C}^{q/2}_{\tilde{T}_{\tilde{k}_{q/2}}-}\} \end{aligned}$$

(recall the $i(k)$'s, $r(k)$'s are defined in subsection 2.2). For $q \in \mathbb{N}^*$, we define

$$\mathcal{B}_0^{sym}(q) = \{F : \mathbb{R}^{qd} \to \mathbb{R}, F \text{ symmetric, bounded,} \\ \int_{x_1,\dots,x_q \in \mathbb{R}^d} F(x_1, \dots, x_q) \tilde{P}_T(dx_q) = 0\} \ .$$

Proposition 4.3. *For $F \in \mathcal{B}_0^{sym}(E^q)$, we have:*

- *for $k < \frac{q}{2}$, $\Delta^{N,k}_{q,T}(F) = 0$,*
- *for q even,*

$$N^{q/2}\mathbb{E}((\eta_T^N)^{\odot q}(F)) \underset{N \to +\infty}{\longrightarrow} \frac{q!}{2^{q/2}(q/2)!}\mathbb{E}\Big(\mathbb{E}(F(\tilde{Z}^1_T, \dots, \tilde{Z}^q_T))|\tilde{C}_t, \tilde{A}) \\ \times \prod_{1 \leq i \leq q/2} \int_0^T \Lambda \tilde{K}^{2i-1}_s \tilde{K}^{2i}_s ds\Big) \ . \tag{4.1}$$

Proof. The convergence to 0 for $k < q/2$ is a consequence of Theorem 3.1. We suppose by now that q is even. In the following computations, we set $R = \sup\{k : \tilde{T}_k \leq T\}$. We have

$$\begin{aligned} \mathbb{E}(G|\mathcal{K}_T) &= \mathbb{E}(\mathbf{1}_{\tilde{T}_1 = T_1} \dots \mathbf{1}_{\tilde{T}_R = T_R} | \mathcal{K}_T) \\ &= \mathbb{E}(\mathbf{1}_{\tilde{T}_1 = T_1} \dots \mathbf{1}_{\tilde{T}_{R-1} = T_{R-1}} \mathbb{E}(\mathbf{1}_{\tilde{T}_R = T_R} | \mathcal{K}_T, \tilde{T}_1 = T_1, \dots, \tilde{T}_{R-1} = T_{R-1}) \ . \end{aligned}$$

On the event $\{\tilde{T}_1 = T_1, \dots, \tilde{T}_{R-1} = T_R\}$, we have $\tilde{K}_t = K_t$, $\forall t : T_{R-1} \leq t < T_R$. And so we have:

$$\begin{aligned} &\mathbb{E}(\mathbf{1}_{\tilde{T}_R = T_R} | \mathcal{K}_T, T_{k-1}, \tilde{T}_1 = T_1, \dots, \tilde{T}_{R-1} = T_{R-1}) \\ &= 1 - \mathbb{P}\Big(\int_{T_{R-1}}^{T_R} \Lambda K_u - \frac{\Lambda K_u (N - K_u)_+}{N-1} du \leq \tilde{U}_k \\ &\qquad | \mathcal{K}_T, \tilde{T}_1 = T_1, \dots, \tilde{T}_{R-1} = T_{R-1}\Big) \\ &\geq 1 - \mathbb{P}\left(\int_{T_{R-1}}^{T_R} \frac{\Lambda (K_T)^2}{N-1} du \leq \tilde{U}_k \middle| \mathcal{K}_T, \tilde{T}_1 = T_1, \dots, \tilde{T}_{R-1} = T_{R-1}\right) \\ &= \exp\left(-\frac{\Lambda (K_T)^2}{N-1}(T_R - T_{R-1})\right) \ . \end{aligned}$$

So, by recurrence

$$\mathbb{E}(G|\mathcal{K}_T) \geq \exp\left(-\frac{\Lambda K_T^2}{N-1} T\right) \ . \tag{4.2}$$

Because of Theorem 3.1, we get:

$$\begin{aligned} N^{q/2}\mathbb{E}((\eta_T^N)^{\odot q}(F)) &= N^{q/2}\mathbb{E}(F(\bar{Z}_T^{N,1},\dots,\bar{Z}_T^{N,q})) \\ &= N^{q/2}\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})) \\ &\underset{N\to+\infty}{\sim} N^{q/2}\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_{L_T=q/2}) \ . \end{aligned}$$

When $\exists k,r,s : N^{k,r}$ jumps at $s, k\in C_s^i, r\in C_s^j$ and $s\in[0,T]$, we say there is a loop between C^i and C^j. We can define in the same way loops between $\tilde{C}^i$, $\tilde{C}^j$. As $F\in\mathcal{B}_0^{sym}$, then $\forall j\in[q]$, we have

$$\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})|C^j \text{ has no loop on } [0,T])=0 \ .$$

Notice also that $[\exists i,\ C^i$ has two loops on $[0,T]$ and $L_T=q/2] \Rightarrow [\ \exists j,\ C^j$ has no loop on $[0,T]]$. As $\frac{q!}{2^{q/2}(q/2)!}$ is the number of ways of partitioning $[q]$ into $q/2$ couples, we get

$$N^{q/2}\mathbb{E}((\eta_T^N)^{\odot q}(F)) \underset{N\to+\infty}{\sim} N^{q/2}\frac{q!}{2^{q/2}(q/2)!}\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_A) \ .$$

Noticing that $\mathcal{L}((Z_T^{N,1},\dots,Z_T^{N,q})|G)=\mathcal{L}((\tilde{Z}_T^{N,1},\dots,\tilde{Z}_T^{N,q})|G)$, we get:

$$\begin{aligned} N^{q/2}\mathbb{E}((\eta_T^N)^{\odot q}(F) \underset{N\to+\infty}{\sim} N^{q/2}\frac{q!}{2^{q/2}(q/2)!}\Big[&\mathbb{E}(F(\tilde{Z}_T^{N,1},\dots,\tilde{Z}_T^{N,q})\mathbf{1}_{\tilde{A}}) \\ &+\mathbb{E}((F(\bar{Z}_T^{N,1},\dots,\bar{Z}_T^{N,q})\mathbf{1}_A\mathbf{1}_{G^c}) \\ &-\mathbb{E}(F(\tilde{Z}_T^{N,1},\dots,\tilde{Z}_T^{N,q})\mathbf{1}_{\tilde{A}}\mathbf{1}_{G^c})\Big] \ . \end{aligned}$$

Notice that, knowing $\mathcal{K}_T$, for $i\neq j$, the number of loops between C^i and C^j on $[0,t]$ is a non-homogeneous Poisson process of intensity $\left(\frac{\Lambda K_t^i K_t^j}{N-1}\right)_{0\le t\le T}$. Notice also that, knowing $\tilde{\mathcal{K}}_T$, for $i\neq j$, the number of loops between $\tilde{C}^i$ and $\tilde{C}^j$ on $[0,t]$ is a non-homogeneous Poisson process of intensity $\left(\frac{\Lambda\tilde{K}_t^i\tilde{K}_t^j}{N-1}\right)_{0\le t\le T}$. So we have:

$$\mathbb{P}(\text{ no loop between } C^i \text{ and } C^j|\mathcal{K}_T)=\exp\left(-\int_0^T\frac{\Lambda K_s^iK_s^j}{N-1}ds\right)=:\alpha(i,j) \ ,$$

$$\mathbb{P}(\text{ no loop between } \tilde{C}^i \text{ and } \tilde{C}^j|\tilde{\mathcal{K}}_T)=\exp\left(-\int_0^T\frac{\Lambda\tilde{K}_s^i\tilde{K}_s^j}{N-1}ds\right)=:\tilde{\alpha}(i,j) \ .$$

We set

$$\begin{aligned} B \ = \ &\{\text{at least one loop between } C^1 \text{ and } C^2\}\cap\dots \\ &\dots\cap\{\text{at least one loop between } C^{q-1} \text{ and } C^q\} \ . \end{aligned}$$

We have

$$\mathbb{P}(A|\mathcal{K}_T)=\mathbb{P}(B|\mathcal{K}_T)-\mathbb{P}(B\backslash A|\mathcal{K}_T) \ , \tag{4.3}$$
$$\mathbb{P}(\text{ at least two loops between } C^1 \text{ and } C^2|\mathcal{K}_T)\le(1-\alpha(1,2))^2 \ ,$$

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and

$$\begin{aligned}\mathbb{P}(B|\mathcal{K}_T) &= \prod_{1\le j\le q/2}(1-\alpha(2j-1,2j))\\ &\le \frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(T\Lambda K_T^{2j-1}K_T^{2j})\\ &\le \frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(T\Lambda \tilde{K}_t^{2j-1}\tilde{K}_T^{2j})\ . \end{aligned} \tag{4.4}$$

So, we have:

$$\begin{aligned} & N^{q/2}\mathbb{E}(|(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_A\mathbf{1}_{G^c}|)\\ &\le N^{q/2}\|F\|_\infty\mathbb{E}(\mathbb{E}(\mathbf{1}_A\mathbf{1}_{G^c}|\mathcal{K}_T))\\ &= N^{q/2}\|F\|_\infty\mathbb{E}(\mathbb{E}(\mathbf{1}_A|\mathcal{K}_T)\mathbb{E}(\mathbf{1}_{G^c}|\mathcal{K}_T))\\ \text{(using (4.2), (4.3), (4.4))}\quad &\le N^{q/2}\|F\|_\infty\mathbb{E}\left(\frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(T\Lambda K_T^{2j-1}K_T^{2j})\right.\\ &\qquad\left.\times\left(1-\exp\left(-\frac{\Lambda(K_T)^2T}{N-1}\right)\right)\right)\\ \text{(using Lemma 4.1)}\quad &\le N^{q/2}\|F\|_\infty\mathbb{E}\left(\frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(T\Lambda \tilde{K}_T^{2j-1}\tilde{K}_T^{2j})\right.\\ &\qquad\left.\times\left(1-\exp\left(-\frac{\Lambda(\tilde{K}_T)^2T}{N-1}\right)\right)\right)\ ,\end{aligned}$$

For a fixed ω,

$$N^{q/2}\frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(t\Lambda\tilde{K}_T^{2j-1}(\omega)\tilde{K}_T^{2j}(\omega))\times\left(1-\exp\left(-\frac{\Lambda(\tilde{K}_T(\omega))^2T}{N-1}\right)\right) \underset{N\to+\infty}{\longrightarrow} 0 \tag{4.5}$$

and

$$\begin{aligned} 0 &\le N^{q/2}\frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(t\Lambda\tilde{K}_T^{2j-1}\tilde{K}_T^{2j})\times\left(1-\exp\left(-\frac{\Lambda(\tilde{K}_T)^2T}{N-1}\right)\right)\\ &\le 2^{q/2}\prod_{1\le j\le q/2}(t\Lambda\tilde{K}_T^{2j-1}\tilde{K}_T^{2j})\end{aligned}$$

which is of finite expectation by (3.3) and (3.2) and Lemma 4.1. So, by dominated convergence:

$$N^{q/2}\mathbb{E}(|(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_A\mathbf{1}_{G^c}|)\underset{N\to+\infty}{\longrightarrow} 0\ .$$

We can show in the same way :

$$N^{q/2}\mathbb{E}(|(F(\tilde{Z}_T^{1},\dots,\tilde{Z}_T^{q})\mathbf{1}_{\tilde{A}}\mathbf{1}_{G^c}|)\underset{N\to+\infty}{\longrightarrow} 0\ .$$

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We have:

$$N^{q/2}\mathbb{E}(F(\tilde{Z}_T^1,\dots,\tilde{Z}_T^q)\mathbf{1}_{\tilde{A}}) = \mathbb{E}(\mathbb{E}(F(\tilde{Z}_T^1,\dots,\tilde{Z}_T^q)|\tilde{\mathcal{K}}_T,\tilde{A}))N^{q/2}\mathbb{P}(\tilde{A}|\tilde{\mathcal{K}}_T))\ ,$$

We set

$$\begin{aligned}\tilde{B} &= \{\text{at least one loop between } \tilde{C}^1 \text{ and } \tilde{C}^2\}\cap\dots\\ &\quad \dots\cap\{\text{at least one loop between } \tilde{C}^{q-1} \text{ and } \tilde{C}^q\}\ .\end{aligned}$$

We decompose

$$\mathbb{P}(\tilde{A}|\tilde{C}_T) = \mathbb{P}(\tilde{B}|\tilde{\mathcal{K}}_T) - \mathbb{P}(\tilde{B}\backslash\tilde{A}|\tilde{\mathcal{K}}_T)\ .$$

We have

$$\begin{aligned}N^{q/2}\mathbb{P}(\tilde{B}|\tilde{\mathcal{K}}_T) &= N^{q/2}\prod_{1\le j\le q/2}(1-\tilde{\alpha}(2j-1,2j))\\ &\xrightarrow[N\to+\infty]{\text{a.s.}} \prod_{1\le i\le q/2}\int_0^T \Lambda\tilde{K}_s^{2i-1}\tilde{K}_s^{2i}ds\ .\end{aligned}$$

We also get, computing very roughly:

$$\begin{aligned}\mathbb{P}(\tilde{B}\backslash\tilde{A}|\tilde{\mathcal{K}}_T) &\le \sum_{1\le r\le q/2}\ \sum_{1\le i_1,\dots i_r\le q/2}\left[\prod_{j\in\{i_1,\dots,i_r\}}(1-\tilde{\alpha}(2j-1,2j))^2\right.\\ &\qquad\left.\times\prod_{j\notin\{i_1,\dots,i_r\}}(1-\tilde{\alpha}(2j-1,2j))\right]\\ &\le \sum_{1\le r\le q/2} C_{q/2}^r\prod_{1\le j\le q/2}\left[\left(T\Lambda\tilde{K}_T^{2j-1}\tilde{K}_T^{2j}\right)^2\vee(T\Lambda\tilde{K}_T^{2j-1}\tilde{K}_T^{2j})\right]\\ &\qquad\times\frac{1}{(N-1)^{q/2+1}}\\ &\le \frac{2^{q/2}}{(N-1)^{q/2+1}}\prod_{1\le j\le q/2}\left[\left(T\Lambda\tilde{K}_T^{2j-1}\tilde{K}_T^{2j}\right)^2\vee(T\Lambda\tilde{K}_T^{2j-1}\tilde{K}_T^{2j})\right]\ ,\end{aligned}$$

so $N^{q/2}\mathbb{P}(\tilde{B}\backslash\tilde{A}|\tilde{C}_T)\xrightarrow[N\to+\infty]{\text{a.s.}} 0$. And so:

$$N^{q/2}\mathbb{P}(\tilde{A}|\tilde{C}_T)\xrightarrow[N\to+\infty]{\text{a.s.}}\prod_{1\le i\le q/2}\int_0^T\Lambda\tilde{K}_s^{2i-1}\tilde{K}_s^{2i}ds\ ,$$

and again by dominated convergence, we get the result.

□

For $F:(\mathbb{R}^d)^q\to\mathbb{R}$, we define

$$(F)_{sym}(x_1,\dots,x_q) = \frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q}F(x_{\sigma(1)},\dots,x_{\sigma(q)})$$

($\mathcal{S}_q$ being the set of permutations of $[q]$). We denote by $\mathcal{I}_q$ the set of partitions of $[q]$ into pairs.

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Corollary 4.4. *For $F \in \mathcal{B}_0^{sym}$ of the form $(f_1 \otimes \cdots \otimes f_q)_{sym}$ and q even,*

$$N^{q/2}\mathbb{E}((\eta_t^n)^{\odot q}(F)) \underset{N\to+\infty}{\longrightarrow} \sum_{I_q\in\mathcal{I}_q} \prod_{\{i,j\}\in I_q} \mathbb{E}(V_T^B(f_i,f_j)) \ ,$$

with

$$V_T^B(f_i,f_j) = \mathbb{E}(f_i(\tilde{Z}_t^1)f_j(\tilde{Z}_T^2)|\tilde{A},\tilde{\mathcal{K}}_t) \times \int_0^T \Lambda \tilde{K}_s^1 \tilde{K}_s^2 ds \ .$$

References

[AN72] Krishna B. Athreya and Peter E. Ney, *Branching processes*, Springer-Verlag, New York, 1972, Die Grundlehren der mathematischen Wissenschaften, Band 196. MR MR0373040 (51 #9242)

[DMPR] P. Del Moral, F. Patras, and S. Rubenthaler, *Tree based functional expansions for Feynman-Kac particle models.*

[DMPR09] ———, ..., In preparation (2009).

[GM94] Carl Graham and Sylvie Méléard, *Chaos hypothesis for a system interacting through shared resources*, Probab. Theory Related Fields **100** (1994), no. 2, 157–173. MR MR1296426 (95j:60165)

[GM97] ———, *Stochastic particle approximations for generalized Boltzmann models and convergence estimates*, Ann. Probab. **25** (1997), no. 1, 115–132.

[GM99] ———, *Probabilistic tools and Monte-Carlo approximations for some Boltzmann equations*, CEMRACS 1999 (Orsay), ESAIM Proc., vol. 10, Soc. Math. Appl. Indust., Paris, 1999, pp. 77–126 (electronic). MR MR1865189 (2003a:82062)