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Piecewise linear density estimation for sampled data

François-Xavier Lejeune*

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Abstract – Nonparametric density estimation is considered for a discretely observed stationary continuous-time process. For each of three given time sampling procedures either random or deterministic, we establish that histograms and frequency polygons can reach the same optimal L_2 -rates as in the independent and identically distributed case. Moreover, thanks to a suitable “high frequency” sampling design, these rates are derived together with a minimized time of observation depending on the regularity of sample paths.

Key words: nonparametric density estimation, histogram, frequency polygon, sampling, mean integrated squared error, rate of convergence.

2000 Mathematics Subject Classification: primary 62G07; secondary 62M.

1 Introduction

Consider a $\mathbb{R}^d$ -valued process $\{X_t, t \in \mathbb{R}\}$, $d \geq 1$, where all X_t ’s have the same unknown marginal density f . The aim of this paper is to study the rates of some nonparametric piecewise linear estimators of f when the process is discretely sampled in time at $t = t_1, \dots, t_n$. During the past three decades, the problem of density estimation for continuous-time observations has been a subject of continued interest in the statistical literature. Especially, it was shown by Castellana and Leadbetter [12] that if a continuous-time process, observed over the time interval $[0, T]$, has enough irregular sample paths, then nonparametric estimators can achieve a mean-square parametric rate of convergence $1/T$. An account of the research in this field may be found e.g. in two complementary monographs by Bosq and Blanke [9] and Kutoyants [19], and in Lejeune [23] for the particular case of piecewise linear estimators.

In practice, however, the whole sample path is not always perfectly observable over a given time period – either due to technical reasons or unavailability of data at all time points. Indeed, most of physical phenomena usually represented by curves generate rather discrete observed values or interpolated ones. Hence it seems more natural to plan an estimation approach based upon n discrete values of the process collected with a suitable time sampling procedure. In the context of nonparametric density estimation, the three sampling procedures considered in the present work have been investigated by Masry [24], Prakasa Rao [25], Wu [30] (random sampling), Bosq [6, 7] and Blanke and Pumo [5] (“high frequency” periodic sampling), among others. As far as we know, most of existing papers – including those cited above – only deal with

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kernel estimation, and none have yet focused a special attention on piecewise linear estimators. In that framework, we are then interested in the rates of histogram and frequency polygon estimators with respect to the mean integrated squared error (MISE) criterion. Here, the chosen histogram-based density estimators have the desirable property of being quickly computed and updated. This is therefore a clear advantage for many applications where typically one has to handle large amounts of data in real time. It is noteworthy that elementary estimators may also be efficient from a theoretical viewpoint. Thus and despite its high simplicity, the frequency polygon – defined in dimension one as the linear interpolant of the mid-points of an equally spaced histogram – is known to be as good as some more sophisticated density estimators in terms of MISE (see Scott [27]).

In this paper, we will show that, under mild conditions, these estimators built with sampled data have at least the same optimal rates $n^{-2/(d+2)}$ (histogram) and $n^{-4/5}$ (univariate frequency polygon) as in the independent and identically distributed (i.i.d.) case. First, we will examine the case of two classical random sampling designs, which are a relevant way to treat the occurrence of low frequency and irregularly spaced measurements. Next, we will investigate a deterministic design that applies when the data are observed at high frequency and during a long time, as in a variety of domains, including econometrics, meteorology, oceanology and many others. Particularly, this sampling design is well-adapted to the continuous-time context since the optimal rates can be derived together with a minimized time of observation depending on the regularity of sample paths (see Bosq [7]). Thanks to this methodology we will furthermore address the important issue of finding an optimal sampling strategy.

The paper is organized as follows. In Section 2 we will review the time sampling procedures and define our framework. Section 3 contains the main assumptions and our results relative to histograms; the behavior of frequency polygons is then studied in Section 4 and a concluding discussion is given in Section 5. Finally, the proofs are postponed until Section 6.

2 Preliminaries and notations

Let $X^T = \{X_t, 0 \leq t \leq T\}$ be a measurable $\mathbb{R}^d$ -valued, $d \geq 1$, continuous-time process on the probability space $(\Omega, \mathcal{F}, P)$, where the X_t 's have a common distribution admitting a density f with respect to the Lebesgue measure over $\mathbb{R}^d$. We suppose that the joint density $f_{(X_s, X_t)}$ of (X_s, X_t) does exist for all $s \neq t$ such that $f_{(X_s, X_t)} = f_{(X_0, X_{|t-s|})} =: f_{|t-s|}$, which is a quite weak stationary condition (see e.g. Bosq [8]). We also denote by g_u the function defined for all $u > 0$ as $g_u := f_u - f \otimes f$ where $(f \otimes f)(y, z) = f(y)f(z)$. Some required asymptotic independence conditions on the process (including α -mixing condition) will be added later with our assumptions. Our purpose is to estimate the function f from n observations collected up to time T by making use of one of the sampling procedures described below.

2.1 Sampling schemes

Let $\mathcal{T}^n = \{t_k, 0 \leq k \leq n\}$ be a strictly increasing sequence of points in time – or *event arrival times* – such that $0 = t_0 < t_1 < \dots < t_n =: T_n$ and $T_n \rightarrow \infty$ as $n \rightarrow \infty$. If $\mathcal{T}^n$ is random, it is also assumed that the processes X^T and $\mathcal{T}^n$ are independent and that all X_{t_k} 's are measurable

with respect to the σ -algebra generated by X^T and $\mathcal{T}^n$. The first two random schemes as defined in Masry [24] are the following.

Renewal sampling – The set of times for observations $\mathcal{T}^n \equiv \mathcal{T}_1^n$ is a renewal type process on $[0, +\infty[$ such that

$$t_0 = 0 \quad \text{and} \quad t_k = \sum_{j=1}^k \tau_j, \quad 1 \leq k \leq n,$$

where $\tau^n = \{\tau_k, 1 \leq k \leq n\}$ is a sequence of positive and i.i.d. random variables – or *inter-arrival times* – generated by a given probability density function $g(t) > 0$ with finite mean δ . Let g^{*k} be the k th fold convolution of g with itself, then $g^{*k}(t)$ is the density function of t_k and we define the *renewal density* h by $h(t) := \sum_{k=1}^{\infty} g^{*k}(t)$. Here and below, the function h is supposed to be bounded by a constant h_0 .

Remark 1. The renewal density is known to satisfy $h(u) \rightarrow \delta^{-1}$ as $u \rightarrow \infty$ (see Cox [14], p. 55), but its explicit expression is generally complicated to obtain. Nevertheless, the boundedness of h is a condition which holds for a large class of sequences τ^n . For the reader convenience, we recall the example in Masry [24] corresponding to the usual situation where τ^n has a Gamma density of type r , i.e.,

$$g(t) = \frac{(r/\delta)(rt/\delta)^{r-1} \exp(-rt/\delta)}{(r-1)!}, \quad r \in \mathbb{N} \setminus \{0\}, \quad \delta > 0, \quad t \geq 0,$$

with mean δ and variance δ^2/r . Thus, if $r = 1$, $h(t) = \delta^{-1}$ for $t \geq 0$ ($\mathcal{T}_1^n$ is a Poisson process) and, if $r = 2$, $h(t) = \delta^{-1}(1 - \exp(-4t/\delta))$ which approaches its limit δ^{-1} monotonically as $t \rightarrow \infty$. In both cases, the value $h_0 = \delta^{-1}$ is clearly appropriate. From $r = 3$, the condition becomes delicate to verify since $h(t)$ oscillates in approaching δ^{-1} . The case $r = 1$ is illustrated e.g. in Ait-Sahalia and Mykland [1] with an example of financial data for which a histogram distribution of the sampling intervals is fitted by an exponential density.

Jittered sampling – First, we assume that the process is regularly observed with a period $\delta > 0$. This sequence $\mathcal{T}^n \equiv \mathcal{T}_2^n$ is then contaminated by an additive noise to model the plausible imperfections of a measurement recording system:

$$t_0 = Z_0 \quad \text{and} \quad t_k = k\delta + Z_k, \quad 1 \leq k \leq n,$$

where $Z^n = \{Z_k, 0 \leq k \leq n\}$ denotes an i.i.d. random sample from a symmetric probability density function $g_J(z)$ over $[-\delta/2, \delta/2]$. In comparison with renewal times, jittered times could be seen as only partially random due to the deterministic component in t_k .

Remark 2. The observations drawn from each of these two random designs are by definition irregularly spaced in time, but the “long-term” expected inter-arrival time between two consecutive random instants is equal to δ in each case.

Finally, we introduce a periodic scheme examined in Bosq [7] for kernel density estimation where the sampling step δ_n is n -decreasing in a deterministic manner.

High frequency sampling – In order to represent the occurrence of high frequency observations during a long time, the sampling instants in $\mathcal{T}^n \equiv \mathcal{T}_3^n$ are defined periodically as

$$t_{0,n} = 0 \quad \text{and} \quad t_{k,n} = k\delta_n, \quad 1 \leq k \leq n,$$

where $\delta_n > 0$ and $\delta_n \rightarrow 0^+$, $T_n = n\delta_n \rightarrow \infty$ as $n \rightarrow \infty$. In the sequel, we will give minimal thresholds δ_n^* over which our estimators converge with the optimal rates of the i.i.d. case. The knowledge of δ_n^* will also help us to minimize the costs of estimation without altering the rates. To explain, observe that two situations may occur in applications. First, if the total time of observation is a given and large enough T_n , the value of a minimal δ_n^* allows to select a maximal number n^* of points in $[0, T_n]$ to estimate f . On the other hand, consider that a maximal and large enough sample size n is available, then we can deduce from δ_n^* a minimal sufficient time $T_n^* = n\delta_n^*$ of observation (see Blanke and Pumo [5]). Furthermore, we will emphasize the convenience of such a framework to sample a continuous-time process. Thus, under the Castellana-Leadbetter's conditions, i.e. $\int_0^\infty \sup_{x,y} |g_u(x,y)| du < \infty$ and $g_u(\cdot, \cdot)$ is continuous at (x, x) for any $u > 0$, Bosq [7] proved that δ_n can be chosen in order to obtain the full rate $1/T_n$ of the pointwise mean squared error of kernel estimators. In that situation, the sampling scheme is said to be *admissible*. Concerning admissible sampling in nonparametric density estimation, let us cite relevant works by Leblanc [20] for wavelets estimators, by Biau [3] for spatial kernel estimators, and by Comte and Merlevde [13] and Blanke [4], respectively, for projection and adaptive kernel estimators.

2.2 Mean integrated squared error

The global accuracy of density estimators can be measured by the *mean integrated squared error* which is the expected squared distance between a density estimator $\hat{f}_n$ and the true density f integrated over $\mathbb{R}^d$:

$$\text{MISE}(\hat{f}_n) = \mathbb{E} \int_{\mathbb{R}^d} \left(\hat{f}_n(x) - f(x) \right)^2 dx.$$

It is also the sum of the *integrated squared bias* (ISB) and the *integrated variance* (IV):

$$\text{ISB}(\hat{f}_n) = \int_{\mathbb{R}^d} \left(\mathbb{E}(\hat{f}_n(x)) - f(x) \right)^2 dx \quad \text{and} \quad \text{IV}(\hat{f}_n) = \int_{\mathbb{R}^d} \mathbb{E} \left(\hat{f}_n(x) - \mathbb{E}(\hat{f}_n(x)) \right)^2 dx.$$

Let us fix the following usual notations: $\mathcal{C}^k(\mathbb{R}^d)$ denotes the set of k -times continuously differentiable functions and $L_k(\mathbb{R}^d)$ the set of functions with integrable k th power over $\mathbb{R}^d$ such that $\|f\|_k = (\int_{\mathbb{R}^d} |f(x)|^k dx)^{1/k}$.

3 Histogram

We primarily examine the *histogram*, which is the oldest and most popular nonparametric estimator. Because of its simplicity, histogram is still widely used in presentation and practice by statisticians. The theoretical properties have been also extensively studied in the i.i.d. case and we may refer e.g. to Scott [28] (Chapter 3) for a background material. For continuous-time delivered observations, both optimal and full rates of MISE and asymptotic normality under

Castellana-Leadbetter's conditions are given in Lejeune [23]. In this section we derive results for observations collected at discretized instants according to the sequences $\mathcal{T}_i^n$, $i = 1, 2, 3$.

3.1 Definitions and assumptions

Prior to the definition of our estimator, we introduce a partition of $\mathbb{R}^d$, say Π_n , into hypercubes of volume h_n^d such that $h_n \rightarrow 0^+$, $nh_n^d \rightarrow \infty$ as $n \rightarrow \infty$:

$$\Pi_n = \{\pi_{nj}, j \in \mathbb{Z}^d\},$$

and

$$\pi_{nj} = \prod_{k=1}^d [b_{jk}, b_{j_k+1}[= \prod_{k=1}^d \left[c_{j_k} - \frac{h_n}{2}, c_{j_k} + \frac{h_n}{2} \right], \quad j = (j_1, \dots, j_d)' \in \mathbb{Z}^d,$$

where $b_j = (b_{j_1}, \dots, b_{j_d})' \in \mathbb{R}^d$, $b_{j_k+1} - b_{j_k} = h_n$ and $c_{j_k} = (b_{j_k} + b_{j_k+1})/2$. Here h_n is the smoothing parameter commonly referred to as the *bin width*. Note that the extension to unequal bin sizes is straightforward with more notations. From now on, we will suppose for any $x \in \mathbb{R}^d$ the existence of an index $j(x, n)$ in $\mathbb{Z}^d$ such that $x \in \pi_{j(x, n)}$ ($=: \pi_{nj}$).

Given Π_n and n discretized observations $X_{t_1}, \dots, X_{t_n}$, the histogram estimator of f is then defined as

$$\hat{f}_n^H(x) = \sum_j \left[\frac{1}{nh_n^d} \sum_{k=1}^n \mathbb{1}_{\pi_{nj}}(X_{t_k}) \right] \mathbb{1}_{\pi_{nj}}(x) =: \sum_j \hat{f}_j \mathbb{1}_{\pi_{nj}}(x), \quad x \in \mathbb{R}^d,$$

where $\mathbb{1}_{\pi_{nj}}$ denotes the indicator function of π_{nj} . In particular, $\hat{f}_n^H$ has a unique value, denoted by $\hat{f}_j$, over each hypercube π_{nj} of Π_n , which explains its high computational advantage.

Let $\mathcal{A}$ and $\mathcal{B}$ be two sub- σ -algebras of $\mathcal{F}$, we introduce the classical strong mixing coefficient defined as

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

Let denote $\sigma(X)$ the σ -algebra of events generated by a random variable X . In the sequel, we will use the definition of a 2- α -mixing process $\{X_t, t \in \mathbb{R}\}$ given in Bosq [8] as

$$\alpha_X^{(2)}(u) := \sup_{t \in \mathbb{R}} \alpha(\sigma(X_t), \sigma(X_{t+u})) \rightarrow 0 \quad \text{as } u \rightarrow \infty.$$

Note that such a condition only for the couples (X_t, X_{t+u}) is less restrictive than the classical one introduced by Rosenblatt [26].

These are now the main assumptions over processes.

Assumptions $\mathbf{A_0}$

- (i) $f \in \mathcal{C}^2(\mathbb{R}^d)$ so that all the partial derivatives are square Riemann-integrable;
- (ii) f is continuous and $\|f\|_\infty = \sup_{y \in \mathbb{R}^d} f(y) < \infty$.

Assumptions $\mathbf{A}_1$ (with renewal and jittered samplings)

- (i) There exists $u_0 > 0$ such that for any $u \geq u_0$: $\sup_{z \in \mathbb{R}^d} |g_u(y, z)| \leq k(y)$ with $k(\cdot)$ a positive, continuous and integrable function defined on $\mathbb{R}^d$;
- (ii) X^T is an arithmetically strongly mixing (ASM) process i.e. there exists $\rho > 2$, $a_0 > 0$ and $u_1 > u_0$ such that for any $u \geq u_1$: $\alpha_X^{(2)}(u) = \alpha(\sigma(X_0), \sigma(X_u)) \leq a_0 u^{-\rho}$.

Assumptions $\mathbf{A}'_1$ (with high frequency sampling)

- (i) There exists $\gamma_0 > 0$ and $u_0 > 0$ such that for any $0 < u \leq u_0$: $\forall y \in \mathbb{R}^d$, $\sup_{z \in \mathbb{R}^d} f_u(y, z) \leq \varphi(y)u^{-\gamma_0}$ with $\varphi(\cdot)$ a positive, continuous and integrable function defined on $\mathbb{R}^d$;
- (ii) There exists a positive, continuous and integrable function $k(\cdot)$ defined on $\mathbb{R}^d$ such that for any $u \geq u_0$: $\forall y \in \mathbb{R}^d$, $\sup_{z \in \mathbb{R}^d} |g_u(y, z)| \leq k(y)\pi(u)$ where $\pi(\cdot)$ is a bounded and ultimately decreasing function which satisfies $\int_{u_1}^{\infty} \pi(u)du < \infty$, $u_1 > u_0$.

The assumptions above are classical in nonparametric estimation with dependent data. A_0 displays some constraints of regularity on the true density f . The condition $A_0(i)$ is specific to the bias treatment, it was previously introduced by Lecoutre [21] to study the multivariate histogram in the i.i.d. case.

The following conditions should take into account the local behavior of sample paths as well as the properties of asymptotic independence of processes (respectively described with the behavior of g_u for u near the origin and for u large). $A_1(i)$ is a mild condition on g_u for intermediate values of u . In particular, it slightly weakens the assumption of boundedness on the conditional density used by Masry [24] and Carbon, Garel, and Tran [10].

$A'_1(i)$ appears to be less usual in density estimation, but it is a typical condition for the continuous-time framework to control the explosive behavior of the joint densities $f_u(\cdot, \cdot)$ in a neighborhood of $u = 0$. Assumptions A'_1 are in the spirit of those made (and widely discussed) by Blanke and Pumo [5]. Here $A'_1(i)$ is used with high frequency sampling to obtain optimal rates together with a short sampling step δ_n depending on a positive known coefficient γ_0 . Roughly speaking, the value of γ_0 is directly linked with the hölderian properties of sample paths and the dimension d : namely, one has $\gamma_0 = d/2$ for a wide class of d -dimensional ergodic diffusion processes and $\gamma_0 = d$ for “smooth” processes (see e.g. Blanke [4] for technical details).

Other assumptions, namely $A_1(ii)$ and $A'_1(ii)$, ensure asymptotic independence between variables distant in time. $A_1(ii)$ involves a mild version of α -mixing which is well-known to be weaker than many dependence structures as ϕ , β or ρ -mixing (see e.g. Doukhan [16]). Finally, admissible high frequency samplings are obtained under $A'_1(ii)$, a quite typical condition in this context.

3.2 Rates of convergence

Using each sampling design defined above, we will now establish the optimal rate of histograms. For the sake of readability, some crucial lemmas which provide upper bounds for the variances and the covariances of $\hat{f}_n^H$ are postponed to the proofs. Let $f'_i := \partial f / \partial x_i$ and define the roughness R of f'_i by its squared L_2 -norm: $R(f'_i) := \int_{\mathbb{R}^d} f'_i(x)^2 dx$. Since the bias of $\hat{f}_n^H$ only

depends on the bin width and the true unknown density f , and not on the dependence structure of the data, we recall the following result given by Lecoutre [21] with multivariate independent observations.

Lemma 3.1. *If Assumption $A_0(i)$ is satisfied then*

$$\text{ISB}(\hat{f}_n^H) = \frac{h_n^2}{12} R_d(f') + o(h_n^2),$$

where $R_d(f') := \sum_{i=1}^d R(f'_i)$.

3.2.1 Renewal and jittered samplings

Let us denote by $\lceil x \rceil$ the smallest integer not less than the real x . The first part of the next theorem gives an asymptotic upper bound for IV. Consequently, from an ad hoc choice of the bin width h_n which balances both ISB and IV terms, we infer that histograms can reach the same optimal rate $n^{-2/(d+2)}$ of convergence to f as in the i.i.d. case.

Theorem 3.2. 1. *Under $A_0(ii)$ and A_1 and if $f^{1-1/p} \in C^1(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ for $1 < p < \rho - 1$, then*

$$\limsup_{n \rightarrow \infty} n h_n^d \text{IV}(\hat{f}_n^H) \leq 1 + C,$$

where $C = 2u_0h_0$ for the renewal sampling and $C = 2 \lceil \frac{u_0}{\delta} \rceil$ for the jittered sampling;

2. *If in addition $A_0(i)$ holds then the choice $h_n = cn^{-1/(d+2)}$, $0 < c < \infty$, yields*

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d} (1 + C),$$

with same constant C .

Remark 3. If $p = \rho - 1$ the rates in Theorem 3.2 remain valid but with larger asymptotic constants (see proofs). Thus, if for instance $\rho \geq 3$, one may choose $p = 2$ provided that $f^{1/2}$ is continuous and integrable.

3.2.2 High frequency sampling

The high frequency model is interesting to find some connections between both discrete and continuous-time frameworks. Here the period δ_n is now a function of the sampling size n so that all observations can be as close in time as desired provided n large enough. Within this setup we also need to check the local condition $A'_1(i)$ on the joint density of (X_0, X_u) for the small values of u , wherein a (known) coefficient γ_0 is linked with the regularity of sample paths. In this framework the previous optimal rate of order $n^{-2/(d+2)}$ is still preserved. Moreover, depending on the value γ_0 , we can derive a minimal δ_n^* (more precisely $\delta_n^*(\gamma_0)$) and then deduce a minimal time of observation of the process T_n^* that ensures this rate.

Theorem 3.3. *According to the value of γ_0 we assume that $\delta_n \geq \delta_n^*(\gamma_0)$ defined as*

$$\delta_n^*(\gamma_0) := d_1 h_n^d \mathbf{1}_{\{\gamma_0 < 1\}} + d_2 h_n^d \ln(h_n^{-d}) \mathbf{1}_{\{\gamma_0 = 1\}} + d_3 h_n^{d/\gamma_0} \mathbf{1}_{\{\gamma_0 > 1\}}, \quad 0 < d_1, d_2, d_3 < \infty. \quad (3.1)$$

1. Then under $A_0(ii)$ and A'_1

$$\limsup_{n \rightarrow \infty} n h_n^d \text{IV}(\hat{f}_n^H) \leq 1 + C_{\gamma_0},$$

where C_{γ_0} is a positive constant which depends upon γ_0 (see its explicit form in proofs);

2. If in addition $A_0(i)$ holds with $h_n = cn^{-1/(d+2)}$, $0 < c < \infty$, then

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d} (1 + C_{\gamma_0}),$$

with same constant C_{γ_0} .

Remark 4. Using A_0 with either A_1 or A'_1 , our results in Theorems 3.2 and 3.3 are similar to those derived with independent variables by Lecoutre [21] in the d -dimensional setup. Thus we retrieve (in limsup) the same optimal rate $n^{-2/(d+2)}$ in terms of MISE. The additional asymptotic constant C or C_{γ_0} in the variance bound arises as a non negligible remainder of the covariance term; it clearly depends on the sampling design in use. Nevertheless, if δ_n is such that $\delta_n/\delta_n^*(\gamma_0) \rightarrow \infty$ as $n \rightarrow \infty$, we can remove C_{γ_0} in Theorem 3.3 to get the exact limiting constant of the i.i.d. case with $h_n = cn^{-1/(d+2)}$, $0 < c < \infty$.

Remembering that $T_n = n\delta_n$ the rate $n^{-2/(d+2)}$ may be easily rewritten in terms of T_n according to the value of γ_0 .

Corollary 3.4. Under A_0 and A'_1 the choice $h_n = cn^{-1/(d+2)}$, $0 < c < \infty$, leads to

$$\text{MISE}(\hat{f}_n^H) = \begin{cases} O(T_n^{-1}) & \text{with } \delta_n = d_1 h_n^d, \ 0 < d_1 < \infty, \text{ if } \gamma_0 < 1; \\ O(T_n^{-1} \ln T_n) & \text{with } \delta_n = d_2 h_n^d \ln(h_n^{-d}), \ 0 < d_2 < \infty, \text{ if } \gamma_0 = 1; \\ O(T_n^{-\frac{2\gamma_0}{2\gamma_0+d(\gamma_0-1)}}) & \text{with } \delta_n = d_3 h_n^{d/\gamma_0}, \ 0 < d_3 < \infty, \text{ if } \gamma_0 > 1. \end{cases}$$

Remark 5. For the special case of irregular paths processes ($\gamma_0 < 1$), we thus observe in Corollary 3.4 a surprising similarity between the best rate of order $1/T_n$ and the $1/T$ -parametric rate encountered in the real continuous-time context. Indeed, the time of observation clearly depends on the value of γ_0 since T_n has to be of order $n^{2/(d+2)}$ ($\gamma_0 < 1$), $n^{2/(d+2)} \ln n$ ($\gamma_0 = 1$) or $n^{(2\gamma_0+d(\gamma_0-1))/((d+2)\gamma_0)}$ ($\gamma_0 > 1$) so as to obtain same efficiency in estimation. Especially, this enlightens the fact that irregular paths processes may be observed less time than more regular ones ($\gamma_0 \geq 1$).

Finally, it may be interesting to indicate the exact limit of the pointwise variance of $\hat{f}_n^H(x)$ in the case $\gamma_0 < 1$. The following proposition is thus obtained as a simple transposition from kernel to histogram estimators of a result by Bosq [7] (Proposition 7.1. (i)).

Proposition 3.1. Let $x \in \mathbb{R}^d$ and assume that

- (i) $\|g_u\|_\infty \leq \pi(u)$ where $(1+u)\pi(u)$ is integrable over $]0, +\infty[$ and $u\pi(u)$ is bounded and ultimately decreasing. Furthermore $g_u(\cdot, \cdot)$ is continuous at (x, x) ;

(ii) $\sup_{(y,z) \in \mathbb{R}^{2d}} \left| \sum_{r=1}^{\infty} \delta_n g_{r\delta_n}(y, z) - \int_0^{\infty} g_u(y, z) du \right| \rightarrow 0$ as $\delta_n \downarrow 0^+$,

then

$$\lim_{n \rightarrow \infty} T_n \text{Var}(\hat{f}_n^H(x)) = 2 \int_0^{\infty} g_u(x, x) du,$$

provided that $\delta_n = o(h_n^d)$.

Remark 6. From Kutoyants [18], the limiting constant is also the minimax bound for mean squared error in the case of ergodic diffusion processes satisfying some conditions of regularity on the trend coefficient and the diffusion coefficient (see Veretennikov [29]).

4 Frequency polygon

Given a (univariate) histogram, the *frequency polygon* results from a natural smoothing with straight lines to get a continuous estimator. However, the gain of this simple linear smoothing is substantial since we immediately improve the weak order h_n^2 inherent to the bias of histograms. The main properties of frequency polygons are gathered in Scott [28] (Chapter 4) within the i.i.d. setup. The mixing case was then treated by Carbon, Garel, and Tran [10], and recently extended to the random fields by Carbon [11]. In continuous-time, Lejeune [22, 23] established both optimal and parametric rates of MISE and asymptotic normality; the extension to the random fields is done in a submitted work by Bensaïd and Dabo-Niang [2]. For the sake of simplicity, we shall confine attention to the real case ($d = 1$ and $\gamma_0 \leq 1$).

For convenience, f' and f'' denote the first and second derivatives of f and we define the roughness of f'' by $R(f'') := \int_{\mathbb{R}} f''(x)^2 dx$.

4.1 Definition and assumptions

Based upon Π_n and $X_{t_1}, \dots, X_{t_n}$, the frequency polygon is simply constructed by connecting the mid-points of the histogram heights with straight line segments

$$\hat{f}_n^{FP}(x) = \sum_j \left[\left(\frac{x - c_j}{h_n} \right) \hat{f}_{j+1} + \left(\frac{c_{j+1} - x}{h_n} \right) \hat{f}_j \right] \mathbf{1}_{[c_j, c_{j+1}]}(x), \quad x \in \mathbb{R}.$$

In the literature we find also alternative definitions which differ from the way of interpolation as e.g. the *edge frequency polygon* introduced by Jones, Samiuddin and Al-Harbey Maatouk [17] or its extended form by Dong and Zheng [15]. All these estimators share the same rates of convergence but with different asymptotic constants.

In agreement with assumptions A_1 and A'_1 in the previous section we will describe the properties of the frequency polygon under the following conditions on f .

Assumptions A'_0

- (i) $f \in \mathcal{C}^2(\mathbb{R})$, $f'' \in L_1(\mathbb{R})$ and $f, f'' \in L_2(\mathbb{R})$;
- (ii) $|f''(x) - f''(y)| \leq l_0 |x - y|^\nu$, $l_0 > 0$, $\nu \in]0, 1]$, for $(x, y) \in \mathbb{R}^2$;
- (iii) f is continuous and $\|f\|_\infty < \infty$.

4.2 Rates of convergence

The ISB contribution is given in Scott [27].

Lemma 4.1. *If Assumptions $A'_0(i)(ii)$ are satisfied then*

$$\text{ISB}(\hat{f}_n^{FP}) = \frac{49}{2880} R(f'') h_n^4 + o(h_n^4).$$

Remark 7. The nice order h_n^4 is much better compared with histograms and familiar for more sophisticated density estimators as kernel estimators. As emphasized earlier the bias term does not depend on the sampling scheme.

4.2.1 Renewal and jittered samplings

Using the analysis on histograms with a new suitable choice of h_n we give the optimal rate of frequency polygons. Note that constants C and C_{γ_0} are unchanged.

Theorem 4.2. 1. *Under $A'_0(iii)$ and A_1 and if $f^{1-1/p} \in C^1(\mathbb{R}) \cap L_1(\mathbb{R})$ for $1 < p < \rho - 1$, then*

$$\limsup_{n \rightarrow \infty} n h_n \text{IV}(\hat{f}_n^{FP}) \leq \frac{2}{3} + C;$$

2. *If in addition $A'_0(i)(ii)$ hold then the choice $h_n = cn^{-1/5}$, $0 < c < \infty$, yields*

$$\limsup_{n \rightarrow \infty} n^{\frac{4}{5}} \text{MISE}(\hat{f}_n^{FP}) \leq \frac{49}{2880} c^4 R(f'') + \frac{1}{c} \left(\frac{2}{3} + C \right).$$

4.2.2 High frequency sampling

Finally, recovering the local properties of sample paths when data become dense in time, we find again the optimal rate while minimizing the time of observation.

Theorem 4.3. *According to the values of γ_0 , we consider optimal choices $\delta_n^*(\gamma_0)$ given by (3.1).*

1. *Then under $A'_0(iii)$ and A'_1*

$$\limsup_{n \rightarrow \infty} n h_n \text{IV}(\hat{f}_n^{FP}) \leq \frac{2}{3} + C_{\gamma_0};$$

2. *If in addition $A'_0(i)(ii)$ hold with $h_n = cn^{-1/5}$, $0 < c < \infty$, then*

$$\limsup_{n \rightarrow \infty} n^{\frac{4}{5}} \text{MISE}(\hat{f}_n^{FP}) \leq \frac{49}{2880} c^4 R(f'') + \frac{1}{c} \left(\frac{2}{3} + C_{\gamma_0} \right).$$

Remark 8. In both Theorems 4.2 and 4.3 we exhibit (in $\limsup$) the same $n^{-4/5}$ -consistency obtained in Scott [27] with i.i.d. observations. The additional asymptotic constant C or C_{γ_0} still stays and relies on the sampling design in use; but, in Theorem 4.3, any choice of δ_n satisfying $\delta_n/\delta_n^*(\gamma_0) \rightarrow \infty$ as $n \rightarrow \infty$ allows to remove C_{γ_0} to get the exact limiting constant of the i.i.d. case with $h_n = cn^{-1/5}$. Finally, note that if we take $p = \rho - 1$ in Theorem 4.2 the rates remain valid up to a decayed constant.

Corollary 4.4. *Under A'_0 and A'_1 the choice $h_n = cn^{-1/5}$, $0 < c < \infty$, leads to*

$$\text{MISE}(\hat{f}_n^{FP}) = \begin{cases} O(T_n^{-1}) & \text{with } \delta_n = d_1 h_n, \ 0 < d_1 < \infty, \text{ if } \gamma_0 < 1; \\ O(T_n^{-1} \ln T_n) & \text{with } \delta_n = d_1 h_n \ln(h_n^{-1}), \ 0 < d_2 < \infty, \text{ if } \gamma_0 = 1. \end{cases}$$

Remark 9. As noticed before real irregular paths processes may be observed less time than more regular ones since T_n has to be of order $n^{4/5}$ ($\gamma_0 < 1$) or $n^{4/5} \ln n$ ($\gamma_0 = 1$) to obtain same efficiency in estimation.

For completeness, the exact limit of the pointwise variance of $\hat{f}_n^{FP}(x)$ follows straightforwardly from Proposition 3.1 (see also Remark 6).

Proposition 4.1. *Under conditions of Proposition 3.1 with $\delta_n = o(h_n)$, one has*

$$\lim_{n \rightarrow \infty} T_n \text{Var}(\hat{f}_n^{FP}(x)) = 2 \int_0^\infty g_u(x, x) du, \quad x \in \mathbb{R}.$$

5 Discussion

In this work we derive the optimal L_2 -rates of two computationally advantageous density estimators in the setup where observations are discretely sampled from a continuous-time process. For practical considerations we have studied three time sampling procedures to properly describe the time occurrences of the real data. Thus, values may be available at low or high frequency but also regularly or irregularly spaced in time. Therefore our main results state that all designs either random or deterministic lead to the optimal rates $n^{-2/(d+2)}$ for histograms and $n^{-4/5}$ ($d = 1$) for frequency polygons, with respect to the MISE convergence, which are those derived in the i.i.d. case. From this result, the frequency polygon is a good alternative to more sophisticated nonparametric density estimators. Particularly, we have focused on a high frequency sampling to reveal some parallels with the idealized continuous-time framework as soon as observations are selected close enough to each other. We then use the local properties of sample paths to have a consistent estimation with a minimal time of experiment. This fact might be explained as follows: irregular sample paths carry much more information than regular ones where the correlation between two successive variables is much stronger. Consequently, we infer that more the paths are irregular – i.e. when $A'_1(i)$ holds with $\gamma_0 < 1$ – more the time of observation would be shortened with a good behavior of the both estimators. Although not presented here, simulations in progress already corroborate our theoretical results in the particular case of two stationary real gaussian processes. As awaited the frequency polygon performs well and appears much closer to kernel estimator than to histogram. To go further in our investigations, it remains to examine the case of non-gaussian processes including, for instance, the cumbersome problem of estimating bimodal densities. The important issue of finding optimal choices for the bin width value is left for future work.

6 Proofs

Throughout this section, we detail the proofs of Theorems 3.2, 3.3, 4.2 and 4.3. In order to do this, some auxiliary lemmas are necessary to derive upper bound expressions for the variance of

$\hat{f}_n^H(x)$, $x \in \pi_{nj}$, which will depend on the sampling scheme being used. Let $\|X\|_q = (\mathbb{E}|X|^q)^{1/q}$ with $1 \leq q < \infty$, then $X \in L_q(P)$ means that $\|X\|_q < \infty$. We recall the following useful covariance inequality as written in Bosq [8] (p. 21).

Lemma 6.1 (Davydov's inequality). *Let $X \in L_q(P)$ and $Y \in L_r(P)$ with $q > 1$, $r > 1$ and $\frac{1}{q} + \frac{1}{r} < 1$, then*

$$|\text{Cov}(X, Y)| \leq 2p \left[2\alpha(\sigma(X), \sigma(Y)) \right]^{1/p} \|X\|_q \|Y\|_r,$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

6.1 Histogram

6.1.1 Variance bounds with random sampling

Lemma 6.2 (renewal sampling). *If $A_0(ii)$ and A_1 hold then we obtain for $1 < p \leq \rho - 1$:*

$$\begin{aligned} nh_n^d \text{Var}(\hat{f}_j) &\leq f(\xi_j)(1 - h_n^d f(\xi_j))(1 + 2u_0 h_0) + 2h_0 k(\dot{\xi}_j) h_n^\varepsilon \\ &\quad + \frac{4p^2(2a_0)^{1/p} h_0}{\rho - p} f(\xi_j)^{1-\frac{1}{p}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}}, \end{aligned} \quad (6.2)$$

with $0 \leq \varepsilon \leq d \left(1 - \frac{1}{\rho-p}\right)$ and $(\xi_j, \dot{\xi}_j) \in \pi_{nj}^2$.

Lemma 6.3 (jittered sampling). *Under the same conditions as in Lemma 6.2 and $1 < p \leq \rho - 1$:*

$$\begin{aligned} nh_n^d \text{Var}(\hat{f}_j) &\leq f(\xi_j)(1 - h_n^d f(\xi_j)) \left(1 + 2 \left\lceil \frac{u_0}{\delta} \right\rceil\right) + 2k(\dot{\xi}_j) \left(h_n^\varepsilon - \left\lceil \frac{u_0}{\delta} \right\rceil h_n^d\right) \\ &\quad + \frac{4p^2(2a_0)^{1/p}}{(\rho - p)\delta^{\frac{p}{p}}}} f(\xi_j)^{1-\frac{1}{p}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} (1 - 2h_n^{d-\varepsilon})^{1-\frac{p}{p}}, \end{aligned} \quad (6.3)$$

with $0 \leq \varepsilon \leq d \left(1 - \frac{1}{\rho-p}\right)$ and $(\xi_j, \dot{\xi}_j) \in \pi_{nj}^2$.

For further use, we give the proofs for the covariances.

Proof of Lemma 6.2 For any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, we suppose the existence of two indexes $j_1(x, n)$ and $j_2(y, n)$ in $\mathbb{Z}^d$ such that $x \in \pi_{j_1(x, n)} (=:\pi_{nj_1})$ and $y \in \pi_{j_2(y, n)} (=:\pi_{nj_2})$. Thus

$$\hat{f}_n^H(x) = \hat{f}_{j_1} = \frac{1}{nh_n^d} \sum_{k=1}^n \mathbb{1}_{\pi_{nj_1}}(X_{t_k}), \quad \hat{f}_n^H(y) = \hat{f}_{j_2} = \frac{1}{nh_n^d} \sum_{k=1}^n \mathbb{1}_{\pi_{nj_2}}(X_{t_k}),$$

and

$$\begin{aligned} nh_n^d \text{Cov}(\hat{f}_{j_1}, \hat{f}_{j_2}) &= \frac{1}{nh_n^d} \sum_{k=1}^n \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_{t_k}), \mathbb{1}_{\pi_{nj_2}}(X_{t_k})) \\ &\quad + \frac{2}{nh_n^d} \sum_{p=1}^{n-1} \sum_{q=p+1}^n \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_{t_p}), \mathbb{1}_{\pi_{nj_2}}(X_{t_q})) =: V_n + C_n. \end{aligned}$$

Set $p_k := P(X_0 \in \pi_{nk})$, $k \in \mathbb{Z}^d$. The “variance term” V_n is easy to compute.

$$V_n = \frac{1}{nh_n^d} \sum_{k=1}^n \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_0)) = \frac{1}{h_n^d} (P(X_0 \in \pi_{nj_1}, X_0 \in \pi_{nj_2}) - p_{j_1}p_{j_2}).$$

Since f is continuous there exists at least one point $\xi_j \in \pi_{nj}$ such that $\int_{\pi_{nj}} f(x)dx = h_n^d f(\xi_j)$. Then if $j_1 \neq j_2$, we get

$$V_n = -\frac{1}{h_n^d} p_{j_1} p_{j_2} = -h_n^d f(\xi_{j_1}) f(\xi_{j_2}),$$

where $(\xi_{j_1}, \xi_{j_2}) \in \pi_{nj_1} \times \pi_{nj_2}$. Otherwise if $j_1 = j_2 = j$:

$$V_n = \frac{1}{h_n^d} p_j (1 - p_j) = f(\xi_j) (1 - h_n^d f(\xi_j)).$$

Let us turn to the “covariance term” C_n . By stationarity and, since $t_p - t_q$ and t_{p-q} are equal in distribution, we have

$$\begin{aligned} C_n &= \frac{2}{nh_n^d} \sum_{r=1}^{n-1} \sum_{p=1}^{n-r} \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_{t_{p+r}-t_p})) \\ &= \frac{2}{h_n^d} \sum_{r=1}^{n-1} \left(1 - \frac{r}{n}\right) \int_0^\infty \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_u)) g^{\star r}(u) du =: C_{n,1} + C_{n,2} + C_{n,3}, \end{aligned}$$

where

$$C_{n,i} := \frac{2}{h_n^d} \sum_{r=1}^{n-1} \left(1 - \frac{r}{n}\right) \int_{E_i} \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_u)) g^{\star r}(u) du, \quad i = 1, 2, 3,$$

with $E_1 = (0, u_0)$, $E_2 = (u_0, h_n^{-d+\varepsilon})$ and $E_3 = (h_n^{-d+\varepsilon}, \infty)$, for some $0 \leq \varepsilon < d$ to be specified later. Recall that $h(u) = \sum_{r=1}^\infty g^{\star r}(u)$, one seeks to bound each covariance subterm. First, by Cauchy-Schwarz inequality and Fubini’s theorem,

$$\begin{aligned} |C_{n,1}| &\leq \frac{2}{h_n^d} \sqrt{\text{Var}(\mathbb{1}_{\pi_{nj_1}}(X_0))} \sqrt{\text{Var}(\mathbb{1}_{\pi_{nj_2}}(X_0))} \int_0^{u_0} h(u) du \\ &\leq 2u_0 h_0 \sqrt{f(\xi_{j_1}) f(\xi_{j_2}) (1 - h_n^d f(\xi_{j_1})) (1 - h_n^d f(\xi_{j_2}))}. \end{aligned}$$

Then $A_1(i)$ and Fubini imply

$$|C_{n,2}| \leq \frac{2}{h_n^d} \int_{u_0}^{h_n^{-d+\varepsilon}} \iint_{\pi_{nj_1} \times \pi_{nj_2}} \sup_{y \in \mathbb{R}^d} |g_u(x, y)| dx dy h(u) du \leq 2h_0 k(\dot{\xi}_{j_1}) h_n^\varepsilon,$$

where $\dot{\xi}_{j_1} \in \pi_{nj_1}$.

Now, it is clear that for n large enough we have $h_n^{-d+\varepsilon} \geq u_1$. So using Davydov's inequality (Lemma 6.1) with mixing condition $A_1(ii)$ and Fubini, for any $(p, q) \in]1, \rho - 1] \times \left[1 + \frac{\rho}{\rho-2}, \infty\right[$ such that $\frac{2}{q} + \frac{1}{p} = 1$:

$$\begin{aligned} |C_{n,3}| &\leq \frac{2}{h_n^d} \int_{h_n^{-d+\varepsilon}}^{\infty} 2p \, 2^{1/p} \|\mathbb{1}_{\pi_{nj_1}}(X_0)\|_q \|\mathbb{1}_{\pi_{nj_2}}(X_u)\|_q (\alpha_X^{(2)}(u))^{1/p} h(u) du \\ &\leq 4p(2a_0)^{1/p} h_0 \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} h_n^{-\frac{d}{p}} \int_{h_n^{-d+\varepsilon}}^{\infty} u^{-\frac{\rho}{p}} du \\ &\leq \frac{4p^2(2a_0)^{1/p} h_0}{\rho - p} \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}}. \end{aligned}$$

Finally, setting $k_1 := 2u_0 h_0$, $k_2 := 2h_0$ and $k_3 := \frac{4p^2(2a_0)^{1/p} h_0}{\rho-p}$, one has

$$\begin{aligned} nh_n^d \text{Cov}(\hat{f}_{j_1}, \hat{f}_{j_2}) &\leq -h_n^d f(\xi_{j_1}) f(\xi_{j_2}) + k_1 \sqrt{f(\xi_{j_1}) f(\xi_{j_2}) (1 - h_n^d f(\xi_{j_1})) (1 - h_n^d f(\xi_{j_2}))} \\ &\quad + k_2 k(\xi_{j_1}) h_n^\varepsilon + k_3 \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}}. \end{aligned} \quad (6.4)$$

We then deduce the lemma by taking $j_1 = j_2 = j$ with the appropriate expression of V_n . It turns out that the covariance is a $O(1/(nh_n^d))$ for any choice of ε pertaining to $\left[0, d\left(1 - \frac{1}{\rho-p}\right)\right]$. ■

Proof of Lemma 6.3 Here the calculus of V_n is exactly the same as in the proof of Lemma 6.2. In fact, the delicate point will consist again in bounding C_n . To do so, we give the common probability density function, say Δ_Z , of all random variables $\{Z_j - Z_i, i < j\}$. Since the variables $\{Z_i, 0 \leq i \leq n\}$ are supposed to be independent and symmetrically distributed, we have $\Delta_Z(t) = g_J^{*2}(t) = \int_{\mathbb{R}} g_J(t-y) g_J(y) dy$ with support over $[-\delta, \delta]$. Let us denote by $[x]$ the largest integer less than or equal to the real x , and set $r_0 := \lceil u_0/\delta \rceil$ and $r_n^1 := \lfloor h_n^{-d+\varepsilon} \rfloor$ for some $0 \leq \varepsilon < d$ to be specified later. Now stationarity implies

$$\begin{aligned} C_n &= \frac{2}{nh_n^d} \sum_{r=1}^{n-1} \sum_{p=1}^{n-r} \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_{t_{p+r}-t_p})) \\ &= \frac{2}{h_n^d} \sum_{r=1}^{n-1} \left(1 - \frac{r}{n}\right) \int_{-\delta}^{\delta} \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_{r\delta+t})) \Delta_Z(t) dt =: C_{n,1} + C_{n,2} + C_{n,3}, \end{aligned}$$

where

$$C_{n,i} := \frac{2}{h_n^d} \sum_{r \in E_i} \left(1 - \frac{r}{n}\right) \int_{-\delta}^{\delta} \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_{r\delta+t})) \Delta_Z(t) dt, \quad i = 1, 2, 3,$$

with $E_1 = \{1, \dots, r_0\}$, $E_2 = \{r_0 + 1, \dots, r_n^1\}$ and $E_3 = \{r_n^1 + 1, \dots, n-1\}$. By Cauchy-Schwarz

we get

$$\begin{aligned} |C_{n,1}| &\leq \frac{2}{h_n^d} \sum_{r=1}^{r_0} \sqrt{\text{Var}(\mathbb{1}_{\pi_{nj_1}}(X_0))} \sqrt{\text{Var}(\mathbb{1}_{\pi_{nj_2}}(X_0))} \int_{-\delta}^{\delta} \Delta_Z(t) dt \\ &\leq 2 \left\lceil \frac{u_0}{\delta} \right\rceil \sqrt{f(\xi_{j_1})f(\xi_{j_2})(1 - h_n^d f(\xi_{j_1}))(1 - h_n^d f(\xi_{j_2}))}. \end{aligned}$$

Then using $A_1(i)$

$$\begin{aligned} |C_{n,2}| &\leq \frac{2}{h_n^d} \sum_{r=r_0+1}^{r_n^1} \int_{-\delta}^{\delta} \Delta_Z(t) \iint_{\pi_{nj_1} \times \pi_{nj_2}} \sup_{y \in \mathbb{R}^d} |g_{r\delta+t}(x, y)| dx dy dt \\ &\leq 2(r_n^1 - r_0) h_n^d k(\dot{\xi}_{j_1}) \int_{-\delta}^{\delta} \Delta_Z(t) dt \\ &\leq 2k(\dot{\xi}_{j_1}) \left(h_n^{-d+\varepsilon} - \left\lceil \frac{u_0}{\delta} \right\rceil \right) h_n^d. \end{aligned}$$

By Davydov's inequality and $A_1(ii)$,

$$|C_{n,3}| \leq \frac{2}{h_n^d} \sum_{r=r_n^1+1}^{n-1} \int_{-\delta}^{\delta} \left| \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_{r\delta+t})) \right| \Delta_Z(t) dt.$$

For any $(p, q) \in]1, \rho - 1] \times \left[1 + \frac{\rho}{\rho-2}, \infty \right]$ such that $\frac{2}{q} + \frac{1}{p} = 1$ and since $\alpha_X^{(2)}(\cdot)$ is arithmetically decreasing, we have

$$\begin{aligned} |C_{n,3}| &\leq \frac{1}{h_n^d} 4p 2^{1/p} h_n^{\frac{2d}{q}} f(\xi_{j_1})^{1/q} f(\xi_{j_2})^{1/q} \sum_{r=r_n^1+1}^{n-1} \int_{-\delta}^{\delta} \left(\alpha_X^{(2)}((r-1)\delta) \right)^{1/p} \Delta_Z(t) dt \\ &\leq h_n^{-\frac{d}{p}} \frac{4p 2^{1/p}}{\delta} \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} \int_{(r_n^1-1)\delta}^{\infty} (\alpha_X^{(2)}(u))^{1/p} du \\ &\leq \frac{4p^2 (2a_0)^{1/p}}{(\rho-p)\delta^{\frac{\rho}{p}}} \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} h_n^{-\frac{d}{p}} (r_n^1 - 1)^{1-\frac{\rho}{p}}. \end{aligned}$$

Now if $p < \rho$ ($1 - \frac{\rho}{p} < 0$) and since $r_n^1 > h_n^{-d+\varepsilon} - 1$ we may write

$$|C_{n,3}| \leq \frac{4p^2 (2a_0)^{1/p}}{(\rho-p)\delta^{\frac{\rho}{p}}} \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} (1 - 2h_n^{d-\varepsilon})^{1-\frac{\rho}{p}},$$

where $(1 - 2h_n^{d-\varepsilon})^{1-\frac{\rho}{p}} \rightarrow 1$ as $n \rightarrow \infty$. Hence we obtain

$$|C_{n,3}| \leq \frac{4p^2 (2a_0)^{1/p}}{(\rho-p)\delta^{\frac{\rho}{p}}} \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} (1 - 2h_n^{d-\varepsilon})^{1-\frac{\rho}{p}}.$$

Finally, setting $k_4 := 2 \lceil \frac{u_0}{\delta} \rceil$ and $k_5 := \frac{4p^2(2a_0)^{1/p}}{(\rho-p)\delta^{\rho/p}}$, one has

$$\begin{aligned} nh_n^d \text{Cov}(\hat{f}_{j_1}, \hat{f}_{j_2}) &\leq -h_n^d f(\xi_{j_1})f(\xi_{j_2}) + k_4 \sqrt{f(\xi_{j_1})f(\xi_{j_2})(1-h_n^d f(\xi_{j_1}))(1-h_n^d f(\xi_{j_2}))} \\ &\quad + 2k(\dot{\xi}_{j_1}) \left(h_n^\varepsilon - \lceil \frac{u_0}{\delta} \rceil h_n^d \right) + k_5 \sqrt{f(\xi_{j_1})^{1-\frac{1}{p}} f(\xi_{j_2})^{1-\frac{1}{p}}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} (1-2h_n^{d-\varepsilon})^{1-\frac{\rho}{p}}, \end{aligned} \quad (6.5)$$

which implies the desired result. The covariance is thus a $O(1/(nh_n^d))$ for any ε in $\left[0, d\left(1 - \frac{1}{\rho-p}\right)\right]$. ■

6.1.2 Proof of Theorem 3.2

Renewal sampling – By integrating over π_{nj} the right-hand side of (6.2) and by summing up over all hypercubes, we first derive an asymptotic upper bound for IV. For some $\varepsilon \in \left[0, d\left(1 - \frac{1}{\rho-p}\right)\right]$,

$$\begin{aligned} nh_n^d \int_{\pi_{nj}} \text{Var}(\hat{f}_j) dx &\leq h_n^d \left\{ f(\xi_j)(1-h_n^d f(\xi_j))\{1+k_1\} \right. \\ &\quad \left. + k_2 k(\dot{\xi}_j) h_n^\varepsilon + k_3 f(\xi_j)^{1-\frac{1}{p}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} \right\}. \end{aligned}$$

Then using the approximation of integral by Riemann sums, i.e.,

$$\sum_j h_n^d f^\kappa(\xi_j) = \|f^\kappa\|_1 + o(1), \quad \kappa = 1 - \frac{1}{p}, 1, 2, \quad \text{and} \quad \sum_j h_n^d k(\dot{\xi}_j) = \|k\|_1 + o(1),$$

one has

$$nh_n^d \text{IV}(\hat{f}_n^H) \leq \left\{ 1 + k_1 + k_2 \|k\|_1 h_n^\varepsilon + k_3 \|f^{1-\frac{1}{p}}\|_1 h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} \right\} (1 + o(1)). \quad (6.6)$$

The two parts of the theorem follow from the choice $h_n = cn^{-1/(d+2)}$, $0 < c < \infty$. So Lemma 3.1 yields

$$\lim_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{ISB}(\hat{f}_n^H) = \frac{c^2}{12} R_d(f'),$$

and combining with (6.6), if $p = \rho - 1$ ($\varepsilon = 0$), we have

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d} \left\{ 1 + k_1 + k_2 \|k\|_1 + k_3 \|f^{1-\frac{1}{p}}\|_1 \right\}.$$

If $p < \rho - 1$ ($\varepsilon > 0$), we improve the asymptotic constant:

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d} \{1 + k_1\}.$$

Jittered sampling – Now, let us integrate over π_{nj} the right-hand side of (6.3):

$$nh_n^d \int_{\pi_{nj}} \text{Var}(\hat{f}_j) dx \leq h_n^d \left\{ f(\xi_j)(1 - h_n^d f(\xi_j))\{1 + k_4\} + 2k(\dot{\xi}_j) \left(h_n^\varepsilon - \left\lceil \frac{u_0}{\delta} \right\rceil h_n^d \right) \right. \\ \left. + k_5 f(\xi_j)^{1-\frac{1}{p}} h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} (1 - 2h_n^{d-\varepsilon})^{1-\frac{\rho}{p}} \right\},$$

for any $\varepsilon \in \left[0, d\left(1 - \frac{1}{\rho-p}\right)\right]$. Then sum up over all indexes j to obtain

$$nh_n^d \text{IV}(\hat{f}_n^H) \leq \left\{ 1 + k_4 + 2\|k\|_1 \left(h_n^\varepsilon - \left\lceil \frac{u_0}{\delta} \right\rceil h_n^d \right) + k_5 \|f^{1-\frac{1}{p}}\|_1 h_n^{\frac{1}{p}\{(d-\varepsilon)(\rho-p)-d\}} \right\} (1 + o(1)).$$

Therefore, if $p = \rho - 1$, the bin width choice $h_n = cn^{-1/(d+2)}$, $0 < c < \infty$, entails

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d} \left\{ 1 + k_4 + 2\|k\|_1 + k_5 \|f^{1-\frac{1}{p}}\|_1 \right\}.$$

If $p < \rho - 1$, we get a better asymptotic constant:

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d} \{1 + k_4\}. \quad \blacksquare$$

6.1.3 Variance bounds with high frequency sampling

The period depends now on the sample size in that $\delta_n \downarrow 0^+$ as $n \rightarrow \infty$. We start by giving a new bound expression for the variance of $\hat{f}_n^H(x)$ which depends upon γ_0 .

Lemma 6.4 (high frequency sampling). *If $A_0(ii)$ and $A'_1(i)(ii)$ hold, then we obtain*

$$nh_n^d \text{Var}(\hat{f}_j) \leq f(\xi_j)(1 - h_n^d f(\xi_j)) + 2\varphi(\dot{\xi}_j) \left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} + \left\{ 2u_0 \|f\|_\infty f(\xi_j) \right. \\ \left. + 2(u_1 - u_0 + \delta_n) k(\ddot{\xi}_j) \sup_{u \in [u_0, u_1]} \pi(u) + 2k(\ddot{\xi}_j) \int_{u_1}^\infty \pi(u) du \left(1 + \frac{\pi(u_1)}{\int_{u_1}^\infty \pi(u) du} \delta_n \right) \right\} h_n^d \delta_n^{-1}, \quad (6.7)$$

with $(\xi_j, \dot{\xi}_j, \ddot{\xi}_j) \in \pi_{nj}^3$ and it entails that the variance is a $O(1/(nh_n^d))$ with the following choices $\delta_n^*(\gamma_0)$ of δ_n :

$$\delta_n^*(\gamma_0) = d_1 h_n^d \mathbf{1}_{\{\gamma_0 < 1\}} + d_2 h_n^d \ln(h_n^{-d}) \mathbf{1}_{\{\gamma_0 = 1\}} + d_3 h_n^{d/\gamma_0} \mathbf{1}_{\{\gamma_0 > 1\}}, \quad 0 < d_1, d_2, d_3 < \infty.$$

Proof of Lemma 6.4 The calculus of V_n remains identical. Now to upper bound C_n , we have to make use of the local assumption $A'_1(i)$. Set $r_n^0 := \lfloor u_0/\delta_n \rfloor$ and $r_n^1 := \lfloor u_1/\delta_n \rfloor$, since X^T is stationary one may write

$$C_n = \frac{2}{h_n^d} \sum_{r=1}^{n-1} \left(1 - \frac{r}{n} \right) \text{Cov}(\mathbf{1}_{\pi_{nj_1}}(X_0), \mathbf{1}_{\pi_{nj_2}}(X_{r\delta_n})) =: C_{n,1} + C_{n,2},$$

where

$$C_{n,1} := \frac{2}{h_n^d} \sum_{r=1}^{r_n^0} \left(1 - \frac{r}{n}\right) \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_{r\delta_n})),$$

$$C_{n,2} := \frac{2}{h_n^d} \sum_{r=r_n^0+1}^{n-1} \left(1 - \frac{r}{n}\right) \text{Cov}(\mathbb{1}_{\pi_{nj_1}}(X_0), \mathbb{1}_{\pi_{nj_2}}(X_{r\delta_n})).$$

First using $A'_1(i)$ we get

$$\begin{aligned} |C_{n,1}| &\leq \frac{2}{h_n^d} \sum_{r=1}^{r_n^0} \iint_{\pi_{nj_1} \times \pi_{nj_2}} \left\{ \sup_{y \in \mathbb{R}^d} f_{r\delta_n}(x, y) + \|f\|_\infty f(x) \right\} dx dy \\ &\leq 2 \sum_{r=1}^{r_n^0} \int_{\pi_{nj_1}} \{ \varphi(x)(r\delta_n)^{-\gamma_0} + \|f\|_\infty f(x) \} dx \\ &\leq 2\varphi(\dot{\xi}_{j_1}) \left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} + 2u_0 \|f\|_\infty f(\xi_{j_1}) h_n^d \delta_n^{-1}, \end{aligned}$$

where $(\xi_{j_1}, \dot{\xi}_{j_1}) \in \pi_{nj_1}^2$. Setting $k_6 := 2u_0 \|f\|_\infty$, we obtain

$$|C_{n,1}| \leq 2\varphi(\dot{\xi}_{j_1}) \left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} + k_6 f(\xi_{j_1}) h_n^d \delta_n^{-1}.$$

Then using $A'_1(ii)$

$$|C_{n,2}| \leq \frac{2}{h_n^d} \sum_{r=r_n^0+1}^{n-1} \iint_{\pi_{nj_1} \times \pi_{nj_2}} \sup_{y \in \mathbb{R}^d} |g_{r\delta_n}(x, y)| dx dy \leq 2h_n^d k(\ddot{\xi}_{j_1}) \left[\sum_{r=r_n^0+1}^{r_n^1} \pi(r\delta_n) + \sum_{r=r_n^1+1}^{n-1} \pi(r\delta_n) \right],$$

where $\ddot{\xi}_{j_1} \in \pi_{nj_1}$. On the one hand, one has

$$\sum_{r=r_n^0+1}^{r_n^1} \pi(r\delta_n) \leq (r_n^1 - r_n^0) \sup_{u \in [u_0, u_1]} \pi(u) \leq (u_1 - u_0) \sup_{u \in [u_0, u_1]} \pi(u) \left(1 + \frac{\delta_n}{u_1 - u_0} \right) \delta_n^{-1}.$$

On the other hand, the monotonicity of $\pi(\cdot)$ implies

$$\sum_{r=r_n^1+1}^{n-1} \pi(r\delta_n) \leq \delta_n^{-1} \sum_{r=r_n^1+1}^{n-1} \delta_n \pi(r\delta_n) \leq \left\{ (u_1 - r_n^1 \delta_n) \pi((r_n^1 + 1) \delta_n) + \int_{u_1}^{\infty} \pi(u) du \right\} \delta_n^{-1}.$$

Setting $k_7 := 2(u_1 - u_0) \sup_{u \in [u_0, u_1]} \pi(u)$ and $k_8 := 2 \int_{u_1}^{\infty} \pi(u) du$, we thus obtain

$$|C_{n,2}| \leq k_7 k(\ddot{\xi}_{j_1}) \left(1 + \frac{\delta_n}{u_1 - u_0} \right) h_n^d \delta_n^{-1} + k_8 k(\ddot{\xi}_{j_1}) \left(1 + \frac{\pi(u_1)}{\int_{u_1}^{\infty} \pi(u) du} \delta_n \right) h_n^d \delta_n^{-1}.$$

Thence

$$\begin{aligned}
nh_n^d \text{Cov}(\hat{f}_{j_1}, \hat{f}_{j_2}) &\leq -h_n^d f(\xi_{j_1})f(\xi_{j_2}) + 2\varphi(\xi_{j_1}) \left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} + k_6 f(\xi_{j_1}) h_n^d \delta_n^{-1} \\
&\quad + k_7 k(\ddot{\xi}_{j_1}) \left(1 + \frac{\delta_n}{u_1 - u_0} \right) h_n^d \delta_n^{-1} + k_8 k(\ddot{\xi}_{j_1}) \left(1 + \frac{\pi(u_1)}{\int_{u_1}^{\infty} \pi(u) du} \delta_n \right) h_n^d \delta_n^{-1}, \quad (6.8)
\end{aligned}$$

which leads to the desired result. Using (6.8), we also deduce the optimal choices $\delta_n^*(\gamma_0)$ of δ_n i.e. the smallest values of δ_n so that C_n is a $O(1)$. These choices are given by (3.1) in accordance with the values of γ_0 . ■

6.1.4 Proof of Theorem 3.3

By integrating over π_{nj} the right-hand side of (6.7):

$$\begin{aligned}
nh_n^d \int_{\pi_{nj}} \text{Var}(\hat{f}_j) dx &\leq h_n^d \left\{ f(\xi_j)(1 - h_n^d f(\xi_j)) + 2\varphi(\xi_j) \left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} + k_6 f(\xi_j) h_n^d \delta_n^{-1} \right. \\
&\quad \left. + k_7 k(\ddot{\xi}_j) \left(1 + \frac{\delta_n}{u_1 - u_0} \right) h_n^d \delta_n^{-1} + k_8 k(\ddot{\xi}_j) \left(1 + \frac{\pi(u_1)}{\int_{u_1}^{\infty} \pi(u) du} \delta_n \right) h_n^d \delta_n^{-1} \right\}.
\end{aligned}$$

Then let us sum up over all indexes j . Since φ is Riemann-integrable, we obtain

$$nh_n^d \text{IV}(\hat{f}_n^H) \leq \left\{ 1 + 2\|\varphi\|_1 \left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} + k_6 h_n^d \delta_n^{-1} + (k_7 + k_8) \|k\|_1 h_n^d \delta_n^{-1} \right\} (1 + o(1)).$$

According to the values of γ_0 , we derive all asymptotic bounds with optimal choices of δ_n :

- if $\gamma_0 < 1$, the choice $\delta_n^*(\gamma_0) = d_1 h_n^d$, $0 < d_1 < \infty$, entails

$$\left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} \leq \frac{1}{d_1} \frac{u_0^{1-\gamma_0}}{1-\gamma_0} \left[1 - \frac{d_1^{1-\gamma_0} \gamma_0}{u_0^{1-\gamma_0}} h_n^{d(1-\gamma_0)} \right];$$

- if $\gamma_0 = 1$, the choice $\delta_n^*(\gamma_0) = d_2 h_n^d \ln(h_n^{-d})$, $0 < d_2 < \infty$, entails

$$\left(\sum_{r=1}^{r_n^0} \frac{1}{r} \right) h_n^d \delta_n^{-1} \leq \frac{1}{d_2} \left[1 - \ln(h_n^{-d}) \ln \left(e \frac{u_0}{d_2} \ln(h_n^{-d}) \right) \right];$$

- if $\gamma_0 > 1$, the choice $\delta_n^*(\gamma_0) = d_3 h_n^{d/\gamma_0}$, $0 < d_3 < \infty$, entails

$$\left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n^d \delta_n^{-\gamma_0} \leq \frac{\gamma_0}{d_3^{\gamma_0} (\gamma_0 - 1)} \left[1 - \frac{1}{\gamma_0 u_0^{\gamma_0-1}} \delta_n^{\gamma_0-1} \right].$$

So setting

$$C_{\gamma_0} := \frac{1}{d_1} \left\{ \frac{2\|\varphi\|_1 u_0^{1-\gamma_0}}{1-\gamma_0} + k_6 + (k_7 + k_8)\|k\|_1 \right\} \mathbf{1}_{\{\gamma_0 < 1\}} + \frac{2\|\varphi\|_1}{d_2} \mathbf{1}_{\{\gamma_0 = 1\}} + \frac{2\|\varphi\|_1 \gamma_0}{d_3^{\gamma_0}(\gamma_0 - 1)} \mathbf{1}_{\{\gamma_0 > 1\}},$$

it remains to choose $h_n = cn^{-1/(d+2)}$, $0 < c < \infty$, as in Theorem 3.2:

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d} (1 + C_{\gamma_0}),$$

and we can also improve our asymptotic constant for any choice of δ_n such that $\delta_n/\delta_n^*(\gamma_0) \rightarrow \infty$ as $n \rightarrow \infty$:

$$\limsup_{n \rightarrow \infty} n^{\frac{2}{d+2}} \text{MISE}(\hat{f}_n^H) \leq \frac{c^2}{12} R_d(f') + \frac{1}{c^d}. \blacksquare$$

6.2 Frequency polygon

6.2.1 Proof of Theorem 4.2

Renewal sampling – First observe that

$$\int_{\mathbb{R}} \text{Var}(\hat{f}_n^{FP}(x)) dx = \sum_j \int_{[c_j, c_{j+1}[} \text{Var}(\hat{f}_n^{FP}(x)) dx,$$

where

$$\begin{aligned} \int_{c_j}^{c_{j+1}} \text{Var}(\hat{f}_n^{FP}(x)) dx &= \frac{1}{h_n^2} \int_{c_j}^{c_{j+1}} \left\{ (x - c_j)^2 \text{Var}(\hat{f}_{j+1}) + (c_{j+1} - x)^2 \text{Var}(\hat{f}_j) \right. \\ &\quad \left. + 2(x - c_j)(c_{j+1} - x) \text{Cov}(\hat{f}_j, \hat{f}_{j+1}) \right\} dx. \end{aligned}$$

For any $j \in \mathbb{Z}$, let us denote by $\overline{V}_j$ (respectively $\overline{C}_{j,j+1}$) an upper bound expression for $nh_n \text{Var}(\hat{f}_j)$ (respectively $nh_n \text{Cov}(\hat{f}_j, \hat{f}_{j+1})$) that is independent of x . We get

$$nh_n \int_{c_j}^{c_{j+1}} \text{Var}(\hat{f}_n^{FP}(x)) dx \leq \frac{h_n}{3} \{ \overline{V}_j + \overline{V}_{j+1} + \overline{C}_{j,j+1} \}. \quad (6.9)$$

Insert now both expressions (6.2) and (6.4) in (6.9), then for $\varepsilon \in \left[0, 1 - \frac{1}{\rho-p}\right]$,

$$\begin{aligned} &nh_n \int_{c_j}^{c_{j+1}} \text{Var}(\hat{f}_n^{FP}(x)) dx \\ &\leq \frac{h_n}{3} \left\{ f(\xi_j)(1 - h_n f(\xi_j))\{1 + k_1\} + k_2 k(\dot{\xi}_j) h_n^\varepsilon + k_3 f(\xi_j)^{1-\frac{1}{p}} h_n^{\frac{1}{p}\{(1-\varepsilon)(\rho-p)-1\}} \right\} \\ &\quad + \frac{h_n}{3} \left\{ f(\xi_{j+1})(1 - h_n f(\xi_{j+1}))\{1 + k_1\} + k_2 k(\dot{\xi}_{j+1}) h_n^\varepsilon + k_3 f(\xi_{j+1})^{1-\frac{1}{p}} h_n^{\frac{1}{p}\{(1-\varepsilon)(\rho-p)-1\}} \right\} \\ &\quad + \frac{h_n}{3} \left\{ -h_n f(\xi_j) f(\xi_{j+1}) + k_1 \sqrt{f(\xi_j) f(\xi_{j+1}) (1 - h_n f(\xi_j)) (1 - h_n f(\xi_{j+1}))} \right. \\ &\quad \left. + k_2 k(\dot{\xi}_j) h_n^\varepsilon + k_3 \sqrt{f(\xi_j)^{1-\frac{1}{p}} f(\xi_{j+1})^{1-\frac{1}{p}}} h_n^{\frac{1}{p}\{(1-\varepsilon)(\rho-p)-1\}} \right\}. \end{aligned}$$

We bound the IV of $\hat{f}_n^{FP}$ by summing up over all indexes j . So for $\varepsilon \in \left[0, 1 - \frac{1}{\rho-p}\right]$,

$$nh_n \text{ IV}(\hat{f}_n^{FP}) \leq \left\{ \frac{2}{3} + k_1 + k_2 \|k\|_1 h_n^\varepsilon + k_3 \|f^{1-\frac{1}{p}}\|_1 h_n^{\frac{1}{p}\{(1-\varepsilon)(\rho-p)-1\}} \right\} (1 + o(1)). \quad (6.10)$$

Now the bin width choice $h_n = cn^{-1/5}$, $0 < c < \infty$, in Lemma 4.1 yields first

$$\lim_{n \rightarrow \infty} n^{\frac{4}{5}} \text{ ISB}(\hat{f}_n^{FP}) = \frac{49}{2880} c^4 R(f''),$$

and combining with (6.10), if $p = \rho - 1$ ($\varepsilon = 0$):

$$\limsup_{n \rightarrow \infty} n^{\frac{4}{5}} \text{ MISE}(\hat{f}_n^{FP}) \leq \frac{49}{2880} c^4 R(f'') + \frac{1}{c} \left\{ \frac{2}{3} + k_1 + k_2 \|k\|_1 + k_3 \|f^{1-\frac{1}{p}}\|_1 \right\}.$$

If $p < \rho - 1$ ($0 < \varepsilon < 1$):

$$\limsup_{n \rightarrow \infty} n^{\frac{4}{5}} \text{ MISE}(\hat{f}_n^{FP}) \leq \frac{49}{2880} c^4 R(f'') + \frac{1}{c} \left\{ \frac{2}{3} + k_1 \right\}.$$

Jittered sampling – The outlines of the proof are unchanged. Insert both expressions (6.3) and (6.5) in (6.9) and sum up over all indexes j , then it follows that for $\varepsilon \in \left[0, 1 - \frac{1}{\rho-p}\right]$,

$$\begin{aligned} nh_n \text{ IV}(\hat{f}_n^{FP}) \leq & \left\{ \frac{2}{3} + k_4 + 2\|k\|_1 \left(h_n^\varepsilon - \left\lceil \frac{u_0}{\delta} \right\rceil h_n \right) \right. \\ & \left. + k_5 \|f^{1-\frac{1}{p}}\|_1 h_n^{\frac{1}{p}\{(1-\varepsilon)(\rho-p)-1\}} (1 - 2h_n^{1-\varepsilon})^{1-\frac{p}{p-1}} \right\} (1 + o(1)). \end{aligned}$$

Take $h_n = cn^{-1/5}$, $0 < c < \infty$, then if $p = \rho - 1$:

$$\limsup_{n \rightarrow \infty} n^{\frac{4}{5}} \text{ MISE}(\hat{f}_n^{FP}) \leq \frac{49}{2880} c^4 R(f'') + \frac{1}{c} \left\{ \frac{2}{3} + k_4 + 2\|k\|_1 + k_5 \|f^{1-\frac{1}{p}}\|_1 \right\}.$$

If $p < \rho - 1$:

$$\limsup_{n \rightarrow \infty} n^{\frac{4}{5}} \text{ MISE}(\hat{f}_n^{FP}) \leq \frac{49}{2880} c^4 R(f'') + \frac{1}{c} \left\{ \frac{2}{3} + k_4 \right\}. \quad \blacksquare$$

6.2.2 Proof of Theorem 4.3

Insert now both expressions (6.7) and (6.8) in (6.9) and sum up over all indexes j , we get

$$nh_n \text{ IV}(\hat{f}_n^{FP}) \leq \left\{ \frac{2}{3} + 2\|\varphi\|_1 \left(\sum_{r=1}^{r_n^0} \frac{1}{r^{\gamma_0}} \right) h_n \delta_n^{-\gamma_0} + k_6 h_n \delta_n^{-1} + (k_7 + k_8) \|k\|_1 h_n \delta_n^{-1} \right\} (1 + o(1)).$$

Then $h_n = cn^{-1/5}$, $0 < c < \infty$, together with the optimal choices δ_n^* of δ_n yield

$$\limsup_{n \rightarrow \infty} n^{\frac{4}{5}} \text{ MISE}(\hat{f}_n^{FP}) \leq \frac{49}{2880} c^4 R(f'') + \frac{1}{c} \left\{ \frac{2}{3} + C_{\gamma_0} \right\},$$

and, if δ_n is such that $\delta_n/\delta_n^*(\gamma_0) \rightarrow \infty$ as $n \rightarrow \infty$, C_{γ_0} is removable in the limiting bound. $\blacksquare$

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