

# GREQAM

Groupement de Recherche en Economie  
Quantitative d'Aix-Marseille - UMR-CNRS 6579  
Ecole des Hautes Etudes en Sciences Sociales  
Universités d'Aix-Marseille II et III

Document de Travail  
n°2007-26

## Credit Market Frictions and the Amplification-Persistence Trade-off

P.A. Pintus

December 2007

DT-GREQAM

# Credit Market Frictions and the Amplification-Persistence Trade-off\*

Patrick A. Pintus<sup>†</sup>

Université de la Méditerranée, GREQAM-IDEP

December 28, 2007

---

\* The author wishes to thank Marty Eichenbaum, David Levine, Jean-Charles Rochet and seminar participants at various places for useful comments and suggestions on the first draft.

<sup>†</sup> GREQAM-IDEP, Centre de la Vieille Charité, 2 rue de la Charité, 13236 Marseille Cedex 2, France. E-mail: [pintus@univmed.fr](mailto:pintus@univmed.fr).

## Abstract

The issue of how endogenous borrowing constraints lead to the amplification and persistence of aggregate shocks is revisited in this paper. Specifically, I show that an *amplification-persistence trade-off* is embodied in the setting proposed by Kiyotaki and Moore (1997). The key point is that while complex unit roots associated with persistence of temporary shocks require the fraction of credit-constrained firms to be small enough, large amplification relies on the opposite condition. Incidentally, I confirm the occurrence of periodic and quasi-periodic cycles around the determinate steady state.

Keywords: imperfect credit markets, complex unit roots, persistence and amplification, endogenous cycles.

*Journal of Economic Literature* Classification Numbers: D92, E32, E44.

# 1 Introduction

It is by now well known that endogenous borrowing constraints act to magnify aggregate shocks to the economy. I show in this paper that, embodied in such a mechanism, is an *amplification-persistence trade-off*. More specifically, I consider a slightly simplified version of Kiyotaki and Moore [8] (KM thereafter). It is proved that while the model aptly replicates any levels of amplification and persistence of temporary shocks, it predicts a negative relationship between both latter features. The key point is that while complex unit roots associated with persistence require the fraction of credit-constrained firms to be small enough, high amplification relies on the opposite condition.

I should stress that this result is directly suggested by the observation of numerical simulations performed in the working paper of KM [7, Figures 3 and 4] (see also Cordoba and Ripoll [2] for related numerical results obtained in a different setting). The contribution of this paper is to provide an analytical proof of the amplification-persistence trade-off. Besides theoretical interest, the relevance of such a result originates from its quantitative implications, as it shows how larger contemporaneous amplification due to credit frictions is associated with lower persistence in the KM's economy. Of course, the existence of such a trade-off does not imply that endogenous credit constraints are quantitatively unimportant. Rather, one should interpret this result as showing that the effects

of a temporary shock, *cumulated* over several periods, may originate from either low amplification and high persistence or the opposite configuration.

In addition to KM [7] and Cordoba and Ripoll [2], a closely related paper is Kocherlakota [9], who studies a framework with homogenous firms. A main lesson one may draw from his analysis is that, for realistic factor shares, both amplification and persistence are expected to be low. This contrasts with the main result of this paper. As one sees from numerical examples presented in Kocherlakota [9, p. 10], both amplification and persistence are small when the capital share is set at realistic values, in his model of endogenous credit constraints. Presumably, results differ in both models mainly because of firms heterogeneity, which is present in KM [7] but absent in Kocherlakota [9].

Moreover, I show that for some parameter values, the KM's economy possesses *complex unit roots*, a feature that empirical studies claim not to be absent from real-world data (see, e.g., Hylleberg *et al.* [6], Grégoir [4]) and which generates hump-shaped impulse response functions with arbitrarily large persistence. In Section 3, I provide some simulations of the model that illustrate this property. In contrast, complex unit roots do not obtain either in Kocherlakota [9], Cordoba and Ripoll [2], or in Freixas and Rochet [3, pp. 180-3], Aghion *et al.* [1], Matsuyama [10]. With the latter group of contributions, my paper shares the result that endogenous credit constraints generate business cycles that persist in the absence of any aggregate shocks to the economy. In particular, a byprod-

uct of the analysis is that the steady state loses (saddle-point) stability through a Hopf bifurcation. Therefore, I confirm the occurrence of periodic and quasi-periodic cycles around the determinate steady state, which has been conjectured by Kiyotaki and Moore [8].<sup>1</sup>

In Section 2, the main result is derived. Some simulations of the model are presented in Section 3. In Section 4, I give some concluding remarks while the final appendix collects some proofs.

## 2 The Kiyotaki-Moore Model without Depreciation

Consider a slightly simplified version of the model - with credit-constrained investment by heterogeneous firms - that has been proposed by KM [8, Section III]. Let us assume that farmers grow trees that do not depreciate over time. In KM's notation, I set  $\lambda = 1$ . This peripheral assumption is adopted for the sake of lightening the analysis. The rate of capital depreciation is bound to be close to zero when the period is short (say, a quarter), so that the  $\lambda = 1$  case is most plausible.

---

<sup>1</sup>It is worth noticing that the Hopf bifurcation does not occur in the basic Lotka-Volterra (predator-prey) model KM [8, p. 235] allude to.

From KM's eqs. (12) and (23)-(24), one learns that the dynamics of the economy with credit-constrained farmers are given by the following three equations.

**Definition 2.1 (Dynamics of the Farming Sector)**

*An intertemporal equilibrium with perfect foresight is a sequence  $(K_t, B_t, q_t)$  of  $\mathbf{R}_{++}^3$ ,  $t = 1, 2, \dots$ , such that, given some  $K_1, B_1 > 0$ :*

$$\begin{cases} K_{t+1} &= (1 - \pi)K_t + \pi[(a + q_{t+1} + \phi)K_t - RB_t]/[\phi + u(K_{t+1})] \\ B_{t+1} &= RB_t + (q_{t+1} + \phi)(K_{t+1} - K_t) - aK_t \\ q_{t+1} &= R[q_t - u(K_t)] \end{cases} \quad (1)$$

where  $u(K) \equiv G'[(\bar{K} - K)/m]/R$ .

In the dynamical system given by eqs. (1),  $\bar{K}$  is the (constant) stock of land supply,  $K$  is the stock of farmers' land,  $B$  is the stock of farmers' debt,  $q$  is the land price (per unit of fruit, the numeraire),  $1 \geq \pi \geq 0$  is the probability that an investment opportunity arises,  $R > 1$  is the gross interest rate,  $m > 0$  is the proportion of non-farmers, and the remaining parameters are such that  $a \geq 0$  and  $\phi \geq 0$ .

The last equation in (1) expresses equilibrium on the land market, while the second and third equality in (1) summarize, respectively, the law of capital accumulation and the budget constraint. KM [8, p. 233] show that eqs. (1) have a unique interior steady state  $(K^*, B^*, q^*)$ . The steady state is independent of

$\pi$ , which turns out to be our main parameter in the foregoing analysis. The parameter  $\pi$  may also be interpreted as the fraction of credit-constrained farmers.

I keep KM's assumption 2' and 3, that is, with  $\lambda = 1$ :

$$c > (1/\beta - 1)(a + \phi)[1 - \beta R(1 - \pi)]/(\beta R\pi), \text{ and } \lim_{s \rightarrow \infty} E_t(R^{-s}q_{t+s}) = 0, \quad (2)$$

where  $c \geq 0$ ,  $1 \geq \beta \geq 0$ . The conditions in (2) respectively ensure that investment dominates consumption and that exploding bubbles in the land price are ruled out. However, I allow violation of KM's assumption 5 (that is,  $\pi > 1 - 1/R$ ) by considering values of  $\pi$  that are small enough, which will be seen to be necessary for complex unit roots to occur.

Linearizing eqs. (1) at  $(K^*, B^*, q^*)$  (see the appendix in KM [7] for some details of the derivation), one gets that  $R > 1$  is an (unstable) eigenvalue while the other two eigenvalues are the roots of the following polynomial:

$$p(x) = x^2 - Tx + D, \text{ with } D = R(1 - \pi)/(1 + \theta\pi/\eta) \text{ and } T = D + 1/(1 + \theta\pi/\eta), \quad (3)$$

where  $1 \geq \theta \equiv a/(a + \phi) \geq 0$ , and  $\eta \geq 0$  is the elasticity of the residual supply of land to the farmers with respect to the user cost, evaluated at steady state (see KM [8, p. 225]). Notice that if the period is interpreted as being short (say, a quarter), then  $R$  is close to one and, therefore, the unstable eigenvalue is close to unit root. As I now show, the two roots of the above polynomial can be either stable or unstable. In fact, the steady state of eqs. (1) undergoes a Hopf bifurcation when  $\pi$  is decreased from one to zero. Note that the choice of  $\pi$  as

the bifurcating parameter is unimportant and that a similar picture would be obtained if, for instance,  $\pi$  was fixed and if  $R$  was increased from one.

**Proposition 2.1 (Local Stability of the Steady State)**

*Consider the steady state  $(K^*, B^*, q^*)$  of eqs. (1) and assume that conditions (2) are satisfied. Then the following holds:*

1. *the steady state is locally a saddle for  $1 \geq \pi > \pi_H$ , where  $\pi_H \equiv \eta(a + \phi)(R - 1)/[\eta(a + \phi)R + a]$ .*
2. *the steady state undergoes a Hopf bifurcation (two complex characteristic roots cross the unit circle) at  $\pi = \pi_H$ .*
3. *the steady state is locally a source when  $\pi_H > \pi \geq 0$ .*

*Proof:* See Appendix A.

Note that local indeterminacy is ruled out under Proposition 2.1, as there is at most as many stable eigenvalues (two) as the number of predetermined variables ( $K$  and  $B$ ). A major implication of the previous result is the following.

**Corollary 2.1 (Complex Unit Roots)**

*Under the assumptions of Proposition 2.1, the dynamics of eqs. (1) near the steady state exhibit complex unit roots when  $\pi = \pi_H \equiv \eta(a + \phi)(R - 1)/[\eta(a +$*

$\phi)R + a]$ , at frequency  $\omega = \arccos\{[2\eta + \theta(2 + 1/R)]/(2\eta + 4\theta)\}$ .

*Proof:* In Proposition 2.1, the steady state is a saddle (case 1) undergoing a Hopf bifurcation (case 2), so that the Jacobian matrix of eqs. (1) has, at steady state, a pair of complex, conjugate eigenvalues with unit modulus when  $\pi = \pi_H$ . It follows that, in case 2, the product of both eigenvalues is equal to one, that is,  $D = 1$ , and that the sum of both eigenvalues is equal to  $T = 2 \cos \omega$ , where  $\omega$  is the argument of the complex eigenvalues. Using the expressions of  $D$  and  $T$  in eqs. (3), one gets that both eigenvalues have unit modulus when  $\pi = \pi_H$  and that  $\omega = \arccos\{[2\eta + \theta(2 + 1/R)]/(2\eta + 4\theta)\}$ .  $\square$

When the length of the period is short (say, a quarter),  $R$  is expected to be not too far from one so that complex unit roots occur for small values of  $\pi_H$ . Therefore, complex unit roots occur at plausible interest rate values when the fraction of credit-constrained farmers is small enough (or, equivalently, if the average time interval between investment is large enough). In this sense, a sufficiently large level of heterogeneity is critical for complex unit roots to occur at realistic parameter values.

KM's assumption 5 (that is,  $\pi > 1 - 1/R$  or, equivalently,  $R(1 - \pi) < 1$ ) ensures that case 1 of Proposition 2.1 prevails and rules out cases 2 and 3 (see KM [7, sec. 4] for a study of the case  $\pi = 1$ , which brings one closer to the basic

model of KM [8, sec. II] where the Hopf bifurcation cannot occur).

To check the stability of the Hopf closed curve, on which lie periodic and quasi-periodic orbits, is a little demanding, as it requires computing higher-order terms in the normal form, the expression of which appears in Appendix A (see eqs. (8)). In Proposition 2.1, either Hopf orbits occur in case 1 and they are repelling, or they coexist with the unstable steady state in case 3 and they are attracting. However, even in case 1 it is important to check for the presence of *unstable* Hopf cycles because they bound the basin of attraction of the steady state, the size of which shrinks at an exponential rate when  $\pi$  tends to  $\pi_H$ . In other words, being “close” to stable unit roots (that is, when  $\pi > \pi_H$  and  $\pi \approx \pi_H$ ) may lead to nonlinear dynamics that are equivalent to the linearized dynamics only in an extremely small neighborhood. This is most important in the presence of aggregate shocks, as for example those discussed by KM. In this case, if exogenous shocks were applied to the economy, one would have to check the size of their support to ensure that the dynamics stay in the basin of attraction of the steady state.

I am now ready to state the main result establishing the amplification-persistence trade-off. *Persistence* is defined as the modulus of the complex eigenvalues, that is,  $\sqrt{D}$ . Alternatively, following KM, persistence is inversely measured by the decay rate  $\delta \equiv 1 - \sqrt{D}$ . In case 1 of Proposition 2.1, the steady state is a saddle that possesses one unstable eigenvalue (that is,  $R > 1$ ) and two, complex

eigenvalues lying inside the unit circle. Therefore, the second assumption in eqs. (2) ruling out exploding bubbles implies that the dynamics are restricted to the stable manifold of the saddle, where they exhibit damped oscillations that decay at rate  $\delta$ . Then persistence is “maximal” when complex unit roots occur, that is, when  $D = 1$  or, equivalently,  $\delta = 0$ . On the other hand, *amplification* is measured by the deviation from steady state occurring after a one-period productivity shock hits the economy, while it is at steady state. For example, let  $\Delta$  denote the (small) initial productivity shock, in percentage terms, and let  $\hat{Y} \equiv (Y - Y^*)/Y^*$  define the next-period deviation of output from steady state, where  $Y^*$  is steady state output. Then the output amplification is  $\hat{Y}/\Delta$ . Analogous definitions hold for the amplification of land price,  $q$ , and of capital  $K$ .

### **Theorem 2.1 (The Amplification-Persistence Trade-off)**

*Assume that the economy has parameter values such that case 1 of Proposition 2.1 holds. Then the dynamics of eqs. (1) near the steady state are such that both the decay rate and the amplification of a temporary, unexpected productivity shock on capital and output increase when  $\pi$  goes up from  $\pi_H$  to one.*

*In other words, there exists an amplification-persistence trade-off.*

*Proof:* From the expressions in eqs. (3), one gets that the decay rate  $\delta \equiv 1 - \sqrt{D}$  is an increasing function of  $\pi$ , as  $\sqrt{D}$ , or to put it differently, persistence, is a decreasing function of  $\pi$ . In particular, the decay rate increases from zero to one when  $\pi$  goes up from  $\pi_H$  to one. I now show that amplification of capital and output increases with  $\pi$ . In case 1 of Proposition 2.1, the steady state is a saddle that possesses one unstable eigenvalue (that is,  $R > 1$ ) and two complex, conjugate eigenvalues lying inside the unit circle. Therefore, the second assumption in eqs. (2) ruling out exploding bubbles implies that the dynamics are restricted to the stable manifold of the saddle. Denote by  $\Delta$  the (small) unexpected, one-period productivity shock, measured in percentage terms, that hits the economy while it is at steady state. On the other hand, denote by hatted variables, the percentage deviation from steady state. For example,  $\hat{q} \equiv (q - q^*)/q^*$  denotes the current deviation of  $q$ , the land price, from its steady state value  $q^*$ . Then linearizing eqs. (1) and restricting the analysis to the (linear) two-dimensional, stable manifold allows one to derive the following expressions:

$$\hat{q}/\Delta = \theta/\eta, \quad \hat{K}/\Delta = \pi\theta\{1 + \theta R/[\eta(R - 1)]\}/(1 + \pi\theta/\eta), \quad (4)$$

where  $1 \geq \theta \equiv a/(a + \phi) \geq 0$ . Moreover, output aggregated over farmers and non-farmers is given by  $Y = (a + c)K + G(\bar{K} - K)$ . At steady state, one has  $Y^* = (a + c - Ra)K^* + Ra\bar{K}$ . Therefore, one has that  $\hat{Y} = \hat{K}(a + c - Ra)/(a +$

$c - Ra + Ra\bar{K}/K^* < \hat{K}$ . Collecting all facts, one gets that:

$$\hat{Y}/\Delta = \pi\theta(a + c - Ra)\{1 + \theta R/[\eta(R - 1)]\}/\{(1 + \pi\theta/\eta)(a + c - Ra + Ra\bar{K}/K^*)\}. \quad (5)$$

Now it is straightforward to show, by using eqs. (4) that while  $\hat{q}/\Delta$  is independent of  $\pi$ , one has that  $d(\hat{K}/\Delta)/d\pi = \theta\{1 + \theta R/[\eta(R - 1)]\}/(1 + \pi\theta/\eta)^2 > 0$ . In other words, capital amplification is an increasing function of  $\pi$ . Finally, noting that both  $\bar{K}$  (by assumption) and  $K^*$  are independent of  $\pi$ , one concludes from eq. (5) and the above finding that output amplification  $\hat{Y}/\Delta$  is also an increasing function of  $\pi$ . This completes the proof of the statement that there exists an amplification-persistence trade-off, as persistence (resp. amplification) decreases (resp. increases) when the share of credit-constrained farmers  $\pi$  goes up from  $\pi_H$  to one.  $\square$

The key point behind this trade-off is that complex unit roots associated with persistence require the fraction of credit-constrained firms to be small enough, so that a low fraction of farmers invest in each period. However, high amplification relies on just the opposite condition, as it occurs only when there are many credit-constrained firms.

### 3 Simulations

So as to illustrate Theorem 2.1, I now turn to numerical examples and simulations. To ease comparison, suppose we adopt the values proposed in KM [8, p. 237], based on quarterly data. That is, we set  $R = 1.01$  (the interest rate equals 1%),  $\eta = 0.1$  (the elasticity of the residual supply of land equals 10%),  $\alpha = 1$  and  $\phi = 20$ . Consistently with the above analysis, we set  $\lambda = 1$  (no tree depreciation). Then one gets that  $\theta \approx 0.05$  and  $\pi_H \approx 0.007$ . Moreover, from the expressions given in the proof of Theorem 2.1, one gets that amplification of land price and of capital are equal to, respectively,  $\hat{q}/\Delta \approx 0.5$  and  $\hat{K}/\Delta \approx 0.015$  when  $\pi = \pi_H$ , that is, when complex unit roots prevail. Finally, we learn from the proof of Theorem 2.1 that output amplification  $\hat{Y}/\Delta$  is bounded above by capital amplification, so that  $\hat{Y}/\Delta < 0.015$ . As expected from the previous analysis, complex unit roots and persistence are associated with low amplification of capital and output. Such an example is illustrated by the time series (simulated with MATLAB) of the deviations from steady state of both land price  $q$  (top panel) and capital  $K$  (bottom panel) when  $\pi = 0.01$  and  $\Delta = 0.01$ , in Figure 1. Notice that although the effect is initially small, the cumulated impact over a long period (say, 40 quarters) is large.

Insert Figures 1 and 2 about here.

Now suppose that  $\pi = 0.99$ , so that almost all farmers invest and are credit-constrained in each period. In this case, the decay rate almost vanishes (that is,  $\delta \approx 0$ ). Then one gets that  $\hat{q}/\Delta \approx 0.5$  is unchanged while  $\hat{K}/\Delta \approx 1.58$ . Although persistence is lost, contemporaneous amplification of capital is multiplied by about 100. This configuration is depicted in Figure 2.

Comparing Figures 1 and 2 suggests that it is important to evaluate *both* amplification and persistence when drawing quantitative implications. In other words, the amplification-persistence trade-off does not imply that endogenous credit constraints are quantitatively unimportant. Rather, one should interpret this result as showing that the effects of a temporary shock, *cumulated* over several periods, may originate from either low amplification and high persistence or the opposite configuration. For example, one can hardly say that an output amplification effect of 1.5 over one period (say, a quarter) is more important than an effect of 0.5 that lasts for three periods.

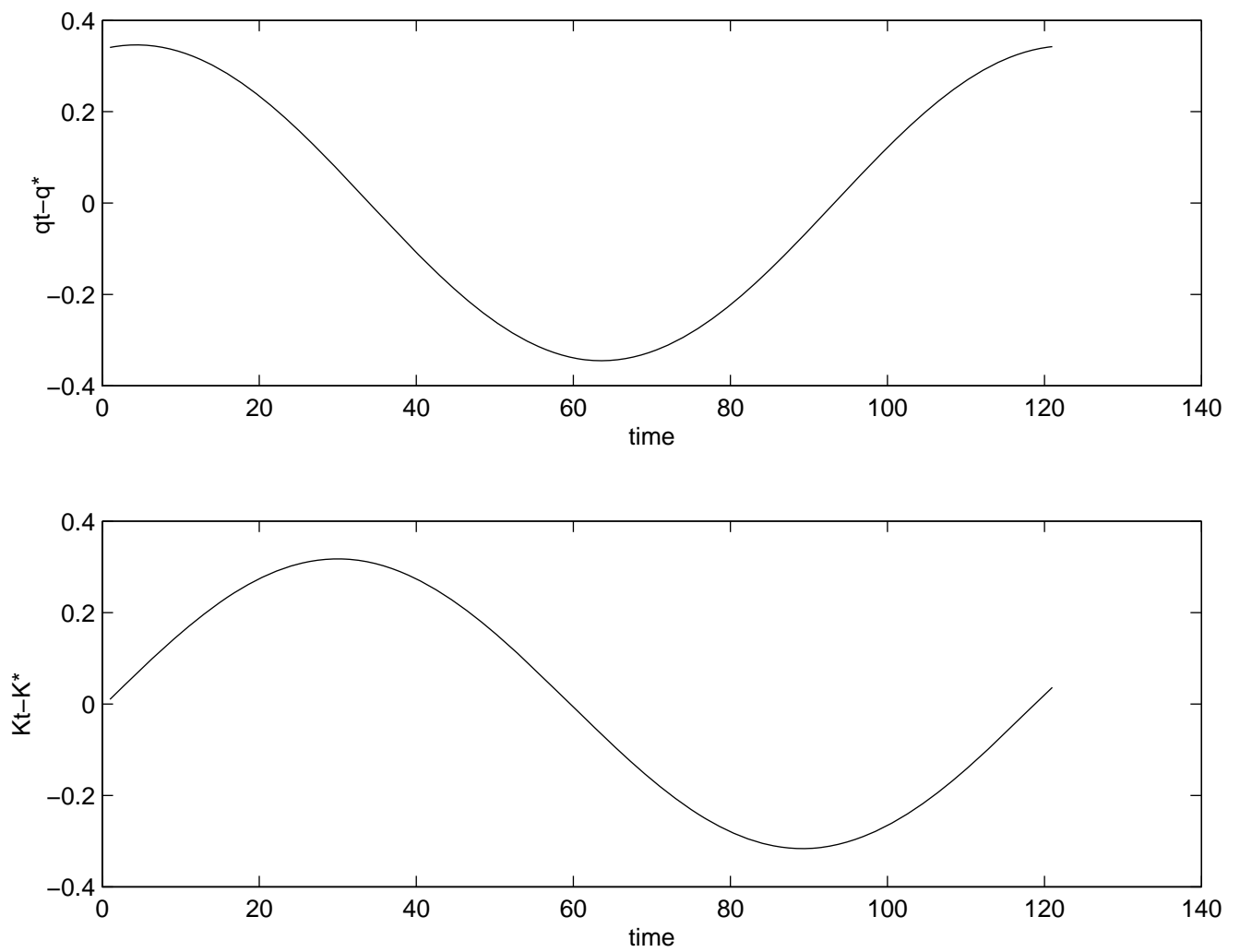


Figure 1: time series of the deviations from steady state of land price  $q$  (top panel) and capital  $K$  (bottom panel), with large persistence and low amplification ( $\pi = 0.01$ ).

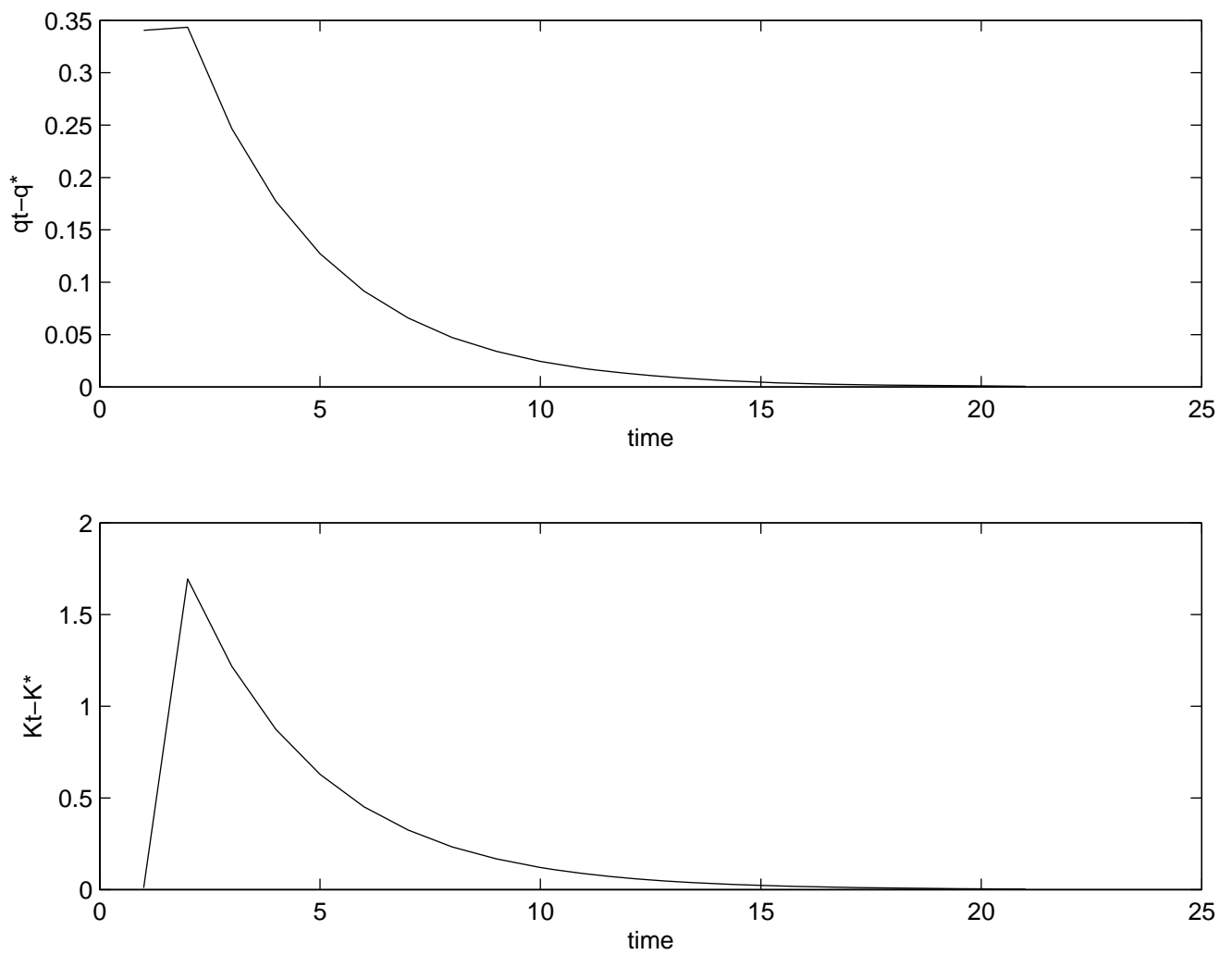


Figure 2: time series of the deviations from steady state of land price  $q$  (top panel) and capital  $K$  (bottom panel), with low persistence and large amplification ( $\pi = 0.99$ ).

## 4 Conclusion

I have considered a slightly simplified version of Kiyotaki and Moore [8] to show that, in such a setting, is embodied an amplification-persistence trade-off. In particular, close to unit-root behavior and persistence are associated with low amplification of temporary productivity shocks at impact. The key point behind this trade-off is that complex unit roots associated with persistence require the fraction of credit-constrained firms to be small enough, so that a low fraction of farmers invest in each period. However, high amplification relies on just the opposite condition, as it occurs only when there are many credit-constrained firms. Of course, the existence of such a result does not imply that endogenous credit constraints are quantitatively unimportant. Rather, one should interpret this result as showing that the effects of a temporary shock, *cumulated* over several periods, may originate from either low amplification and high persistence or the opposite configuration. As a byproduct of the analysis, it is shown that the steady state loses (saddle-point) stability through a Hopf bifurcation, which creates business cycles that persist in the absence of any shock to the economy.

Though more tedious, a similar analysis could be conducted to cover the case with tree depreciation (that is,  $\lambda < 1$ ) and would deliver analog outcomes provided that  $\lambda$  is not too small. More importantly, relaxing the linearity assumptions regarding preferences and technology is not expected to alter the main

result. As a matter of fact, Cordoba and Ripoll [2] report some numerical examples confirming that the amplification-persistence trade-off remains valid in a version of Kiyotaki and Moore [8] with concave utility and production functions. However, endogenous movements of the interest rate are expected to alter the conditions leading to complex unit roots, persistence and amplification. Finally, although I conjecture it is likely to be the case, it remains to be seen if the amplification-persistence trade-off that is underlined in this paper also holds in alternative business-cycle model with financial market frictions.

## A Proof of Proposition 2.1

In this appendix, I provide a proof of Proposition 2.1. From Kiyotaki and Moore [7, p. 33], one gets the expression of the characteristic polynomial associated with the Jacobian of eqs. (1), that is,  $P(x) = (x - R)p(x)$ , where  $p(x) = x^2 - Tx + D$  and:

$$D = R(1 - \pi)/[1 + \pi a/(\eta(a + \phi))] \text{ and } T = D + 1/[1 + \pi a/(\eta(a + \phi))], \quad (6)$$

Therefore,  $R > 1$  is an unstable eigenvalue, while the other two are either stable or unstable, as the next Lemma shows.

**Lemma A.1**

Consider the steady state  $(K^*, B^*, q^*)$  of eqs. (1) and assume that conditions (2) are satisfied. Then the following holds:

1. the steady state is locally a saddle for  $1 \geq \pi > \pi_H$ , where  $\pi_H \equiv \eta(a + \phi)(R - 1)/[\eta(a + \phi)R + a]$ .
2. the steady state is locally a source when  $\pi < \pi_H$ .

*Proof:* As already noticed, the eigenvalue  $R > 1$  is unstable, so what remains to be shown is that the two roots of  $p(x) = x^2 - Tx + D$  are both either stable or unstable. A straightforward way to proceed is analyzing how  $T$  and  $D$  vary when  $\pi$  is decreased from one to zero. That is, define:

$$D(\pi) = R(1 - \pi)/[1 + \pi a/(\eta(a + \phi))] \text{ and } T(\pi) = D(\pi) + 1/[1 + \pi a/(\eta(a + \phi))], \quad (7)$$

where  $a, \phi, \eta \geq 0$  and  $R > 1$ . One easily gets that  $D(\pi)$  is a strictly decreasing function of  $\pi$ , with  $D(1) = 0$  and  $D(0) = R > 1$ . Therefore, there exists a unique  $1 > \pi_H > 0$  such that  $D(\pi_H) = 1$ , where  $\pi_H \equiv \eta(a + \phi)(R - 1)/[\eta(a + \phi)R + a]$ . Moreover, one has that  $0 < T(1) < 1$ ,  $0 < T(\pi_H) < 2$ ,  $T(0) = 1 + D(0)$  and  $0 < T(\pi) < 1 + D(\pi)$  for all  $0 < \pi < 1$ . This proves that  $p(x)$  has two stable (resp. unstable) roots if  $1 > \pi > \pi_H$  (resp.  $\pi < \pi_H$ ).  $\square$

The last steps consist in showing that the steady state undergoes a Hopf bifurcation at  $\pi = \pi_H$  and this requires appealing to the Center Manifold Theorem for Maps and to the Hopf Bifurcation Theorem for Maps. An application of center manifolds theory helps to reduce the dimension of the dynamics to the number of eigenvalues crossing the unit circle - two in our case. Consider  $F : N \times I \rightarrow \mathbf{R}^3$  a family of difference equations, obtained from eqs. (1) after translating the steady state to the origin, where  $N$  is an open set of  $\mathbf{R}^3$  containing the origin,  $I$  is an open interval of  $\mathbf{R}$  containing  $\pi_H$ ,  $F$  is  $C^r$  with  $r \geq 2$ . Define  $A_\pi$  as the Jacobian matrix of  $F$  evaluated at the steady state. From the proof of Lemma A.1, one knows that  $A_{\pi_H}$  has complex eigenvalues with modulus equals to one. By a suitable linear change of variables,  $A_{\pi_H}$  can be brought into real canonical form, that is,  $A_{\pi_H} = \text{diag} \{C, U\}$ , with  $C$  corresponding to the (two-dimensional) center eigenspace  $E^c$  and  $U$  corresponding to the (one-dimensional) unstable eigenspace  $E^u$ . Defining  $X$  as an element of  $\mathbf{R}^3$ , it can be written as  $X^c + X^u$ ,  $X^c$  in  $E^c$  and  $X^u$  in  $E^u$ . Therefore, analyzing the local bifurcation of  $F$  is equivalent to studying the local behavior of the following map:

$$\begin{cases} X^c \mapsto CX^c + G(X^c, X^u) \\ X^u \mapsto UX^u + H(X^c, X^u), \end{cases} \quad (X^c, X^u) \in \mathbf{R}^2 \times \mathbf{R}, \quad (8)$$

where the  $C^r$  functions  $G$  and  $H$ , with  $r \geq 2$ , vanish at the steady state, for all  $\pi$ , and have zero partial derivatives at the steady state when  $\pi = \pi_H$ . I can then apply the following theorem.

**Theorem A.1 (Center Manifold Theorem)** (*Wiggins [11, p. 205]*)

*There exists a  $C^r$  center manifold for (8) which can be locally represented as a graph as follows*

$$W^c(0) = \{(X^c, X^u) \in \mathbf{R}^2 \times \mathbf{R} \mid X^u = h(X^c), |X^c| < \delta, h(0) = 0, Dh(0) = 0\} \quad (9)$$

*for  $\delta$  sufficiently small. Moreover, the dynamics of (8) restricted to the center manifold is, for  $u$  sufficiently small, given by the two-dimensional map*

$$u \mapsto Cu + G(u, h(u)), \quad u \in \mathbf{R}^2. \quad (10)$$

Appealing to two additional theorems (e.g. Wiggins [11, Theo. 2.1.4, 2.1.5], one can further deduce both that the zero solution of (10) is stable (resp. unstable) when the zero solution of (8) is stable (resp. unstable), and how to compute the center manifold from  $h$ . Next, I state the final theorem.

**Theorem A.2 (Hopf Bifurcation Theorem)** (*Guckenheimer and Holmes [5, p. 162]*)

*Let  $f_\pi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be a one-parameter family of mappings which has a smooth family of fixed points at which the eigenvalues are complex conjugate,  $x(\pi)$ ,  $\bar{x}(\pi)$ .*

*Assume:*

$$|x(\pi)| = 1 \text{ but } x^j(\pi_H) \neq 1, \text{ for } j = 1, 2, 3, 4. \quad (11)$$

$$d|x(\pi_H)|/d\pi \equiv d \neq 0. \quad (12)$$

Then there is a smooth change of coordinates  $g$  so that the expression of  $gf_\pi g^{-1}$  in polar coordinates has the form

$$gf_\pi g^{-1}(r, \nu) = (r(1 + d(\pi - \pi_H) + kr^2), \nu + m + nr^2) + \text{high-order terms} \quad (13)$$

If, in addition,  $k \neq 0$ , then there is a two-dimensional surface  $\Sigma$  in  $\mathbf{R}^2 \times \mathbf{R}$  having quadratic tangency with the plane  $\mathbf{R}^2 \times \{\pi_H\}$  which is invariant for  $f_\pi$ . If  $\Sigma \cap \mathbf{R}^2 \times \{\pi_H\}$  is larger than a point, then it is a simple closed curve.

*Proof:* From the proof of Lemma A.1, one can check that condition (11) is met. That is, an eigenvalue equals to  $-1$  is ruled out because  $T > 0$  when  $D = 1$ . An eigenvalue equals to  $1$  is ruled out because  $T < 2$  when  $D = 1$ . An eigenvalue equals to  $(1 + i\sqrt{3})/2$  is ruled out because  $T > 1$  when  $D = 1$ . Finally, an eigenvalue equals to  $i$  is ruled out because  $T > 0$  when  $D = 1$ . Moreover, there exists a neighborhood  $N$  of  $\pi_H$  and a uniquely defined, differentiable function  $|x_H(\pi)|$  defined on  $N$  such that  $|x_H(\pi_H)| = 1$  and  $d|x_H(\pi)|/d\pi < 0$ . The last inequality is obtained from the fact that  $|x| = \sqrt{D}$ , which implies  $d|x_H(\pi)|/d\pi = 0.5[D(\pi)]^{-3/2}D'(\pi) < 0$  for all  $\pi$ , which proves that condition (12) holds. Finally,  $k \neq 0$  is generically satisfied by our parameterized family of maps. □

This completes the proof of Proposition 2.1. □

## References

- [1] Aghion, P., Banerjee, A., Piketty, T., 1999. Dualism and macroeconomic volatility. *Quarterly Journal of Economics* 114, 1359-97.
- [2] Cordoba, J.-C., Ripoll, M., 2004. Credit cycles redux. *International Economic Review* 45, 1011-46.
- [3] Freixas, X., Rochet, J.-C., 1997. *Microeconomics of banking*. MIT Press, Cambridge.
- [4] Grégoir, S., 2006. Efficient tests for the presence of a pair of complex conjugate unit roots in real time series. *Journal of Econometrics* 130, 45-100.
- [5] Guckenheimer, J., Holmes, P., 1983. *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*. Springer-Verlag, New-York.
- [6] Hylleberg, S., Engle, R.F., Granger, C.W.J., Yoo, B.S., 1990. Seasonal integration and cointegration. *Journal of Econometrics* 44, 215-38.
- [7] Kiyotaki, N., Moore, J., 1995. Credit cycles. NBER working paper series n. 5083.
- [8] Kiyotaki, N., Moore, J., 1997. Credit cycles. *Journal of Political Economy* 105, 211-48.
- [9] Kocherlakota, N., 2000. Creating business cycles through credit constraints. *Federal Reserve Bank of Minneapolis Quarterly Review* 24, 2-10.
- [10] Matsuyama, K., 2007. Credit traps and credit cycles. *American Economic Review* 97, 503-16.
- [11] Wiggins, S., 1990. *Introduction to applied nonlinear dynamical systems and chaos*. Texts in Applied Mathematics n. 2. Springer-Verlag, New-York.