



*Functiones et Approximatio*  
XXXIX.2 (2008), 261–271

## EQUIDISTRIBUTION MODULO 1 AND SALEM NUMBERS

CHRISTOPHE DOCHE, MICHEL MENDÈS FRANCE, JEAN-JACQUES RUCH

À Władysław Narkiewicz à l'occasion  
de son 70<sup>e</sup> anniversaire

**Abstract:** Let  $\theta$  be a Salem number. It is well-known that the sequence  $(\theta^n)$  modulo 1 is dense but not equidistributed. In this article we discuss equidistributed subsequences. Our first approach is computational and consists in estimating the supremum of  $\lim_{n \rightarrow \infty} n/s(n)$  over all equidistributed subsequences  $(\theta^{s(n)})$ . As a result, we obtain an explicit upper bound on the density of any equidistributed subsequence. Our second approach is probabilistic. Defining a measure on the family of increasing integer sequences, we show that relatively to that measure, almost no subsequence is equidistributed.

**Keywords:** Salem number, Equidistribution modulo 1,  $J_0$  Bessel function.

### 1. Subsequences

Let  $u = (u(n))$  be an infinite sequence of real numbers. A subsequence  $u \circ s = (u(s(n)))$  is said to have density  $d \leq 1$  if as  $n$  increases  $n/s(n) \rightarrow d$ . Suppose the sequence  $u$  is dense (mod 1). Answering a question of one of us in 1973, Y. Dupain and J. Lesca [6] established that the set of densities  $d$  of equidistributed (mod 1) subsequences of  $u$  is a closed interval  $[0, d_0]$  where  $d_0 \leq 1$  depends on  $u$ . They also showed how to compute  $d_0$ . For  $0 \leq x \leq 1$ , define the *repartition function*

$$f(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{n < N \mid \{u(n)\} < x\}$$

where  $\{u(n)\}$  is the fractional part of  $u(n)$ . We only consider those  $x$  where  $f(x)$  and its derivative  $f'(x)$  both exist, i.e. almost everywhere. Y. Dupain and J. Lesca proved that  $d_0 = \inf_x f'(x)$ .

A particularly striking example of such an instance concerns the distribution (mod 1) of the powers of Salem numbers  $\theta > 1$ . A Salem number [10] (see also [3]) is a real algebraic integer whose algebraic conjugates other than  $\theta$  all lie in the unit disc  $|z| \leq 1$  with one conjugate at least on the boundary  $|z| = 1$ . It is then known that one and only one of these conjugates  $\theta^{-1}$  is inside the disc while the others are on the boundary. The degree  $2t$  of  $\theta$  is necessarily even and at least equal to 4.

---

**Mathematics Subject Classification:** 11K06, 11J71.

Denote the different conjugates by  $\theta, \theta^{-1}, \exp(\pm 2i\pi\omega_1), \dots, \exp(\pm 2i\pi\omega_{t-1})$ . The sum of all conjugates of an algebraic integer is an integer and therefore for all  $n \in \mathbb{N}$ ,

$$\theta^n + \theta^{-n} + 2 \sum_{j=1}^{t-1} \cos 2\pi n \omega_j \equiv 0 \pmod{1}$$

so that the distribution of  $\theta^n \pmod{1}$  is essentially that of  $-2 \sum_{j=1}^{t-1} \cos 2\pi n \omega_j$ . Ch. Pisot and R. Salem [9] observed that  $1, \omega_1, \dots, \omega_{t-1}$  are  $\mathbb{Z}$ -linearly independent so that, according to Kronecker, the  $(t-1)$  dimensional sequence  $(\omega_1 n, \dots, \omega_{t-1} n)$  is equidistributed in  $(\mathbb{R}/\mathbb{Z})^{t-1}$ . As a consequence, the sequence  $(\theta^n)$  is therefore clearly dense  $\pmod{1}$ . Furthermore, for all  $k \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \exp 2i\pi k \theta^n &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \prod_{j=1}^{t-1} \exp(-2i\pi k \cdot 2 \cos 2\pi n \omega_j) \\ &= \left( \int_0^1 \exp(-4i\pi k \cos 2\pi x) dx \right)^{t-1} \\ &= J_0(4\pi k)^{t-1} \neq 0 \end{aligned} \tag{1.1}$$

where  $J_0(\cdot)$  is the Bessel function of the first kind of index 0.

Since  $|J_0(\alpha)| < 1$  for all real  $\alpha \neq 0$ , the above limit tends to 0 as  $t \rightarrow \infty$ . Y. Dupain and J. Lesca conclude that for large degrees  $t$ , the sequence  $(\theta^n \pmod{1})$  is close to being equidistributed, a fact that S. Akiyama and Y. Tanigawa [1] make very explicit in their article. This is quite remarkable since even though for almost all real  $\tau > 1$ ,  $(\tau^n)$  is equidistributed  $\pmod{1}$ , no explicit  $\tau$  is known (J. F. Koksma [8]).

We know the existence of  $d_0 < 1$  (and quite obviously  $d_0 > 0$ ) such that  $s(n) \sim \frac{1}{d_0} n$  and  $(\theta^{s(n)})$  equidistributed  $\pmod{1}$ . We shall see later on that those sequences are rare. But we can already guess why these sequences  $s(n)$  are exceptional. This is a consequence of our first rather trivial theorem.

**Theorem 1.1.** *If  $s(n)$  is an increasing sequence of integers such that  $(\theta^{s(n)})$  is equidistributed  $\pmod{1}$ , then there exists an irrational  $x$  such that  $x s(n)$  is not equidistributed  $\pmod{1}$ .*

**Proof.** We note that

$$\theta^{s(n)} \equiv -2 \sum_{j=1}^{t-1} \cos 2\pi \omega_j s(n) - \theta^{-s(n)} \pmod{1}.$$

The  $(t-1)$  dimensional sequence  $(\omega_1 s(n), \dots, \omega_{t-1} s(n))$  is not equidistributed in  $(\mathbb{R}/\mathbb{Z})^{t-1}$  since if it were,  $(\theta^{s(n)})$  would not be equidistributed  $\pmod{1}$ . Therefore there exist integers  $h_1, \dots, h_{t-1}$  not all 0 such that

$$h_1 \omega_1 s(n) + \dots + h_{t-1} \omega_{t-1} s(n)$$

is not equidistributed (mod 1). The theorem is established with

$$x = \sum_{j=1}^{t-1} h_j \omega_j . \quad \blacksquare$$

Next, we develop a method to approximate  $d_0$  for the sequence  $(\theta^n \pmod{1})$ , where  $\theta$  is a Salem number of degree  $2t$ . The results indicate that  $d_0$  tends to 1 very quickly as  $t$  tends to infinity. A key result in this approach is the study of the minimum of a cosine series on  $]0, 1[$ . Under certain conditions, we show that the minimum is always attained at  $x = 1/2$ , cf. Theorem 2.1.

### 2. Explicit Computations of $d_0$

The repartition function is explicitly determined for a Salem number of degree 4, cf. [5]. Namely,

$$f(x) = \frac{5}{2} - \frac{1}{\pi} \left( \arccos \frac{x-2}{2} + \arccos \frac{x}{2} + \arccos \frac{x-1}{2} + \arccos \frac{x+1}{2} \right) .$$

It follows that

$$f'(x) = \frac{1}{2\pi} \left( \frac{1}{\sqrt{1 - (\frac{x}{2} - 1)^2}} + \frac{1}{\sqrt{1 - (\frac{x-1}{2})^2}} + \frac{1}{\sqrt{1 - (\frac{x}{2})^2}} + \frac{1}{\sqrt{1 - (\frac{x+1}{2})^2}} \right) .$$

A direct study of  $f'(x)$  shows that it attains its minimum for  $x = \frac{1}{2}$  and gives the exact value of  $d_0$ , i.e.

$$\frac{1}{\pi} \left( \frac{4}{\sqrt{7}} + \frac{4}{\sqrt{15}} \right) = 0.809988350 \dots \tag{2.1}$$

For a Salem number of degree  $2t$  with  $t > 2$ , we want to estimate the corresponding  $d_0$ . First, let us show the following lemma.

**Lemma 2.1.** *Let  $\theta$  be a Salem number of degree  $2t$ , then the repartition function  $f(x)$  of the sequence  $(\theta^n)$  modulo 1 satisfies*

$$f'(x) = 1 + 2 \sum_{k=1}^{\infty} J_0(4k\pi)^{t-1} \cos 2\pi kx$$

on  $]0, 1[$ , for all  $t \geq 2$ .

**Proof.** We have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \exp 2i\pi k \theta^n = \int_0^1 \exp 2i\pi kx \, dx$$

where  $\nu$  is the repartition function  $f(x)$ . According to Y. Dupain [5] the measure  $d\nu = f'(x) dx$  is absolutely continuous. It follows from (1.1) that

$$J_0(4\pi k)^{t-1} = \int_0^1 \exp 2i\pi kx f'(x) dx .$$

We can associate with  $f'(x)$  its Fourier series

$$\sum_{k \in \mathbb{Z}} J_0(4\pi k)^{t-1} \exp(-2i\pi kx) = 1 + 2 \sum_{k=1}^{\infty} J_0(4\pi k)^{t-1} \cos 2\pi kx. \quad (2.2)$$

If this series converges uniformly, then its sum is continuous and equals  $f'(x)$ . The lemma is clear for  $t > 3$ , since  $J_0(x) = O(x^{-\frac{1}{2}})$  and we even have equality on  $[0, 1]$ . For  $t = 2$  and 3, we need the following result.

**Lemma 2.2.** *The sequence  $(J_0(4\pi k))$  is positive for all  $k > 0$  and strictly decreasing.*

**Proof.** In [1, Lemma 2], it is shown that

$$J_0(2\pi k) = \frac{1}{\pi\sqrt{k}} \left( \frac{1}{\sqrt{2}} - \frac{1}{16\sqrt{2}\pi k} + R \right), \text{ with } |R| \leq \frac{9}{512\pi^2 k^2}.$$

It is straightforward to deduce that

$$0 \leq \frac{1}{2\pi\sqrt{k}} - J_0(4\pi k) \leq \frac{1}{61\pi^2 k^{\frac{3}{2}}}. \quad (2.3)$$

This proves the first part of the lemma. Now

$$\frac{1}{2\pi} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) \geq \frac{1}{8\pi k^{\frac{3}{2}}} > \frac{2}{61\pi^2 k^{\frac{3}{2}}}.$$

This shows that

$$\frac{1}{2\pi\sqrt{k}} - \frac{1}{61\pi^2 k^{\frac{3}{2}}} > \frac{1}{2\pi\sqrt{k+1}} - \frac{1}{61\pi^2 (k+1)^{\frac{3}{2}}}$$

which implies that  $J_0(4\pi k) > J_0(4\pi(k+1))$ , for  $k > 0$ . ■

We deduce that the series (2.2) is uniformly convergent on the compact  $[\varepsilon, 1-\varepsilon]$ , for any  $\varepsilon > 0$  and therefore  $f'(x)$  is equal to this series on  $]0, 1[$ . ■

A consequence of Lemma 3.2 is that  $d_0$  only depends on  $t$  and satisfies

$$d_0 = \inf_{x \in ]0, 1[} \left( 1 + 2 \sum_{k=1}^{\infty} J_0(4k\pi)^{t-1} \cos 2\pi kx \right) .$$

Next let us recall a definition we shall use later.

**Definition 2.1.** Let  $(b_k)$  be a sequence of real numbers and let  $\Delta^0 b_k = b_k$  and  $\Delta^n b_k = \Delta^{n-1} b_k - \Delta^{n-1} b_{k+1}$ , for all  $n > 0$ . The sequence  $(b_k)$  is said to be totally monotone if  $\Delta^n b_k \geq 0$  for all  $k$ , and  $n = 0, 1, 2, \dots$

By a famous result of Hausdorff [7], the total monotonicity of  $(b_k)$  is equivalent to the existence of a nonnegative measure  $\mu$  on  $[0, 1]$  such that the  $b_k$ 's are the moments of  $\mu$ , i.e.

$$b_k = \int_0^1 u^k d\mu .$$

**Example 2.1.** Let  $s$  be a real positive number. The sequence  $(b_k)$  defined by

$$b_k = \frac{1}{(k + 1)^s}$$

for all  $k \geq 0$  is totally monotone.

**Theorem 2.1.** Let  $(a_k)$  be a sequence of nonnegative real numbers (except maybe for  $a_0$ ). Assume that  $(a_{k+1})$ ,  $k \geq 0$  is totally monotone, then the function

$$g(x) = \sum_{k=0}^{\infty} a_k \cos 2\pi kx$$

is well-defined and decreasing on the interval  $]0, 1/2[$ . As a corollary,  $g(x)$  attains its minimum for  $x = \frac{1}{2}$ .

**Proof.** Let us introduce

$$h(x) = \sum_{k=1}^{\infty} a_k \cos 2\pi kx = \sum_{k=0}^{\infty} b_k \cos 2\pi(k + 1)x .$$

Since,  $g$  and  $h$  only differ by  $a_0$ , it is enough to study  $h$  to prove the theorem on  $g$ . Since  $(b_k) = (a_{k+1})$ ,  $\Delta b_k \geq 0$ , for all  $k$ . So the sequence  $(b_k)$  is decreasing and this shows that the series  $h(x)$  is convergent for all  $x \in ]\varepsilon, 1 - \varepsilon[$ , for all  $\varepsilon > 0$ . Since  $h(x) = h(1 - x)$ , it is enough to study  $h$  on  $]0, 1/2[$ .

Since the  $b_k$ 's are the moments of a certain nonnegative measure  $\mu$ , we obtain

$$\begin{aligned} h(x) &= \sum_{k=0}^{\infty} b_k \cos 2\pi(k + 1)x \\ &= \sum_{k=0}^{\infty} \int_0^1 u^k \cos 2\pi(k + 1)x d\mu \\ &= \Re \int_0^1 \frac{e^{2i\pi x}}{1 - e^{2i\pi x}u} d\mu. \end{aligned}$$

The last equality being justified by the nonnegativity of  $\mu$ . It follows that

$$h(x) = \int_0^1 \frac{\cos 2\pi x - u}{1 + u^2 - 2u \cos 2\pi x} d\mu .$$

To show that  $h(x)$  is decreasing on  $]0, 1/2]$ , evaluate  $h(x) - h(y)$  for  $0 < x \leq y \leq 1/2$ . Let

$$j_x(u) = \frac{\cos 2\pi x - u}{1 + u^2 - 2u \cos 2\pi x} .$$

Then reducing to the same (positive) denominator, we see that the numerator of  $j_x(u) - j_y(u)$  is  $(\cos 2\pi x - \cos 2\pi y)(1 - u^2)$  which is nonnegative for all  $u \in [0, 1]$ .

Since  $\mu$  is a nonnegative measure, we deduce that  $h(x) \geq h(y)$  whenever  $x \leq y \leq 1/2$  and that  $h(x) \geq h(1/2)$  for all  $x \in ]0, 1/2]$ . These results apply trivially to the function  $g$ . ■

**Corollary 2.1.** *Let  $s > 0$ . Then the series*

$$g(x) = a_0 + \sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{k^s}$$

*is decreasing on  $]0, 1/2]$  and satisfies*

$$g(x) \geq a_0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} .$$

**Remark.** It is possible to compute  $g(1/2)$  very efficiently following the method explained in [4]. For instance, for the sequence  $(a_k)$  defined by a given  $a_0$  and  $a_k = 1/\sqrt{k}$ , for  $k \geq 1$ , we have that

$$g(x) \geq g(1/2) = a_0 - 0.6048986434216303702472659142359554997597625451 \dots$$

All the digits in the last equality are correct as can be established knowing the first 60  $a_k$ 's.

Unfortunately, we are not able to show that the sequence  $(J_0(4\pi k)^{t-1}), k > 0$  is totally monotone, though the extensive numerical computations of its first  $n$ -th forward differences seem to indicate that this is the case. Based on the case  $t = 2$  and also on direct computations of  $f'(x)$  for various  $x$ , we conjecture that  $\inf_x f'(x) = f'(1/2)$  for  $t \geq 2$ . However, to be totally rigorous, we cannot directly apply Theorem 2.1 to obtain the value of  $d_0$ . Nevertheless, this result will give an approximation of  $d_0$ , for  $t > 2$ .

The idea is to apply (2.3) to deduce that

$$\left| J_0(4\pi k)^{t-1} - \frac{1}{(2\pi\sqrt{k})^{t-1}} \right| \leq \frac{1}{61\pi^2 k^{\frac{3}{2}}} \frac{(t-1)}{(2\pi\sqrt{k})^{t-2}} .$$

It follows that

$$\left| f'(x) - 1 - 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{(2\pi\sqrt{k})^{t-1}} \right| \leq \sum_{k=1}^{\infty} \frac{2}{61\pi^2 k^{\frac{3}{2}}} \frac{(t-1)}{(2\pi\sqrt{k})^{t-2}},$$

which, combined with Theorem 2.1, implies that for all  $x \in ]0, 1[$

$$f'(x) \geq \underbrace{1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2\pi\sqrt{k})^{t-1}}}_{S_1} - \underbrace{\sum_{k=1}^{\infty} \frac{2}{61\pi^2 k^{\frac{3}{2}}} \frac{(t-1)}{(2\pi\sqrt{k})^{t-2}}}_{S_2}.$$

The main contribution, i.e.  $S_1$ , can be obtained using the acceleration convergence method explained in [4], whereas the second series  $S_2$  is simply (up to a constant) an evaluation of the  $\zeta$  function at the point  $(t + 1)/2$ . This gives a lower bound for  $d_0$ . An upper bound is given by  $d_0 \leq f'(1/2)$ , where  $f'(1/2)$  is bounded, for any  $K$  even, by the truncated alternating series

$$1 + 2 \sum_{k=1}^K (-1)^k J_0(4\pi k)^{t-1}.$$

The convergence is quite slow for  $t = 3$  so that we fixed  $K = 2.10^6$  to obtain a relevant upper bound. Much less terms are necessary for larger  $t$ . A conjectured value  $d_0^*$  is also given relying on the assumption that  $d_0 = f'(1/2)$  and on the computation of  $f'(1/2)$  using [4]. The method seems to converge and at most the first 10 terms are sufficient to give a result with an error less than  $10^{-10}$ . Also, we checked for  $t = 2$  that the value given in (2.1) is, up to several hundred digits, equal to the one computed with this approach.

Note that if the sequence  $(J_0(4\pi k)^{t-1})$ , defined for  $k > 0$  is totally monotone, then both assumptions are valid, and therefore  $d_0 = d_0^*$ . All the figures are given in Table 1.

Table 1: Lower bound, upper bound, and conjectured value of  $d_0$

t	$S_1$	$S_2$	$S_1 - S_2$	$f'(1/2)$	$d_0^*$
3	0.964884753	0.000869699	0.964015054	0.965745539	0.965745543
4	0.993830708	0.000112882	0.993717825	0.994046008	0.994046007
5	0.998944571	0.000016098	0.998928472	0.998991788	0.998991787
6	0.999822887	0.000002401	0.999820485	0.999832498	0.999832497
7	0.999970695	0.000000367	0.999970328	0.999972560	0.999972559
8	0.999995201	0.000000056	0.999995144	0.999995551	0.999995550
9	0.999999220	0.000000008	0.999999211	0.999999285	0.999999284
10	0.999999874	0.000000001	0.999999872	0.999999886	0.999999885

In the next section we shall define the notion of "almost all" increasing sequences of integers  $(s(n))$ . For almost all sequences  $(s(n))$  and for all irrational

hal-00353834, version 1 - 16 Jan 2009

numbers  $x$ ,  $(xs(n))$  is equidistributed. This already shows how exceptional those sequences  $(s(n))$  are for which  $(\theta^{s(n)})$  is equidistributed.

Furthermore R. Salem [11] demonstrated that if  $(s(n))$  is any increasing sequence such that  $s(n) = O(n)$ , then the Hausdorff dimension of the set of  $x$  for which  $(xs(n))$  is not equidistributed (mod 1), vanishes. The  $x$ 's in Theorem 1.1 are therefore "rare" if indeed  $s(n) \sim \frac{1}{d_0}n$ .

### 3. Metrical Results

Let  $S$  be the family of finite or infinite strictly increasing sequences of positive integers. To each  $s = (s(n)) \in S$  corresponds a unique sequence  $\chi \in D = \{0, 1\}^{\mathbb{N}}$  (characteristic sequence) and conversely:

$$\chi(n) = \begin{cases} 1 & \text{if } n \in s, \\ 0 & \text{if not.} \end{cases}$$

Any measure on  $D$  lifts to a measure on  $S$ .

Let  $0 < d < 1$ . Put  $m\{1\} = d$  and  $m\{0\} = 1 - d$ . Then  $\mu = \prod m$  is a probability measure on  $D$  to which corresponds a probability measure on  $S$  which we still denote by  $\mu$  or  $\mu_d$  if we wish to emphasize the parameter  $d$ .

**Theorem 3.1.** *Consider the polynomial  $P(X) = \sum_{\ell=0}^{\nu} a_{\ell}X^{\ell}$  where at least one of the coefficients  $a_{\ell}$ ,  $1 \leq \ell \leq \nu$  is irrational. Then for  $\mu$ -almost all sequences  $s \in S$ ,  $P(s) = (P(s(n)))$  is equidistributed (mod 1).*

**Theorem 3.2.** *If  $\theta$  is a Salem number then  $\mu$ -almost no sequence  $(\theta^{s(n)})$  is equidistributed (mod 1). More generally, if  $P$  is any positive integer valued polynomial,  $\theta^{P(s)} = (\theta^{P(s(n))})$  is  $\mu$ -almost never equidistributed (mod 1).*

We have seen in Section 1 that there exists a  $d_0 \in ]0, 1[$  for which no sequence  $s = (s(n))$  exists such that  $s(n) \sim \frac{1}{d}n$  ( $d > d_0$ ) and  $(\theta^{s(n)})$  equidistributed (mod 1). For  $d \leq d_0$  there do exist  $d$ -density equidistributed subsequences  $(\theta^{s(n)})$  but they are  $\mu_d$ -rare.

**Remark.** For  $d \in [0, 1]$  let  $T(d)$  be the family of increasing sequences  $(s(n))$  of density  $d$  such that  $(\theta^{s(n)})$  is equidistributed (mod 1). We know that  $T(d) = \emptyset$  as long as  $d > d_0$ . Could it be true that as  $d$  decreases to 0 the family  $T(d)$  "increases in size"? Could one devise a way to show that this is so, e.g. by defining a fractal dimension adapted to the question?

### 4. Proof of Theorem 3.1

A sequence  $\chi \in \{0, 1\}^{\mathbb{N}}$  is said to be  $d$ -normal if all finite words  $w = w_1 \dots w_{\ell} \in \{0, 1\}^{\ell}$  occur in  $\chi$  with the frequency  $d^k(1-d)^{\ell-k}$  where  $k$  is the number of 1's in  $w$ . It is well known that  $\mu_d$ -almost all  $\chi$  are  $d$ -normal. For such a sequence

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} (\chi(n) - d) = 0$$

and more generally, for all  $k \geq 1$  and all integers  $h_1 \leq \dots \leq h_k$  where at least one couple  $h_i < h_{i+1}$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \prod_{i=1}^k (\chi(n + h_i) - d) = 0.$$

A sequence  $Y$  is said to be *uncorrelated* if for all  $k \geq 1$  and all integers  $h_1 \leq \dots \leq h_k$  where at least one couple  $h_i < h_{i+1}$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \prod_{i=1}^k Y(n + h_i) = 0.$$

If  $\chi \in \{0, 1\}^{\mathbb{N}}$  is  $d$ -normal, then as remarked above,  $\chi - d$  is uncorrelated.

**Lemma 4.1.** *For all real polynomials  $P$  and all uncorrelated sequences  $Y$*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} Y(n) \exp 2i\pi P(n) = 0.$$

**Proof.** The result is obviously true if  $\deg P = 0$ . We now argue by induction and assume the truth of the lemma for all  $P$  with  $\deg P = \nu - 1 \geq 0$ . Let  $Q$  be any polynomial of degree  $\nu$  and let  $h \geq 1$  be an arbitrary integer. Put  $f(n) = Y(n) \exp 2i\pi Q(n)$  and consider the correlation

$$\begin{aligned} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \overline{f(n)} f(n + h) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} Y(n) Y(n + h) \exp 2i\pi(Q(n + h) - Q(n)). \end{aligned}$$

The product  $Z(n) = Y(n)Y(n + h)$  is again uncorrelated and the polynomial  $P(n) = Q(n + h) - Q(n)$  is of degree  $\nu - 1$ . Therefore

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \overline{f(n)} f(n + h) = 0$$

for all  $h \geq 1$ . A classical result (see J. Bass [2]) then implies

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} f(n) = 0. \quad \blacksquare$$

We now prove Theorem 3.1. Suppose  $P(X) = \sum_{\ell=0}^{\nu} a_{\ell} X^{\ell}$  where at least one of the coefficients  $a_1, \dots, a_{\nu}$  is irrational. Consider the exponential mean

$$\begin{aligned} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} \exp 2i\pi h P(s(n)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\ell < s(N)} \chi(\ell) \exp 2i\pi h P(\ell) \end{aligned}$$

where  $h \geq 1$  is an integer, and where  $\chi$  is the characteristic function of  $s$ .

For  $\mu = \mu_d$ -almost all  $s$ ,  $s(N) \sim \frac{1}{d}N = L$ . The theorem will be established if for  $L \rightarrow \infty$

$$\frac{1}{L} \sum_{\ell < L} \chi(\ell) \exp 2i\pi h P(\ell) \rightarrow 0 .$$

The above average can be decomposed into two parts

$$\frac{1}{L} \sum_{\ell < L} (\chi(\ell) - d) \exp 2i\pi h P(\ell) + \frac{d}{L} \sum_{\ell < L} \exp 2i\pi h P(\ell) .$$

For  $\mu_d$ -almost all  $s$ ,  $\chi - d$  is uncorrelated and therefore the first average converges to 0. As for the second average, it converges to 0 because the sequence is well known to be equidistributed (mod 1) [12].

### 5. Proof of Theorem 3.2

Let  $P(X) = \sum_{\ell=0}^{\nu} a_{\ell} X^{\ell}$ ,  $a_{\nu} > 0$ , be a polynomial which takes integer values when  $X$  runs through  $\mathbb{N}$ . If  $s \in S$ ,

$$\theta^{P(s(n))} \equiv -2 \sum_{j=1}^{t-1} \cos 2\pi \omega_j P(s(n)) + o(1)$$

if  $P$  is nonconstant (if  $P$  is constant the theorem is trivial). The  $(t - 1)$  polynomials  $\omega_1 P, \dots, \omega_{t-1} P$  all have irrational coefficients. According to Theorem 3.1, the sequences  $(\omega_j P(s(n)))$  are  $\mu_d$ -almost surely equidistributed (mod 1) and more to the point, for all  $\underline{h} = (h_1, \dots, h_{t-1}) \in \mathbb{Z}^{t-1} \setminus \{0\}$  the sequences  $\underline{h} \underline{\omega} P(s)$  are equidistributed (mod 1). Here  $\underline{h} \underline{\omega} P(s)$  is the scalar product of  $\underline{h}$  and  $\underline{\omega} = (\omega_1, \dots, \omega_{t-1})$ . Therefore the  $(t - 1)$  dimensional sequence  $(\omega_1 P(s), \dots, \omega_{t-1} P(s))$  is equidistributed in  $(\mathbb{R}/\mathbb{Z})^{t-1}$  and as in the first section, we conclude that

$$\frac{1}{N} \sum_{n < N} \exp 2i\pi k P(s(n)) \xrightarrow{N \rightarrow \infty} J_0(4k\pi)^{t-1} \neq 0 .$$

### 6. A Final Remark

All our arguments are based on the fact that  $\theta^n$  is essentially a finite sum of  $\cos 2\pi \omega_j n$ . We could probably extend some of our results to the study of sequences  $u = (u(n))$  of the type

$$u(n) = \sum_{j=1}^t F(n\omega_j) .$$

**Acknowledgement.** The authors had long discussions with Vitaly Bergelson and with Bahman Saffari whom we would very much like to thank warmly.

## References

- [1] S. Akiyama and Y. Tanigawa, *Salem numbers and uniform distribution modulo 1*, Publ. Math. Debrecen **64**(3-4) (2004), 329–341.
- [2] J. Bass, *Suites uniformément denses, moyennes trigonométriques, fonctions pseudo-aléatoires*, Bull. Soc. Math. France **87** (1959), 1–64.
- [3] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J.-P. Schreiber, *Pisot and Salem numbers*, Birkhäuser Verlag, Basel 1992. With a preface by David W. Boyd.
- [4] H. Cohen, F. Rodriguez-Villegas, and D. Zagier, *Convergence acceleration of alternating series*, Experiment. Math. **9**(1) (2000), 3–12.
- [5] Y. Dupain, Répartition et discrétion. PhD thesis, Université Bordeaux I, 1978.
- [6] Y. Dupain and J. Lesca, Répartition des sous-suites d'une suite donnée. Acta Arith., **23** (1973), 307–314.
- [7] F. Hausdorff, *Momentprobleme für ein endliches Intervall*, Math. Z. **16**(1) (1923), 220–248.
- [8] J. F. Koksma, *Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins*, Compositio Math. **2** (1935), 250–258.
- [9] Ch. Pisot and R. Salem, *Distribution modulo 1 of the powers of real numbers larger than 1*, Compositio Math. **16** (1964), 164–168.
- [10] R. Salem, *Power series with integral coefficients*, Duke Math. J. **12** (1945), 153–172.
- [11] R. Salem, *Uniform distribution and capacity of sets*, Comm. Sémin. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] (Tome Supplémentaire) (1952), 193–195.
- [12] Hermann Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*. Math. Ann. **77**(3) (1916), 313–352.

**Addresses:** Christophe Doche: Division of ICS, Building E6A Room 360, Macquarie University, NSW 2109 Australia, Michel Mendès France, Jean-Jacques Ruch: Laboratoire A2X, Université Bordeaux I, 351, cours de la Libération, F-33405 Talence Cedex France

**E-mail:** doche@ics.mq.edu.au, michel.mendes-france@math.u-bordeaux1.fr, jean-jacques.ruch@math.u-bordeaux1.fr, mmf@math.u-bordeaux.fr, ruch@math.u-bordeaux.fr

**Received:** 18 December 2007; **revised:** 16 July 2008